Formulation of Approximate Equations for Modeling Moist Deep Convection on the Mesoscale

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ABSTRACT

The aim of this report is to formulate a set of approximate equations which can be applied to the numerical modeling of precipitating, deep, convective, mesoscale systems in the atmosphere.

In Chapter 2, a set of approximate equations describing deep convection in a moist atmosphere is derived. Liquid water is parameterized into only two categories: cloud water and rain. It is shown that the natural dependent variable appearing in the equation for the conservation of energy is the potential temperature θ_L, introduced by Betts (1973).

In Chapter 3, dimensional arguments are used to obtain a parameterization scheme for the precipitation in mesoscale convective models. By following Kessler (1967), the liquid water is divided into two categories: cloud water and rain. The physical basis of the Marshall-Palmer spectrum is discussed in order to predict the parameters associated with rain drops. The parameterization is consistent with the turbulent nature of the motion in clouds and it is valid in deep convection.

In Chapter 4, the equations derived in Chapter 2 and 3 are averaged, and a system of equations describing the mean behavior of a precipitating, deep, convective mesoscale system is developed. A second-order closure approximation is made for turbulence quantities so that equations are obtained for the first two moments of the dependent variables.

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LIST OF SYMBOLS - ENGLISH

C	Drag coefficient
C_p	Specific heat at constant pressure of mixture of dry air and water vapor
C_{pa}	Specific heat at constant pressure for dry air
C_{pv}	Specific heat at constant pressure for water vapor
C_w	Specific heat of liquid water
D	Effects of molecular diffusion on entropy change of air
E_c	Mean collection efficiency
E_{cr}	Effective collection efficiency between a raindrop and cloud droplets
e	Partial pressure due to water vapor
e_s	Saturation vapor pressure
e_{sr}	A constant used in 4.29
F	Ventilation coefficient
f	Coriolis frequency vector
f_c	Mean collision frequency for cloud droplets which become raindrops after colliding
g	Acceleration due to gravity
$H(x)$	Heaviside unit step function
h	Function defined in 4.26
K	Thermal conductivity of air
L	Latent heat of condensation
M	Downward flux of rain relative to air
$\dot{m}_a$	Rate of increase of mass of a single raindrop by collection of cloud water
m_i	Momentum vector defined in 2.29
$\underset{\sim}{m}$	Momentum vector

N_c	Mean cloud droplet concentration
N_r	Raindrop concentration
P	Total atmospheric pressure
P_o	Reference state pressure
P_r	1000 mb reference pressure
p_s	Saturation vapor pressure
Q	Effect of relative motion between air and rain on r
$\overline{q}^2$	Turbulent intensity
q_{so}	Reference state saturation mixing ratio
R	Gas constant for dry air and water vapor mixture
R_a	Gas constant for dry air
R_c	Rate of condensation of water
R_e	Reynolds number
R_r	Rate of change of rain density due to motion of the rain relative to the air
R_v	Gas constant for water vapor
r	Potential density
r_b	Maximum drop radius
r_c	Mean cloud droplet radius
r_{cm}	Mean droplet radius corresponding to minimum water density ρ_{cm} and concentration N_c
r_m	Characteristic radius associated with the breakup of raindrops
S	Entropy of the gas
S_a	Rate of increase of rain due to the accretion of rain with cloud water
S_c	Rate of conversion of cloud water into rain
S_e	Rate of evaporation of rain
T	Temperature

T_i	Time scale defined in 2.16
T_o	Reference state temperature
T_s	A constant used in 2.11
t	Time
$\underset{\sim}{u}$	Velocity vector
V_b	Terminal velocity of a raindrop
V_c	Mean relative terminal velocity of droplets
V_j	Velocity of air
x_i=1,2,3	Cartesian coordinates

LIST OF SYMBOLS - GREEK

α_4	Parameter defined in 2.18
α_s	A constant in 2.11
ε	Rate of dissipation of turbulent energy per unit mass
γ	Molecular diffusivity for rainwater
γ_m	Molecular diffusivity for momentum
γ_o	Surface tension of water
γ_r	Effective molecular diffusivity of rain
γ_t	Molecular diffusivity for total water substance
γ_θ	Molecular diffusivity for heat
μ	Viscosity of air
μ_o	Viscosity of air defined with respect to base state
ρ	Air density
ρ_a	Dry air density
ρ_c	Cloud water density
ρ_c'	Perturbation cloud liquid water density
ρ_{cm}	Minimum mean water density below which there is no conversion
ρ_o	Reference state density
ρ_r	Rain water density
ρ_r'	Perturbation rainwater density
ρ_s	Saturation vapor density
ρ_t	Total water density
ρ_t'	Perturbation total water density
ρ_v	Water vapor density
ρ_w	Total liquid water density
ρ_{wt}	Density of water

σ	Supersaturation of the ambient air
σ_c	Standard deviation of cloud liquid water density
τ_i	Time scale defined in 2.17
θ	Potential temperature
θ_L	Liquid water potential temperature
$\theta_{(r)}$	Raindrop spectral density
ν_m	Molecular diffusivity of momentum
ν_r	Molecular diffusivity of rainwater density
ν_t	Molecular diffusivity of total water density
ν_θ	Molecular diffusivity of heat
χ	Average saturation excess or deficit

1. INTRODUCTION

One of the most difficult meteorological systems to numerically model is the moist, deep, convective, mesoscale system. A numerical model of such a system should simulate the processes of condensation on cloud droplets, formation of precipitation by warm-cloud and ice-phase processes, the time and spatial distribution of total water substance including its redistribution by precipitation and evaporation of precipitation in the sub-cloud layer. These processes influence the dynamics of a mesoscale convective system through the latent heat of condensation and freezing and through water loading, by altering the buoyancy field of the mesoscale convective system. Furthermore, the time scale of this system is often sufficiently long (two to six hours) for radiational processes to be important.

A mesoscale convective system frequently evolves from a turbulent planetary boundary layer to a moist cumulus layer, to precipitating cumulus congestus, to isolated cumulonimbi, and perhaps ultimately to precipitating squall lines; these, in turn, interact with regional and synoptic scale meteorological systems. Thus a broad range of meteorological eddies should be considered. Even the most advanced-level computers, however, permit the explicit representation by numerical prediction techniques of eddies over a scale range of only two orders of magnitude. Thus, for example, even the next generation computers, such as the NCAR/CRAY system, will only permit an explicit representation of mesoscale convective systems over a horizontal-scale range of 1.0 to 100.0 km.

The major problems in modeling cumulus/mesoscale (horizontal scales of 1.0 to 100.0 km) systems are thus assocaited with interfacing model predictions on these scales with regional- and synoptic-scale meteorological systems and parameterization of cloud microphysical processes and cumulus-scale eddies (horizontal scales of 0.01 to 1.0 km) on the cumulus/meso-scale. The major thrust of this report is toward the latter, namely: the parameterization of cumulus - and sub-cumulus-scale eddies and the parameterization of cloud microphysical processes.

In Chapter 2 we discuss the general formulation of an approximate set of momentum, thermodynamic energy, and water continuity equations. The objective is to formulate a system with as few dependent variables as possible and to choose thermodynamic variables which minimize both the computational effort and the likelihood of obtaining errors in the numerical evaluation of the thermodynamic energy equation.

In Chapter 3, a simple, first-order warm-cloud precipitation parameterization is derived. The derived system has the virtue of being consistent to a first-order level throughout and, moreover, can be easily adapted to the second-order turbulence theory discussed in Chapter 4.

In Chapter 4, the system of equations is averaged and a second-order turbulence model is derived. The intended application of the turbulence model is to the parameterization of thermal- and cumulus-scale eddies (0.01 to 1.0 km) which may develop from a planetary boundary layer mesoscale system or be imbedded in a cumulonimbus/mesoscale convective system. The turbulence theory represents the first such theory in which turbulent energy and eddy momentum transport can be generated by cloud diabatic processes (condensation and evaporation) as well as the loading of cloud condensate.

2. GENERAL FORMULATION

2.1 Introduction

There has been a considerable amount of work on the development of a consistent approximation scheme which models the variations of the thermodynamic variables in the atmosphere in a reasonably realistic and yet computationally simple manner. Particular attention has been given to the elimination of unnecessary sound waves from models of convection (Ogura and Phillips, 1962). Dutton and Fichtl (1969) also analyze the momentum equations and the energy equation for the behavior of a dry atmosphere. The present work follows that of Dutton and Fichtl in order to derive a consistent set of equations which is applicable to deep convection in a moist atmosphere. Ice is not included explicitly in the parameterization of precipitation, restricting the precipitation to supercooled rain. With the development of an appropriate parameterization, however, the ice phase could be included in the equations in a manner similar to that of rain.

2.2 Decomposition of thermodynamic variables

We consider the motion in a three-dimensional region using the Cartesian co-ordinates (x_1, x_2, x_3) where x_3 increases vertically upward. Each thermodynamic variable is decomposed into its reference state (denoted by the subscript o) and a deviation from the reference state (denoted by a prime). The reference state variables are functions of x_3 only. Thus the pressure, temperature and density at any point are respectively

$$p = p_o(x_3) + p'(\underset{\sim}{x}, t); \quad \rho = \rho_o(x_3) + \rho'(\underset{\sim}{x}, t);$$
$$T = T_o(x_3) + T'(\underset{\sim}{x}, t) \; ; \qquad (2.1)$$

where t is time. The fundamental assumption of the analysis is that any induced deviation from a reference state is small; for example

$$|\rho'/\rho_o| << 1. \tag{2.2}$$

Consequently Dutton and Fichtl (1969) always neglect the ratio of a deviation to a reference state quantity when compared with unity. This is equivalent to ensuring that only the leading order terms in the deviation quantities are retained in the (ρ_c, ρ_r) model equations. There is no reference state for condensate water, however, because fluctuations in water densities are comparable with their overall magnitude.

In order to satisfy (2.2), the reference state pressure is taken to be hydrostatic, i.e.

$$\partial p_o/\partial x_3 = - \rho_o g \tag{2.3}$$

where g is the gravitational acceleration. We further assume that the air can be treated as a perfect gas and that the reference state air is dry; thus

$$p_o = R_a \rho_o T_o \tag{2.4}$$

where R_a is the gas constant for dry air. By specifying the reference state of the temperature,

$$T_o = T_o(x_3) \ , \tag{2.5}$$

and the surface pressure p_o prior to the formation of an active disturbance, equations (2.3) - (2.4) can be used to find p_o and ρ_o. Generally only a small error will be introduced by taking p_o to be the initial observed pressure field (which includes the effects of water vapor) rather than calculating it formally. The function T_0 (x_3) could be determined from a suitable sounding of the atmosphere. We note, however, that the use of explicit functional forms (polynomials, say) for the reference state variables and their derivatives would result in some convenience in the numerical computation of the behavior of the disturbed atmosphere.

2.3 Potential temperature

The potential temperature θ is defined by

$$\theta = T(p_r/p)^{R_a/C_{pa}} , \qquad (2.6)$$

where P_r = 1000 mb and C_{pa} is the specific heat at constant pressure for dry air. The reference state of θ is given from (2.3) - (2.5) and (2.6) as

$$\theta_o = T_o(p_r/p_o)^{R_a/C_{pa}}. \qquad (2.7)$$

It is convenient to use θ rather than the absolute temperature as a variable in the model equations. Thus, to the first order in deviation quantities, we may write

$$T'/T_o = \theta'/\theta_o + (R_a/C_{pa})p'/p_o . \qquad (2.8)$$

Because deep convection is considered, the pressure deviation must be included in (2.8) (Dutton and Fichtl, 1969).

2.4 Saturation vapor density

When considering the conservation of water in Section 2.6 below, it is necessary to calculate the saturation vapor density ρ_s for the atmosphere. This is related to the saturation vapor pressure e_s by the equation of state

$$e_s = R_v \rho_s T \tag{2.9}$$

where R_v is the gas constant for water vapor. The latent heat of vaporization L is related to e_s by the Clausius-Clapeyron equation

$$de_s/dT = Le_s/(R_v T^2) \; . \tag{2.10}$$

Taking L to be constant, we find from (2.9) - (2.10) that to the first order in T'/T_o

$$\rho_s = \rho_{so} \{1 + (L/(R_v T_o) - 1)(T'/T_o)\} \tag{2.11}$$

where $\rho_{so} = e_{so}/R_v T_o(x_3)$,

and $$e_{so} = e_{sr} \exp\{\alpha_s \; [1 - T_s/T_o(x_3)]\} \; .$$

The constants e_{sr}, α_s, T_s [$\alpha_s = L/(R_v T_s)$] are chosen to give a good approximation to the observed values of e_{so}. Although the method of approximation is to neglect the ratio of a deviation to a reference state quantity when compared with unity, the term in T'/T_o is retained formally in (2.11) because $L/R_v T_o$ is of order 20 rather than unity. This is also consistent with the policy of retaining no more than the leading terms in deviation quantities. Ogura and Phillips (1962) assert that the higher order terms are required in (2.11) - in fact they take the exponential of T'/T_o. However, even for deep convective storms, where the deviation term in (2.11) may become as large as 1%, the present representation ought to be adequate. It must be recognized, however, that the limits of the above approximations may be approached in the tops of severe storms where Sinclair (1973) reported values of $|T'/T_o|$ exceeding 5%.

With $L = 2.5154 \times 10^{10}$ erg gm^{-1}, $P_s = 12.272$ mb, $\alpha_s = 19.2484$ and $T_s = 283.16$ K, there is less than 1.7% error in the expression for p_{so} used in (2.11) over the range $-30^oC \le T_o \le 20^oC$; L varies by less than 2.5% over this range.

Using (2.8) to eliminate the absolute temperature, we obtain from (2.11) the saturation vapor density correct to the first order in deviation quantities:

$$\rho_s = \rho_{so} \{1 + (L/R_v T_o - 1)[\theta'/\theta_o + (R_a/C_{pa})p'/p_o]\} \ . \qquad (2.12)$$

2.5 Equations of motion

If u_i is the velocity in the x_i-direction then Dutton and Fichtl (1969) show that the equation of mass conservation for deep convection reduces to

$$\partial(\rho_o u_j)/\partial x_j = 0 \ . \qquad (2.13)$$

The full equation for the conservation of momentum is

$$\rho du_i/dt - \rho\varepsilon_{ijk} f_k u_j + \partial p'/\partial x_i + \rho' g \delta_{i3} = \mu \{\nabla^2 u_i + \frac{1}{3} \partial^2 u_j/\partial x_i \partial x_j\} \ , \qquad (2.14)$$

where $\underset{\sim}{f}$ is the Coriolis frequency vector and μ is the viscosity of the air. Equation (2.3) has been used to eliminate the contribution of the reference state hydrostatic pressure. The neglect of ρ'/ρ_o compared with unity implies that ρ can be replaced by ρ_o in the first two terms of (2.14).

Dutton and Fichtl (1969) assert that the full contribution of the viscous terms ought to be retained for deep convection. However, u_i

can be decomposed into a mean part and a random fluctuation about the mean. If we apply the Dutton and Fichtl analysis to the formulation of a deep convection model with explicit vertical resolution of 2.0 km or greater, then the vertical length scale of the predicted mean corresponds to deep convection, while that of the fluctuating component corresponds to shallow convection. Thus the turbulent fluctuating velocity is incompressible. On the other hand, the Reynolds number of the mean motion is so large that the effect of molecular diffusion on it can be neglected. Since molecular diffusion will only affect the turbulent fluctuating part of the motion field which is assumed incompressible, it is therefore consistent to ignore the contribution of the $1/3\ \partial^2 u_j/\partial x_i x_j$ term in (2.14). The momentum equation for the total flow for deep convection may therefore be written in the form

$$du_i/dt - \varepsilon_{ijk} f_k\ u_j + \rho_o^{-1}\ \partial p'/\partial x_i + (\rho'/\rho_o) g\ \delta_{i3} = \nu_m \nabla^2 u_i\ , \tag{2.15}$$

where ν_m is the molecular diffusivity of momentum. The equations for the mean flow will be developed in Part II.

2.6 Conservation of water

The total water density ρ_t' in the atmosphere consists of the water vapor density ρ_v' together with the total liquid water density ρ_w'; thus

$$\rho_t' = \rho_v' + \rho_w'\ . \tag{2.16}$$

Primed quantities are used to signify that these water densities are deviation quantities because the reference state of the atmosphere is dry. (If ice is present then it contributes to ρ_w'.) Because of the complexity of calculating the complete development of the water droplet spectral density, it is usual (following Kessler, 1967) to parameterize the precipitation process by decomposing the total liquid water into cloud water (ρ_c') and rain (ρ_r') such that

$$\rho_w' = \rho_c' + \rho_r' , \tag{2.17}$$

where the cloud water is assumed to move with the air while rain falls at an appropriate terminal velocity relative to the air. The amount of supersaturation in clouds is observed to be small. Hence we assume that any water vapor in excess of the saturation vapor density condenses instantaneously into cloud water; i.e.

$$\rho_c' = (\rho_t' - \rho_r' - \rho_s)\ H\ (\rho_t' - \rho_r' - \rho_s) \tag{2.18}$$

where H(x) is the Heaviside unit step function.

The equations for the conservation of total water and rain are readily found to be

$$\partial\rho_t'/\partial t + \partial(u_j\rho_t')/\partial x_j - \partial M/\partial x_3 = \nu_t\ \nabla^2\rho_t' , \tag{2.19}$$

$$\partial\rho_r'/\partial t + \partial(u_j\rho_r')/\partial x_j - \partial M/\partial x_3 = \nu_r\ \nabla^2\rho_r' + S_c + S_a - S_e , \tag{2.20}$$

where ν_t and ν_r are the effective molecular diffusivities of total water and rain, respectively M is the downward flux of rain relative to the air; S_c is the rate of conversion of cloud water into rain ("autoconversion"); S_a is the rate of increase of rain due to the accretion of rain with cloud water; S_e is the rate of evaporation of rain. The last four quantities must be determined from some parameterization of the precipitation process. In any parameterization scheme, these terms are given as functions of the ensemble or grid point average of ρ_r', ρ_c', ρ_v' and ρ_s together with some reference state variables. We note that ρ_v' and ρ_w'

can be eliminated from (2.18) - (2.20) so that the conservation of water can be described in terms of the three variables ρ_t', ρ_c' and ρ_r'.

2.7 Equation of state

The constituent gases of the atmosphere are assumed to be two perfect gases: dry air and water vapor. If p_a and p_v' are the respective partial pressures of dry air and water vapor then the equations of state for the gases are

$$p_a = R_a \rho_a T \quad \text{and} \quad e_v' = R_v \rho_v' T , \tag{2.21}$$

where $\rho_a = \rho - \rho_t'$ is the dry air density. Thus the equation of state for the combination of the gases is

$$p = (\rho - \rho_t') R_a T \{1 + (R_v/R_a) \rho_v' / (\rho - \rho_t')\} . \tag{2.22}$$

Using (2.1), (2.4), (2.8), (2.16) and (2.17), we find that to the first order in deviation quantities equation (2.22) yields

$$\rho'/\rho_o = (1 - R_a/C_{pa})(p'/p_o) - \theta'/\theta_o - (R_v/R_a - 1)(\rho_t'/\rho_o) + (R_v/R_a)(\rho_c' + \rho_r')/\rho_o . \tag{2.23}$$

Equation (2.23) allows the density deviation, which is required in the momentum equation (2.15), to be calculated.

It should be noted that the density deviation ρ' is the total density deviation including air, water vapor and condensate. It is assumed that the condensate is falling at terminal velocity and, therefore, transmitting its weight to the air.

2.8 Thermodynamic energy equation

The application of the first law of thermodynamics to a unit mass of dry air containing water vapor and condensate gives

$$\{C_{pa}dT - (R_aT/p_a)dp_a\} + (\rho_v'/\rho_a)\{C_{pv}dT - (R_vT/p_v')dp_v'\}$$
$$+ (\rho_w'/\rho_a)C_wdT = TdS \tag{2.24}$$

where C_{pv} is the specific heat at constant pressure for water vapor; C_w is the specific heat of liquid water; S is the entropy of the gas. It is assumed in (2.24) that the gases and liquid water are in thermal equilibrium. The entropy of the system is increased by the condensation of water vapor and by the molecular diffusion of heat into the system. We therefore have

$$dS/dt = (L/T)\ \{d(\rho_w'/\rho_a)/dt - Q\} + D, \tag{2.25}$$

where Q is the rate of change of the rain mixing ratio due to the rain falling relative to the air, and D represents the effects of molecular diffusion. The function Q is determined from the precipitation parameterization.

Equation (2.21), (2.24) and (2.25) can be manipulated readily to yield

$$(C_p/T)dT/dt - (R/p)dp/dt - d(L\rho_w'/T\rho_a)/dt + LQ/T = D, \tag{2.26}$$

where $C_p = C_{pa}\ \{1 + (C_{pv}/C_{pa})(\rho_v'/\rho_a) - (L/C_{pa}T - C_w/C_{pa})(\rho_w'/\rho_a)\}$,

$$R = R_a\ \{1 + (R_v/R_a)(\rho_v'/\rho_a)\}.$$

Now both C_{pv}/C_{pa} and R_v/R_a are of order 2 while $(L/C_{pa}T - C_w/C_{pa})$ is of order 5. Moreover, the two deviation terms in the expression for C_p are of opposite sign. We therefore neglect the deviation terms in C_p and R because they are small compared with unity. Also we have

$$T^{-1}dT = d(\ell nT) = d(\ell n(T_o+T'))$$
$$= d(\ell nT_o + \ell n(1+T'/T_o))$$
$$= d(\ell nT_o) + d(T'/T_o)$$

to the first order in the deviation. Thus, by using (2.7), (2.8) and (2.17), neglecting second order terms in the deviations and including molecular diffusion explicitly, the equation for the conservation of energy in deep convection reduces to

$$d(\theta_L'/\theta_o)/dt + u_3\partial(\ell n\theta_o)/\partial x_3 + (L/C_{pa}T_o)Q = \nu_\theta\nabla^2(\theta_L'/\theta_o) \tag{2.27}$$

where $$\theta'/\theta_o = \theta_L'/\theta_o + (L/C_{pa}T_o)(\rho_c'+\rho_r')/\rho_o , \tag{2.28}$$

and ν_θ is the effective molecular diffusivity of heat. We note that the temperature θ_L, first introduced by Betts (1973), is in fact the naturally occuring dependent variable in the energy equation. If ice were included in the system then additional terms, involving the latent heats of fusion and sublimation, must be included in (2.25). These terms would in turn produce extra terms in (2.28); that is, θ_L would be generalized.

2.9 Conclusion

We now have a consistent set of equations which ought to describe validly the behavior of a moist atmosphere in deep convection. For three-dimensional motion, there are seven differential equations [namely (2.13), (2.15), (2.19), (2.20) and (2.27)] for the dependent variables $\underset{\sim}{u}$, p', ρ_t', ρ_r' and θ_L'. The cloud water density ρ_c' is specified by the functional equation (2.18), while the secondary variables ρ_s, ρ' and θ'

are related linearly to the primary dependent variables by equations (2.12), (2.23), and (2.28).

The differential equations represent the conservation of dynamic variables if we introduce the variables

$$m_i = \rho_o u_i \quad \text{and} \quad r' = \rho_c \theta_l'/\theta_o \ . \tag{2.29}$$

It is seen that $\underset{\sim}{m}$ is the momentum vector while r is a potential density. Then the complete set of model equations is

$$\partial m_j/\partial x_j = 0 \ ,$$

$$\partial m_i/\partial t + \partial(u_j m_i)/\partial x_j + \partial p'/\partial x_i + \rho' g \, \delta_{i3} = \rho_o \nu_m \nabla^2 u_i \ ,$$

$$\partial r'/\partial t + \partial(u_j r')/\partial x_j + m_3 \partial(\ln\theta_o)/\partial x_3$$

$$+ (L/C_{pa}T_o) \, \rho_o Q = \nu_\theta \nabla^2 r' \ , \tag{2.30}$$

$$\partial \rho_t'/\partial t + \partial(u_j \rho_t')/\partial x_j - \partial M/\partial x_3 = \nu_t \nabla^2 \rho_t' \ ,$$

$$\partial \rho_r'/\partial t + \partial(u_j \rho_r')/\partial x_j - \partial M/\partial x_3 = \nu_r \nabla^2 \rho_r' + S_c + S_a - S_e \ ,$$

$$\rho_c' = (\rho_t' - \rho_r' - \rho_s)H(\rho_t' - \rho_r' - \rho_s) \ ,$$

with the secondary variables $\underset{\sim}{u}$, ρ_s and ρ' given in terms of the primary variables by

$$u_i = m_i/\rho_o \ ,$$

$$\rho_s = (\rho_{so}/\rho_o) \{\rho_o + (L/R_v T_o - 1) [r' + (p'/C_{pa}T_o) \tag{2.31}$$

$$+ (L/C_{pa}T_o)(\rho_c' + \rho_r')] \} \ ,$$

$$\rho' = (1 - R_a/C_{pa})(p'/R_aT_o) - r' - (R_v/R_a - 1)\rho_t'$$

$$- (L/C_{pa}T_o - R_v/R_a)(\rho_c' + \rho_r') .$$

Any other thermodynamic variable can be derived from this set of equations.

The dependent variables in these model equations are random functions of time and space because the motion is invariably turbulent. Any solution of the system clearly involves some averaging process (for example, an average over a finite-difference mesh or an ensemble average). In Chapter 4 of this work, we therefore consider a second-order closure approximation scheme for the first two moments of the dependent variables.

3. PARAMETERIZATION OF RAIN

3.1 Introduction

With the development of rather sophisitcated numerical models of the dynamics of the atmosphere (for example, Deardorff, 1974a,b and Chapter 4 of this paper), there is a need for precipitation parameterization schemes which are compatible with such models. Computation time and storage limitations appear at present to preclude the calculation of the behavior of the complete water drop spectral density in any cloud model which accounts for the fluid motion in a realistic manner. Hence there have been presented several parameterization schemes which allow precipitation to be included in a cloud model with a relatively small increase in overall computation time and storage.

Most schemes are based essentially on the work of Kessler (1969) which divides the total liquid water into two parts: cloud water, which is assumed to move with the air, and rain, which is allowed to translate relative to the air. Berry (1968) gives some justification to this division by showing that the nature of the collision process between drops produces a natural break in the spectral density at a radius of about 40 μm. The second assumption of Kessler is that the spectral density of the rain is given by the Marshall-Palmer spectrum. These two assumptions are made in the present parameterization. However, the physical basis of the Marshall-Palmer spectrum is examined in order to determine the behavior of the parameters associated with it, and this leads to slightly different results from those of Kessler.

Perhaps the most arbitrary feature of Kessler's parameterization is the assumed form of the rate of conversion of cloud water directly into rain ("autoconversion'). An alternate expression for this, which has been used by Liu and Orville (1969) and Simpson and Wiggert (1969) is given by Berry (1968) who estimates the time T required for the sixth-moment radius of the spectral density to reach 40 μm under the action of droplet coalescence. His calculations are made for a liquid water density of 1 gm m^{-3}, and then he generalizes his formula to an arbitrary water density. Cotton (1972) found that the Berry (1968) parameterization consistently resulted in the formation of precipitation too low in the cloud. He suggested that since the precipitation formation process takes a certain amount of time for the droplet population to broaden to the extent that rapid conversion takes place, one should parameterize the auto-conversion rate to be a function of liquid water content and the 'age' of a parcel of droplets. Using the Berry (1967) stochastic droplet collection model, he developed a set of empirical auto-conversion formulas which described the auto-conversion rate as a function of liquid water content and the Lagrangian 'age' of the droplet population. Unfortunately, the need to estimate the Lagrangian time scale of a population of droplets is particularly inconvenient in multi-dimensional models. In addition, the formulation of the time dependence in the parameterization scheme was deduced from constant liquid water content stochastic collection numerical experiments which have questionable applicability to the atmosphere. In the present work, we assume that the cloud water spectral density has an asymptotic self-similar form and so the rate of conversion can be deduced from dimensional arguments.

Berry and Reinhardt (1974) use detailed calculations of the temporal development of the droplet spectral density due to coalescence in order to obtain a parameterization of rain. Their scheme is consequently more sophisticated than any dimensionally-based scheme such as the present one. However, they require the calculation and storage of three time-dependent variables (namely, the rain density and two characteristic mass scales), whereas the simpler schemes involve only the rain density. The physical limitation of their parameterization is that the effect of neither evaporation nor drop disintegration is included. It is not obvious that these effects can be parameterized simply by the addition of further terms to their equations.

3.2 Water conservation equations

Following Kessler (1969), we divide the total liquid water density ρ_w into two components such that

$$\rho_w = \rho_c + \rho_r \, , \tag{3.1}$$

where ρ_c is the cloud water density and ρ_r is the rain density. Cloud water is assumed to follow fluid particle paths while rain has a downward motion relative to the fluid due to its greater terminal velocity. The total water density ρ_t is the sum of the total liquid water density and the water vapor density ρ_v, i.e.

$$\rho_t = \rho_w + \rho_v \, . \tag{3.2}$$

Because the degree of supersaturation in clouds is observed to be small, it is usual to assume that any water vapor in excess of the saturation vapor density condenses instantaneously into cloud water; thus

$$\rho_c = (\rho_t - \rho_r - \rho_s) \; H \; (\rho_t - \rho_r - \rho_s) \tag{3.3}$$

where ρ_s is the saturation vapor density and H(x) is the Heaviside step function. The Heaviside step function is defined as

$$H(x) = \begin{Bmatrix} 1 & x \geq 0 \\ 0 & x < 0 \end{Bmatrix}$$

In a Cartesian coordinate system (x_1, x_2, x_3) with x_3 increasing vertically upward, the conservation of total water is given by the equation

$$\frac{\partial \rho_t}{\partial t} + \frac{\partial}{\partial x_j}(V_j \rho_t) - \frac{\partial M}{\partial x_3} = \nu_t \nabla^2 \rho_t \ ; \tag{3.4}$$

where V_j is the air velocity in the x_j-direction; M is the downward flux of rain relative to the air; ν_t is the effective molecular diffusivity of total water; t is time. Equation (3.4) implies that there are no sources or sinks of total water. The flux out of any given volume is induced only by the fluid and molecular motions, except that the rain also falls under the influence of gravity.

The conservation of rain is given by

$$\frac{\partial \rho_r}{\partial t} + \frac{\partial}{\partial x_j}(V_j \rho_r) - \frac{\partial M}{\partial x_3} = \nu_r \nabla^2 \rho_r + S_c + S_a - S_e \ , \tag{3.5}$$

where ν_r is the effective molecular diffusivity of rain; S_c is the rate of conversion of cloud water into rain; S_a is the rate of increase of rain due to the accretion of rain with cloud water; S_e is the rate of evaporation of rain. The last three terms of (3.5) represent the sources and sinks of rain.

The equation for the conservation of energy includes the source due to the release of latent heat by the condensation of water. It is therefore necessary to have an expression for the rate of condensation of water R following a fluid particle; that is

$$R_c = \frac{d\rho_w}{dt} = R_r \ , \tag{3.6}$$

where R_r is the rate of change of rain density due to its settling at terminal velocity.

Equations (3.1) - (3.6) together with the equations for the conservation of mass (or volume), momentum and energy and the equation of state form a closed system provided that the terms M, S_c, S_a, S_e and R_r can be expressed in terms of the other variables. Physical and dimensional arguments are used below in order to derive such expressions. Because the motion in a cloud is turbulent, all the dependent variables are random and hence the parameterized terms also are random. This random nature is taken into account in the present parameterization.

3.3 Conversion of cloud water into rain

The term S_c in (3.5) accounts for the initial generation of rain by the coalescence of two cloud water droplets to yield a rain drop. This process may therefore be expressed in the form

$$S_c = f_c \; \rho_c \; H(\bar{\rho}_c - \rho_{cm}) \; , \qquad (3.7)$$

where f_c is the mean collision frequency for cloud water droplets which become rain drops after colliding, H(x) is the Heaviside unit step function and so ρ_{cm} is a minimum mean cloud water density below which there is no conversion. The overbar denotes the mean value of a random variable. (Because the dependent variables are generally random functions of both time and position, the mean value is interpreted strictly as an ensemble average.) Since f_c and $H(\bar{\rho}_c - \rho_{cm})$ are average quantities, the random fluctuations of S_c are proportional to those of ρ_c, and so moments and covariances involving S_c are determined easily.

Because the boundary between cloud water and rain drops occurs at a radius of about 40 μm, condensation plays a negligible role in the final conversion process which is controlled by droplet coalescence. The initial development of the droplet distribution is, however, due to condensation alone and so there is no conversion until the droplets are large enough for coalescence to be significant. We assume that this transition occurs when the mean droplet radius is equal to r_{cm}. Hence

$$\rho_{cm} = (4\pi/3)\rho_{wt}\, r_{cm}^3\, N_c \ , \tag{3.8}$$

where ρ_{wt} is the density of water and N_c is the mean cloud water droplet concentration. The effect of coalescence on the droplet distribution becomes comparable with that of condensation when the droplets are larger than about 10 μm (Manton, 1974). When r_{cm} = 10 μm, N_c = 100 cm^{-3} and ρ_{wt} = 1.00 gm cm^{-3}, (3.8) gives a critical mean cloud water density of 0.42 gm m^{-3} which is comparable with the value of 1 gm m^{-3} suggested by Kessler (1967).

We assume that, once the mean cloud water density is greater than ρ_{cm}, the cloud water spectral density attains an asymptotic self-similar form so that the mean collision frequency can be written as

$$f_c = \pi r_c^2\, E_c V_c N_c \tag{3.9}$$

where r_c is the mean cloud droplet radius, E_c is the mean collection efficiency for droplets involved in the conversion process and V_c is the mean relative terminal velocity of the droplets.

Any cloud droplet can be converted into a rain drop and so the characteristic scales in (3.9) are representative of the whole cloud water spectral density. Thus the relative velocity V_c is characterized by the terminal velocity of a droplet of radius r_c, and this is given adequately by Stokes law:

$$V_c = (2/9)(\rho_{wt}\, g r_c^2/\mu_o) \tag{3.10}$$

where ρ_{wt} is the density of water, g is the gravitational acceleration, and μ_o is the viscosity of the air. [Subscript o on a thermodynamic variable indicates that it is a reference state value in the sense of Dutton and Fichtl (1969) and it is therefore not a random variable.] The mean mass of a cloud water droplet is $\bar{\rho}_c/N_c$. Thus the characteristic radius r_c is given by

$$r_c^3 = (3/4\pi)(\bar{\rho}_c/\rho_{wt} N_c) \ . \quad (3.11)$$

The collector droplet in any collision which produces conversion has a radius between 20 μm and 25 μm. The behavior of the linear collision efficiency for such droplets is shown by Scott and Chen (1970). By averaging those data, and assuming that the mean coalescence efficiency is unity, it is found that the mean collection efficiency for droplets associated with conversion may be represented by

$$E_c = 0.55 \ . \quad (3.12)$$

Putting (3.10) - (3.12) into (3.9), we see that the mean collision frequency is

$$f_c = \alpha_1 \ (\rho_{wt} \ g/\mu_o N_c{}^{1/3})(\bar{\rho}_c/\rho_{wt})^{4/3} \quad (3.13)$$

with $\alpha_1 = (2/9)(3/4)^{4/3} \ E_c/\pi^{1/3} = 0.057$.

Thus the rate of conversion depends weakly upon the cloud water droplet concentration; the fewer the number of droplets for a given cloud water density, the larger is the mean droplet size and hence the larger the conversion rate because fewer consecutive collisions are required for a droplet to be converted. We assume that N_c is an environmentally determined parameter which can be climatologically derived. The conversion rate increases also with decreasing viscosity, that is as the terminal velocity increases. This variation is not large: μ_0 decreases by about 12% as the temperature decreases from 20^0C to -20^0C.

Taking $N_c = 100\ cm^{-3}$, $\mu_0 = 1.72 \times 10^{-4}\ gm\ cm^{-1}\ sec^{-1}$, $\rho_{wt} = 1.00$ $gm\ cm^{-3}$ and $g = 980\ cm\ sec^{-2}$, we find that the mean conversion rate when $\bar{\rho}_c$ is greater than 0.42 gm m^{-3} is

$$\bar{S}_c = 7.00\times10^{-4}\ \bar{\rho}_c^{\ 7/3}\ gm\ m^{-3}\ sec^{-1} \tag{3.14}$$

where $\bar{\rho}_c$ is in gm m^{-3}. The present conversion rate, given by (3.14), is proportional to $\bar{\rho}_c^{\ 7/3}$ and so it increases more rapidly than that of Kessler (1967) for which $S_c \propto \rho_c$. Liu and Orville (1969) suggest that Kessler's critical cloud water density, below which S_c is zero, may be taken as zero and this gives

$$\bar{S}_c = 10^{-3}\ \bar{\rho}_c\ gm\ m^{-3}\ sec^{-1}\ . \tag{3.15}$$

By comparing (3.14) with (3.15), it is seen that (3.14) gives lower values for $\bar{S}_c$ than does (3.15) only when $\bar{\rho}_c$ is less than 1.31 gm m^{-3}. The present parameterization therefore produces an initially more gradual conversion rate than does Kessler's; but at high cloud water densities it produces a much more rapid conversion. Thus the cloud water density is prevented from becoming excessively large. Soong and Ogura (1973) note that if the rate constant in Kessler's formulation is reduced much below $10^{-3}\ sec^{-1}$ then excessive values of ρ_c are found at cloud top in model calculations.

3.4 Marshall-Palmer spectral density

Observations of rainfall show that the rain drop spectral density, $\phi(r)$ where r is the drop radius, tends to behave exponentially (Marshall and Palmer, 1948), and it therefore may be written in the form

$$\phi(r_\rho) = (N_r/r_m) \exp(-r_\rho/r_m) \ , \qquad (3.16)$$

where N_r is the rain drop concentration and r_m is a characteristic radius for the distribution. Because the rain drop concentration is generally quite low, the spectral density cannot strictly be defined in a statistically meaningful manner as a local random variable. Indeed, the agreement between observed spectral densities and (3.16) increases with the sample volume of the observations (Cotton, 1975c). However, we assume that a meaningful estimation of ϕ can be obtained from an average over a volume with a length scale that is small compared with the length scale of the turbulent fluctuations. Thus ϕ is treated as a locally random variable. (If it is assumed that ϕ, and hence the following parameterization, can be interpreted only in an average sense then ρ_r must be replaced by $\overline{\rho_r}$ in all the equations below).

From (3.16), the rain density is

$$\rho_r = \int_0^\infty (4\pi/3)\ r_\rho^{\ 3}\ \rho_{wt}\ \phi(r_\rho)dr_\rho = 8\pi\rho_{wt}\ M_r r_m^3. \qquad (3.17)$$

It is usual to assume that N_r/r_m is a constant determined by the environment: Kessler (1967) suggests a value of $10^7 m^{-4}$. Equation (3.42) then implies that r_m is proportional to $\rho_r^{\ 0.25}$, and therefore that there is no constant characteristic length scale associated with the formation of the distribution. But this is not so. [We note that Cotton (1975) points out that there is apparently no physical foundation for the constancy of N_r/r_m.]

Experiments by Blanchard and Spencer (1970) suggest that the Marshall-Palmer distribution is an asymptotic spectral density produced by a balance between coalescence, tending to increase the size of an individual drop, and drop breakup, tending to reduce the drop size. Breakup occurs when the surface tension force, which keeps a drop coherent, becomes less than the aerodynamic or collision forces acting on the drop. Hence there is a maximum drop radius r_b for which these forces balance; that is

$$\gamma_o r_b \sim \rho_o V_b^2 C r_b^2 \qquad (3.18)$$

where γ_0 is the surface tension of water, ρ_0 is the air density, V_b is the terminal velocity of the drop and C is the drag coefficient. The term on the right hand side of (3.19) represents both the aerodynamic force on a drop and the rate of change of momentum during a collision between two drops. The terminal velocity of rain drops is discussed in Section 3.5 below, it thus follows that $\rho_0 V_b^2 C \sim \rho_{wt} g r_b$. Thus there is a constant radius associated with the rain distribution, namely $r_b \sim (\gamma_0/\rho_{wt} g)^{\frac{1}{2}}$. If (3.16) does represent an asymptotic self-similar spectral density then we expect r_m to vary with r_b and hence

$$r_m = \alpha_2 (\gamma_o/\rho_{wt} g)^{1/2} , \qquad (3.19)$$

where α_2 is a constant. The surface tension of water is a rather weak function of temperature (γ_0 decreases by about 4% as the temperature increases from -10^oC to 10^oC) and of dissolved impurities (a 10% solution of NaCl increases γ_0 by about 4%). We therefore expect r_m to be a constant, essentially independent of the detailed environment.

Considering the data of Hudson (1963) and Mueller and Sims 1966), Blanchard and Spencer (1970) show that the spectral densities of rain falling at about 200 mm hr^{-1} are essentially independent of geographical location; i.e. the distribution is not affected by local environmental changes. They also plot the logarithm of ϕ as a function of r for rainfall rates between 25 mm hr^{-1} and 1550 mm hr^{-1}. For r greater than about 1 mm, all the distributions are roughly linear and parallel. But we see from (3.16) that $-1/r_m$ is proportional to the slope of these distributions, and so it is found that r_m is approximately 0.27 mm. Taking γ_0 = 74.22 dyne cm^{-1}, we see that the constant associated with the characteristic radius of the rain spectral density is

$$\alpha_2 = 0.098 \; . \tag{3.20}$$

For precipitation rates in the range 90 to 722 mm hr^{-1}, Blanchard and Spencer find that the median volume diameter of the rain distributions is essentially constant and equal to 3.2 mm. The median volume radius for the spectral density (3.16) is 3.67 r_m which yields a median volume diameter of 2.0 mm when r_m = 0.27 mm. The reason for the discrepancy between the observed median volume diameter and that obtained from the slope of the distributions is that the observed high rainfall spectra do not follow (3.16) at radii less than about 1 mm. The observed spectral density of small drops is much less than that given by the Marshall-Palmer spectrum. Although this phenomenon might be real (due to collection by the larger drops), it is possibly a result of the method of measurement. Cotton (1975c) states that photographic and impaction techniques tend to underestimate the number of small drops present in a sample.

It appears that the distribution (3.16) with r_m given by (3.19) and (3.20) ought to give a representation of the asymptotic spectral density when drop breakup is the dominant controlling mechanism.

Blanchard and Spencer (1970) state that breakup becomes effective once there are drops with radii greater than about 1.5 mm and that this occurs for rainfall rates greater than about 25 mm hr^{-1}. Using the expression (3.23) derived in Section 3.5 below relating the rain mass flux to the rain density, we note that this lower limit to the validity of the present parameterization corresponds to a rain density of about 1.17 gm m^{-3}.

3.5 Mass flux of rain

In order to obtain expressions for M in (2.5) and R_r in (2.6), the terminal velocity of rain drops must be determined. When a drop of radius r falls at its terminal velocity V, the weight of the drop is balanced by the aerodynamic drag. This is represented by the equation

$$(r\pi/3)r_\rho^3\ \rho_{wt}g = 0.5\ \pi r_\rho^2\ \rho V^2 C\ ,$$

where the drag coefficient C is a function of the Reynolds number, $2\rho_0 Vr/\mu_0$. The Reynolds number of the motion is very large and so we expect, as a first approximation at least, V to be independent of viscosity; that is, we expect C to be constant (Spilhaus, 1948). For drops with radii in the range 0.5 mm to 2.9 mm, the data of Gunn and Kinzer (1949) yield an average value for C of 0.587. Thus the terminal velocity of rain drops is given approximately by

$$V(r_\rho) = \alpha_3(\rho_{wt}/\rho_0)^{\frac{1}{2}}(gr_\rho)^{\frac{1}{2}} \qquad (3.21)$$

with α_3 = 2.13. Equation (3.21) predicts the terminal velocity with an accuracy of at least 10% over the stated range which includes all the rain in the data of Blanchard and Spencer (1970) that fit the asymptotic distribution (3.16). The equation accounts for the increase in terminal velocity with decreasing density which could be significant in a deep cloud.

The rain mass flux relative to the air is

$$M = \int_0^\infty (4\pi/3) r_\rho^3 \, \rho_{wt} \, V(r) \phi(r_\rho) dr_\rho \qquad (3.22)$$

Putting (3.16), (3.17) and (3.21) into (3.22), we readily find that

$$M = V_o \rho_{r_\rho} \qquad (3.23)$$

where $V_o = \frac{1}{6} \Gamma(4.5) \; V(r_m) = 1.94 \; V(r_m)$.

We see that the effective terminal velocity of the rain V_o is independent of the rain density. When $r_m = 0.27$ mm and $\rho_o = 1.28 \times 10^{-3}$ gm cm^{-3}, this velocity is 5.93 m sec^{-1}. Kessler (1967), assuming N_r/r_m to be constant, finds the effective terminal velocity to be given by 5.15 $\rho_r^{0.125}$ m sec^{-1} (with ρ_r in gm m^{-3}), and this coincides with the present result when $\rho_r = 3.09$ gm m^{-3}. However, Kessler's velocity varies slowly with ρ_r : it increases from 5.15 m sec^{-1} to 7.49 m sec^{-1} as ρ_r increases from 1 gm m^{-3} to 20 gm m^{-3}.

The rate of change of rain density due to the motion of the rain relative to the air is

$$R_r = \int_0^\infty V(r_\rho) \frac{\partial}{\partial x_3} \{(4\pi/3) r_\rho^3 \, \rho_{wt} \, \phi(r_\rho) dr_\rho\} \; .$$

Because r_m is essentially constant, the differential operator is readily interchanged with the integral and we find from (3.16), (3.17), (3.21) and (3.22) that

$$R_r = V_o \; \partial\rho_r / \partial x_3 \; . \qquad (3.24)$$

3.6 Accretion of cloud water by rain

The rate of increase in mass of a single rain drop of radius r due to the collection of cloud water is (Kessler, 1967)

$$\dot{m}_a = \pi r_\rho^2 E_{cr} V \rho_c , \qquad (3.25)$$

where E_{cr} is the effective collection efficiency between the rain drop and cloud water droplets which are assumed to have zero terminal velocity relative to that of the rain drop. Kessler suggests that E_{cr} may be approximated by

$$E_{cr} = 1 , \qquad (3.26)$$

independent of r. Hence the total rate of increase in rain densities due to accretion is found from (3.16), (3.17), (3.21), (3.25) and (3.26) to be

$$S_a = \int_0^\infty \phi(r)\, \dot{m}_a(r_\rho) dr_\rho = \alpha_4 \, (\rho_{wt}/\rho_o)^{\frac{1}{2}} \, (g/r_m)^{\frac{1}{2}} \, (\rho_r \rho_c/\rho_{wt}), \qquad (3.27)$$

where $\alpha_4 = \frac{1}{8}\alpha_3 \, E_{cr} \, \Gamma \, (3.5) = 0.884$.

Equation (3.27) implies that S_a is simply proportional to the product of the cloud water and rain densities; in particular, with $\rho_o = 1.28 \times 10^{-3}$ gm cm^{-3}

$$S_a = 4.71 \times 10^{-3} \, \rho_c \rho_r \text{ gm m}^{-3} \text{ sec}^{-1}$$

where ρ_c and ρ_r are in gm m^{-3}. This may be compared with Kessler s expression, namely

$$S_a = 5.22 \times 10^{-3} \, \rho_c \rho_r^{0.875} \text{ gm m}^{-3} \text{ sec}^{-1} \qquad (3.28)$$

We see that the present result gives lower accretion rates than does Kessler's when ρ_r is less than 2.28 gm m^{-3}; however, the differences are not large at moderate rain densities.

By decomposing the densities into their respective mean values and fluctuations about their means, the mean accretion rate is given simply from (3.27) by

$$\overline{S}_a = \alpha_4 \; (\rho_{wt}/\rho_o)^{1/2} \; (g/r_m)^{1/2} \; (\overline{\rho}_c\overline{\rho}_r + \overline{\rho_c'\rho_r'})/\rho_{wt}$$

where $\overline{\rho_c'\rho_r'}$ is the covariance of the cloud water and rain densities. Similarly, higher order moments and covariances can be evaluated readily in terms of the covariances of the dependent variables. The moments of S_a for parameterizations involving non-integral powers of ρ_r, such as Kessler's, cannot be determined without the addition of assumptions regarding the probability distributions of ρ_r and ρ_c.

3.7 Evaporation of rain

The rate of change of radius of a rain drop due to evaporation as it falls relative to the air is (Mason, 1957)

$$dr_\rho/dt = (\sigma/r_\rho)(1+F\ Re^{\frac{1}{2}})/(L^2\rho_{wt}/DR_vT^2 + \rho_{wt}\ R_vT/Dp_s)$$

where σ is the supersaturation of the ambient air; Re is the Reynolds number of the relative motion; L is the latent heat of vaporization of water; K is the thermal conductivity of air; R_v is the gas constant for water vapor; T is the temperature; D is the diffusivity of water vapor in air; ρ_s is the saturation vapor pressure at temperature T. The ventilation coefficient is approximated by

$$F = 0.21 \qquad (3.29)$$

for Re $\gtrsim$ 200, that is for r $\gtrsim$ 0.45 mm. Moreover, (3.21) implies that Re is proportional to $r^{3/2}$ and hence $RFe^{1/2} >> 1$ for rain drops. Using the equation

$$\sigma p_s = (\rho_v - \rho_s) R_v T$$

and assuming that deviations from the reference states of thermodynamic variables are small (Dutton and Fichtl, 1969), we find that the rate of change of mass of a rain drop due to evaporation may be approximated by

$$\dot{m}_e = 4\pi r \, D_{eo} \, (\rho_v - \rho_s) F \, (2\rho_o V r_\rho / \mu_o)^{1/2} \qquad (3.30)$$

where $$D_{eo} = D_o/(1 + L_o^2 D_o p_{so}/K_o R_v^2 T_o^3) \; .$$

The rate of decrease in rain density due to evaporation is given from (3.16), (3.17), (3.21), (3.29) and (3.30) by

$$S_e = -\int_0^\infty \dot{m}_e(r_\rho)\phi(r_\rho)dr = \alpha_s(\rho_{wt}/\rho_o)^{1/4}(\rho_o^2 r_m^3 g/\mu_o^2)^{1/4} \; (D_{eo}/r_m^2)$$

$$\times \; (\rho_s - \rho_v)\rho_r/\rho_{wt} \qquad (3.31)$$

where $$\alpha_s = (\alpha_3/2)^{1/2} \; F\Gamma(2.75) = 0.349.$$

The rate of evaporation is seen to be proportional to $(\rho_s - \rho_v)\rho_r$, that is to integral powers of the random dependent variables. The moments of S_e can therefore be calculated easily in terms of the moments of ρ_s, ρ_v and ρ_r. The effective diffusivity D_{eo} increases from 0.113 to 0.187 $cm^2 \; sec^{-1}$ as the temperature decreases from 0^oC to $-2o^oC$ in a standard atmosphere. Thus, for deep convection at least, the variation of D_{eo} ought to be taken into account.

When $D_{eo} = 0.113$ cm sec^{-1}, $\rho_o = 1.28 \times 10^{-3}$ gm cm^{-3} and μ_o = 1.72×10^{-4} gm cm^{-1} sec^{-1}, (3.31) yields

$$S_e = 2.91\times10^{-4} \ (\rho_s-\rho_v)\rho_r \ \text{gm m}^{-3} \ \text{sec}^{-1} \ .$$

with ρ_s, ρ_v and ρ_r in gm m^{-3}. On the other hand, Kessler (1967) uses the parameterization

$$S_e = 5.44\times10^{-4} \ (\rho_s-\rho_v)\rho_r^{0.65} \ \text{gm m}^{-3}\text{sec}^{-1}.$$

Because the present result explicitly accounts for the Reynolds number dependence of the evaporation rate from a falling drop, it produces higher values for S_e when the rain density is greater than 5.97 gm m^{-3}.

3.8 Conclusion

Physical and dimensional arguments are used above to obtain a parameterization of the effects of rain on the dynamics and thermodynamics of a cloud model. In particular, it is shown that at high rain densities the characteristic radius r_m of the Marshall-Palmer rain drop spectrum is a constant. Comparison with the results of Kessler (1967), who assumes that r_m is not constant, shows that the present parameterization yields lower rates of accretion and evaporation at rain densities less than a few gm m^{-3}.

An expression for the rate of conversion of cloud water into rain is derived and is found to be proportional to the 7/3-power of the mean cloud water density ρ_c. Thus it increases more rapidly than the rate suggested by Kessler which is linear in ρ_c. The present parameterization includes the effect of the cloud water droplet concentration.

4. THE AVERAGE EQUATIONS

4.1 Introduction

There has been much progress in the development of finite-difference numerical models of turbulent flows. The simulation of turbulence is achieved by choosing a spatial mesh size which is somewhat smaller than the scale of the energy-containing eddies. By assuming that the unresolved (sub-grid scale) fluctuations lie in the Kolmogorov inertial sub-range of fluctuations, the known properties of the latter can be used to predict the behavior of the former (Lilly, 1967). Both first order (Deardorff, 1972) and second order (Deardorff, 1974a) closure approximations have been used in the simulation of atmospheric turbulence.

Measurements in the atmosphere during strongly unstable conditions show the existence of plumes or thermals with horizontal scales of about 200 m (Warner and Telford, 1967). This suggests that it is necessary to select a spatial mesh size no larger than some tens of meters in order to have the sub-grid scale fluctuations in a numerical model lie within the inertial sub-range. [we note that the existence of a turbulence spectral density proportional to $\kappa^{-5/3}$ is necessary but not sufficient for the existence of isotropic turbulence (Weiler and Burling, 1967).] On the other hand, there is some evidence that the development of a cumulus cloud field is strongly coupled with the prevailing mesoscale (and perhaps synoptic scale) system [Pielke (1974), Cotton (1975a,b), Cotton, Pielke and Gannon (1976)]. This implies that the motion in a region with a horizontal scale of some tens of kilometers must be studied in order to resolve this coupling. Computation time and storage limitations make it impossible at present to use a finite-difference mesh fine enough to resolve the small scale turbulence in a region large enough to resolve the coupling between the mesoscale and cloud scale systems.

As with most practical problems in turbulence, however, it is not really necessary to predict the detailed structure of the turbulence in the cloud field. Prediction of the mean values (and perhaps the variances) of the dependent variables is itself a worthwhile objective. Since the work of Reynolds (1901), most advances in the solution of practical problems in turbulence have come from attempts to predict the mean motion rather than the complete random field. In this chapter, model equations for the first two moments of the dependent variables are derived from the set of approximate equations discussed in Chapter 2. A second order closure approximation is used to predict the behavior of the turbulence covariances.

Since it is believed that the scale range of 0.1 to 1.0 km is a region of strong kinetic energy generation and transport in a precipitating, moist convective system, the use of both a simple first order closure approximation and coarse horizontal resolution as is frequently done [see Takada (1971), Schlesinger (1973a,b; 1975), Wilhelmson (1974) and Miller and Pearce (1974)] is highly inadequate for the simulation of such systems.

A large cloud system often penetrates most of the troposphere and so the process to be modelled is deep convection in the sense that the depth of the convection layer is comparable with the scale height of the atmosphere. By following Dutton and Fichtl (1969), the model equations derived in Chapter 2 are valid in deep convection. Although the equations are for use in a deep convection model, ice is not included in the parameterization of the precipitation at this time. This restricts the precipitation to at most supercooled rain. With the development of an appropriate parameterization, however, the ice phase could be included in the equations in a manner similar to that of rain [see for example, Wisner, et.al., (1972); Cotton, (1972)].

4.2 Model equations

It is seen in Chapter 2 that the behavior of a moist atmosphere can be described in terms of the momentum vector $\underset{\sim}{m}$ the pressure p, the density variable r, the total water density ρ_t, the rain density ρ_r and the cloud water density ρ_c. It should be noted that the prime notation on r', ρ_r', ρ_c', ρ_t' will be replaced by r, ρ_r, ρ_c, ρ_t in this section. The potential density variable r is related to the potential temperature θ by

$$r = \rho_o\theta'/\theta_o - (L/C_{pa}T_o)(\rho_c+\rho_r) \qquad (4.1)$$

where ρ is the total density of the air, L is the latent heat of vaporization of water, C_{pa} is the specific heat at constant pressure for dry air, T is absolute temperture. The subscript o denotes the reference state of a thermodynamic variable, while a prime denotes the deviation of the local mean of a thermodynamic variable from its reference state. [We see from (4.1) that $\theta_o r/\rho_o$ is the deviation of the temperature variable θ_L introduced by Betts (1973).] If the reference state temperature $T_o(x_3)$ is specified from some horizontal average of the atmosphere then p_o, ρ_o and θ_o are given by

$$\begin{aligned} \partial p_o/\partial x_3 &= -\rho_o g \ , \\ p_o &= R_a\rho_o T_o \ , \qquad (4.2) \\ \theta_o &= T_o(p_r/p_o)^{R_a/C_{pa}} \end{aligned}$$

where (x_1, x_2, x_3) are Cartesian coordinates with x_3 increasing vertically upward; g is the gravitational acceleration; R_a is the gas constant for dry air; p_r = 1000 mb. The model equations are found to be

$$\partial m_j/\partial x_j = 0 , \tag{4.3}$$

$$\partial m_i/\partial t + \partial(u_j m_i)/\partial x_j - \varepsilon_{ijk} f_k m_j + \partial p'/\partial x_i + \rho' g \delta_{i3} = \rho_o \nu_m \nabla^2 u_i , \tag{4.4}$$

$$\partial r/\partial t + \partial(u_j r)/\partial x_j + m_3 \, \partial(\ln \theta_o)/\partial x_3 + (L/C_{pa} T_o)\rho_o Q = \nu_\theta \nabla^2 r , \tag{4.5}$$

$$\partial \rho_t/\partial t + \partial(u_j \rho_t)/\partial x_j - \partial M/\partial x_3 = \nu_t \nabla^2 \rho_t , \tag{4.6}$$

$$\partial \rho_r/\partial t + \partial(u_j \rho_r)/\partial x_j - \partial M/\partial x_3 = \nu_r \nabla^2 \rho_r + S_c + S_a - S_e , \tag{4.7}$$

$$\rho_c = (\rho_t - \rho_r - \rho_s) H(\rho_t - \rho_r - \rho_s) , \tag{4.8}$$

where t is time; H(x) is the Heaviside unit set function; $\underset{\sim}{f}$ is the Coriolis frequency vector; M and Q account for the relative motion between the air and the rain; S_c, S_a and S_e represent the conversion of cloud water into rain, the accretion of rain with cloud water and the evaporation of rain; ν_m, ν_θ, ν_t and ν_r are effective molecular diffusivities. The velocity vector $\underset{\sim}{u}$, the density deviation ρ' and the saturation vapor density ρ_s are given by

$$u_i = m_i/\rho_o , \tag{4.9}$$

$$\rho' = (1 - R_a/C_{pa})(p'/R_a T_o) - r - (R_v/R_a - 1)\rho_t - (L/C_{pa} T_o - R_v/R_a)(\rho_c + \rho_r), \tag{4.10}$$

$$\rho_s = q_{so}\{\rho_o + (L/R_vT_o - 1)[r + (p'/C_{pa}T_o) + (L/C_{pa}T_o)(\rho_c + \rho_r)]\} , \tag{4.11}$$

where the reference state saturation mixing ratio is

$$q_{so} = (P_s/p_o)\exp\{\alpha_s(1 - T_s/T_o)\} ;$$

$$P_s = 12.272\text{mb},\ \alpha_s = 19.2484 \quad \text{and} \quad T_s = 283.16\ \text{K}.$$

The dependent variables in (4.2) - (4.11) are random functions of space and time. In order to derive equations for the mean values of these variables we decompose them into their mean values and turbulent fluctuations about their means. For example, the momentum is written as

$$m_i = \overline{m}_i(\underset{\sim}{x},t) + m_i''(\underset{\sim}{x},t)$$

where an overbar denotes the ensemble average of a quantity and a double prime denotes a random fluctuation about the mean.

It should be noted that we are interpreting the averaging operator as an ensemble average as opposed to the subgrid scale averaging operator employed by Deardorff (1974b). As long as the observed mean flow over the scales of interest vary relatively smoothly in space and time, a distinction between the two averaging operators is of little consequence. However, a cumulus-convective environment is often characterized by large variability both on the scales one wishes to represent as a mean flow and on the fluctuating field. Using a grid scale averaging operator then places an often impossible burden on the finite difference operator which must represent such a highly variable field on the scale of one or two times the truncation scale. Use of the ensemble averaging operator, on the other hand, allows the modeller to choose a grid size small enough to resolve such a variable mean field. The turbulence is then defined and modelled with respect to such an observed mean field and not with respect to the chosen grid resolution of the numerical scheme.

Certainly the application of an ensemble averaging operator to a numerical model on the cumulus/mesoscale is a new one. Perhaps the most similar application of an ensemble-averaging operator is that of Arakawa and Shubert (1974) who applied it to the parameterization of the effects of cumulus-cloud on the large-scale environment. In our application, the ensemble average may be thought of as the average of cumulus-cloud properties (including their variances and covariances) which have been observed for a given set of mesoscale properties, such as convergence, static stability, vertical shear of the horizontal wind, and surface-layer fluxes of heat, moisture and momentum. Each set of these properties (which so define an ensemble realization) would be expected to yield a different family of turbulence characteristics. It is, thus, not the intent of this modeling effort to represent explicitly, by numerical prediction techniques, individual cumulus towers. Instead, only the gross statistical effects of such cumulus towers (i.e. fluxes, kinetic energy) on the mesoscale environment are sought. Regardless of whether the cumulus clouds grow as distinct, visible entities in a weak mesoscale convergence field or as imbedded towers in a cumulonimbus-mesoscale system, only the gross statistical properties are desired.

Because deep convection is considered, the length scales of mean and fluctuating quantities are different. Clearly the vertical scale for mean variables is the total depth of the solution domain which is comparable with the scale height of the atmosphere. On the other hand, the length scale of the turbulent fluctuations in the atmosphere is generally no larger than a few hundred meters. We therefore assume that mean quantities satisfy the restrictions of deep convection while fluctuations behave like disturbances in shallow convection, in the sense of Dutton and Fichtl (1969). The primary consequences of this are that the turbulent velocity fluctuations may be treated as incompressible and that the fluctuating pressure can be neglected when compared with the fluctuating density. Thus the term $u_j'' \, \partial\rho_o/\partial x_j$ can be neglected in (4.3) and (4.4) yielding

$$\partial m_j/\partial x_j \simeq \partial \overline{m}_j/\partial x_j + \rho_o \, \partial u_j''/\partial x_j ,$$
$$\frac{\partial}{\partial x_j}(u_j m_i) \simeq m_i \frac{\partial u_j}{\partial x_j} + m_j \frac{\partial u_i}{\partial x_j} + u_i \, \overline{u}_j \frac{\partial \rho_o}{\partial x_j} \tag{4.12}$$

Similarly, p'' is neglected in (4.10) and (4.11) so that

$$\rho'' = - r'' - (R_v/R_a - 1)\rho_t'' - (L/C_{pa}T_o - R_v/R_a)(\rho_c'' + \rho_r''), \tag{4.13}$$

$$\rho_s'' = q_{so}(L/R_vT_o - 1)\,\{r'' + (L/C_{pa}T_o)(\rho_c'' + \rho_r'')\} . \tag{4.14}$$

Equations (4.12) - (4.14) are used in the determination of the covariances of the random variables.

The quantities M, Q, S_c, S_a and S_e appearing in (4.6) - (4.7) are derived from some parameterization of the precipitation process. Such a parameterization, which takes into account the random nature of the dependent variables and is valid for deep convection, is given in Chapter 3. By interpreting the spectral density of the rain drops as a strictly averaged quantity, it is found that

$$M = V_o\overline{\rho}_r \ , \quad Q = V_o\partial(\overline{\rho}_r/\rho_o)/\partial x_3 \ ,$$

$$S_c = 0.057(\rho_{wt} g/\mu_o N_c{}^{1/3})(\overline{\rho}_c/\rho_{wt})^{4/3}\ \rho_c\ H(\overline{\rho}_c - \rho_{cm}),$$

$$\equiv A_o(\overline{\rho}_c/\rho_{wt})^{4/3}\ \rho_c\ H(\overline{\rho}_c - \rho_{cm})\ , \qquad (4.15)$$

$$S_a = 0.884(\rho_{wt}/\rho_o)^{1/2}(g/r_m)^{1/2}(\overline{\rho}_r/\rho_{wt})\rho_c \equiv B_o(\overline{\rho}_r/\rho_{wt})\rho_c\ ,$$

$$S_e = 0.349(\rho_{wt}/\rho_o)^{1/4}(\rho_o{}^2 r_m{}^3 g/\mu_o{}^2)^{1/4}(D_{eo}/r_m{}^2)(\overline{\rho}_r/\rho_{wt})(\rho_c+\rho_s+\rho_r-\rho_t)\ ,$$

$$\equiv C_o(\overline{\rho}_r/\rho_{wt})(\rho_c+\rho_s+\rho_r-\rho_t)\ ,$$

where

$$V_o = 4.13(\rho_{wt}/\rho_o)^{1/2}(g r_m)^{1/2}\ ,$$

$$D_{eo} = D_o/(1+L^2 D_o \rho_o q_{so}/K_o R_v{}^2 T_o{}^3)\ ,$$

$$\rho_{cm} = (4\pi/3)\rho_{wt}\ r_{cm}^3\ N_c\ ,$$

ρ_{wt} is the density of water, N_c is the mean concentration of cloud water droplets, r_m = 0.027 cm is a characteristic radius associated with the breakup of rain drops, r_{cm} = 10 μm is a critical droplet radius below which the droplet distribution is controlled by condensation alone, μ is the viscosity of air, K is the thermal conductivity of air and D is the diffusivity of water vapor in air. Equations (4.2) - (4.15) form a closed system from which the first two moments of the dependent variables are now calculated.

4.3 Mean values of dependent variables

The Reynolds number of the mean motion is very large and so the effects of molecular diffusion on the mean dependent variables can be neglected. Using (4.9) - (4.11) and (4.15), we now take the mean values of equations (4.3) - (4.7) to obtain

$$\partial \overline{m}_j/\partial x_j = 0\ , \qquad (4.16)$$

$$\partial \overline{m}_i/\partial t + \partial(\overline{u}_j\ \overline{m}_i)/\partial x_j + \partial(\overline{u_j''m_i''})/\partial x_j - \varepsilon_{ijk}f_k\ \overline{m}_j + \overline{\partial p'/\partial x_i} + \overline{\rho'}\ g\ \delta_{i3} = 0\ , \tag{4.17}$$

$$\partial \overline{r}/\partial t + \partial(\overline{u}_j\ \overline{r})/\partial x_j + \partial(\overline{u_j''r''})/\partial x_j + \overline{m}_3\ \partial(\ell n\theta_o)/\partial x_3 + (L/C_{pa}T_o)\rho_o V_o\ \partial(\overline{\rho}_r/\rho_o)/\partial x_3 = 0\ , \tag{4.18}$$

$$\partial\overline{\rho}_t/\partial t + \partial(\overline{u}_j\ \overline{\rho}_t)/\partial x_j + \partial(\overline{u_j''\rho_t''})/\partial x_j - \partial(V_o\overline{\rho}_r)/\partial x_3 = 0, \tag{4.19}$$

$$\partial\overline{\rho}_r/\partial t + \partial(\overline{u}_j\ \overline{\rho}_r)/\partial x_j + \partial(\overline{u_j''\rho_r''})/\partial x_j - \partial(V_o\overline{\rho}_r)/\partial x_3 = A_o(\overline{\rho}_c/\rho_{wt})^{4/3}\ \overline{\rho}_c\ H(\overline{\rho}_c-\rho_{cm}) + B_o(\overline{\rho}_r/\rho_{wt})\overline{\rho}_c - C_o(\overline{\rho}_r/\rho_{wt})(\overline{\rho}_c+\overline{\rho}_r+\overline{\rho}_s-\overline{\rho}_t), \tag{4.20}$$

where $\overline{u}_i = \overline{m}_i/\rho_o$, (4.21)

$$\overline{\rho'} = (1-R_a/C_{pa})(\overline{p'}/R_aT_o) - \overline{r} - (R_v/R_a-1)\overline{\rho}_t - (L/C_{pa}T_o-R_v/R_c) \times (\overline{\rho}_c+\overline{\rho}_r)\ , \tag{4.22}$$

$$\overline{\rho}_s = q_{so}\ \{\rho_o + (L/R_vT_o-1)\ [\overline{r} + (\overline{p'}/C_{pa}T_o) + (L/C_{pa}T_o)(\overline{\rho}_c+\overline{\rho}_r)]\} \tag{4.23}$$

Although each density ρ_t, ρ_r and ρ_s is strictly non-negative, the difference $\rho_t-\rho_r-\rho_s$ can have any real value. In order to determine the moments of ρ_c, we therefore introduce the assumption that the zero-mean random variable $\rho_t''-\rho_r''-\rho_s''$ is normally distributed with variance

$$\sigma_c^2 = \overline{(\rho_t''-\rho_r''-\rho_s'')^2}\ . \tag{4.24}$$

Thus the average value of (4.23) yields

$$\overline{\rho}_c = (\overline{\rho}_t-\overline{\rho}_r-\overline{\rho}_s)h + (2\pi)^{-1/2}\ \sigma_c\ \exp\{-\ (\overline{\rho}_t-\overline{\rho}_r-\overline{\rho}_s)^2/2\sigma_c^2\} \tag{4.25}$$

where $$h = \frac{1}{2}\{1 + \mathrm{erf}[(\bar{\rho}_t - \bar{\rho}_r - \bar{\rho}_s)/2^{1/2}\sigma_c]\} \ . \tag{4.26}$$

The function h behaves as a Heaviside unit step function as σ_c approaches zero; i.e., $h = H(\bar{\rho}_t - \bar{\rho}_r - \bar{\rho}_s)$ as $\sigma_c \rightarrow 0$. By accounting for the turbulent fluctuations, equation (4.25) implies that the mean cloud water density is non-zero (but small) even when the air is not saturated in the mean.

Figure 1 illustrates the dependence of $\bar{\rho}_c/\sigma_c$ on $(\bar{\rho}_t - \bar{\rho}_r - \bar{\rho}_s)/\sigma_c$. Also shown in Figure 1 is the variation of h as a function of $(\bar{\rho}_t - \bar{\rho}_r - \bar{\rho}_s)/\sigma_c$. The function h, which may be interpreted to represent the probability of cloud at a given point, is a unique function of $(\bar{\rho}_t - \bar{\rho}_r - \bar{\rho}_s)/\sigma_c$. Figure 2 illustrates the predicted variation in average cloud liquid water mass density $\bar{\rho}_c$ as a function of σ_c and for various values of the average saturation deficit or excess χ. For a cloud which is exactly saturated on the average ($\bar{\chi} = 0$), the theory predicts that $\bar{\rho}_c$ varies linearly with σ_c. As the cloudy region becomes increasingly subsaturated ($\bar{\chi} < 0$), $\bar{\rho}_c$ remains near zero until σ_c becomes sufficiently large after which $\bar{\rho}_c$ again varies linearly as a function of σ_c. The same behavior can be seen for a cloud with an average saturation excess ($\bar{\chi} > 0$) except that $\bar{\rho}_c \simeq \bar{\chi}$ for small values of σ_c and varies linearly with σ_c for larger values of σ_c.

Equations (4.16) to (4.26) completely specify the average behavior of a precipitating convective system provided that the pressure perturbation field is diagnosed and the turbulent covariances and variances are modelled. The conventional technique for evaluating the pressure field in three dimensions for an anelastic system is to take the divergence of (4.17) and solve for p' using either numerical techniques such as relaxation or some form of direct method of inverting an elliptic equation.

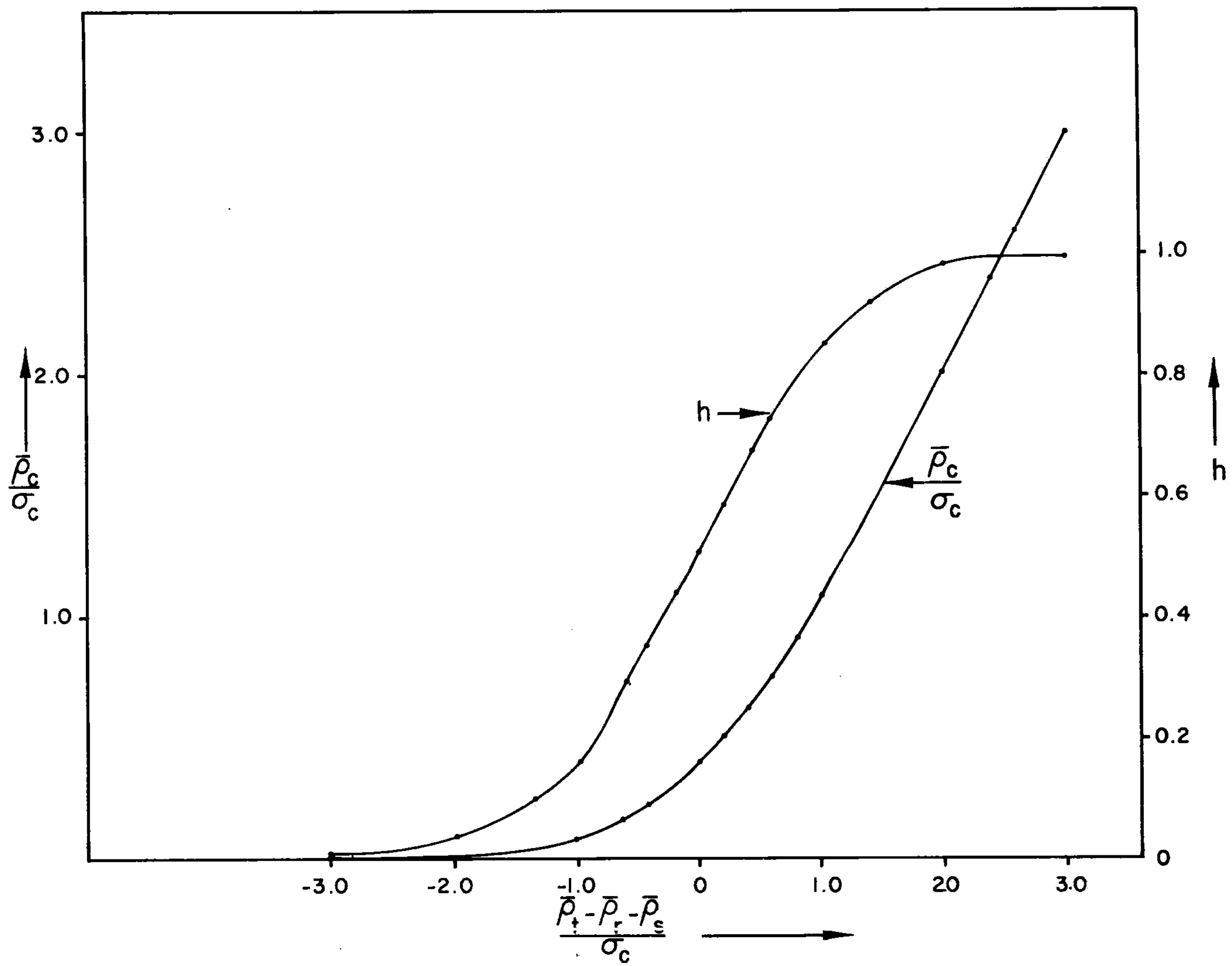

Figure 4.1 Variation of $\overline{\rho}_c/\sigma_c$ and h as a function of $(\overline{\rho}_t - \overline{\sigma}_r - \overline{\rho}_s)/\sigma_c$.

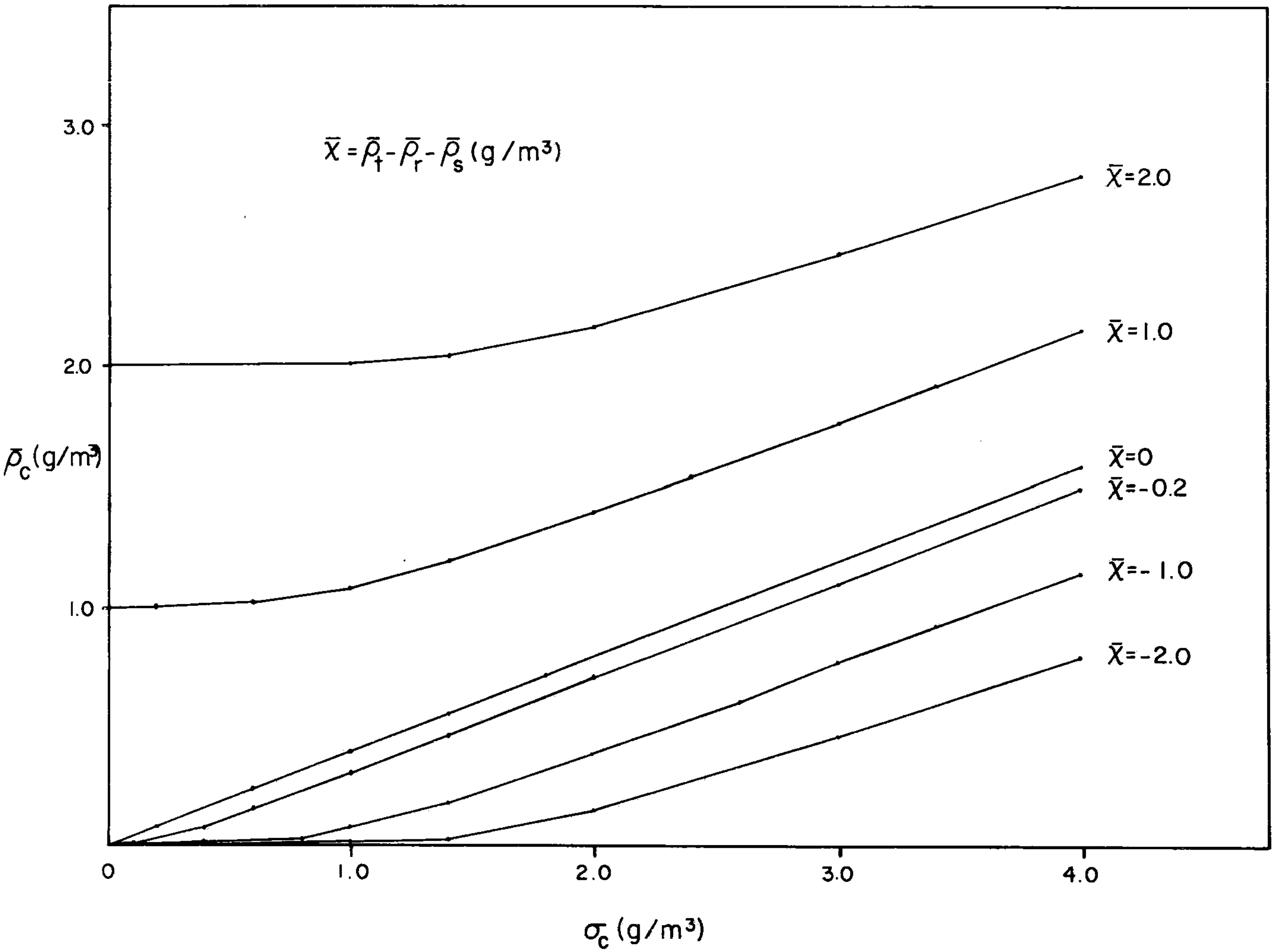

Figure 4.2 Variation of mean cloud liquid water content $\overline{\rho_c}$ as a function of σ_c for various values of average saturation deficit ($\chi < 0$) or excess ($\chi > 0$).

Swarztrauber (1976) has recently reviewed a number of such direct method techniques including cyclic reduction, Fourier analysis and a combination of the two for a variety of boundary conditions. In the present work, the turbulence covariances and variances are modelled by forming equations for the second moments and then closure is obtained by relating the third moments to the first and second.

4.4 Turbulent fluxes

The second moments appearing in the mean equations (4.16) - (4.19) are the turbulent fluxes of momentum, potential density, total water and rain. Equations for these covariances are readily found from (4.3) - (4.7) and (4.9) - (4.15) to be

$$\partial(\overline{u_k''m_i''})/\partial t + \partial(\bar{u}_j\ \overline{u_k''m_i''})/\partial x_j - (\varepsilon_{ijn}f_n\ \overline{m_j''u_k''} + \varepsilon_{kjn}f_n\ \overline{m_j''u_i''}) + \overline{m_k''u_j''}\ \partial\bar{u}_i/\partial x_j + \overline{m_i''u_j''}\ \partial\bar{u}_k/\partial x_j + \rho_o\ \partial(\overline{u_i''u_k''u_j''})/\partial x_j + \partial(\overline{p''u_k''})/\partial x_i + \partial(\overline{p''u_i''})/\partial x_k - \overline{p''(\partial u_i''/\partial x_k + \partial u_k''/\partial x_i)} + g(\overline{\rho''u_i''}\delta_{k3} + \overline{\rho''u_k''}\delta_{i3}) = -2\rho_o\nu_m\ \overline{(\partial u_i''/\partial x_j)(\partial u_k''/\partial x_j)}\ , \qquad (4.27)$$

$$\partial(\overline{u_i''r''})/\partial t + \partial(\bar{u}_j\ \overline{u_i''r''})/\partial x_j - \varepsilon_{ijk}f_k\ \overline{u_j''r''} + \overline{r''u_j''}\ \partial\bar{u}_i/\partial x_j + \overline{u_i''u_j''}\ \partial\bar{r}/\partial x_j + \overline{u_i''m_3''}\ \partial(\ell n\theta_o)/\partial x_3 + \partial(\overline{u_j''u_i''r''})/\partial x_j + \rho_o^{-1}\ \partial(\overline{p''r''})/\partial x_i - \rho_o^{-1}\ \overline{p''\partial r''/\partial x_i} + (g/\rho_o)\ \overline{\rho''r''}\ \delta_{i3} = -(\nu_m+\nu_\theta)\ \overline{(\partial u_i''/\partial x_j)(\partial r''/\partial x_j)}\ , \qquad (4.28)$$

$$\partial(\overline{u_i''\rho_t''})/\partial t + \partial(\bar{u}_j \overline{u_i''\rho_t''})/\partial x_j - \varepsilon_{ijk}f_k \overline{u_j''\rho_t''} + \overline{\rho_t''u_j''} \partial\bar{u}_i/\partial x_j + \overline{u_i''u_j''} \partial\bar{\rho}_t/\partial x_j + \partial(\overline{u_j''u_i''\rho_t''})/\partial x_j + \rho_o^{-1} \partial(\overline{p''\rho_t''})/\partial x_i - \rho_o^{-1} \overline{p''\partial\rho_t''/\partial x_i} + (g/\rho_o) \overline{\rho''\rho_t''} \delta_{i3} = -(\nu_m+\nu_t) \overline{(\partial u_i''/\partial x_j)(\partial\rho_t''/\partial x_j)}, \quad (4.29)$$

$$\partial(\overline{u_i''\rho_r''})/\partial t + \partial(\bar{u}_j \overline{u_i''\rho_r''})/\partial x_j - \varepsilon_{ijk}f_k \overline{u_j''\rho_r''} + \overline{\rho_r''u_j''} \partial\bar{u}_i/\partial x_j + \overline{u_i''u_j''} \partial\bar{\rho}_r/\partial x_j + \partial(\overline{u_j''u_i''\rho_r''})/\partial x_j + \rho_o^{-1}\partial(\overline{p''\rho_r''})/\partial x_i - \rho_o^{-1} \overline{p''\partial\rho_r''/\partial x_i} + (g/\rho_o) \overline{\rho''\rho_r''} \delta_{i3} = -(\nu_m+\nu_r)\overline{(\partial u_i''/\partial x_j)(\partial\rho_r''/\partial x_j)} + A_o(\bar{\rho}_c/\rho_{wt})^{4/3} \overline{u_i''\rho_c''} H(\bar{\rho}_c-\rho_{cm}) + B_o(\bar{\rho}_r/\rho_{wt}) \overline{u_i''\rho_c''} - C_o(\bar{\rho}_r/\rho_{wt})(\overline{u_i''\rho_c''} + \overline{u_i''\rho_s''} + \overline{u_i''\rho_r''} - \overline{u_i''\rho_t''}) , \quad (4.30)$$

where $$\overline{u_i''u_k''} = \overline{m_i''u_k''}/\rho_o = \overline{m_k''u_i''}/\rho_o , \quad (4.31)$$

$$\overline{\rho''u_i''} = -\overline{r''u_i''} - (R_v/R_a-1) \overline{\rho_t''u_i''} - (L/C_{pa}T_o-R_v/R_a) (\overline{\rho_c''u_i''} + \overline{\rho_r''u_i''}) \quad (4.32)$$

$$\overline{\rho_s''u_i''} = q_{so}(L/R_vT_o-1) \{\overline{r''u_i''} + (L/C_{pa}T_o)(\overline{\rho_c''u_i''} + \overline{\rho_r''u_i''})\} , \quad (4.33)$$

$$\overline{\rho''r''} = -\overline{r''^2} - (R_v/R_a-1) \overline{\rho_t''r''} - (L/C_{pa}T_o-R_v/R_a)(\overline{\rho_c''r''} + \overline{\rho_r''r''}), \quad (4.34)$$

$$\overline{\rho''\rho_t''} = -\overline{r''\rho_t''} - (R_v/R_a-1)\,\overline{\rho_t''^2} - (L/C_{pa}T_o-R_v/R_a)\,(\overline{\rho_c''\rho_t''} + \overline{\rho_r''\rho_t''}), \tag{4.35}$$

$$\overline{\rho''\rho_r''} = -\overline{r''\rho_r''} - (R_v/R_a-1)\,\overline{\rho_t''\rho_r''} - (L/C_{pa}T_o-R_v/R_a)\,(\overline{\rho_c''\rho_r''} + \overline{\rho_r''^2}). \tag{4.36}$$

As in all equations for average quantities, the molecular diffusion of the turbulent fluxes is neglected in equations (4.27) - (4.30).

In order to obtain a closed system of equations, we require all the terms in the equations to depend upon only first or second moments of the variables. Turbulent dissipation occurs at scales small enough for local isotropy to apply, and yet the rate of dissipation is controlled by the large scale motions. Thus, following Lumley (1970) and Mellor (1973), we take

$$2\nu_m\overline{(\partial u_i''/\partial x_j)(\partial u_k''/\partial x_j)} = \tfrac{1}{3}(\overline{q^2}/T_1)\delta_{ik}, \qquad \overline{(\partial u_i''/\partial x_j)(\partial s''/\partial x_j)} = 0 \;, \tag{4.37}$$

where $\overline{q^2} = \overline{u_i''^2}$; T_1 is proportional to the external time scale of the turbulence; s" denotes any fluctuating scalar variable.

The action of the pressure fluctuations working against the fluctuating rate of strain causes the turbulence to approach a state of isotropy. Rotta (1951) shows that this can be modelled simply in the form

$$-\overline{p''(\partial u_i''/\partial x_k + \partial u_k''/\partial x_i)} = (\rho_o/\tau_1)(\overline{u_i''u_k''} - \tfrac{1}{3}\,\overline{q^2}\,\delta_{ik}), \tag{4.38}$$

and so analogously we take

$$- \overline{p'' \partial r''/\partial x_i} = \left(\frac{1}{\tau_2} \overline{r'' u_i''} \right), \quad - \overline{p'' \partial \rho_t''/\partial x_i} = (\rho_o/\tau_3) \overline{\rho_t'' u_i''}$$

$$- \overline{p'' \partial \rho_r''/\partial x_i} = (\rho_o/\tau_4) \overline{\rho_r'' u_i''} \quad , \qquad (4.39)$$

where τ_i (i = 1, ...,4) is proportional to the internal time scale of the turbulence. Lilly (1967) suggests that a term proportional to the mean rate of deformation ought to be added to the right hand side of (4.38). However, this term is derived on the basis of rapid deformation theory which is not obviously applicable to atmospheric flows where the deformations are controlled internally. Rapid deformations could be produced perhaps by rapid variations in the mean stratification. Indeed, Deardorff (1974b) finds that such additional terms become important near inversions in the atmosphere. On the other hand, Lumley and Khajeh-Nouri (1973) give some theoretical basis to the suggestion that the pressure - strain rate correlation does not depend explicitly upon the mean strain rate. Moreover, Mellor (1973) includes a mean strain rate term in (4.38), but he finds that the constant of proportionality is very small when his model is adjusted to predict the behavior of the atmospheric equilibrium surface layer.

The third order turbulent transport terms in the equations for the second moments are analogous to the Reynolds stress terms in the equations of mean motion. In equilibrium situations, the turbulent transport terms in (4.27) and (4.28) can possibly be neglected (for example, Mellor, 1973). Deardorff (1974a) finds that, although they contribute significantly to the budgets of turbulent energy and of the mean square fluctuations in temperature and water vapor, they are not important terms in the equations for the turbulent fluxes. We approximate the turbulent transport terms in (4.27) by

$$\rho_o \partial(\overline{u_i''u_k''u_j''})/\partial x_j + \partial(\overline{p''u_k''})/\partial x_i + \partial(\overline{p''u_i''})/\partial x_k = - \frac{1}{3} c_1 \, \delta_{ik} \, \partial\{\overline{q^2} \, T \, \partial(\rho_o \, \overline{q^2})/\partial x_j\}/\partial x_j \; , \qquad (4.40)$$

where T is the external time scale of the turbulence and c_1 is a constant. This representation corresponds simply to the diffusion of turbulent energy with an effective diffusivity of $c_1 \, \overline{q^2} \, T$. There is an equal contribution to each component of $\overline{q^2}$ which implies that the pressure-velocity correlation terms dominate the left hand side of (4.40). Turbulent transport is neglected in the velocity covariance equations; similarly, we take

$$\partial(\overline{u_j''u_i''s''})/\partial x_j + \rho_o^{-1} \, \partial(\overline{p''s''})/\partial x_i = 0. \qquad (4.41)$$

The closure approximations (4.38) - (4.41) completely specify the dissipation and third order terms in (4.27) - (4.30) provided that the turbulent time scales are suitably given (as shown in Section 4.6 below). The present closure scheme can be modified readily to obtain diagnostic equations for the turbulent fluxes of heat and water and for the velocity covariances. This is achieved by neglecting the time derivative terms which correspond to assuming that the turbulent fluxes are in complete local equilibrium, in the sense that the local production terms precisely balance the local dissipation terms. In any case, alternate turbulence parameterization schemes can be adopted without changing the essence of the equations.

Both the variables $\rho_t''-\rho_r'-\rho_s''$ and u_i'' can have any real value. Thus, consistently with the treatment of $\overline{\rho}_c$ in Section 4.3 we assume that the joint probability distribution function of $\rho_t''-\rho_r''-\rho_s''$ and u_i'' is bivariate normal. The turbulent flux of cloud water is therefore found from (4.8) and (4.26) to be

$$\overline{\rho_c''u_i''} = (\overline{\rho_t''u_i''} - \overline{\rho_r''u_i''} - \overline{\rho_s''u_i''})h. \tag{4.42}$$

4.5 Scalar covariances

We now have equations for the mean value and the turbulent fluctuation of each dependent variable. Equations (4.24) and (4.34) - (4.10), however, involve the covariances of the scalars r, ρ_t, ρ_r and ρ_c. The covariance equations for r, ρ_t and ρ_c are obtained from (4.5) - (4.7) and (4.14) - (4.15); thus, neglecting molecular diffusion,

$$\partial(\overline{r''^2})/\partial t + \partial(\overline{u}_j\ \overline{r''^2})/\partial x_j - \overline{r''^2}\ \overline{u}_3\ \partial(\ln\rho_o)/\partial x_3$$
$$+ 2\ \overline{r''u_j''}\ \partial\overline{r}/\partial x_j + 2\ \overline{r''m_3''}\ \partial(\ln\theta_o)/\partial x_3 + \partial(\overline{u_j''r''^2})/\partial x_j$$
$$= -2\nu_\theta\ \overline{(\partial r''/\partial x_j)^2}\ , \tag{4.43}$$

$$\partial(\overline{\rho_t''^2})/\partial t + \partial(\overline{u}_j\ \overline{\rho_t''^2})/\partial x_j - \overline{\rho_t''^2}\ \overline{u}_3\ \partial(\ln\rho_o)/\partial x_3 +$$
$$2\ \overline{\rho_t''u_j''}\ \partial\overline{\rho}_t/\partial x_j + \partial(\overline{u_j''\rho_t''^2})/\partial x_j =$$
$$-2\nu_t\ \overline{(\partial\rho_t''/\partial x_j)^2}\ , \tag{4.44}$$

$$\partial(\overline{\rho_r''^2})/\partial t + \partial(\overline{u}_j\ \overline{\rho_r''^2})/\partial x_j - \overline{\rho_r''^2}\ \overline{u}_3\ \partial(\ln\rho_o)/\partial x_3 +$$
$$2\ \overline{\rho_r''u_j''}\ \partial\overline{\rho}_r/\partial x_j + \partial(\overline{u_j''\rho_r''^2})/\partial x_j = -2\nu_r\ \overline{(\partial\rho_r''/\partial x_j)^2} +$$
$$-2A_o(\overline{\rho}_c/\rho_{wt})^{4/3}\ \overline{\rho_c''\rho_r''}\ H(\overline{\rho}_c-\rho_{cm}) + 2B_o(\overline{\rho}_r/\rho_{wt})\ \overline{\rho_c''\rho_r''} -$$

$$2C_o\ (\bar{\rho}_r/\rho_{wt})(\overline{\rho_c''\rho_r''} + \overline{\rho_s''\rho_r''} + \overline{\rho_r''^2} - \overline{\rho_t''\rho_r''}), \qquad (4.45)$$

$$\partial(\overline{r''\rho_t''})/\partial t + \partial(\bar{u}_j\ \overline{r''\rho_t''})/\partial x_j - \overline{r''\rho_t''}\ \bar{u}_3\ \partial(\ell n\rho_o)/\partial x_3$$

$$+ \overline{r''u_j''}\ \partial\bar{\rho}_t/\partial x_j + \overline{\rho_t''u_j''}\ \partial\bar{r}/\partial x_j + \overline{\rho_t''m_3''}\ \partial(\ell n\theta_o)/\partial x_3 +$$

$$\partial(\overline{u_j''\rho_t''r''})/\partial x_j = -(\nu_\theta+\nu_t)\ \overline{(\partial\rho_t''/\partial x_j)(\partial r''/\partial x_j)}, \qquad (4.46)$$

$$\partial(\overline{r''\rho_r''})/\partial t + \partial(\bar{u}_j\ \overline{r''\rho_r''})/\partial x_j - \overline{r''\rho_r''}\ \bar{u}_3\ \partial(\ell n\rho_o)/\partial x_3 +$$

$$\overline{r''u_j''}\ \partial\bar{\rho}_r/\partial x_j + \overline{\rho_r''u_j''}\ \partial\bar{r}/\partial x_j + \overline{\rho_r''m_3''}\ \partial(\ell n\theta_o)/\partial x_3 +$$

$$\partial(\overline{u_j''\rho_r''r''})/\partial x_j = -(\nu_\theta+\nu_r)\ \overline{(\partial r''/\partial x_j)(\partial\rho_r''/\partial x_j)} +$$

$$A_o(\bar{\rho}_c/\rho_{wt})^{4/3}\ \overline{\rho_c''r''}\ H(\bar{\rho}_c-\rho_{cm}) + B_o(\bar{\rho}_r/\rho_{wt})\ \overline{\rho_c''r''} -$$

$$C_o(\bar{\rho}_r/\rho_{wt})(\overline{\rho_c''r''} + \overline{\rho_s''r''} + \overline{\rho_r''r''} - \overline{\rho_t''r''}), \qquad (4.47)$$

$$\partial(\overline{\rho_r''\rho_t''})/\partial t + \partial(\bar{u}_j\ \overline{\rho_r''\rho_t''})/\partial x_j - \overline{\rho_r''\rho_t''}\ \bar{u}_3\ \partial(\ell n\rho_o)/\partial x_3 +$$

$$\overline{\rho_t''u_j''}\ \partial\bar{\rho}_r/\partial x + \overline{\rho_r''u_j''}\ \partial\bar{\rho}_t/\partial x_j + \partial(\overline{u_j''\rho_r''\rho_t''})/\partial x_j =$$

$$-(\nu_t+\nu_r)\overline{(\partial\rho_t''/\partial x_j)(\partial\rho_r''/\partial x_j)} + A_o(\bar{\rho}_c/\rho_{wt})^{4/3}\ \overline{\rho_c''\rho_t''}$$

$$H(\bar{\rho}_c-\rho_{cm}) + B_o(\bar{\rho}_r/\rho_{wt})\ \overline{\rho_c''\rho_t''} - C_o(\bar{\rho}_r/\rho_{wt})\ (\overline{\rho_c''\rho_t''} +$$

$$\overline{\rho_s''\rho_t''} + \overline{\rho_r''\rho_t''} - \overline{\rho_t''^2}), \qquad (4.48)$$

where $\overline{\rho_s''r''} = q_{so}\ (L/R_vT_o-1)\ \{\overline{r''^2} + (L/C_{pa}T_o)$

$$(\overline{\rho_c''r''} + \overline{\rho_r''r''})\}, \qquad (4.49)$$

$$\overline{\rho_s''\rho_t''} = q_{so}\ (L/R_vT_o-1)\ \{\overline{r''\rho_t''} + (L/C_{pa}T_o)$$

$$(\overline{\rho_t''\rho_c''} + \overline{\rho_t''\rho_r''})\}, \qquad (4.50)$$

$$\overline{\rho_s''\rho_r''} = q_{so}\ (L/R_v T_o - 1)\ \{\overline{r''\rho_r''} + (L/C_{pa}T_o)$$

$$(\overline{\rho_r''\rho_c''} + \overline{\rho_r''^2}\} \ . \tag{4.51}$$

The dissipation and third order terms in (4.43) - (4.48) are parameterized by similar closure approximations to those of Section 4.4. Thus the dissipation terms are modelled so that

$$2\nu_\theta\ \overline{(\partial r''/\partial x_j)^2} = \overline{r''^2}/T_2\ ,\ 2\nu_t\ \overline{(\partial \rho_t''/\partial x_j)^2} = \overline{\rho_t''^2}/T_3\ ,$$

$$2\nu_r\ \overline{(\partial \rho_r''/\partial x_j)^2} = \overline{\rho_r''^2}/T_4\ ,$$

$$(\nu_\theta + \nu_t)\ \overline{(\partial r''/\partial x_j)(\partial \rho_t''/\partial x_j)} = \overline{\rho_t'' r''}/T_5\ ,$$

$$(\nu_\theta + \nu_r)\ \overline{(\partial r''/\partial x_j)(\partial \rho_r''/\partial x_j)} = \overline{\rho_r'' r''}/T_6\ ,$$

$$(\nu_t + \nu_r)\ \overline{(\partial \rho_t''/\partial x_j)(\partial \rho_r''/\partial x_j)} = \overline{\rho_t''\rho_r''}/T_7$$

where T_i $(i = 2,\ldots,7)$ is proportional to the external time scale of the turbulence. Consistently with (4.40) - (4.41), we assume that turbulent transport gives rise to the diffusion of all scalar covariances with an effective diffusivity proportional to $\overline{q^2}\,T$. Hence

$$\overline{u_j''r''^2} = -\ c_2\ \overline{q^2}\ T\ \partial\overline{r''^2}/\partial x_j\ ,\ \overline{u_j''\rho_t''^2} = -\ c_3\ \overline{q^2}\ T\ \partial\overline{\rho_t''^2}/\partial x_j ,$$

$$\overline{u_j''\rho_r''^2} = -\ c_4\ \overline{q^2}\ T\ \partial\overline{\rho_r''^2}/\partial x_j\ ,\ \overline{u_j''\rho_t'' r''} = -\ c_5\ \overline{q^2}\ T$$

$$\partial(\overline{\rho_t'' r''})/\partial x_j ,\ \overline{u_j''\rho_r'' r''} = -\ c_6\ \overline{q^2}\ T\ \partial(\overline{\rho_r'' r''})/\partial x_j\ ,$$

$$\overline{u_j''\rho_t''\rho_r''} = -\ c_7\ \overline{q^2}\ T\ \partial(\overline{\rho_t''\rho_r''})/\partial x_j\ , \tag{4.53}$$

where c_i $(i = 2,\ldots,7)$ is a constant.

The covariances of ρ_c'' with the scalars r'', ρ_c'' and ρ_r'' are specified by assuming that $\rho_t'' - \rho_r'' - \rho_s''$ and the fluctuations in each scalar s'' have a joint probability distribution function which is bivariate normal. This cannot be strictly correct because the scalar fluctuations are bounded below by the negative of their mean values (that is $s'' \geq -\bar{s}$ because $s \geq 0$). However, provided that the standard deviation of s'' $[(\overline{s''^2})^{\frac{1}{2}}]$ is not much larger than the mean value of s ($\bar{s}$), the error associated with this approximation is expected to be small. Thus we find from (4.8) and (4.26) that

$$\overline{\rho_c''r''} = (\overline{r''\rho_t''} - \overline{r''\rho_r''} - \overline{r''\rho_s''})h \; ,$$

$$\overline{\rho_c''\rho_t''} = (\overline{\rho_t''^2} - \overline{\rho_t''\rho_t''} - \overline{\rho_t''\rho_s''})h \; , \qquad (4.54)$$

$$\overline{\rho_c''\rho_r''} = (\overline{\rho_r''\rho_t''} - \overline{\rho_r''^2} - \overline{\rho_r''\rho_s''})h \; .$$

The variance of $\rho_t'' - \rho_r'' - \rho_s''$ is seen from (4.4) and (4.24) to be

$$\sigma_c^2 = \overline{\rho_t''^2} + \overline{\rho_r''^2} + \overline{\rho_s''^2} -$$

$$2\,\overline{\rho_t''\rho_r''} - 2\,\overline{\rho_t''\rho_s''} + 2\,\overline{\rho_r''\rho_s''} \qquad (4.55)$$

where $\overline{\rho_s''^2} = q_{so}^2\,(L/R_vT_o-1)^2\,\{\overline{r''^2} + 2(L/C_{pa}T_o)(\overline{r''\rho_c''}$

$+ \overline{r''\rho_r''}) + (L/C_{pa}T_o)^2\,(\overline{\rho_c''^2} + 2\overline{\rho_c''\rho_r''} + \overline{\rho_r''^2}\,)\}.$

The variance of ρ_c can be calculated from the assumption, used in (4.25), that $\rho_t'' - \rho_r'' - \rho_s''$ is normally distributed with variance σ_c^2. It is found that

$$\overline{\rho_c''^2} = \{(\bar{\rho}_t-\bar{\rho}_r-\bar{\rho}_s)^2 + \sigma_c^2\}h$$

$$+ (2\pi)^{-\frac{1}{2}} \sigma_c(\overline{\rho}_t-\overline{\rho}_r-\overline{\rho}_s) \exp\{-(\overline{\rho}_t-\overline{\rho}_r-\overline{\rho}_s)^2/2\sigma_c^2\} - \overline{\rho}^2. \qquad (4.56)$$

4.6 Time scales of turbulence

The parameterization of the dissipation and third order terms in the equations for the second moments of the dependent variables involves the introduction of turbulent time scales. Although the internal and external scales of turbulence are actually distinct (for example, Lumley and Khajeh-Nouri, 1973), we assume that the turbulence can be characterized by a single time scale T. Thus the external time scales T_i, appearing in equations (4.37) and (4.52), are given by

$$T_i = T/a_i \quad (i = 1,\ldots,7) \qquad (4.57)$$

where a_i is a constant. The turbulent transport terms (4.39) and (4.53) also involve the external time scale T together with the constants c_i (i = 1,...,7). The internal time scale of the turbulence occurs in the pressure correlation terms (4.38) - (4.39), and so we take

$$\tau_i = T/b_i \quad (i = 1,\ldots,4) \qquad (4.58)$$

where b_i is a constant.

Lumley (1970) suggests that $T = \overline{q^2}/\varepsilon$ where ε is the rate of dissipation of turbulent energy per unit mass. But this involves the introduction of a further partial differential equation for ε. When modelling an equilibrium boundary layer flow, it is common (for example, Mellor, 1973) to take T proportional to $x_3\ (q^2)^{-\frac{1}{2}}$. However, such approximations cannot be applied justificably to more general flows. In the absence of any density stratification, T would possibly be determined by the mean strain rate. Thus a scalar invariant form for T, which accounts for the effect of stratification, is given by

$$T^{-2} = (\partial \bar{u}_i/\partial x_j)^2 \ \psi^2(\eta) \tag{4.59}$$

where $\eta = -\ \overline{\rho'' u_3''}\ g/\left(\overline{m_i'' u_j''}\ \partial \bar{u}_i/\partial x_j\right)$ and $\psi^2(0) = 1$.

The dimensionless parameter η is seen from (4.1) to be the ratio of the rate of suppression of turbulence by the density stratification to the rate of production of turbulent energy by the working of the Reynolds stresses against the mean rate of strain field. We expect the turbulent time scale to decrease with increasing instability, and so the dimensionless function ψ ought to be a monotonic decreasing function of η.

The form of $\psi(\eta)$ and the constants a_i, b_i and c_i are obtained by comparing the predictions of the model equations with observations of the constant flux surface layer in the atmosphere. When applied to a constant flux layer, the present model is similar to that of Manton and Cotton (1977), except that turbulent diffusion is neglected in the latter. The transport of scalar quantities in a turbulent flow involves a common mechanism, namely the direct mixing of air masses (Monin and Yaglom, 1971). This similarity of the behavior of scalars suggests that

$$\begin{aligned} &a_2 = a_3 = a_4 = a_5 = a_6 = a_7 \ , \\ &b_2 = b_3 = b_4 \ , \\ &c_2 = c_3 = c_4 = c_5 = c_6 = c_7 \ . \end{aligned} \tag{4.60}$$

Under neutral stability conditions, So and Mellor (1972) and Klebanoff (1955) observe that $\overline{-u_1''u_3''/q^2}$ equals 0.16 when the steady mean wind is in the x_1-direction; Champagne et.al. (1970) find a value of

0.17. The model is consistent with the observations of So and Mellor if

$$b_1 = 1.69 \quad \text{and} \quad a_1 = 2b_1/(2+3b_1{}^2) = 0.32. \tag{4.61}$$

Businger et.al. (1971) find that $(\overline{u_1''u_3''}\ \partial\bar{\rho}/\partial x_3)/(\overline{\rho''u_3''}\ \partial\bar{u}_1/\partial x_3)$ approaches 0.74 as $\overline{\rho''u_3''}$ approaches zero, which implies that

$$b_2 = 1.25\ . \tag{4.62}$$

As $\overline{\rho''u_3''} \to +\infty$, Businger et.al. observe that the limiting Richardson number $(-g\ \partial\bar{\rho}/\partial x_3)/\rho_0\ (\partial\bar{u}_1/\partial x_3)^2$, above which turbulence is suppressed by the stable stratification, is equal to 0.21 and that $(\overline{u_1''u_3''}\ \partial\bar{\rho}/\partial x_3)/(\overline{\rho''u_3''}\ \partial\bar{u}_1/\partial x_3)$ approaches unity. Thus η has a limiting value of 0.21 in steady equilibrium conditions, and the model then yields

$$a_2 = 0.78\ . \tag{4.63}$$

The diffusion constant c_1 and c_2 are found by requiring the model to agree with the observations of Wyngaard et.al. (1971) under steady unstable conditions. In particular, they find that as $\overline{\rho''u_3''} \to -\infty$,

$$\overline{u_3''^2} = 3.6\ (-Kx_3 g\ \overline{\rho''u_3''}/\rho_0)^{2/3}\ ,$$

$$\overline{\rho''^2} = 0.90\ \{(\overline{\rho''u_3''})^2\ \rho_0/Kx_3 g\}^{2/3}\ ,$$

$$g\ \overline{\rho''^2}/\overline{m_3''u_3''}\ \partial\bar{\rho}/\partial x_3 = 1\ ,$$

where the von Karman constant $\kappa = 0.35$. These results agree with the predictions of the model equations when

$$c_1 = 0.48 \quad \text{and} \quad c_2 = 3.37. \tag{4.64}$$

The assumed values of the constants in (4.60) - (4.64) are not sufficient to specify completely the time scale function ψ. However, its limiting behavior in stable and unstable conditions is determined. It is found that

$$\psi^2 = 0.44 \quad \text{when} \quad \eta = 0.21$$

and (4.65)

$$\psi^2 \sim -3.85\eta \quad \text{as} \quad \eta \to -\infty \, .$$

Although η is never greater than 0.21 under steady equilibrium conditions, it could presumably attain any real value during general unsteady conditions. A monotonic decreasing function satisfying (4.59) and (4.65) is

$$\psi^2 = \begin{cases} 1-3.85\eta, & \eta \lesssim 0 \, . \\ \exp(-3.91\eta), & \eta > 0 \end{cases} \tag{4.66}$$

The turbulent time scale T is given by (4.59) and (4.66) provided that $(\partial \overline{u}_i/\partial x_j)^2$ is finite. If the air is stably stratified and there is no mechanical generation of turbulence, then any turbulence is suppressed rapidly. Hence we take, consistently with (4.66),

$$1/T \to 0 \quad \text{as} \quad (\partial \overline{u}_i/\partial x_j)^2 \to 0 \quad \text{with} \quad \eta > 0. \tag{4.67}$$

On the other hand, turbulence can be sustained by an unstable stratification in the absence of any mechanical generation; that is in free convection. The asymptotic form of T as $\eta \to -\infty$ is found from (4.59) and (4.66) by replacing the turbulent fluxes in η by their equivalent eddy diffusivity representations. Hence we take

$$\overline{\rho''u_3''} = -\ K_b\ \partial\overline{\rho}/\partial x_3\ ,$$
$$\overline{u_i''u_j''} = -\ K_m(\partial\overline{u}_i/\partial x_j + \partial\overline{u}_j/\partial x_i)\ , \tag{4.68}$$

where K_b and K_m are the turbulent diffusivities of mass and momentum. In a constant flux surface layer, the model equations predict that

$$K_b/K_m = (\overline{\rho''u_3''}\ \partial\overline{u}_1/\partial x_3)/(\overline{u_1''u_3''}\ \partial\overline{\theta}/\partial x_3) = 2.22. \tag{4.69}$$

Thus, from (4.59), (4.65), (4.68) and (4.69), we find that for free convection

$$1/T^2 \rightarrow +8.56\ (g/\theta_o)(\partial\overline{\theta}/\partial x_3) \quad \text{as}$$
$$(\partial\overline{u}_i/\partial x_j)^2 \rightarrow 0 \text{ with } \eta < 0. \tag{4.70}$$

We assume that (4.70), derived for a constant flux layer, is valid for all free convection situations. Equations (4.59), (4.66), (4.67) and (4.70) specify the turbulent time scale under all conditions.

4.7 Conclusion

Equations (4.16) - (4.26), (4.27) - (4.42), (4.43) - (4.56) and (4.57) - (4.70) form a closed system which describes the first two moments of the dependent variables in deep convection. When the mean motion is three-dimensional, there are 28 partial differential equations; only 27 of these contain time derivatives since only the spatial derivative of $\overline{p'}$ occurs in (4.17). However, if the turbulent fluxes of heat and water and the velocity covariances are assumed to be in complete local equilibrium (as discussed in Section 4.4), then these variables are found diagnostically and the number of partial differential equations involving time derivatives is reduced to 15.

The algebraic equations for the moments involving ρ_c are non-linear and so they cannot be solved explicitly. Hence an iterative procedure must be used to obtain these variables. The turbulent time scale T, given by (4.59) and (4.66), is a non-linear function of the turbulent mass flux and the velocity covariances. Therefore, if diagnostic equations are used to find the turbulent fluxes then the solution procedure must be iterative.

5. CONCLUDING REMARKS

In Chapters 2, 3 and 4, we have derived a set of approximate equations for modeling precipitating, moist, deep convection. The intended application of this system of equations is to the modeling of mesoscale convective systems having horizontal scales of 1.0 to 100.0 km. In Chapter 3, we have derived a first-order parameterization of warm-cloud precipitation processes. This system must be generalized to an ice-phase system in the future. In Chapter 4, the system of equations is averaged and closed with a second-order, turbulent transport model. The turbulent transport model is designed to parameterize the effects of thermals and cumulus towers on the mesoscale. Such cumulus towers can be either isolated towers growing from a weak-planetary-boundary-layer, mesoscale disturbance or be imbedded within an intense mesoscale cumulonimbus system. The turbulence parameterization is being tested in simple, one-dimensional, horizontally homogeneous models before it is coded into the fully three-dimensional system. An objective of the first stage of evaluation is to determine the relative merits (computational expediency versus accuracy) of the fully time-dependent, turbulence-transport equations versus a semi-diagnostic system.

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REFERENCES

Arakawa, A., and W. H. Shubert, 1974: Interaction of cumulus cloud ensemble with the large-scale environment, 1, J. Atmos. Sci., 31, 674-701.

Berry, E. X., 1967: Cloud droplet growth by collection, J. Atmos., Sci., 24, 688-701.

Berry, E. X., 1968: Modification of the Warm Rain Process, Proc. First Nat. Conf. on Weather Modification, Albany, New York, Am. Meteor. Soc., 81-88.

Berry, E. X., R. L. Reinhardt, 1974: An analysis of cloud droplet growth by collection: Part IV. A new parameterization, J. Atmos. Sci., 31, 2127-2135.

Betts, A. K., 1973: Non-precipitating cumulus convection and its parameterization, Quart. J. Roy. Met. Soc., 99, 178-196.

Blanchard, D. C. and A. T. Spencer, 1970: Experiments on the generation of raindrop size distributions by breakup, J. Atmos. Sci., 27, 101.

Businger, J. A., J. C. Wyngaard, Y. Izumi and E. F. Bradley, 1971: Flux-profile relationships in the atmospheric surface layer, J. Atmos. Sci., 28, 181-189.

Champagne, F. H., V. G. Harris, and S. Corrsin, 1970: Experiments on nearly homogeneous turbulent shear flow, J. Fluid Mech., 41, 81-139.

Cotton, W. R., 1972: Numerical simulation of precipitation of development in supercooled cumuli, 1, Mon. Weather Rev., 100, 757-763.

Cotton, W. R., 1972: Numerical simulation of precipitation development in supercooled cumuli, 2, Mon. Weather Rev., 100, 764-784.

Cotton, W. R., 1975a: Data requirements for verification and development of models on the cumulus scale, Third Symp. on Meteor. Observations and Instrumentation, Am. Meteor., Soc., Boston, Mass., 1-4.

Cotton, W. R., 1975b: Modeling of Convective Storms, OPEN SESAME: Proc. of the opening meeting at Boulder, Colo., Sept. 4-6, 1974.

Cotton, W. R., 1975c: Theoretical cumulus dynamics, Rev. Geophys. and Space Phy., 13(2), 419-448.

Cotton, W. R., R. A. Pielke and P. T. Gannon, 1976: Numerical experiments on the cumulus scale, J. Atmos. Sci., 33, 252-261.

Deardorff, J. W., 1972: Numerical investigation of neutral and unstable planetary boundary layers, J. Atmos. Sci., 29, 91-115.

Deardorff, J. W., 1974a: Three-dimensional numerical study of the height and mean structure of a heated planetary boundary layer, Boundary-Layer Meteor., 7, 81-106.

Deardorff, J. W., 1974b: Three-dimensional numerical study of turbulence in an entraining mixed layer, Boundary-Layer Meteor., 7, 199.

Dutton, J. A. and G. H. Fichtl, 1969: Approximate equations of motions for gases and liquids, J. Atmos. Sci., 26, 241-254.

Gunn, R. and G. D. Kinzer, 1949: The terminal velocity of fall for waterdrops in stagnant air, J. Meteor., 6, 243-248.

Hudson, N. W., 1963: Rhodesian J. Agric. Res., 1, 6.

Kessler, E. III, 1969: On the distribution and continuity of water substance in atmospheric circulation, Meteor. Mon. Amer. Meteorol. Soc., 10, 84.

Klebanoff, P. S., 1955: Characteristics of turbulence in a boundary layer with zero pressure gradient, NACA Rept. 1247.

Lilly, D. K., 1967: The representation of small-scale turbulence in numerical simulation experiments, in Proceedings of the IBM Scientific Computing Symposium on Environmental Sciences, IBM Form 320-1951, Thomas J. Watson Research Center, Yorktown, NY.

Liu, J. Y. and H. D. Orville, 1969: Numerical modeling of precipitation and cloud shadow effects on mountain-induced cumuli, J. Atmos. Sci., 26, 1283-1298.

Lumley, J. L., 1970: Toward a turbulent constitutive relation, J. Fluid Mech., 41, 413-434.

Lumley, J. L., and B. Khajeh-Nouri, 1974: Computational modeling of turbulent transport, Proc. Second IUTAM-IUGG Symp. Turbulent Diffusion in Environmental Pollution, Charlottesville, Virginia, Advances in Geophysics, 18A, 169-192.

Manton, M. J., 1974: Tellus, 26, 489.

Manton, M. J. and W. R. Cotton, 1977: Parameterization of the atmospheric surface layer, J. Atmos. Sci., Feb.

Marshall, J. S. and W. McK. Palmer, 1948: The distribution of raindrops with size, J. Meteor., 5, 165-166.

Mason, B. J., 1957: Physics of Clouds, Oxford Press, 481 pp.

Mellor, G. L., 1973: Analytic prediction of the properties of stratified planetary surface layers, J. Atmos. Sci., 30, 1061-1069.

Miller, M. J. and R. P. Pearce, 1974: A three-dimensional primitive equation model of cumulonimbus convection, Quart. J. Roy. Meteorol. Soc., 100, 133-154.

Monin, A. S. and A. M. Yaglon, 1971: Statistical Fluid Mechanics, Vol. 1, M.I.T. Press.

Mueller, E. A. and A. L. Sims, 1966: Relationships between reflectivity, attenuation and rain fall rates derived from drop size spectra, Tech. Rept. ECOM-00032-F, Illinois State Water Survey, Contract DA-28-043 AMC-00032(E), 112 pp.

Ogura, Y., and N. A. Phillips, 1962: Scale analysis of deep and shallow convection in the atmosphere. J. Atmos. Sci., 19, 173.

Pielke, R. A., 1974: A three-dimensional numerical model of the sea breezes over south Florida, Mon. Weather Rev., 102, 115-139.

Reynolds, O., 1901: Papers on Mechanical and Physical Subjects, Vol. 2, Cambridge University Press, 535-577.

Rotta, J. C., 1951: Statische theorie nichthomogener turbulenz, Z. Phys., 129, 547-572.

Schlesinger, R. E., 1973a: A numerical model of deep moist convection, 1, Comparative experiments for variable ambient moisture and wind shear, J. Atmos. Sci., 30, 835-856.

Schlesinger, R. E., 1973b: A numerical model of deep moist convection, 2, A proto-type experiment and variations upon it, J. Atmos. Sci., 30, 1374-1391.

Schlesinger, R. E., 1975: A three-dimensional numerical model of an isolated deep convective cloud: Preliminary results. J. Atmos. Sci., 32, 934-957.

Sinclair, P. C., 1973: Severe storm air velocity and temperature structure deduced from penetrating aircraft. Proceeding of the AMS 8th Severe Local Storms Conference, Denver, Colorado, October 1973, pp. 25-32.

So, R. M. C. and G. L. Mellor, 1972: An experimental investigation of turbulent boundary layers along curved surfaces, NASA CR-1940.

Swarztrauber, P., 1976: The methods of cyclic reduction, Fourier Analysis, cyclic reduction - Fourier Analysis for the Discrete Solution of Poisson's Equation on a Rectangle, submitted for publication, Reviews of SIAM.

Takada, T., 1971: Numerical simulation of a precipitating convective cloud: The formation of a "long-lasting" cloud, J. Atmos. Sci., 28, 350-376.

Warner, J. and J. W. Telford, 1967: Convection below cloud base, J. Atmos. Sci., 24, 374.

Weiler, H. S. and R. W. Burling, 1967: Direct measurements of stress and spectra of turbulence in the boundary layer over the sea, J. Atmos. Sci., 24, 653-664.

Wilhelmson, R., 1974: The life cycle of a thunderstorm in three-dimensions, J. Atmos. Sci., 31, 1629-1651.

Wisner, C., H. D. Orville and C. Myers, 1972: A numerical model of a hail-bearing cloud, J. Atmos. Sci., 29, 1160-1181.

Wyngaard, J. C., O. R. Cote and Y. Izumi, 1971: Local free convection, similarity, and the budgets of shear stress and heat flux, J. Atmos. Sci., 28, 1171-1182.