STOCHASTIC PROPERTIES OF LAKE OUTFLOWS

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August 1966

HYDROLOGY PAPERS
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ACKNOWLEDGMENT

Graduate research assistant R. I. Jeng began the research for this paper while studying towards his Master's Degree, under the advice and supervision of Dr. V. M. Yevdjevich, Professor of Civil Engineering, Colorado State University. Afterwards, the study was continued by both writers, and the final results are given in this paper.

The writers wish to acknowledge the financial support of the U. S. National Science Foundation in the research leading to this hydrology paper. In addition, the National Center for Atmospheric Research, Boulder, Colorado, has helped this study by allowing free computer time. Finally, the writers wish to express their appreciation to Dr. M. M. Siddiqui, Professor of Mathematics and Statistics, for his valuable suggestions in the analytical approach used in this paper.

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ABSTRACT

This paper presents the study of outflow characteristics affected by properties of natural lakes when inflows are log-normally distributed. The general differential equation for water storage, based on the continuity equation, is used in this study, with properties of inflows and lakes given. For the purposes of this investigation, lake properties are described by the storage capacity function and by the outflow rating curve function. Independent inflows are described by the log-normal distribution function with two parameters: the mean, and the standard deviation of logarithms. Moreover, sequential mathematical models are derived for outflows by integrating the storage differential equation under the assumption that the average inflow of a natural lake is equal to the average outflow. Data of inflows generated on a CDC 3600 computer consisted of 10,000 independent standard normal numbers, with mean zero and variance unity. Then these numbers were transformed to a log-normal distribution with various values of standard deviation of logarithms, I_v.

The digital computer produced the independent log-normal numbers, solutions of outflow generating equations, parameters of the outflow distribution, and the first ten serial correlation coefficients of outflow series. Two hundred and ten outflow sequences were generated. They represented the following combinations: (1) five values of I_v, the index of variability of inflows (I_v = 0.15, 0.25, 0.40, 0.60, 0.90); (2) Seven values of n, the ratio between the powers for storage function and the outflow rating function (n = 1/2, 1/3, 3/4, 1, 3/2, 2, 3); and (3) six values of d, a lumped dimensionless parameter descriptive of inflow, lake properties, and time interval, Δt, used for the finite difference integration of the differential equation (d = 0.3, 1.0, 3.0, 10, 30, 100).

The properties of log-normal distribution related to this study are given in summary form in Chapter II. Chapter III has as its subject the properties of natural lakes, mathematical model for generation of outflow sequences and its parameters. Outflow characteristics, analytical solution for n = 1, and the relationship of parameters of outflow and inflow are presented in Chapter IV. A worthwhile further study would be to find analytically the exact distribution function of outflow for given lake and inflow conditions, for any value of n.

Equations for the ratio of statistical parameters of outflow to those of inflow, as function of d for n equal to the values other than unity, are empirically found, and are based on the equations for n = 1 which are analytically obtained. Those empirical equations show that the statistical parameters of outflows from natural lakes are less than and converge to those of inflows as the parameter d increases to infinity.

STOCHASTIC PROPERTIES OF LAKE OUTFLOWS[1]

By: Raymond I. Jeng[2] and Vujica M. Yevdjevich[3]

CHAPTER I

INTRODUCTION

In many water resources development problems, natural storage and the characteristics of the regulated outflow from natural lakes are important assets in addition to those of storage by reservoirs or other artificial controls. The reservoir storage problems have been extensively studied in the past for various probability distributions and for the time dependence of the inflow variable. Because the reservoir inflow and outflow sequences are complicated in practice, no general analytical solutions for reservoir regulation problems have yet been developed. The regulation problems of natural lakes look simpler to solve than those of reservoirs. Unlike outflow from a reservoir which is regulated by a hydraulic structure and subject to varying water demand, the outflow from a natural lake is only governed: by the type of inflow, by the lake outflow rating curve, and by storage properties of that lake.

Although a theoretical probability distribution function and time dependence equation may be used to describe mathematically the characteristics of observed outflow for natural lakes, an alternative approach is used in this study. Characteristics of outflow distribution and of its time dependence are described in terms of the most important parameters of inflow, outflow rating curve, and lake storage. The outflow sequences are obtained mainly by the data generation method (Monte Carlo method) starting from inflow and lake characteristics. In particular, natural lakes with independent inflows log-normally distributed are studied. The CDC 3600 computer was used: (a) to solve the storage differential equation in finite difference form; (b) to evaluate parameters of the outflow distribution; and (c) to compute the first ten autocorrelation coefficients from the generated series of outflow as the empirical time dependence. Generated data consisted of 10,000 independent numbers normally distributed, with mean zero and variance unity, which have been transformed to log-normal distribution with various parameters.

Listed below are the assumptions and approaches that were used in this study in deriving the outflow characteristics from given inflow and natural lake properties:

1. The mathematical model of outflow sequences was derived from the general storage differential equation obtained by using the outflow rating curve and the storage capacity function of a lake for given distribution of independent inflows. Next, the two relationships of rating curve and storage capacity were assumed to be power functions of the water depth above the level of zero outflow. Last, the average outflow was assumed to equal to the average inflow. Therefore, the evaporation and seepage from the lake were neglected.

2. Seven parameters described inflow, rating curve and storage function of lakes. They were reduced to three parameters and were introduced in the storage finite difference equation. The ranges of these parameters were selected to cover the practical cases.

3. The mathematical functions for variance, skewness, and excess of the outflow distribution, as well as for the serial correlation coefficients of outflows, for the parameter n (to be defined in later text) being unity, are theoretically derived from the sequential mathematical model for given values of the other two parameters. The working assumption in this study was that inflows are mutually independent and independent of previous outflows.

4. Equations of the same type as for $n = 1$ are fitted to relationships of outflow parameters for the values of n different from unity, as obtained from generated time series.

[1] A small version of this paper is submitted to Lake Garda Symposium (October 1966) of the International Association for Scientific Hydrology.

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CHAPTER II

SUMMARY OF PROPERTIES OF LOG-NORMAL DISTRIBUTION

1. Definition and expression for log-normal distribution. The log-normal distribution (1, 2)* herein is defined as the distribution of a variable whose logarithms obey the normal law of probability. It is the nature of this distribution to allow certain properties of the log-normal function to be derived immediately from those of the normal distribution. Nevertheless, there are still some features of the log-normal function which have no analogies in normal theory. Many hydrologic variables approximately follow the log-normal distribution.

In this paper, a positive variable X $(0 < X < \infty)$ - with $Y = \ln X$ normally distributed with mean μ and σ^2 - is log-normally distributed.

Furthermore, the probability density of X is obtained as:

$$f(X) = \frac{1}{X} \frac{1}{\sqrt{2\pi}\,\sigma_n} e^{-(\ln X - \mu_n)^2/2\sigma_n^2} \qquad 2.1$$

with μ_n the mean of logarithms of X (or the logarithm of geometric mean) and σ_n^2 the variance of ln X. The range of X is from 0 to $+\infty$, while the mean of X is μ and the variance of X is σ^2.

2. Relationship between parameters. Parameters μ_n and σ_n, and μ and σ are moment parameters and are usually estimated from samples available by the method of moments.

The relationship between them is:

$$\mu = e^{\mu_n + \frac{1}{2}\sigma_n^2} \qquad 2.2$$

and

$$\sigma^2 = e^{2\mu_n + \sigma_n^2}\,(e^{\sigma_n^2} - 1)\,. \qquad 2.3$$

If one lets $e^{\sigma_n^2} - 1 = \eta^2$, then $\sigma^2 = \mu^2\eta^2$, and $\eta = \sigma/\mu$, with η the coefficient of variation of the distribution.

The coefficient of skewness, γ_1, is

$$\gamma_1 = \eta^3 + 3\eta\,, \qquad 2.4$$

and the coefficient of excess, γ_2, is

$$\gamma_2 = \eta^2(\eta^6 + 6\eta^4 + 15\eta^2 + 16). \qquad 2.5$$

Similarly, the parameters μ_n and σ_n, expressed by μ and σ, are

$$\mu_n = \ln \frac{\mu^2}{\sqrt{\mu^2 + \sigma^2}} = \ln \frac{\mu}{\sqrt{1 + \eta^2}} \qquad 2.6$$

and

$$\sigma_n = [\ln(1 + \eta^2)]^{1/2}. \qquad 2.7$$

3. Properties of log-normal distribution as related to this study. The following properties of log-normal distribution were used in this study:

(1) Any log-normal distribution with the two parameters, μ_n and σ_n, may be reduced to a log-normal distribution with one-parameter in the following way. By reducing the variable X to a new variable $K = X/\bar{X}$, the modular coefficients are obtained. If they are log-normally distributed, their standard deviation of logarithms, or s(ln K), is a sufficient parameter to describe the distribution. The symbol I_v, the index of variability, was used in the following text as the main parameter of log-normal inflow distributions.

(2) The independent numbers which are normally distributed, with mean of zero and variance of unity, can easily be transformed to modular coefficients which are log-normally distributed, with mean of unity and the standard deviation of logarithms of modular coefficients being the index of variability, I_v. Therefore, it is easy to obtain by the data generation method (Monte Carlo method) sequences which are sufficiently long and are log-normally distributed with given parameter I_v.

(3) The annual inflows into natural lakes very often satisfy the two conditions set up for this study: (a) they are approximately log-normally distributed; and (b) they are approximately independent in sequence.

In case the inflows are serially correlated, the new parameters of time dependence of inflows must be used besides the three parameters described in the text below.

* The figures in parentheses ( ) designate the references which are given at the end of the text.

CHAPTER III

GENERATION OF OUTFLOW SEQUENCES

1. Basic storage equation. The characteristics of outflow from a natural lake depend on the following three groups of parameters: (a) the outflow rating curve parameters which depend on the shape of outlet cross-section, and on the slope and roughness of the outflowing river reach; (b) the parameters of storage function which relate the storage capacities of natural lakes to the lake levels or depths; and (c) the parameters of inflows. The sequences of outflow are best derived from the above three types of parameters by using the continuity equation. It expresses the basic relation between inflow, outflow and storage, and is given as (6, 8):

$$P - Q = \frac{dS}{dt} \qquad 3.1$$

where P is the inflow discharge, Q is the outflow discharge, S is storage volume above a given level, t is time, and dS/dt is the rate of storage change with time. The dynamic partial differential equation of unsteady water movement needs not be used for the natural lakes, because the velocities through lakes are very small and dynamic effects are negligible. Therefore, it is sufficient to use the continuity equation 3.1.

2. Properties of natural lakes. The most important properties of lakes treated here were the storage function and the outlet rating function. Both were expressed as approximations by the same type of mathematical function of water depth as referred to the datum of zero outflow.

The relation between the lake volume and water depth can very often be expressed as a good approximation by the following power function (6, 8, 9)

$$S = a H^m \qquad 3.2$$

with S the storage volume, and H the depth of water above the datum of zero outflow. The reference level of eq. 3.2 can also be the lowest possible level of the lake. However, the datum of zero outflow has a big advantage, namely to allow the use of the same datum as for the rating curve. The parameters a and m of eq. 3.2 depend on the datum selected.

The outflow rating curve is defined here as the relation of the outflowing discharge to the water depth in the lake above the datum of zero outflow. It can often be fitted by a power function of the type (6, 8)

$$Q = b H^r \qquad 3.3$$

with Q the outflowing discharge and H the depth of water. The datum of eq. 3.3 is the level of outlet cross-section with zero outflow, and it is the same as the datum selected for eq. 3.2. Parameters b and r of eq. 3.3 depend on: the shape of the outlet, channel roughness, water surface slope of the outflow river reach, and the type of flow. Whenever the storage function is mentioned in this study, it will be subject to the condition of the datum being the level of zero outflow. In some cases, this modified relation fits the upper points of the storage curve better than it fits the whole curve from the lowest to the highest level of a body of water. In practice, owing to existing problems of sedimentation, the coefficients a and m of eq. 3.2 change with time. They will be considered constants for a given lake to make the problem mathematically tractable. In case the parameters a, m, b, and r change with time, the ordinary differential equation, eq. 3.1, becomes a partial differential equation.

3. Derivation of mathematical model for generation of outflow sequences. From eq. 3.3, the expression $H = (Q/b)^{1/r}$ is obtained. Substituting this expression in eq. 3.2, then

$$S = a(Q/b)^{m/r} = a/b^n \, Q^n = \frac{1}{c} Q^n \qquad 3.4$$

where $n = m/r$ and $c = b^n/a$.

The differentiation of eq. 3.4 gives $dS = \frac{1}{c} n \, Q^{n-1} \, dQ$, and using $dS = (P - Q)\,dt$ from eq. 3.1, then

$$\frac{1}{c} n \, Q^{n-1} \, dQ = (P - Q)\, dt.$$

The general differential equation for the generation of outflows is (6)

$$\frac{dQ}{dt} - \frac{c}{n} Q^{-n+1} P + \frac{c}{n} Q^{-n+2} = 0. \qquad 3.5$$

This equation can be expressed as a difference equation by letting:

$$dQ = \Delta Q = Q_{i+1} - Q_i, \text{ and } dt = \Delta t,$$

where $Q = \frac{1}{2}(Q_{i+1} + Q_i)$, with Q_i, actual outflow at i^{th} time and Q_{i+1} at $(i+1)^{th}$ time, and $P = \frac{1}{2}(P_{i+1} + P_i)$, with P_i actual inflow at i^{th} time, and P_{i+1} at $(i+1)^{th}$ time. If the modular coefficient $X_i = P_i/P_o$ and $Y_i = Q_i/Q_o$ were used here instead of P_i and Q_i, then $P_i = X_i P_o$, $Q_i = Y_i Q_o$, where P_o is the average inflow discharge, and Q_o is the average outflow discharge. The average inflow equals the average outflow of a lake for a sufficiently long term if the evaporation, the precipitation on the lake, and the water seepage outside the outflowing river are neglected, so that $P_o = Q_o$.

Accordingly, eq. 3.5 becomes:

$$Y_{i+1} - Y_i - \frac{c}{n} \Delta t \; P_o^{-n+1} \left(\frac{1}{2}\right)^{-n+2} (Y_{i+1} + Y_i)^{-n+1}$$

$$(X_{i+1} + X_i - Y_{i+1} - Y_i) = 0.$$

In addition, if one lets

$$(\frac{1}{2})^{-n+2} c \ \Delta t \ P_o^{-n+1} = d ,$$

which is a dimensionless parameter, then the result becomes

$$Y_{i+1} = Y_i + \frac{d}{n} (Y_{i+1} + Y_i)^{-n+1} (X_{i+1} + X_i - Y_{i+1} - Y_i) . \qquad 3.6$$

This is the mathematical model which was used in this study for the generation of outflow sequences. To sum up, the difference between eqs. 3.5 and 3.6 is in their dimensions; the latter is dimensionless while the former has dimensions of L^3/T^2.

4. Description of parameters in the sequential mathematical model. Originally there were seven parameters involved in the derivation of sequential mathematical model: (a) Two parameters, a and m, which are in the storage function; (b) Two parameters, b and r, which are in the outflow rating curve function; (c) One parameter, Δt, used for changing the differential equation to the difference equation; (d) The parameter, P_o, the average of streamflow introduced by using the modular coefficients; and (e) The parameter, I_v, the index of variability of inflow, used for specifying the characteristics of inflow. In case the inflows are time dependent, the additional parameters are those of the time dependence mathematical models. However, only the independent inflows are treated in this paper. As a result, the above seven parameters were reduced to only three parameters which were used in the mathematical model for the generation of outflow sequences. These three basic parameters are: n, I_v, and d. They are defined as follows:

(a) The parameter n. The parameter n is defined as m/r, or as the ratio of powers for storage function and outflow rating curve function. This parameter describes the ratio of the rate-of-change of the lake storage and rate-of-change of the outflow rating curve, with the depth of the lake above the datum of zero outflow. Its values depend entirely upon the storage function and outflow rating curve function. They range from 1/4 to 4 in nearly all cases for the free surface outflow.

The parameter m in eq. 3.2 ranges within the limits 1 to 5; it depends mostly on the range of levels, and also on the shape of the natural lake. For a high reference level and a small range of levels, m is usually 1.0 to 1.5 and rarely greater than 2. For the highest range of levels, it is 2 to 5. The range of the parameter r of eq. 3.3 is usually small, 1.5 to 3.0. The parameters a and b have wide ranges of variation, the variation for b depending on the dimensions used for Q and H, and on the outlet river reach cross-sections, river water surface slope, and bed roughness.

For n < 1, a natural lake has a larger value of r than of m. This indicates that the slopes of lake shores are steep and that the outlet cross-section has slow rising banks. For n > 1, the value of m is larger than r. This kind of lake has flat lake banks and steep outlet cross-section banks. Thus, the value of n, indicating the ratio of the rate-of-change of lake (storage function) and outlet rating curve with depth, can be used as an index, whether the ratio of bank slopes of a natural lake and of its outflow cross-section is large or small. Table 1 gives values of n for seven lakes in the Unites States, which are taken as examples. The practical range of n is between 0.50 - 1.00 in these cases.

(b) The parameter d. This parameter, $d = (\frac{1}{2})^{-n+2} (b^n/a) \Delta t \ P_o^{-n+1}$, represents the relationship of average inflow, time unit selected, and the properties of the storage function and the outflow rating curve. It is dimensionless. The longer time unit selected, the larger is the value of d. The effect of mean inflow, P_o, on the value of d depends on the value of n.

For the mathematical model of outflow sequences, when d tends to zero, eq. 3.6 becomes $Y_{i+1} = Y_i$. Or, if the outflow is assumed equal to inflow initially, the outflow is constant. For a very small value of d, regardless of the value of n and the inflow characteristics, the outflow is nearly constant. For large values of d, the characteristics of the sequence of outflows will be very much influenced by the value of n and the characteristics of inflows. For a given value of n, the differences between the outflow and inflow characteristics depend on the value of d. The differences will be less as the value of d increases. Also, the differences increase as the value n increases. This dimensionless and always positive parameter, d, has a relatively large range of values. Seven lakes have been analyzed for finding the values of d, and they are given in Table 1. If Δt equals a year, the seven values of d range from 0.628 to 25.2. The range from 0.3 to 100 is investigated in this paper.

(c) The parameter I_v. This parameter, the index of variability (3) is the standard deviation of logarithms of the modular coefficients of inflows. Therefore, the symbol I_v describes the characteristics of inflow.

Because the modular coefficients were used in this paper, I_v has the property of being dimensionless. As the coefficient of variation equals the index of variability, the mean of logarithms of the modular coefficients of stream flow is equal to one half of the index of variability. This makes the variance of the modular coefficients a function of the variance of the logarithms of modular coefficients only.

From the properties of the log-normal distribution, the stream flow has large values of variance, standard deviation, skewness, and excess if I_v is a large value. Since these parameters can be expressed by the index of variability, it is possible to determine the shape of the inflow distribution and its characteristics by using the value of the index of variability only. As soon as the average inflow P_o and the index of variability I_v are known for log-normally distributed inflows, the inflow conditions are uniquely defined.

Finally the range of this parameter for annual river flows is from zero to unity, rarely exceeding the unity. Figure 1 gives the frequency density, $f(I_v)$, and frequency distribution, $F(I_v)$, for annual flow of 140 rivers taken from Colorado State University Hydrology Paper No. 1 (7). For the purposes of this study, five values of I_v are studied:

0.15, 0.25, 0.40, 0.60, and 0.90, covering, thus, the practical range for annual river flows.

Table 1 Values of parameters n and d for seven lakes in the United States given as examples

No.	Name	State	Average Discharge (c.f.s.)	n	d
1	Flathead Lake	Montana	11,620	0.61	5.02
2	Coeur d'Alene Lake	Idaho	6,023	0.662	13.4
3	Henry's Lake	Idaho	47.2	0.554	2.75
4	Priest Lake	Idaho	1,250	0.778	12.80
5	Pend Oreille Lake	Idaho	21,550	0.855	25.2
6	Bear Lake	Utah	300	0.65	0.628
7	Clear Lake	California	503	0.733	1.49[1]
	Clear Lake	California	503	0.64	1.175[2]

[1] The 1901-1903 outflow rating curve was used.

[2] The 1905-1909 outflow rating curve was used.

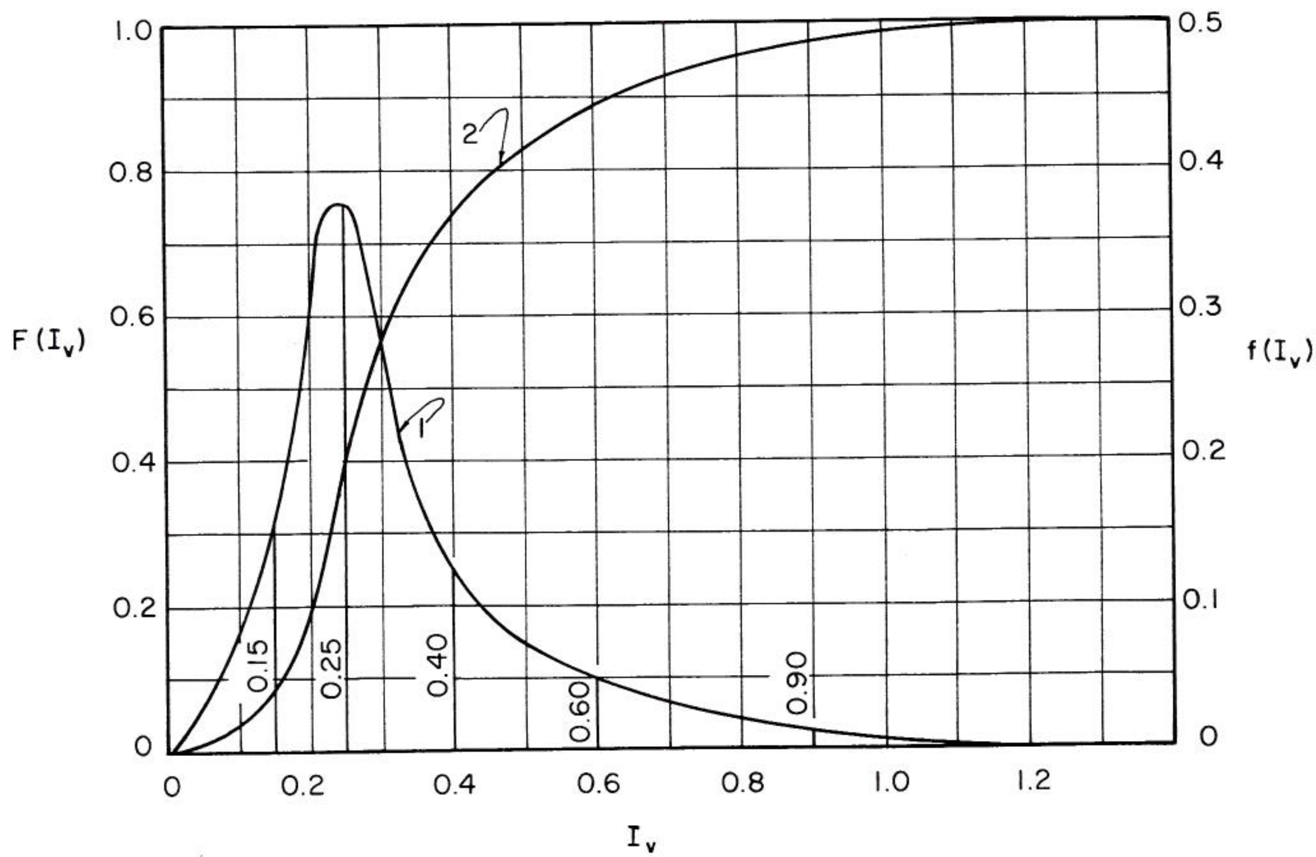

Fig. 1 Frequency density (1) and frequency distribution (2) of the index of variability (I_v) for 140 river gaging stations taken from CSU Hydrology Paper No. 1.

CHAPTER IV

CHARACTERISTICS OF OUTFLOWS

The exact distribution function of outflow for given inflows and lake characteristics was not arrived at in this paper. Rather than solve the general differential equation 3.5 analytically, the difference equation, eq. 3.6, was used to determine the outflows by the data generation method. First, a total of 10,000 independent values of inflow were generated for each run. Then the distributions of outflow and their dependence in sequence were determined from the 10,000 outflows generated by eq. 3.6. Next, the parameters which describe the outflow distribution, such as mean, variance, standard deviation, skewness, excess, coefficient of variation, and the serial correlation coefficients for the first ten lags, were computed on a CDC 3600 digital computer. The 10,000 independent numbers used for generation of inflows were normally distributed with mean zero and variance unity. They were transformed to log-normal distributions with various I_v values.

1. Analysis of equation for generating outflow sequences. The equation for outflow sequences depends partly upon the value of n. Therefore, seven values of n, 1/3, 1/2, 3/4, 1, 3/2, 2, and 3, were analyzed as separate cases.

For the value n equals unity, which is the most simple case, eq. 3.6 becomes linear, or

$$Y_{i+1} = [Y_i + d(X_{i+1} + X_i - Y_i)]/(1 + d) \quad 4.1$$

in which Y_{i+1} can easily be solved since Y_i, X_{i+1}, X_i, and d are all known.

For n equals 2, eq. 3.6 becomes

$$Y_{i+1}{}^2 + \frac{d}{2} Y_{i+1} - Y_i{}^2 + \frac{d}{2} Y_i - \frac{d}{2} (X_{i+1} + X_i) = 0. \quad 4.2$$

Equation 4.2 is quadratic, and Y_{i+1} can be immediately found in

$$Y_{i+1} = -\frac{d}{4} + \frac{1}{2} \sqrt{\frac{d^2}{4} + 4Y_i{}^2 + 2d(X_{i+1} + X_i - Y_i)} \quad 4.3$$

The positive value of the square root was chosen because the outflows of the natural lake are not negative in the majority of practical cases.

2. Generation of independent log-normal inflows. When the inflows are log-normally distributed, the modular coefficients of inflows are also log-normal. Hence, a new variable $Z_i = \ln X_i$ is normally distributed with mean $E[\ln X] = \mu_z$ and the variance $I_v{}^2$. The independent standard normal variable, t_i, is obtained by the transformation:

$$t_i = (\ln X_i - \mu_z)/I_v = (Z_i - \mu_z)/I_v . \quad 4.4$$

Since X_i is the modular coefficient, its mean is unity. By using the relationship between the parameters of the log-normal distribution and corresponding normal distribution, eqs. 2.2, 2.3, 2.6, and 2.7 give

$$\overline{X} = e^{\mu_z + \frac{1}{2} \text{var } Z}$$

and

$$\text{var } X = e^{2\mu_z + \text{var } Z} (e^{\text{var } Z} - 1).$$

Because $\overline{X} = 1$ and $\text{var } Z = I_v{}^2$, then $\mu_z = -\frac{1}{2} I_v{}^2$.

If one substitutes these expressions in eq. 4.4, the values of X_i are

$$X_i = e^{Z_i} = e^{I_v t_i - \frac{1}{2} I_v{}^2} \quad 4.5$$

where $Z_i = I_v t_i + \mu_z$, with t_i representing the independent standard normal numbers. Similarly

$$\text{Var } X = e^{I_v{}^2} - 1 , \quad 4.6$$

or the variance of modular coefficients of inflows can be expressed in terms of I_v only.

The 10,000 independent numbers normally distributed with mean zero and variance unity, assuming the initial outflow is equal to initial inflow, were generated on the CDC 3600 computer and were used as t_i numbers in eq. 4.5 to obtain X_i numbers. Sequences of inflows were then obtained for five different values of I_v, 0.15, 0.25, 0.40, 0.60, 0.90. Sequences of outflows were obtained by using eqs. 4.1 and 4.3 for the two values n = 1 and n = 2. For other values of n, eq. 3.6 was used. Six different values of d, 0.3, 1, 3, 10, 30, and 100 were used for generating the outflows. The generated outflow sequences, each n = 10,000 long, represent various combinations of n, I_v, and d, or 7 x 5 x 6 = 210 different cases.

3. Determination of outflow characteristics. Parameters of the outflow, Y_i, computed from 10,000 values of outflow are:

(1) The mean

$$\overline{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i \quad 4.7$$

(2) The variance

$$s_y{}^2 = \frac{1}{N} \sum_{i=1}^{N} (Y_i - \overline{Y})^2 \quad 4.8$$

(3) The coefficient of variation

$$C_v = \eta = \frac{s_y}{\overline{Y}} \qquad 4.9$$

(4) The skewness coefficient

$$\gamma_1 = \sum_{i=1}^{N} (Y_i - \overline{Y})^3 / N s_y^{\,3} \qquad 4.10$$

(5) The excess coefficient

$$\gamma_2 = \frac{\sum_{i=1}^{N} (Y_i - \overline{Y})^4}{N s_y^{\,4}} - 3 \qquad 4.11$$

and

(6) The serial correlation coefficients

$$r_k = \frac{\sum_{i=1}^{N-k} (Y_i Y_{i+k}) - \frac{1}{N-k} \sum_{i=1}^{N-k} Y_i \sum_{i=1}^{N-k} Y_{i+k}}{\left[\sum_{i=1}^{N-k} Y_i^{\,2} - \frac{1}{N-k}\left(\sum_{i=1}^{N-k} Y_i\right)^2\right]^{1/2}} \times$$

$$\times \left[\sum_{i=1}^{N-k} Y_{i+k}^{\,2} - \frac{1}{N-k}\left(\sum_{i=1}^{N-k} Y_{i+k}\right)^2\right]^{-1/2} \qquad 4.12$$

with r_k the k^{th}-lag serial correlation coefficient and k the time lag.

4. Analytical solution for n = 1. Only for n equals unity, the parameters of the outflow distribution are determined analytically from the equation of outflow sequences when the inflows are mutually independent and independent of previous outflows. For n = 1, eq. 4.1 can be written as:

$$Y_{i+1} = \frac{1-d}{1+d} Y_i + \frac{d}{1+d} (X_{i+1} + X_i) . \qquad 4.13$$

Taking the expected value of both sides,

$$E(Y_{i+1}) = \frac{1-d}{1+d} E(Y_i) + \frac{d}{1+d} [E(X_{i+1}) + E(X_i)] \text{ or } \frac{2d}{1+d} E(Y) = \frac{2d}{1+d},$$

so that $E(Y) = 1$. Thus, the expected value of the modular coefficients of outflows equals unity. This result satisfies the assumption that the average outflow equals the average inflow. Multiplying eq. 4.13 by X_{i+1} and taking the expected values, one finds that

$$E[X_{i+1} Y_{i+1}] = \frac{1-d}{1+d} E[X_{i+1} Y_i] + \frac{d}{1+d} [E(X_{i+1}^{\,2}) + E(X_{i+1} X_i)] . \qquad 4.14$$

According to the assumptions that X_{i+1} and Y_i are independent, one concludes that $E(X_{i+1} Y_i) = E(X_{i+1}) E(Y_i) = 1$; that X_{i+1} and X_i are independent, so one concludes that $E(X_{i+1} X_i) = 1$ and $\text{var } X + 1 = E(X^2)$. Then eq. 4.14 becomes

$$E[X_{i+1} Y_{i+1}] = \frac{1-d}{1+d} + \frac{d}{1+d} [\text{var } X + 2] =$$

$$= 1 + \frac{d}{1+d} \text{ var } X , \qquad 4.15$$

and from eq. 4.13, the following expression was derived

$$\text{var } Y_{i+1} = \left(\frac{1-d}{1+d}\right)^2 \text{var } Y_i + \left(\frac{d}{1+d}\right)^2 [2 \text{ var } X + 2 \text{ cov}(X_{i+1}, X_i)] + 2\frac{d(1-d)}{(1+d)^2} \text{cov}(Y_i, X_{i+1} + X_i) \qquad 4.16$$

Because $\text{cov}(X_{i+1}, X_i) = 0$ and $\text{cov}(Y_i, X_{i+1}) = 0$, then $\text{cov}(X_i, Y_i) = E(X_i Y_i) - E(X_i) E(Y_i) = \frac{d}{1+d} \text{var } X$. By using these relationships, eq. 4.16 gives

$$\frac{\text{var } Y}{\text{var } X} = \frac{d}{1+d} \qquad 4.17$$

or

$$\text{var } Y = \frac{d}{1+d} (e^{I_v^{\,2}} - 1). \qquad 4.18$$

To find the skewness and excess coefficients of outflow, the expected values for X^2, X^3, X^4, Y^2, Y^3, Y^4, XY^2, X^2Y, X^3Y, and XY^3 must be determined. By using the moment generating function, then

$$E(X^t) = e^{(\overline{\ln X}) t + \frac{1}{2} I_v^{\,2} t^2} = e^{\frac{1}{2} I_v^{\,2} t (t-1)} . \qquad 4.19$$

$E(X^2)$, $E(X^3)$, and $E(X^4)$ can be obtained immediately from eq. 4.19 as

$$E(X^2) = e^{I_v^{\,2}} ; \quad E(X^3) = e^{3I_v^{\,2}} ; \quad E(X^4) = e^{6I_v^{\,2}} .$$

In a similar way, the following expected values were determined:

$$E(X^2Y) = \frac{1}{1+d} (e^{I_v^{\,2}} + d e^{3I_v^{\,2}})$$

$$E(XY^2) = \frac{1-d}{(1+d)^2} + \frac{3d e^{I_v^{\,2}}}{(1+d)^2} + \left(\frac{d}{1+d}\right)^2 e^{3I_v^{\,2}}$$

$$E(Y^3) = \frac{1}{(1+d)^2 (3+d^2)} [3(1-d)(1+d^2) + 3d(3-d+2d^2) e^{I_v^{\,2}} + d^2(4-d+d^2) e^{3I_v^{\,2}}]$$

and

$$E(Y^2) = \frac{1 + d e^{I_v^{\,2}}}{1+d} .$$

Further, the expression for the skewness coefficient is

$$\gamma_1 = \frac{M_3}{s_y^{\,3}} = \left[\left(\frac{d}{1+d}\right)(e^{I_v^{\,2}} - 1)\right]^{-3/2} [E(Y^3) - 3E(Y^2) + 2] . \qquad 4.20$$

By using the above expected values for Y^3 and Y^2, the ratio of the skewness coefficient of outflows, $\gamma_1 = \gamma_1(Q)$, and the skewness coefficient of inflows, $\gamma_1(P)$, becomes

$$\frac{\gamma_1(Q)}{\gamma_1(P)} = \frac{(4 - d + d^2)\sqrt{d}}{(3 + d^2)\sqrt{1 + d}} \qquad 4.21$$

with $\gamma_1(P)$ given as a function of $\eta = C_v$ by eq. 2.4, with $\eta = (e^{I_v^2} - 1)^{1/2}$. Therefore, the ratio of eq. 4.21 is independent of I_v, and is expressed only in terms of d.

Similarly, for the coefficient of excess, the following expected values were needed:

$$\underline{E(XY^3)} = \left(\frac{d}{1 + d}\right)^3 e^{6I_v^2} + \frac{d^2(13 + 3d^2)}{(1 + d)^3(3 + d^2)} e^{3I_v^2} + \frac{3d^2}{(1 + d)^3} e^{2I_v^2} + \frac{3d(1 - d)(6 - d + d^2)}{(1 + d)^3(3 + d^2)} e^{I_v^2} + 3\left(\frac{1 - d}{1 + d}\right)^3 \frac{1}{3 + d^2}$$

$$\underline{E(X^2Y^2)} = \left(\frac{d}{1 + d}\right)^2 e^{6I_v^2} + \frac{2d}{(1 + d)^2} e^{3I_v^2} + \frac{d}{(1 + d)^2} e^{2I_v^2} + \frac{(1 - d)}{(1 + d)^2} e^{I_v^2}$$

$$\underline{E(X^3Y)} = \frac{1}{1 + d} e^{3I_v^2} + \frac{d}{1 + d} e^{6I_v^2} \quad \text{and}$$

$$\underline{E(Y^4)} = \frac{d^3(2 - d + d^2)}{(1 + d^2)(1 + d)^3} e^{6I_v^2} + \frac{4d^2(4 - 3d + 7d^2 - d^3 + d^4)}{(1 + d)^3(3 + d^2)(1 + d^2)} e^{3I_v^2} + \frac{3d^2(1 - d + 2d^2)}{(1 + d)^3(1 + d^2)} e^{2I_v^2} + \frac{6d(1 - d)(3 - 2d + 8d^2 - 2d^3 + d^4)}{(1 + d)^3(3 + d^2)(1 + d^2)} e^{I_v^2} .$$

The expression for the excess coefficient is

$$\gamma_2 = \frac{E(Y^4) - 4E(Y^3) + 6E(Y^2) - 3}{s_y^4} - 3 . \qquad 4.22$$

The ratio of excess coefficients of outflows, $\gamma_2(Q)$, and inflows, $\gamma_2(P)$, versus the parameter d, for n = 1, is obtained by the solution of eq. 4.22 and this ratio is:

$$\frac{\gamma_2(Q)}{\gamma_2(P)} = \frac{d(2 - d + d^2)}{(1 + d)(1 + d^2)} . \qquad 4.23$$

The correlogram of outflow for n = 1 may be obtained by using the same method. Equation 4.13 may be written in terms of Y_{i-k} and X_1, X_{i-1}, $X_{i-2}, \ldots, X_{i-k}$ as

$$Y_i = \left(\frac{1 - d}{1 + d}\right)^k Y_{i-k} + \left(\frac{1 - d}{1 + d}\right)^{k-1} \frac{d}{1 + d} X_{i-k} + \left(\frac{1 - d}{1 + d}\right)^{k-2} \frac{d}{1 + d}\left(1 + \frac{1 - d}{1 + d}\right) X_{i-k+1} + \ldots\ldots + \frac{d}{1 + d} X_i . \qquad 4.24$$

Multiplying eq. 4.24 by Y_{i-k} and taking expectations, eq. 4.24 becomes

$$E(Y_i Y_{i-k}) = \left(\frac{1 - d}{1 + d}\right)^k \left(\frac{d}{1 + d} \operatorname{var} X + 1\right) + \left(\frac{1 - d}{1 + d}\right)^{k-1} \frac{d}{1 + d}\left(1 + \frac{d}{1 + d} \operatorname{var} X\right) + \frac{1}{1 + d}\left(1 - \left(\frac{1 - d}{1 + d}\right)^{k-1}\right) + \frac{d}{1 + d} = \left(\frac{1 - d}{1 + d}\right)^{k-1} \frac{d}{(1 + d)^2} \operatorname{var} X + 1 . \qquad 4.25$$

The serial correlation coefficient ρ_k is then

$$\rho_k = \frac{E(Y_i Y_{i-k}) - E(Y_i)E(Y_{i-k})}{[\operatorname{var}(Y_i)\ \operatorname{var}(Y_{i-k})]^{1/2}} = \frac{\left(\frac{1 - d}{1 + d}\right)^{k-1} \frac{d}{(1 + d)^2} \operatorname{var} X}{\left(\frac{d}{1 + d}\right) \operatorname{var} X} ,$$

which expression gives finally

$$\rho_k = \frac{(1 - d)^{k-1}}{(1 + d)^k} , \quad \text{for } k \geq 1 , \qquad 4.26$$

with $\rho_k = 1$, for k = 0. Specifically for k = 1, the first serial correlation coefficient becomes

$$\rho_1 = \frac{1}{1 + d} . \qquad 4.27$$

For k = 2, it becomes

$$\rho_2 = \frac{1 - d}{(1 + d)^2} . \qquad 4.28$$

The result given by eq. 4.26 shows that the serial correlation coefficients for n = 1 depend on the value of d and k only; they are independent of the value I_v. When d is greater than 1, eq. 4.26 shows that the serial correlation coefficients become negative whenever the value of k is even, because d is always positive.

Equations 4.17, 4.21, 4.23 and 4.26 were used to compute the ratios of variance, skewness coefficients, excess coefficient of outflow and inflow, respectively, as well as the serial correlation coefficients of outflows for n = 1. The results are given in Table 2, all as functions of d.

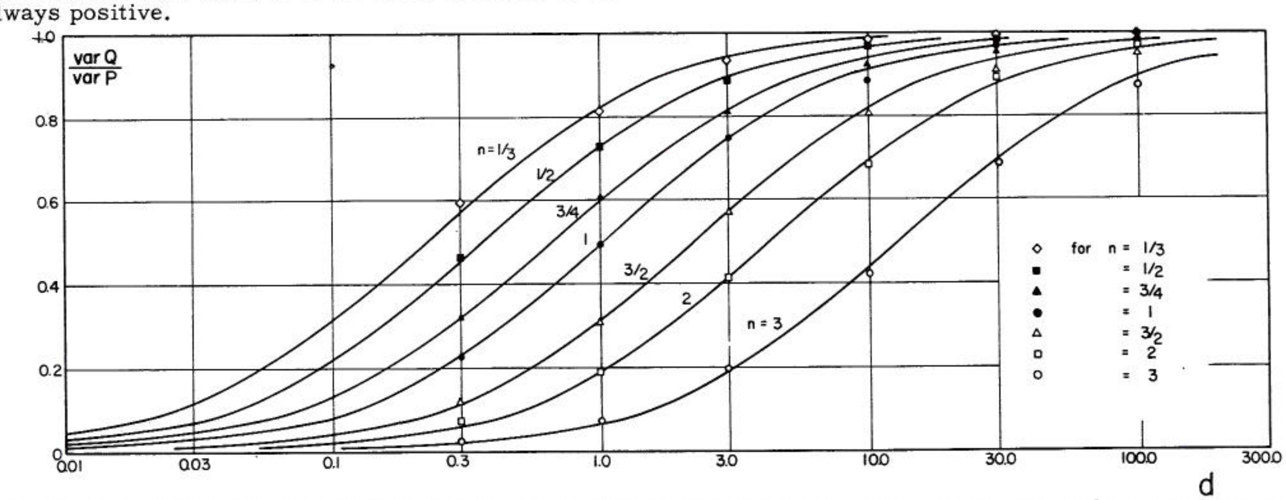

Fig. 2 Relationship of ratios of variance of lake outflows and lake inflows to the dimensionless parameter d, for various values of the parameter n.

5. Variance of outflows for various values of n. Values of the ratio of variances of outflow and inflow obtained by the data generation method are close to a constant value for given values of parameters d and n. Accordingly, these results show that the values of this ratio are independent of the value of I_v.

The values of this ratio shown in Fig. 2 were the average values over the various values of I_v for given parameters d and n. These average values of ratio were then used to find the distance of a horizontal transposition of curves of eq. 4.17 in order to fit well the points obtained for the values of n different from unity.

Figure 2 shows the relationship of the ratio of variances of outflow and inflow, var Q/var P, to the dimensionless parameter d, for seven values of the parameter n. The averages of 210 points, given as points in Fig. 2, represent the results of the data generation method. Likewise, the lines represent either the exact relationship, which is the case only for n = 1, or the fitted curves, which is the case for all other six values of n. In addition, eq. 4.17 gave for n = 1: var Q/var P = d/(1 + d). It was found that horizontal transpositions of the curve of eq. 4.17 fit well the points obtained for the six values of n different from unity. However, the position of each line for a given n was obtained by the least square method of fitting the line of eq. 4.17 to the points. For var Q/var P = 0.5 and for n = 1, the value of d is unity. This same value of 0.5 determines the special value of d, denoted here by d_m, with d_m then obtained for each six n values different from unity.

The logarithms of n and d_m (with an arbitrarily added value of 2), or (2 + ln n) versus (2 + ln d_m), are plotted in Fig. 3 in log-log paper (for the double-log relationship). A least square straight line fit to points in Fig. 3 gives the relationship of d_m, and n. This relationship should be considered only as an experimental curve which approximates closely

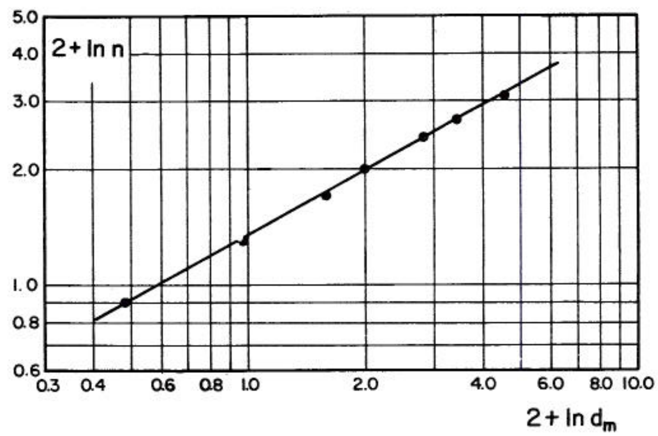

Fig. 3 Relationship of the parameters n and d_m with d_m the value of the parameter d for the ratio of variance of outflow and inflow being 0.5.

the true relationship. The fitted curve is

$$2 + \ln d_m = 2(1 + \tfrac{1}{2} \ln n)^{1.78}. \qquad 4.29$$

The constant 2 is introduced in order to avoid the negative values in taking the logarithms. The selection of this constant does not change the results, but changes the coefficient before ln n (the coefficient is ½ before ln n, for the constant 2). For $n \geq 0.10$, the constant 2 is sufficient to avoid the negative values. For $n < 0.10$, a larger constant should be used. With this value of d_m, then

$$\frac{\text{var } Q}{\text{var } P} = \frac{d}{d + 0.135\, e^{2(1 + ½ \ln n)^{1.78}}}, \qquad 4.30$$

with the variance of X (which is equal to P/P_o) given by eq. 4.6.

Table 2 Ratios of variance, skewness and excess coefficient of outflow and inflow, and serial correlation coefficients (ρ_1 to ρ_{10}) of outflow, for n = 1 and various values of d, computed from analytical expressions, eqs. 4.17, 4.21, 4.23, and 4.26.

d	$\frac{Var(Q)}{Var(P)}$	$\frac{\gamma_1(Q)}{\gamma_1(P)}$	$\frac{\gamma_2(Q)}{\gamma_2(P)}$	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}
0.01	0.010	0.133	0.020	0.990	0.970	0.951	0.932	0.914	0.895	0.878	0.860	0.843	0.827
0.03	0.029	0.226	0.057	0.971	0.914	0.861	0.811	0.763	0.719	0.677	0.637	0.600	0.565
0.05	0.048	0.287	0.093	0.952	0.862	0.780	0.706	0.639	0.578	0.523	0.473	0.428	0.387
0.07	0.065	0.335	0.126	0.935	0.812	0.706	0.614	0.533	0.464	0.403	0.350	0.304	0.265
0.10	0.091	0.392	0.172	0.909	0.744	0.609	0.498	0.408	0.333	0.273	0.223	0.183	0.149
0.30	0.231	0.589	0.379	0.769	0.414	0.223	0.120	0.065	0.035	0.019	0.010	0.005	0.003
0.50	0.333	0.666	0.467	0.667	0.222	0.074	0.025	0.008	0.003	0.001	0.000	0.000	0.000
0.70	0.412	0.698	0.495	0.588	0.104	0.018	0.003	0.001	0.000	0.000	0.000	0.000	0.000
1.00	0.500	0.707	0.500	0.500	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.00	0.750	0.722	0.600	0.250	-0.125	0.063	-0.031	0.016	-0.008	0.004	-0.002	0.001	-0.001
5.00	0.833	0.783	0.705	0.167	-0.111	0.074	-0.049	0.033	-0.022	0.015	-0.010	0.007	-0.004
7.00	0.875	0.828	0.770	0.125	-0.094	0.071	-0.053	0.040	-0.030	0.022	-0.017	0.013	-0.009
10.00	0.909	0.870	0.828	0.091	-0.074	0.061	-0.050	0.041	-0.033	0.027	-0.022	0.018	-0.015
30.00	0.968	0.952	0.937	0.032	-0.030	0.028	-0.026	0.025	-0.023	0.022	-0.020	0.019	-0.018
50.00	0.980	0.970	0.961	0.020	-0.019	0.018	-0.018	0.017	-0.016	0.016	-0.015	0.014	-0.014
70.00	0.986	0.979	0.972	0.014	-0.014	0.014	-0.013	0.013	-0.012	0.012	-0.012	0.012	-0.011
100.00	0.990	0.984	0.980	0.010	-0.010	0.010	-0.010	0.009	-0.009	0.009	-0.009	0.009	-0.008
150.00	0.993	0.990	0.987	0.007	-0.007	0.007	-0.007	0.007	-0.007	0.007	-0.006	0.006	-0.006
200.00	0.995	0.993	0.990	0.005	-0.005	0.005	-0.005	0.005	-0.005	0.005	-0.005	0.005	-0.005

Table 3 Variance of outflow for various values of n, d, and I_v

d	n \ I_v	0.15	0.25	0.4	0.6	0.9	d	n \ I_v	0.15	0.25	0.4	0.6	0.9
0.3	1/3	0.014	0.038	0.105	0.249	0.460	10	1/3	0.022	0.063	0.173	0.412	0.862
	1/2	0.010	0.030	0.082	0.195	0.359		1/2	0.022	0.062	0.170	0.404	0.868
	3/4	0.008	0.021	0.057	0.136	0.231		3/4	0.021	0.061	0.160	0.380	0.708
	1	0.005	0.015	0.039	0.099	0.243		1	0.020	0.056	0.156	0.353	1.020
	3/2	0.003	0.008	0.021	0.051	0.087		3/2	0.019	0.053	0.141	0.322	0.571
	2	0.002	0.005	0.012	0.029	0.074		2	0.016	0.044	0.121	0.273	0.714
	3	0.001	0.002	0.005	0.010	0.020		3	0.010	0.028	0.074	0.164	0.272
1.0	1/3	0.019	0.053	0.135	0.342	0.596	30	1/3	0.022	0.063	0.175	0.417	0.996
	1/2	0.017	0.047	0.120	0.308	0.509		1/2	0.023	0.063	0.170	0.413	0.932
	3/4	0.014	0.039	0.108	0.242	0.409		3/4	0.020	0.062	0.767	0.396	0.802
	1	0.015	0.032	0.084	0.210	0.587		1	0.022	0.061	0.164	0.417	1.155
	3/2	0.007	0.020	0.055	0.122	0.206		3/2	0.021	0.059	0.159	0.370	0.649
	2	0.005	0.013	0.033	0.080	0.196		2	0.020	0.056	0.151	0.398	1.109
	3	0.002	0.005	0.013	0.031	0.058		3	0.016	0.046	0.121	0.260	0.420
3.0	1/3	0.021	0.059	0.163	0.398	0.692	100	1/3	0.022	0.064	0.171	0.422	0.983
	1/2	0.020	0.056	0.160	0.373	0.668		1/2	0.022	0.064	0.171	0.434	0.960
	3/4	0.019	0.052	0.144	0.327	0.559		3/4	0.022	0.064	0.164	0.417	0.899
	1	0.017	0.048	0.126	0.322	0.950		1	0.022	0.063	0.171	0.434	1.316
	3/2	0.013	0.037	0.101	0.226	0.391		3/2	0.022	0.063	0.162	0.401	0.702
	2	0.010	0.027	0.070	0.167	0.424		2	0.022	0.061	0.168	0.436	1.536
	3	0.005	0.013	0.034	0.076	0.143		3	0.020	0.058	0.153	0.365	0.579

For given values of n and d, eq. 4. 30 gives the ratio of variance of outflow and inflow. As expected, the variance of outflow was always smaller than the variance of the corresponding inflow. If another constant, for instance a, is used instead of 2 in eq. 4. 30, this expression becomes

$$\frac{\text{var } Q}{\text{var } P} = \frac{d}{d + e^{a(1+\frac{1}{a}\ln n)^{1.78}} - a} \quad , \qquad 4.31$$

with a selected in such a way that no logarithms will have negative numbers. The computed values of variance of outflow, from generated sequences, are presented in Table 3, as functions of n, d and I_v.

6. Skewness coefficient of outflow for various values of n. Figure 4 gives the points which are obtained for ratio of the skewness coefficient of outflow to the skewness coefficient of inflow. They were computed by the data generation method, a total of 210 cases, as functions of the parameter d for various values of n. All 210 cases are plotted in Fig. 4. For n = 1, the analytical approach yielded the solution of eq. 4. 21.

Figure 4 shows a relatively good agreement for n = 1 between the curve of eq. 4. 21, and the points for n = 1 obtained from 10, 000 generated outflows. The simple horizontal transposition of the curve of eq. 4. 21 was used to fit curves for six values of n, different from unity. The fit is not especially good, because the third statistical moment was used to compute these ratios. Even for n = 10, 000 the third moment is not very accurate. More accurate estimates of $\gamma_1(P)$ and $\gamma_1(Q)$, arrived at by generating a much larger number of inflows and outflows than the 10, 000 used in this study, would likely show that the transposition of the analytical curve for n = 1 would fit much better the points for n different from unity, than is the case in this study with only 10, 000 numbers.

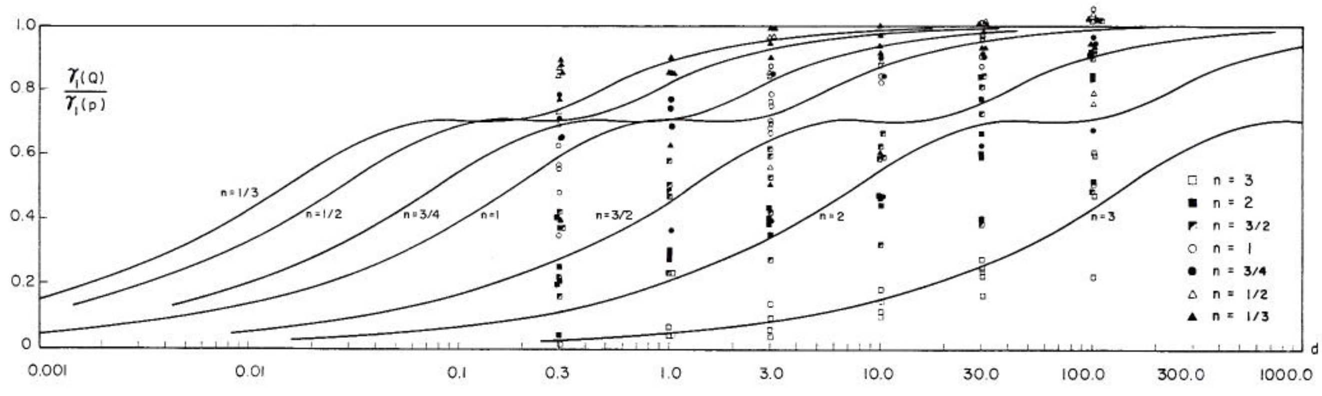

Fig. 4 Relationship of ratios of skewness coefficient of lake outflows and lake inflows to the dimensionless parameter d, for various values of the parameter n.

Analysis of the relationship of $\gamma_1(Q)$ and the coefficient of variation of Q shows that the distributions of outflows do not satisfy eq. 2.4. This means that the distributions of outflows are not log-normally distributed when the inflows are log-normally distributed. The converse is also true. For example, Table 4 shows a comparison of computed values of $\gamma_1(Q)$ from generated outflow sequences with $\gamma_1'(Q)$ which is obtained by eq. 2.4, assuming the outflow is log-normally distributed. The ratio $\frac{\gamma_1(Q)}{\gamma_1'(Q)}$ is either greater or smaller than unity, showing that the outflows usually are not log-normally distributed when the inflows are.

In case d = 1, eq. 4.21 gives $\gamma_1(Q)/\gamma_1(P) = 1/\sqrt{2}$. The same procedure as used above for the variance ratio, or the least square fit, was used for the ratio of skewness coefficients to obtain the empirical relationship. The values of these ratios of skewness coefficient of outflow and inflow obtained by the data generation method have large sampling fluctuations for different values of I_v and for various values of n and d. However, eq. 4.21 shows that the ratio should be independent of I_v.

All 210 values of the ratio are shown in Fig. 4. The simple horizontal transposition of the curve of eq. 4.21 was used as the procedure to fit the points obtained for the values of n different from unity.

The horizontal transposition and least square fit to points in Fig. 4 gave the values of d_m, which are plotted in Fig. 5. A least square straight line fit to points in Fig. 5 gave the following relationship between d_m and n:

$$3 + \ln d_m = 3\left(1 + \frac{1}{3}\ln n\right)^{3.812} . \qquad 4.32$$

The constant a = 3 is used for eq. 4.32 to avoid the negative numbers in taking the logarithms. The ratio of skewness coefficients of outflow and inflow is

$$\frac{\gamma_1(Q)}{\gamma_1(P)} = \frac{4 - \frac{d}{d_m} + \left(\frac{d}{d_m}\right)^2}{\left[3 + \left(\frac{d}{d_m}\right)^2\right]\sqrt{\frac{d_m + d}{d}}} . \qquad 4.33$$

In addition, the computed values of skewness coefficients of outflow, from generated sequences, are presented in Table 5, as functions of n, d, and I_v.

7. Excess coefficient of outflow for various values of n. The ratio of excess coefficient of outflow and inflow to the parameter d, for n = 1, is given by eq. 4.23. Because the fourth moments of inflow and outflow used in computing this ratio, even for the sample size of 10,000, were not of such accuracy as to allow a derivation of a sufficiently accurate empirical equation, the values of $\gamma_2(Q)/\gamma_2(P)$ are subject to very large sampling fluctuation.

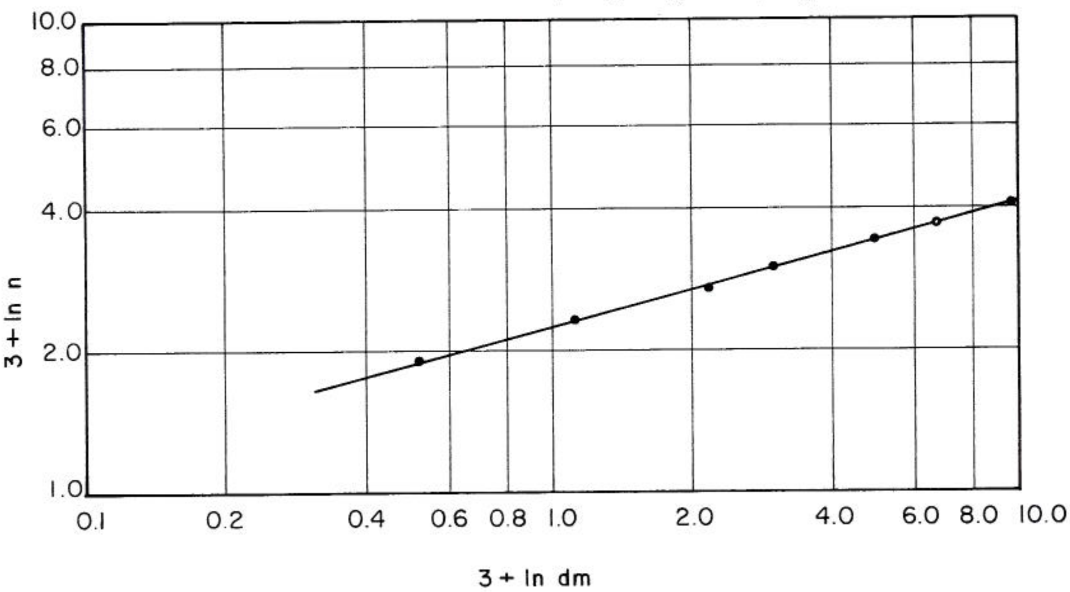

Fig. 5 Relationship of the parameters n and d_m with d_m the value of the parameter d for the ratio of skewness coefficient of outflow and inflow being $1/\sqrt{2}$

The 210 points do not show that the ratio $\gamma_2(Q)/\gamma_2(P)$ was independent of I_v, though the large sampling variation of the ratio may mask the real relationship which may be independent of I_v, as it was shown to be the case for the ratios of variances and skewness coefficient, and as it will be shown to be for the serial correlation coefficients of outflows.

Figure 6 gives the computed points of the ratio $\gamma_2(Q)/\gamma_2(P)$, as well as the theoretical curve for n = 1 given by eq. 4.23. In case this ratio is equal to 0.50, the value d_m = 1, as it was for the variance ratio. Similarly as above, a least square straight line fit to points in Fig. 7 gives the relationship of d_m and n:

$$2 + \ln d_m = 2\left(1 + \frac{1}{2}\ln n\right)^{3.060} \qquad 4.34$$

As a result, eq. 4.25 becomes

Table 5 Skewness coefficients of outflow for various values of n, d, and I_v

d	n \ I_v	0.15	0.25	0.4	0.6	0.9	d	n \ I_v	0.15	0.25	0.4	0.6	0.9
0.3	1/3	0.371	0.656	1.120	1.520	1.673	10	1/3	0.404	0.736	1.273	1.865	2.583
	1/2	0.352	0.653	1.078	1.467	1.588		1/2	0.393	0.720	1.255	1.809	2.846
	3/4	0.289	0.592	0.836	1.417	1.487		3/4	0.371	0.686	1.125	1.788	2.009
	1	0.213	0.495	0.719	1.248	1.934		1	0.372	0.635	1.075	1.641	2.878
	3/2	0.097	0.318	0.466	0.726	0.683		3/2	0.260	0.505	0.781	1.185	1.339
	2	0.018	0.189	0.511	0.410	0.867		2	0.195	0.357	0.586	0.878	1.402
	3	-0.054	0.088	1.062	0.037	-0.053		3	0.051	0.139	0.200	0.291	0.456
1.0	1/3	0.375	0.676	1.072	1.678	2.634	30	1/3	0.434	0.703	1.335	2.088	3.950
	1/2	0.343	0.626	1.008	1.550	1.925		1/2	0.430	0.698	1.326	2.058	3.818
	3/4	0.314	0.584	0.947	1.362	1.557		3/4	0.421	0.687	1.256	1.809	2.762
	1	0.294	0.507	0.904	1.499	3.170		1	0.428	0.685	1.111	2.067	4.095
	3/2	0.216	0.439	0.645	0.939	1.005		3/2	0.371	0.612	1.072	1.451	1.637
	2	0.118	0.224	0.385	0.685	1.186		2	0.339	0.505	0.768	1.174	1.713
	3	0.018	0.180	0.086	0.083	-0.147		3	0.093	0.184	0.318	0.555	0.892
3.0	1/3	0.437	0.678	1.272	1.881	2.170	100	1/3	0.412	0.788	1.177	2.075	3.311
	1/2	0.423	0.646	1.224	1.691	2.374		1/2	0.412	0.786	1.174	2.450	3.116
	3/4	0.386	0.579	1.087	1.562	1.674		3/4	0.411	0.783	1.211	2.122	2.815
	1	0.307	0.521	0.855	1.504	2.991		1	0.409	0.751	1.322	2.102	5.226
	3/2	0.272	0.369	0.759	1.058	1.155		3/2	0.398	0.757	1.165	1.768	2.020
	2	0.169	0.305	0.455	0.865	1.505		2	0.362	0.679	1.154	1.640	2.341
	3	0.061	-0.027	0.123	0.079	0.249		3	0.260	0.446	0.632	0.928	0.909

$$\frac{\gamma_2(Q)}{\gamma_2(P)} = \frac{d(2d_m^2 - d\,d_m + d^2)}{(d + d_m)(d_m^2 + d^2)} , \qquad 4.35$$

with the value of d_m given by eq. 4.34.

The scatter of points in Fig. 6 is great. Whether this scatter comes from the sampling errors, from the use of analogy with the variance ratio for eq. 4.34, or from the possibility that the ratio of eq. 4.35 is not truly independent of I_v, or from all of these three factors working simultaneously, it is not possible to state in this study.

Finally, the computed excess coefficients of outflow, for generated sequences, are presented in Table 6, as functions of n, d and I_v.

8. Serial correlation coefficients of outflow for various values of n. By using the same procedure for the serial correlation coefficients, ρ_k, of outflow, as was used for the ratio of variances, the theoretical curve of ρ_k for outflow for n = 1, shown in eq. 4.26, is transposed along the d-axis to fit the points (by the least square method) for n different from unity. Figure 8 gives the relationship of $(2 + \ln n)$ and $(2 + \ln d_m)$, with d_m the value of d for $\rho_1 = 0.50$. The least square fitted straight line of Fig. 8 gives the relationship,

$$2 + \ln d_m = 2\left(1 + \frac{1}{2}\ln n\right)^{1.84} \qquad 4.36$$

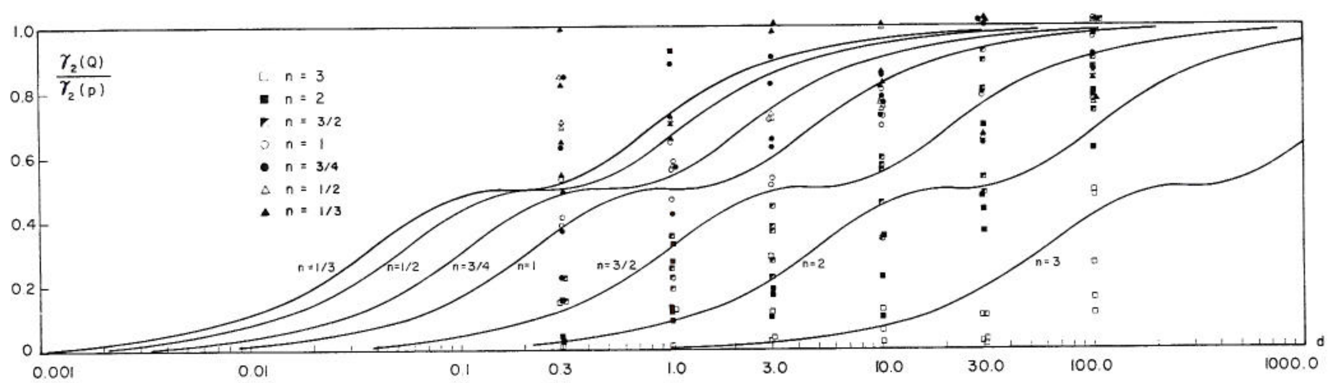

Fig. 6 Relationship of the ratios of excess of lake outflows and lake inflows to the dimensionless parameter d, for various values of the parameter n.

Table 4 Comparison of computed $\gamma_1(Q)$ with $\gamma_1'(Q)$ being obtained by eq. 2.4, assuming the outflow is log-normally distributed.

n	d	$I_v = 0.15$			$I_v = 0.25$			$I_v = 0.40$			$I_v = 0.60$			$I_v = 0.90$		
		$\gamma_1(Q)$	$\gamma_1'(Q)$	$\frac{\gamma_1(Q)}{\gamma_1'(Q)}$	$\gamma_1(Q)$	$\gamma_1'(Q)$	$\frac{\gamma_1(Q)}{\gamma_1'(Q)}$	$\gamma_1(Q)$	$\gamma_1'(Q)$	$\frac{\gamma_1(Q)}{\gamma_1'(Q)}$	$\gamma_1(Q)$	$\gamma_1'(Q)$	$\frac{\gamma_1(Q)}{\gamma_1'(Q)}$	$\gamma_1(Q)$	$\gamma_1'(Q)$	$\frac{\gamma_1(Q)}{\gamma_1'(Q)}$
	0.30	0.352	0.311	1.132	0.653	0.527	1.239	1.078	0.885	1.218	1.467	1.417	1.035	1.588	2.147	0.740
	1.00	0.343	0.393	0.873	0.626	0.666	0.940	1.008	1.102	0.915	1.550	1.841	0.842	1.925	2.764	0.696
	3.00	0.423	0.432	0.979	0.646	0.729	0.886	1.224	1.258	0.973	1.691	2.051	0.824	2.374	3.201	0.742
$\frac{1}{2}$	10.00	0.393	0.447	0.879	0.721	0.767	0.940	1.255	1.308	0.959	1.809	2.153	0.840	2.846	3.706	0.768
	30.00	0.430	0.450	0.956	0.698	0.767	0.910	1.326	1.324	1.002	2.058	2.199	0.936	3.818	4.050	0.943
	100.00	0.412	0.448	0.920	0.786	0.776	1.013	1.174	1.312	0.895	2.450	2.256	1.086	3.116	4.068	0.766
	0.30	0.371	0.353	1.051	0.656	0.595	1.103	1.120	1.009	1.110	1.520	1.629	0.933	1.673	2.510	0.667
	1.00	0.375	0.416	0.901	0.676	0.706	0.958	1.072	1.168	0.918	1.678	1.962	0.855	2.634	3.059	0.861
	3.00	0.437	0.441	0.991	0.678	0.745	0.910	1.272	1.288	0.988	1.881	2.132	0.882	2.170	3.314	0.655
$\frac{1}{3}$	10.00	0.404	0.450	0.898	0.736	0.773	0.952	1.273	1.317	0.967	1.865	2.181	0.855	2.583	3.716	0.695
	30.00	0.434	0.451	0.962	0.703	0.769	0.914	1.335	1.327	1.006	2.088	2.212	0.944	3.950	4.223	0.935
	100.00	0.412	0.448	0.920	0.788	0.777	1.014	1.177	1.314	0.896	2.075	2.220	0.935	3.311	4.124	0.803
	0.30	0.289	0.260	1.112	0.592	0.441	1.342	0.836	0.726	1.152	1.417	1.170	1.211	1.487	1.699	0.875
	1.00	0.314	0.358	0.877	0.584	0.605	0.965	0.947	1.019	0.929	1.362	1.630	0.836	1.557	2.467	0.631
	3.00	0.386	0.415	0.930	0.579	0.700	0.827	1.087	1.191	0.913	1.562	1.929	0.810	1.674	2.902	0.577
$\frac{3}{4}$	10.00	0.371	0.441	0.841	0.686	0.756	0.907	1.125	1.265	0.889	1.788	2.108	0.848	2.009	3.273	0.614
	30.00	0.421	0.449	0.938	0.688	0.764	0.901	1.256	1.294	0.971	1.809	2.152	0.841	2.762	3.678	0.751
	100.00	0.411	0.447	0.919	0.783	0.775	1.010	1.211	1.285	0.942	2.122	2.211	0.960	2.815	3.895	0.723
	0.30	0.213	0.219	0.973	0.495	0.370	1.338	0.719	0.597	1.204	1.248	0.986	1.266	1.934	1.643	1.177
	1.00	0.294	0.323	0.910	0.507	0.539	0.941	0.904	0.895	1.010	1.499	1.489	1.007	3.170	2.785	1.138
	3.00	0.307	0.396	0.775	0.521	0.665	0.783	0.855	1.117	0.765	1.504	1.892	0.795	2.991	3.822	0.783
1	10.00	0.372	0.436	0.853	0.635	0.727	0.873	1.075	1.253	0.858	1.641	2.050	0.800	2.878	4.087	0.704
	30.00	0.428	0.449	0.953	0.685	0.761	0.900	1.111	1.280	0.868	2.067	2.234	0.925	4.095	4.467	0.917
	100.00	0.409	0.452	0.905	0.751	0.774	0.970	1.322	1.314	1.006	2.102	2.255	0.932	5.226	4.886	1.070
	0.30	0.097	0.160	0.605	0.318	0.273	1.165	0.466	0.443	1.052	0.726	0.700	1.037	0.683	0.991	0.689
	1.00	0.216	0.259	0.834	0.439	0.432	1.016	0.645	0.715	0.902	0.939	1.109	0.847	1.005	1.630	0.617
	3.00	0.272	0.349	0.779	0.369	0.588	0.628	0.759	0.983	0.772	1.058	1.559	0.679	1.155	2.333	0.495
$\frac{3}{2}$	10.00	0.260	0.411	0.633	0.505	0.704	0.717	0.781	1.176	0.664	1.185	1.886	0.628	1.339	2.693	0.497
	30.00	0.371	0.438	0.847	0.612	0.746	0.820	1.072	1.262	0.849	1.451	2.054	0.706	1.639	2.996	0.546
	100.00	0.398	0.444	0.896	0.757	0.769	0.984	1.165	1.276	0.913	1.768	2.162	0.818	2.020	3.315	0.609
	0.30	0.018	0.120	0.150	0.190	0.204	0.931	0.511	0.333	1.535	0.410	0.522	0.785	0.867	0.852	1.018
	1.00	0.118	0.205	0.576	0.224	0.337	0.665	0.385	0.552	0.697	0.685	0.884	0.775	1.186	1.430	0.829
	3.00	0.169	0.299	0.565	0.305	0.494	0.617	0.455	0.817	0.557	0.865	1.300	0.665	1.505	2.218	0.679
2	10.00	0.195	0.386	0.505	0.357	0.643	0.555	0.586	1.094	0.536	0.878	1.757	0.500	1.402	3.159	0.444
	30.00	0.339	0.429	0.790	0.505	0.728	0.694	0.768	1.227	0.626	1.174	2.170	0.541	1.713	4.330	0.396
	100.00	0.362	0.446	0.812	0.679	0.763	0.890	1.154	1.299	0.888	1.640	2.262	0.725	2.342	5.546	0.422
	0.30	-0.054	0.071	-0.756	0.088	0.118	0.746	1.062	0.212	5.009	0.037	0.301	0.123	-0.053	0.456	-0.116
	1.00	0.018	0.127	0.142	0.180	0.218	0.826	0.086	0.349	0.246	0.083	0.547	0.152	-0.147	0.790	-0.186
	3.00	0.061	0.203	0.300	-0.027	0.342	-0.079	0.123	0.554	0.222	0.079	0.858	0.092	0.249	1.281	0.194
3	10.00	0.051	0.302	0.169	0.139	0.509	0.273	0.200	0.837	0.239	0.291	1.273	0.229	0.456	1.795	0.254
	30.00	0.093	0.383	0.243	0.184	0.651	0.283	0.318	1.083	0.294	0.555	1.643	0.338	0.692	2.175	0.318
	100.00	0.260	0.425	0.612	0.446	0.739	0.604	0.632	1.233	0.513	0.928	1.962	0.473	0.909	2.529	0.359

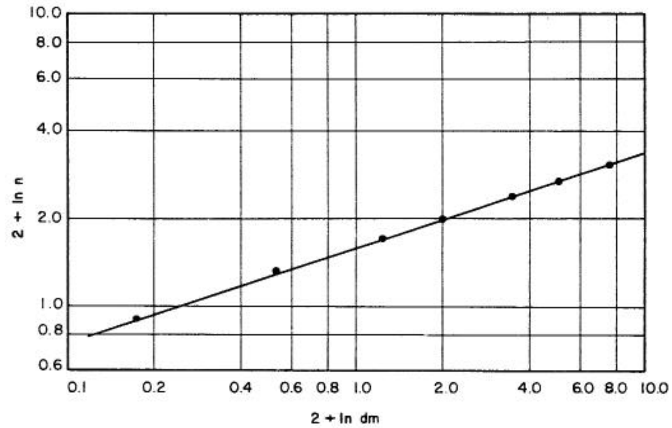

Fig. 7 Relationship of the parameter n and d_m with d_m the value of the parameter d for the ratio of excess of outflow and inflow being 0.5.

Table 6 Excess coefficients of outflow for various values of n, d, and I_v.

d	n \ I_v	0.15	0.25	0.4	0.6	0.9	d	n \ I_v	0.15	0.25	0.4	0.6	0.9
0.3	1/3	0.147	0.622	2.299	3.609	3.401	10	1/3	0.226	0.826	2.841	5.444	10.910
	1/2	0.123	0.663	1.981	3.317	2.949		1/2	0.223	0.791	2.799	4.998	1.538
	3/4	0.067	0.604	1.048	3.204	2.937		3/4	0.215	0.736	2.205	5.654	5.615
	1	0.008	0.517	1.090	2.280	6.992		1	0.215	0.775	0.157	4.612	14.034
	3/2	-0.085	0.216	0.429	1.021	0.564		3/2	0.166	0.549	1.276	2.281	2.066
	2	-0.178	0.041	2.607	0.208	3.150		2	0.031	0.345	0.999	1.518	3.467
	3	-0.262	-0.145	6.205	-0.131	0.044		3	0.019	0.124	0.071	-0.113	-0.279
1	1/3	0.214	0.810	1.974	4.355	19.200	30	1/3	0.350	0.637	3.134	8.190	35.843
	1/2	0.17	0.688	1.803	3.838	7.477		1/2	0.349	0.629	3.107	7.973	3.548
	3/4	0.125	0.559	1.598	2.815	3.090		3/4	0.339	0.618	2.957	5.298	14.326
	1	0.103	0.445	1.570	3.874	24.536		1	0.292	0.756	1.939	7.725	38.568
	3/2	0.067	0.340	0.729	1.270	1.155		3/2	0.267	0.517	2.258	3.203	3.222
	2	0.082	0.094	0.370	0.842	3.259		2	0.207	0.420	1.031	3.194	8.671
	3	0.297	0.895	0.360	0.068	0.000		3	0.032	0.034	0.299	0.195	-0.070
3	1/3	0.318	0.715	2.925	6.57	6.930	100	1/3	0.250	1.137	2.171	7.957	20.456
	1/2	0.312	0.700	2.781	4.751	11.742		1/2	0.252	1.136	2.166	1.520	1.752
	3/4	0.270	0.627	2.308	4.155	3.472		3/4	0.257	1.135	2.566	9.361	1.276
	1	0.102	0.514	1.435	4.766	19.054		1	0.228	0.923	3.389	7.969	66.721
	3/2	0.133	0.218	1.053	1.853	1.322		3/2	0.268	1.099	2.445	4.927	5.705
	2	-0.049	0.167	0.287	1.271	3.951		2	0.185	0.764	2.714	5.307	19.960
	3	0.085	0.030	0.099	-0.103	0.407		3	0.147	0.460	0.750	1.063	0.414

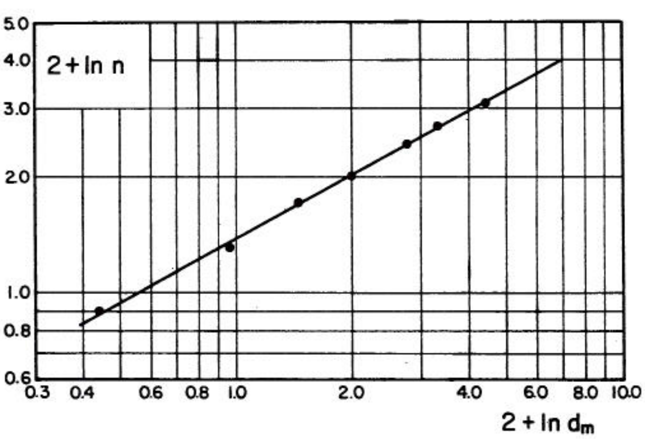

Fig. 8 Relationship of the parameter n and d_m with d_m the values of the parameter d for the first serial correlation coefficient of outflow being 0.5.

with the same explanations for the use of the constant 2 as were given above for the ratio of variances. It must be noted that the constant 1.84 is close to 1.78 which is found for the ratio of variances.

With eq. 4.36, eq. 4.26 for ρ_k becomes

$$\rho_k = \frac{\left(1 - \frac{d}{d_m}\right)^{k-1}}{\left(1 + \frac{d}{d_m}\right)^{k}} \qquad 4.37$$

with d_m given by eq. 4.36.

Because the variance and serial correlation coefficients are both derived from the second statistical moments, the relationship of n and d_m given in eqs. 4.29 and 4.36 can be combined together and considered as a unique relationship. A least square straight line fitted for both the variance and the serial correlation coefficients as shown in Fig. 9 gives the relationship of d_m and n:

$$2 + \ln d_m = 2(1 + \frac{1}{2}\ln n)^{1.81} \qquad 4.38$$

Therefore, eq. 4.38 is recommended for use instead of eq. 4.29 and 4.36.

For the special cases, ρ_1 (the first ten serial correlation coefficient), the obtained points by the generation of 10,000 outflows are given in Fig. 10, the uppermost graph. The line for n = 1 gives $\rho_1 = 1/(1 + d)$, and this line is transposed horizontally to fit the observed points for six n values different from unity. The fit in Fig. 10 is very good. For n = 1 and k = 2, eq. 4.38 gives ρ_2, and the second graph in Fig. 10 gives the fit. Similarly, Fig. 10 also gives the fits for ρ_3 through ρ_5, and Fig. 11 for ρ_6 through ρ_{10}.

The values of serial correlation coefficients obtained by the data generation method are very close to a constant value for given parameters d and n. According to these results, this value is independent of I_v, as shown by eq. 4.26. Moreover, the values of serial correlation coefficients shown in Figs. 10 and 11 were the average values over the various values of I_v. For eq. 4.26, the distances have been horizontally transposed for various values of k in Figs. 10 and 11. This was done according to the relationship between d_m and n given by eq. 4.38. All fits are good.

In Table 7, the computed values of first serial correlation coefficient of outflow, for generated sequences, are presented as functions of n, d, and I_v. The computed average serial correlation coefficients of outflow, from generated sequences, for all values of I_v are presented in Table 8, as function of n and d.

For any value of n and I_v, listed in Table 9 are the maximum negative values of serial correlation coefficients for even lags k, and the second maximum values for uneven lag κ for various values of d and for n = 1. For n equal to the values other than unity, the corresponding value of d/d_m should be used instead of d only, with d_m obtained from eq. 4.38.

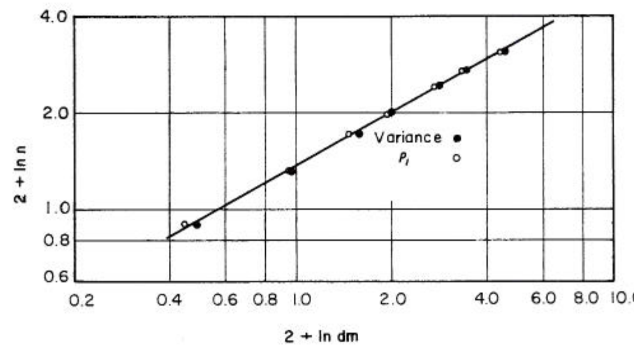

Fig. 9 A least square line fitted to the relationship of the parameter n and d_m for both variance and first serial correlation coefficient.

Table 7 First serial correlation coefficient of outflow for various values of n, d, and I_v

d	n \ I_v	0.15	0.25	0.4	0.6	0.9	d	n \ I_v	0.15	0.25	0.4	0.6	0.9
0.3	1/3	0.415	0.428	0.395	0.383	0.367	10	1/3	0.008	0.000	0.006	0.010	0.004
	1/2	0.541	0.553	0.524	0.513	0.492		1/2	0.022	0.015	0.018	0.026	0.017
	3/4	0.676	0.687	0.670	0.681	0.666		3/4	0.047	0.041	0.050	0.049	0.050
	1	0.767	0.771	0.763	0.780	0.765		1	0.096	0.106	0.078	0.108	0.078
	3/2	0.876	0.881	0.879	0.887	0.881		3/2	0.166	0.159	0.164	0.165	0.186
	2	0.930	0.932	0.933	0.935	0.937		2	0.287	0.296	0.262	0.285	0.284
	3	0.975	0.974	0.979	0.974	0.972		3	0.543	0.529	0.535	0.553	0.559
1	1/3	0.179	0.169	0.187	0.174	0.177	30	1/3	0.002	0.007	0.002	0.012	0.004
	1/2	0.266	0.255	0.274	0.261	0.273		1/2	0.006	0.012	0.006	0.014	0.002
	3/4	0.390	0.380	0.385	0.382	0.389		3/4	0.015	0.020	0.014	0.024	0.013
	1	0.504	0.501	0.512	0.499	0.508		1	0.039	0.052	0.033	0.056	0.026
	3/2	0.686	0.684	0.693	0.702	0.697		3/2	0.060	0.063	0.056	0.066	0.069
	2	0.806	0.804	0.810	0.821	0.833		2	0.123	0.132	0.102	0.087	0.016
	3	0.925	0.928	0.925	0.925	0.905		3	0.270	0.263	0.241	0.286	0.347
3	1/3	0.072	0.08	0.089	0.050	0.055	100	1/3	0.016	0.005	-0.007	-0.002	-0.002
	1/2	0.112	0.119	0.129	0.086	0.094		1/2	0.018	0.007	-0.005	-0.000	-0.001
	3/4	0.179	0.185	0.179	0.177	0.195		3/4	0.020	0.009	0.021	0.009	-0.000
	1	0.260	0.242	0.255	0.251	0.250		1	-0.002	0.006	0.011	-0.003	0.017
	3/2	0.414	0.418	0.420	0.423	0.447		3/2	0.034	0.023	0.034	0.016	0.009
	2	0.575	0.568	0.581	0.604	0.660		2	0.025	0.034	0.032	0.008	-0.135
	3	0.803	0.800	0.806	0.798	0.777		3	0.118	0.093	0.082	0.105	0.181

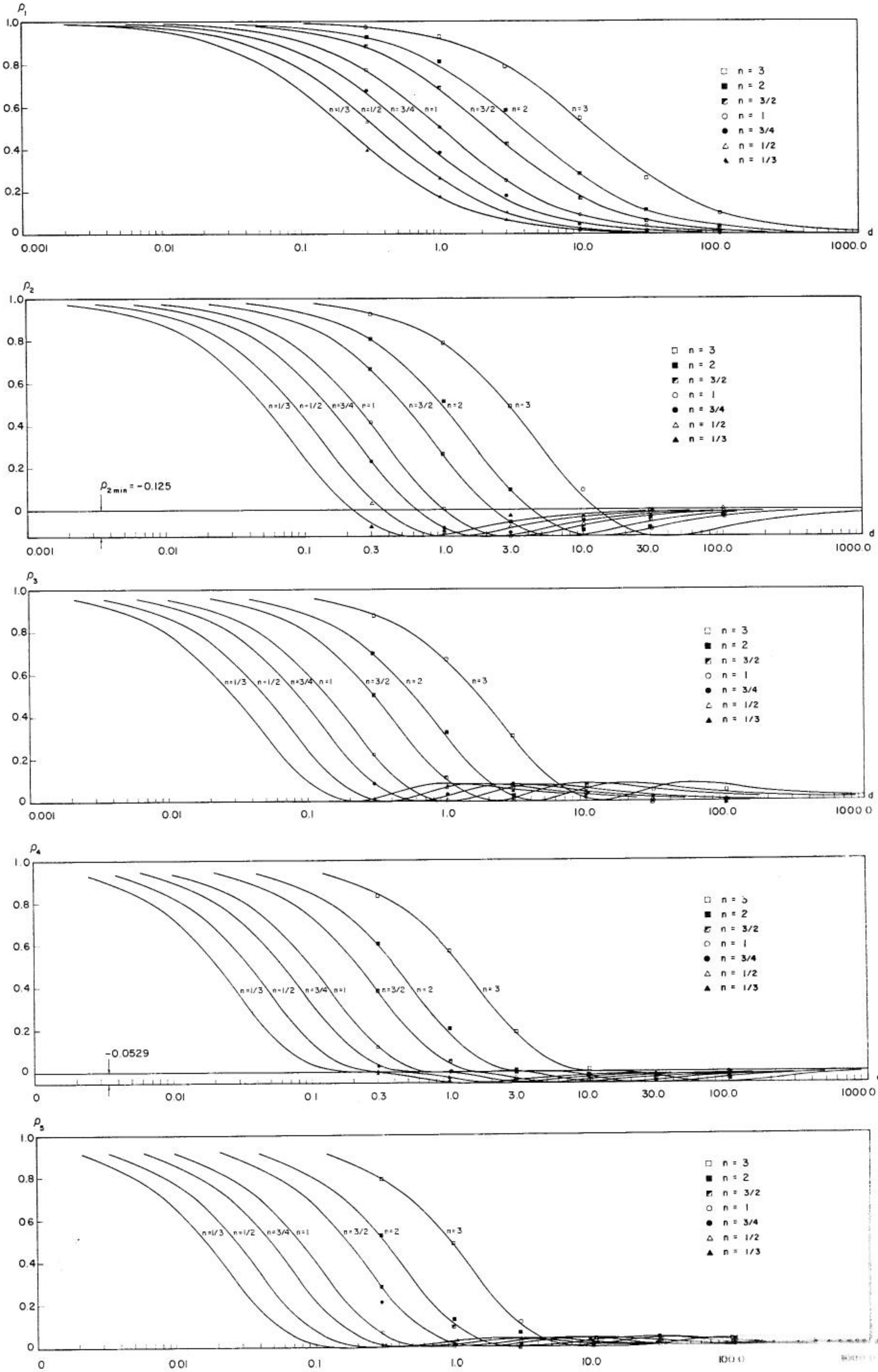

Fig. 10 Relationships of the first through fifth serial correlation coefficient of lake outflows to the dimensionless parameter d, for various values of the parameter n

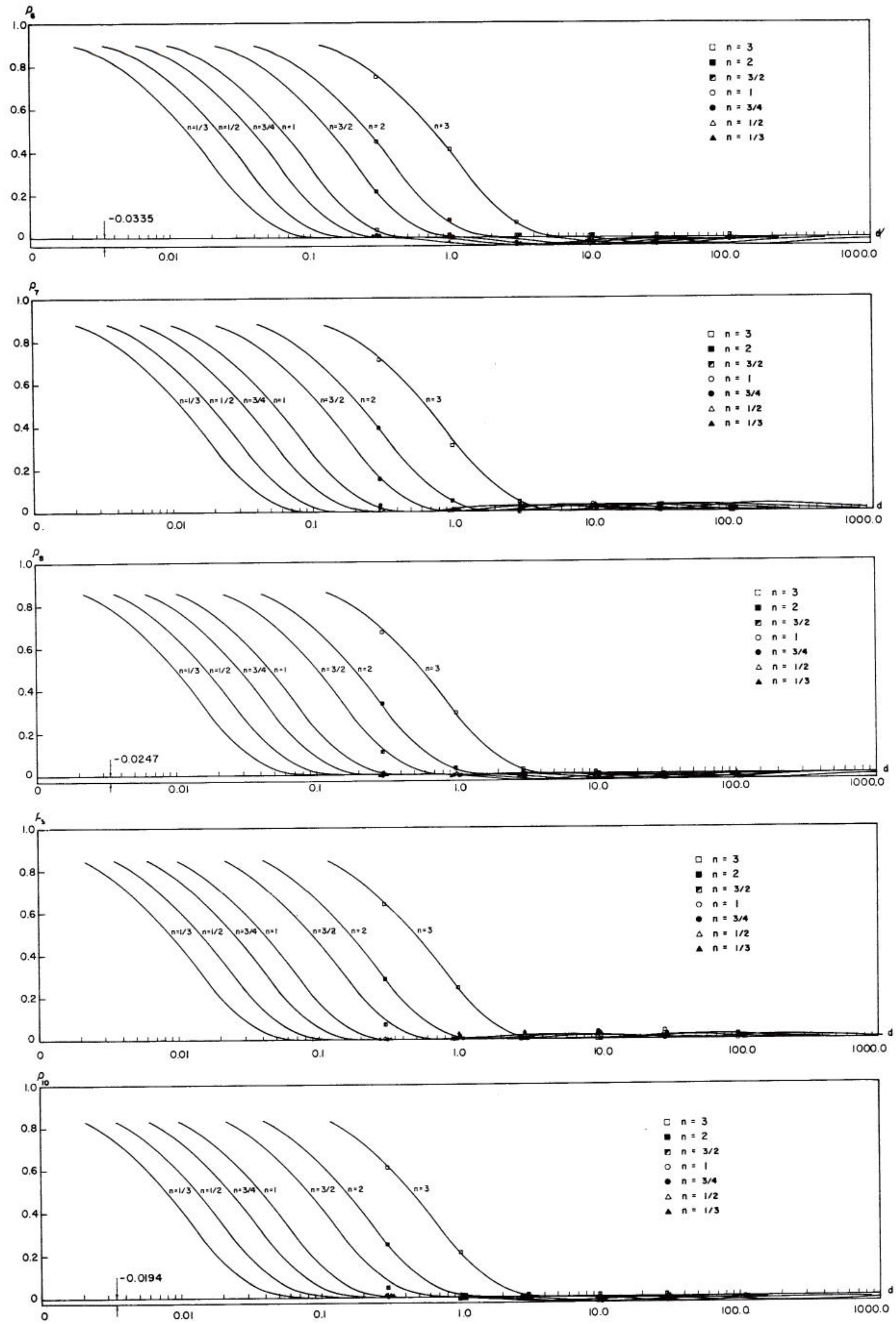

Fig. 11 Relationships of the sixth through tenth serial correlation coefficient of lake outflows to the dimensionless parameter d, for various values of the parameter n

Table 8 Average serial correlation coefficients of outflow for various values of n and d.

d	n \ r	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}
0.3	1/3	0.398	-0.077	0.008	-0.004	0.011	0.007	-0.002	-0.002	0.004	0.004
	1/2	0.533	0.037	-0.000	0.002	0.012	0.008	-0.000	-0.003	0.000	0.001
	3/4	0.676	0.237	0.080	0.032	0.021	0.012	0.003	-0.002	-0.006	-0.009
	1	0.769	0.414	0.223	0.122	0.066	0.037	0.019	0.008	0.002	-0.000
	3/2	0.883	0.668	0.505	0.382	0.289	0.215	0.155	0.109	0.073	0.045
	2	0.933	0.809	0.701	0.606	0.524	0.453	0.390	0.336	0.289	0.248
	3	0.975	0.926	0.879	0.835	0.793	0.752	0.713	0.675	0.639	0.606
1	1/3	0.177	-0.098	0.077	-0.038	0.025	0.021	0.011	-0.009	0.027	-0.006
	1/2	0.266	-0.108	0.066	-0.023	0.011	-0.007	0.003	-0.005	-0.002	-0.003
	3/4	0.385	-0.081	0.037	0.006	0.001	0.001	-0.001	-0.004	-0.006	-0.003
	1	0.505	0.008	0.005	0.002	0.001	0.001	0.000	-0.001	-0.004	-0.006
	3/2	0.692	0.271	0.114	0.050	0.019	0.007	0.000	-0.005	-0.008	-0.004
	2	0.815	0.515	0.325	0.204	0.128	0.080	0.049	0.029	0.008	0.002
	3	0.926	0.789	0.673	0.573	0.486	0.411	0.347	0.292	0.246	0.208
3	1/3	0.069	-0.023	0.051	-0.036	0.030	-0.030	0.026	-0.019	0.016	-0.016
	1/2	0.108	-0.074	0.072	-0.414	0.035	-0.032	0.029	-0.013	0.012	-0.011
	3/4	0.183	-0.106	0.084	-0.034	0.051	-0.021	0.019	-0.013	0.004	-0.004
	1	0.251	-0.128	0.058	-0.029	0.022	-0.002	0.005	-0.000	-0.003	-0.006
	3/2	0.424	-0.055	0.012	0.005	-0.011	-0.007	-0.000	-0.004	-0.003	0.000
	2	0.582	0.097	0.012	0.002	0.007	0.009	0.006	0.004	0.008	0.007
	3	0.797	0.490	0.307	0.191	0.116	0.070	0.040	0.020	0.006	-0.002
10	1/3	0.006	-0.025	0.018	-0.015	0.017	-0.016	0.021	-0.009	0.018	-0.018
	1/2	0.020	-0.035	0.028	-0.025	0.027	-0.026	0.024	-0.018	0.025	-0.021
	3/4	0.048	-0.046	0.038	-0.039	0.037	-0.031	0.027	-0.020	0.039	-0.027
	1	0.093	-0.072	0.057	-0.047	0.036	-0.032	0.031	-0.023	0.017	-0.017
	3/2	0.168	-0.087	0.075	-0.047	0.039	-0.018	0.016	-0.003	0.020	-0.012
	2	0.283	-0.094	0.046	-0.020	0.005	-0.009	0.005	-0.002	-0.002	-0.003
	3	0.544	0.100	0.034	0.012	0.011	0.008	0.007	0.010	0.010	0.005
30	1/3	0.005	-0.009	0.008	-0.011	0.013	-0.003	0.005	-0.003	0.015	-0.006
	1/2	0.008	-0.010	0.011	-0.016	0.016	-0.005	0.008	-0.007	0.018	-0.007
	3/4	-0.017	-0.026	0.023	-0.020	0.018	-0.006	0.016	-0.010	0.024	-0.016
	1	0.041	-0.010	0.031	-0.024	0.024	-0.025	0.022	0.025	0.029	-0.012
	3/2	0.063	-0.046	0.052	-0.034	0.039	-0.015	0.029	-0.019	0.032	-0.017
	2	0.119	-0.080	0.066	-0.044	0.040	-0.024	0.024	-0.028	0.020	-0.006
	3	0.265	-0.084	0.053	-0.022	0.010	0.007	0.004	-0.001	0.003	0.002
100	1/3	-0.002	-0.003	0.004	-0.000	0.005	-0.004	-0.003	-0.001	0.003	0.001
	1/2	0.012	0.011	0.002	-0.010	0.005	-0.001	-0.003	-0.015	0.001	-0.004
	3/4	0.012	-0.017	0.008	-0.010	0.005	-0.008	0.002	-0.016	0.003	-0.006
	1	0.006	-0.008	0.019	-0.007	0.015	-0.009	0.005	-0.010	0.017	0.007
	3/2	0.023	-0.013	0.016	-0.019	0.015	-0.018	0.012	-0.021	0.012	-0.009
	2	0.030	-0.027	0.039	-0.030	0.027	-0.021	0.013	-0.020	0.025	-0.027
	3	0.099	-0.033	0.050	-0.029	0.028	-0.016	0.014	-0.019	0.010	-0.007

Table 9 Values of minimum ρ_k for even k and second maximum of ρ_k for uneven k, with the corresponding value of parameter d, for n = 1

k	d	ρ_k
2	3	-0.1250
3	5	0.0740
4	7	-0.0529
5	9	0.0410
6	11	-0.0335
7	13	0.0283
8	15	-0.0247
9	17	0.0217
10	19	-0.0194

CHAPTER V

DISCUSSIONS AND CONCLUSIONS

1. Discussions of results. Results obtained by the analytical solution for $n = 1$ and by the data generation method on a digital computer for n different from unity show that the variance, the skewness, and the excess coefficients of the outflow from natural lakes are less than those of the inflow for any value of n and d. These parameters of outflow distribution increase and converge to those of the inflow distribution as the parameter d increases to infinity. Large values of n give small values of ratios of variance, skewness, and excess coefficients of outflow and inflow, and have a rapid rate of convergence to zero with an increase of n. It can be said that the effects of a natural lake, with small value of d and a large value of n, on the outflow distribution are such that the outflow is closer to the normal distribution than the inflow, if the inflow is log-normally distributed. Hypothetically, it can be stated that ouflow from a natural lake has a smaller range of fluctuations than the inflow because of the storage effect. Therefore, the values of the variance, the skewness, and excess coefficients of the outflow distribution tend to be smaller than those of the inflow distribution.

The method used in this study to obtain the stochastic properties of lake outflow, for given lake inflow and lake storage and outlet conditions, was a combination of analytical and data generation methods. For $n = 1$, all parameters were obtained by the analytical approach, becuase in that case the storage differential equation in finite difference form could be solved in closed form for the stochastic variables involved. For n different from unity, even for integer numbers of n, the parameters could not be obtained by the analytical approach, because solutions in closed form were not obtained. By analogy with other differential equations, for which the closed form solutions are not available, and the numerical finite differences equations are used for the approximate solutions, the similar approach was used in this study. The basic differential equation in this study was an ordinary differential equation of stochastic variables. The approximate solutions were obtained by transforming this equation into a finite differences equation, and by using the data generation method (Monte Carlo Method) to obtain sequences of stochastic variables. The combination of these two approaches produced the solutions in the approximate form for values of n different from unity. To check the general correctness of analytical solutions, and the degree of deviation of approximate solutions from exact solutions as obtained by the data generation method, the solutions for parameters in case $n = 1$ were obtained also by the data generation method. The agreement between the exact and the approximate procedure for parameters in case $n = 1$ was very good.

2. Conclusions. The following conclusions were advanced:

(1) By knowing the approximate power function relationships of lake storage and lake outflow rating curve to the water depths above the level of zero outflow, the relationship of outflow to inflow characteristics and vice versa can be established. For the limited case of $n = 1$, it was established analytically; or for all cases, by the data generation method (Monte Carlo Method). This allows for simple determination of outflow characteristics for given inflow characteristics, or vice versa.

(2) The assumption of independent log-normally distributed inflows is not necessary for establishing the inflow-outflow relations, though it was used in this paper to illustrate the method.

(3) Once the three parameters (n, d, I_V) of the inflow and of the lake characteristics are known, the relationships given in this paper permit a straight forward determination of outflow characteristics. These can be described by the variance, the coefficient of skewness, and eventually by the coefficient of excess as well as by the serial correlation coefficients.

(4) Though the parameters of outflow are derived theoretically only for $n = 1$, it may be expected that they could be derived for the other values of n, or for any value of n.

(5) For accurate values of the skewness and excess coefficients of outflow, the sample size of 10,000 generated outflows seems to be inadequate, mainly because of serial correlation, but larger samples may be inexpensively generated to improve their accuracy.

(6) Simple statistical tests may be performed to test whether the original generated numbers are normally distributed and independent. These tests were not carried out in this study though they were used in other similar investigations.

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Key Words: Hydrology, Log-normal distribution, Generation of outflow and inflow sequences, Outflow-inflow relationship for natural lakes, Natural Lakes.

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