NON-VACUUM ADaM FIELD EQUATIONS*

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The canonical version of the vacuum Einstein field equations formulated ten years ago by Arnowitt, Deser, and Misner (ADaM) [1] has stimulated several attempts to quantize certain cosmological models, most notably Misner's so-called Mixmaster Universe [2]. Some researchers have begun recently to extend these methods to non-vacuum spacetimes; for example, Nutku earlier at this conference described the canonical theory of a scalar field in Schwarzschild spacetime. The purpose of this talk is to generalize the ADaM field equations to include an arbitrary stress-energy tensor. This is not a "first step" toward a canonical formulation of the full non-vacuum field equations; rather, it is simply a possible starting point.

Essentially, the ADaM field equations are a linear combination of Einstein's $G_{\mu\nu} = 0$ equations that is particularly well-suited to a "three-plus-one split" of spacetime, i.e., a division of spacetime into three-dimensional spacelike sections labelled by the parameter time. The metric of each section is the spacelike part of the metric for all of spacetime:

$$g_{ij} \equiv {}^4g_{ij} . \qquad (1a)$$

(Superscript "4" denotes quantities referred to the full four-dimensional spacetime, while no superscript implies three-dimensional quantities. Latin indices run from

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1 to 3, Greek from 0 to 3. Signature is - 2.) ADaM replace the remaining four metric components - which give information on how one hypersurface fits into the next 3 - with: a three-scalar

$$N \equiv (-\,{}^4g^{oo})^{-\frac{1}{2}} \tag{1b}$$

and a covariant three-vector

$$N_i \equiv {}^4g_{oi} \,. \tag{1c}$$

The ADaM field equations are derived from the usual variational principle,

$$\delta I = \delta \int {}^4R(-\,{}^4g)^{\frac{1}{2}}\, d\,{}^4x = 0 \,. \tag{2}$$

Were one to use $\{{}^4g^{\mu\nu}\}$ as the set of independent variables, one would obtain $G_{\mu\nu} = 0$ from Eq. (2) [4] . Using the ADaM variables $\{N, N_i, g_{ij}\}$, on the other hand, gives the ADaM equations.

To obtain the non-vacuum equations, let L be the Lagrangian for the non-gravitational fields. Then Eq. (2) generalizes to

$$\delta I = \delta \int ({}^4R + 2\,\kappa\, L)(-\,{}^4g)^{\frac{1}{2}}\, d\,{}^4x = 0 \,. \tag{3}$$

Using $\{{}^4g^{\mu\nu}\}$ as the variables gives [5]

$$G_{\mu\nu} = \kappa\, T_{\mu\nu} \,, \tag{4}$$

where

$$T_{\mu\nu} = L\,{}^4g_{\mu\nu} - 2\,\frac{\partial L}{\partial\,{}^4g^{\mu\nu}} + \frac{2}{(-\,{}^4g)^{\frac{1}{2}}}\left[(-\,{}^4g)^{\frac{1}{2}}\,\frac{\partial L}{\partial\,{}^4g^{\mu\nu}{}_{,\beta}}\right]_{,\beta} \,. \tag{5}$$

The non-vacuum ADaM equations follow from Eq. (3) if one uses the set $\{a_{\alpha\beta}\}$ of ADaM variables, defined by

$$a_{oo} \equiv (-\,{}^4g^{oo})^{-\frac{1}{2}};\; a_{oi} \equiv {}^4g_{oi};\; a_{io} \equiv {}^4g_{io}\;;\; a_{ij} \equiv {}^4g_{ij} \,. \tag{6}$$

It is convenient in what follows to ignore the symmetry of $a_{\alpha\beta}$ and ${}^4g_{\mu\nu}$. For instance, variations of a_{oi} will be taken while holding a_{io} fixed. The final results will, of course, be symmetrized.

Because the transformation from $\{{}^4g^{\mu\nu}\}$ to $\{a_{\alpha\beta}\}$ is nonsingular and does not involve derivatives of ${}^4g^{\mu\nu}$ or explicit dependence upon the spacetime coordinates, the equations obtained from varying $a_{\alpha\beta}$ will be the linear combination

$$0 = \frac{\delta I}{\delta a_{\alpha\beta}} = \frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{\alpha\beta}} \frac{\delta I}{\delta\, {}^4g^{\mu\nu}} \tag{7}$$

of the equations obtained from varying ${}^4g^{\mu\nu}$. We therefore need only find $\partial\, {}^4g^{\mu\nu}/\partial a_{\alpha\beta}$, in which it is understood that the derivative is taken holding all other $a_{\gamma\delta}$ fixed. This is the key to the difference between Einstein and ADM: it means, for example, that $\partial\, {}^4g^{01}/\partial a_{01}$ is not the same as $\partial\, {}^4g^{01}/\partial\, {}^4g_{01} = -{}^4g^{00}\, {}^4g^{11}$, because in the first case one holds $\{{}^4g^{00}, {}^4g_{02}, {}^4g_{03}, {}^4g_{ij}\}$ fixed while in the second case one holds $\{{}^4g_{00}, {}^4g_{02}, {}^4g_{03}, {}^4g_{ij}\}$ fixed. Bearing this in mind, we write down the equations of transformation:

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{ij}} = -\, {}^4g^{\mu i}\, {}^4g^{\nu j} + {}^4g^{o\mu}\, {}^4g^{o\nu} N^i N^j \quad ; \tag{8a}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{oi}} = -\, {}^4g^{o\mu}\, {}^4g^{\nu i} - {}^4g^{o\mu}\, {}^4g^{o\nu} N^i \quad ; \tag{8b}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{io}} = -\, {}^4g^{\mu i}\, {}^4g^{\nu o} - {}^4g^{o\mu}\, {}^4g^{o\nu} N^i \quad ; \tag{8c}$$

$$\frac{\partial\, {}^4g^{\mu\nu}}{\partial a_{oo}} = 2\, {}^4g^{o\mu}\, {}^4g^{o\nu} N \quad . \tag{8d}$$

It is straightforward to use Eqs. (7) and (8) to find the non-vacuum ADM field equations. (Here π^{ij} is the momentum canonical to g_{ij}, defined by Eq. (9c) below. Indices on it and N_i are raised and lowered by the three-dimensional metric, covariant differentiation with respect to which is denoted by a slash, "|".)

$$-\, g^{\frac{1}{2}}\left[{}^3R + g^{-1}(\tfrac{1}{2}\pi^2 - \pi^{ij}\pi_{ij})\right] = -\, 2\kappa N^2 g^{\frac{1}{2}} T^{oo} \quad ; \tag{9a}$$

$$-\pi^{ij}{}_{|j} = \kappa N g^{\frac{1}{2}}(T^{oi} + N^i T^{oo}) \quad ; \tag{9b}$$

$$\partial_t g_{ij} = 2Ng^{-\frac{1}{2}}(\pi_{ij} - \tfrac{1}{2}g_{ij}\pi) + N_{i|j} + N_{j|i} \quad ; \tag{9c}$$

$$\begin{aligned}\partial_t \pi^{ij} = &- Ng^{\frac{1}{2}}({}^3R^{ij} - \tfrac{1}{2}g^{ij}\,{}^3R) + \tfrac{1}{2}Ng^{-\frac{1}{2}}g^{ij}(\pi^{mn}\pi_{mn} - \tfrac{1}{2}\pi^2) \\ &-2Ng^{-\frac{1}{2}}(\pi^{im}\pi_m{}^j - \tfrac{1}{2}\pi\pi^{ij}) + g^{\frac{1}{2}}(N^{|ij} - g^{ij}N^{|m}{}_{|m}) \\ &+ (\pi^{ij}N^{|m})_{|m} - N^i{}_{|m}\pi^{mj} - N^j{}_{|m}\pi^{mi} \\ &+ \kappa Ng^{\frac{1}{2}}(T^{ij} - T^{oo}N^iN^j) \quad .\end{aligned} \tag{9d}$$

I wish to remark on a few features of these equations. First, as we would expect, they do not contain L, since they are simply a linear combination of Eqs. (4). This means they can be used even if a Lagrangian is not available. Second, Eqs. (9) are instructive in understanding even the ADaM vacuum equations, since the particular linear combination used by ADaM is manifest. And third, the equations contain $T^{\mu\nu}$, the contravariant components of the four-dimensional stress-energy tensor. In many situations (e.g., scalar field) one might feel that the covariant components, $T_{\mu\nu}$, are physically more meaningful in a 3 + 1 split, in which case one can rewrite the equations as follows. Using the unit normal to the three-hypersurface, $n^\alpha = -N\,{}^4g^{o\alpha}$, one can define a "preferred" energy and momentum density for the matter:

$$\mathcal{E} \equiv n^\alpha n^\beta\,{}^4T_{\alpha\beta} \quad , \tag{10a}$$

$$\mathcal{P}_i \equiv n^\alpha\,{}^4T_{\alpha i} \quad . \tag{10b}$$

Then the stress tensor in the hypersurface is

$$\mathcal{T}_{ik} = {}^4T_{ik} \quad . \tag{10c}$$

In terms of these quantities, the relevant parts of Eqs. (9) become

$$-2\kappa N^2 g^{\frac{1}{2}} T^{oo} = -2\kappa g^{\frac{1}{2}} \mathcal{E} \quad ; \qquad (11a)$$

$$\kappa N g^{\frac{1}{2}} (T^{oi} + N^i T^{oo}) = -\kappa g^{\frac{1}{2}} \mathcal{P}^i \quad ; \qquad (11b)$$

$$\kappa N g^{\frac{1}{2}} (T^{ij} - N^i N^j T^{oo}) = \kappa g^{\frac{1}{2}} (N \mathcal{T}^{ij} + N^i \mathcal{P}^j + N^j \mathcal{P}^i) \quad , \qquad (11c)$$

where all indices on $\mathcal{P}$ and $\mathcal{T}$ are raised by the three-dimensional metric.

Steps toward a full canonical theory could well begin here. One method would be to specify in advance the motion of the matter in terms of the metric tensor (e.g., homogeneous cosmology), and then to solve the constraint Eqs. (9a,b) by analogy with vacuum ADaM. A more general approach must include a canonical formulation for the fields present in spacetime. In any case, the basic gravitational constraints and dynamical equations will be Eqs. (9).

REFERENCES

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5. ibid., § 94.

GENERAL RELATIVITY AS A DYNAMICAL SYSTEM ON THE MANIFOLD $\mathcal{A}$ OF RIEMANNIAN METRICS WHICH COVER DIFFEOMORPHISMS

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1. Introduction

In this paper we consider the geometrodynamical formulation of general relativity, due most recently to Arnowitt, Deser, and Misner [2], DeWitt [3] and Wheeler [8], from the point of view of manifolds of maps (function spaces) and infinite-dimensional geometry.

Hydrodynamics is approached from this point of view by Arnold [1] and by Ebin-Marsden [4]; in Fischer-Marsden [5, 6] the function spaces appropriat for a dynamical formulation of general relativity are introduced. We hope that our approach will clarify the basic dynamical structure of the Einstein equations and the associated infinite-dimensional geometry in a spirit analogous to that which has been done in hydrodynamics.

The key to our approach is the group $\mathcal{D}$ = Diff(M) of smooth (C^∞) diffeomorphisms of a fixed 3-dimensional manifold M. For hydrodynamics one concentrates on $\mathcal{D}_\mu$, the volume preserving diffeomorphisms [4]. For relativity one uses the manifold $\mathcal{A}$ of Riemannian metrics which cover diffeomorphisms. We begin with a description of $\mathcal{A}$.

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2. The Manifold $\mathcal{A}$ and the Einstein System

Let M be a fixed (no changes in topology) closed (compact without boundary) 3-dimensional oriented smooth manifold, and let

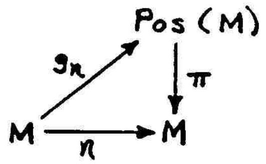
$\mathcal{M}$ = Riem (M) = manifold of smooth Riemannian (positive-definite) metrics on M;

$\mathcal{D}$ = Diff(M) = the group (under composition) of smooth orientation-preserving diffeomorphisms of M; and

$S_2(M)$ = vector space of smooth symmetric 2-covariant tensor fields on M.

Note that $S_2(M)$ is a linear space and that in any decent topology, M is an open convex cone in $S_2(M)$.

Let π : Pos(M) $\to$ M denote the tensor bundle of symmetric positive definite bilinear forms so that $\pi^{-1}(m)$ = space of inner products on $T_m M$. A <u>Riemannian metric</u> g_η <u>which covers a diffeomorphism</u> $\eta \in \mathcal{D}$ is a smooth map $g_\eta : M \to Pos(M)$ such that the following diagram commutes:



(that is, $\pi \circ g_\eta = \eta \in \mathcal{D}$). Thus g_η assigns to each point m $\in$ M an inner product of the tangent space $T_{\eta(m)}M$. We let $\mathcal{A}$ denote the manifold of all such maps for all $\eta \in \mathcal{D}$. $\mathcal{A}$ is the <u>manifold of Riemannian metrics which cover diffeomorphisms</u>. One can prove that $\mathcal{A}$ has the structure of a smooth infinite dimensional manifold, cf. [4, § 2]; we shall not require this structure.

There is a natural projection $\bar{\pi} : \mathcal{A} \to \mathcal{D}$ defined by $\bar{\pi}(g_\eta) = \pi \circ g_\eta = \eta \in \mathcal{D}$. Also, if $g_\eta \in \mathcal{A}$, observe that $g_\eta \circ \eta^{-1} \in \mathcal{M}$ is an "ordinary" Riemannian metric for M. Now $\mathcal{A}$ is diffeomorphic to $\mathcal{D} \times \mathcal{M}$ by the map

$$\Phi_R : \mathcal{Q} \to \mathcal{D} \times \mathcal{M} \; ; \quad g_\eta \mapsto (\eta, g_\eta \circ \eta^{-1}) \, ,$$

(Φ_R = right translation) with inverse

$$\Phi_R^{-1} : \mathcal{D} \times \mathcal{M} \to \mathcal{Q} \; ; \quad (\eta, g) \mapsto g \circ \eta \, .$$

Thus information on $\mathcal{Q}$ can be transferred to $\mathcal{D} \times \mathcal{M}$ and vice-versa via the mapping Φ_R . It is convienient to think of $\mathcal{D} \times \mathcal{M}$ as a realization of $\mathcal{Q}$.

Let $\mathcal{T} = C^\infty(M;\mathbb{R})$ = the vector space of smooth real-valued functions $\xi : M \to \mathbb{R}$ (scalar fields or 0-covariant tensor fields on M).

We will refer to $\mathcal{T}$ as the relativistic time-translation group. Note that the constant functions on M form a subgroup of $\mathcal{T}$ which is isomorphic to R, the classical time-translation group. The manifold $\mathcal{T} \times \mathcal{Q} \approx \mathcal{T} \times \mathcal{D} \times \mathcal{M}$ is the proper configuration space for a geometrodynamical formulation of general relativity as we now explain. We will be concerned with the propagation of initial Cauchy data $(g_0, h_0) \in \mathcal{M} \times S_2(M)$ off some 3-dimensional hypersurface M of, a yet to be constructed, Ricci-flat (vacuum) space-time V_4. Here $h = \dot{g} = \partial g/\partial t$ is the velocity canonically conjugate to the configuration fields g. As g_t is determined only up to its isometry class, the evolution is determined only up to an arbitrary curve $\eta_t \in \mathcal{D}$ of diffeomorphisms called the actual shift (with $\eta_0 = id_M$ = e = the identity diffeomorphism); that is, g_t and $(\eta_t^{-1})^* g_t$ are isometric evolutions, where

$$(\eta_t^{-1})^* g_t(m) \cdot (Y_m, Z_m) = g_t \circ \eta_t^{-1}(m) \cdot (T_{\eta_t^{-1}}(Y_m), T_{\eta_t^{-1}}(Z_m)), \quad Y_m, Z_m \in T_m M,$$

is the "push-forward" of a covariant tensor field. Moreover, one is free to specify on M an arbritrary system of clock rates, or equivalently of clock settings, given as a curve $\xi_t \in \mathcal{T}$ of time functions (the clock settings) with $\xi_0 = \underline{0}$ = the zero function on M (all clocks start at high noon). This arbitrariness or degenency is reflected in the evolution equations as follows:

The Einstein System: Let M be a closed oriented 3-dimensional manifold. Let X_t be an arbitrary time-dependent vector field called the shift vector field and N_t an arbitrary positive scalar field called the lapse function; $N_t(m) > 0$

for all $(t,m) \in \mathbb{R} \times M$. Let g be a given Riemannian metric on M, and let k be a given symmetric 2-covariant tensor field on M such that

$$\delta(k - (\mathrm{Tr}\, k)g) = 0 ,$$

$$\tfrac{1}{2}\left((\mathrm{Tr}\, k)^2 - k\cdot k\right) + 2R(g) = 0 .$$

The problem is to find a time-dependent metric field g_t on M such that g_t and the supplementary variable

$$k_t = \frac{1}{N_t}\left(\frac{\partial g_t}{\partial t} + L_{X_t} g_t\right) ,$$

satisfy:

(i) the given initial conditions: $(g_o, k_o) = (g, k)$,

(ii) the evolution equation

$$\frac{\partial k_t}{\partial t} = S_{g_t}(k_t) - 2N_t \,\mathrm{Ric}(g_t) + 2\,\mathrm{Hess}(N_t) - L_{X_t} k_t .$$

Our notation is the following:

δk = divergence of $k = (\delta k)_i = k_i{}^j{}_{|j}$ ($|j$ = covariant derivative with respect to the time-dependent metric g) ,

$\mathrm{Tr}k$ = Trace $k = g^{ij}k_{ij} = k^i{}_i$,

$k\cdot k$ = dot product for symmetric tensors = $k_{ij}k^{ij}$,

$k\times k$ = cross-product for symmetric tensors = $k_{i\ell}k^{\ell}{}_j$,

$S_g(k) = k\times k - \frac{1}{2}(\mathrm{Tr}k)k = k_{i\ell}k^{\ell}{}_j - \frac{1}{2}(g^{mn}k_{mn})k_{ij}$ = DeWitt spray on $\mathcal{M}$,

$L_{X_t} g_t$ = Lie derivative of g_t with respect to the time-dependent vector field $X_t = X_{i|j} + X_{j|i}$,

$L_{X_t} k_t$ = Lie derivative of $k_t = X^{\ell}k_{ij|\ell} + k_{i\ell}X^{\ell}{}_{|j} + k_{j\ell}X^{\ell}{}_{|i}$,

$\mathrm{Ric}(g_t)$ = Ricci curvature tensor formed from $g_t = R_{ij} =$

$$\Gamma^k_{ij,k} - \Gamma^k_{ki,j} + \Gamma^k_{ij}\Gamma^{\ell}_{k\ell} - \Gamma^{\ell}_{ik}\Gamma^k_{\ell k} ,$$

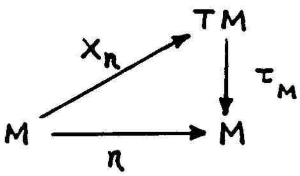
$R(g_t)$ = Scalar curvature = R^k_k ,

$\mathrm{Hess}(N)$ = Hessian of N = double covariant derivative = $N_{|i|j}$.

We now explain how the Einstein system, the lapse function N_t, the shift vector field X_t, and the configuration space $\mathcal{T} \times \mathcal{D} \times \mathcal{M}$ are interrelated (see Fischer-Marsden [5] for more details).

3. <u>The Geometry of the Shift Vector Field</u>

Let $\mathcal{D}$ = Diff(M), the group of all smooth orientation preserving diffeomorphisms of M. Now $\mathcal{D}$ is a manifold modeled on a Frechet space; (see Ebin-Marsden [4] and related references for the structure of $\mathcal{D}$). The tangent space $T_\eta \mathcal{D}$ at a point $\eta \in \mathcal{D}$ is the manifold of smooth maps $X_\eta : M \to TM$ which cover η, that is, such that the following diagram commutes:



where τ_M denotes the canonical projection of TM to M. To see this let $\eta_t \in \mathcal{D}$ be a curve in $\mathcal{D}$, $\eta_0 = \eta$, so that $\left.\frac{d\eta_t}{dt}\right|_{t=0}$ represents a tangent vector in $T_\eta \mathcal{D}$. But for $m \in M$ fixed, $\sigma(t) = \eta_t(m)$ is a curve in M with $\sigma(0) = \eta_t(m)$ and with tangent $\sigma'(0) = \left.\frac{d\eta_t}{dt}(m)\right|_{t=0} \in T_{\eta(m)}M$. Thus $\frac{d\eta_t}{dt}$ is a map from M to TM covering η.

We refer to X <u>as a vector field which covers</u> η, so that $T\mathcal{D}$ is the <u>manifold of vector fields covering diffeomorphisms</u>. In particular, $T_e\mathcal{D} = \mathfrak{X}(M)$ = the vector space of smooth vector fields on M = the Lie algebra of $\mathcal{D}$.

As with the manifold $\mathcal{Q}$, there is a natural projection $\bar{\tau} : T\mathcal{D} \to \mathcal{D}$ defined by $\bar{\tau}(X_\eta) = \tau_M \circ X_\eta = \eta \in \mathcal{D}$.

Let $R_{\eta_1} : \mathcal{D} \to \mathcal{D}$ denote right translation by η_1; ($R_{\eta_1}(\eta) = \eta\eta_1$). Then

$$TR_{\eta_1} : T\mathcal{D} \to T\mathcal{D} \quad ; \quad X_\eta \mapsto X_\eta \circ \eta_1 \;,$$

so that for $X_\eta \in T_\eta\mathcal{D}$, $TR_{\eta^{-1}}(X_\eta) = X_\eta \circ \eta^{-1} \in T_e\mathcal{D}$ is an "ordinary" vector field on M, called <u>the pull-back</u> of X_η <u>by right translation</u>.

Now let $X_t : M \to TM$ be a time-dependent vector field on M. Then the flow η_t of X_t with η_0 = identity is a smooth curve in $\mathcal{D}$ (as X_t is time-dependent, η_t is not a one-parameter subgroup of $\mathcal{D}$) which satisfies

$$\frac{d\eta_t}{dt} = X_t \circ \eta_t \ , \quad \text{or} \quad \frac{d\eta_t}{dt} \circ \eta_t^{-1} = X_t \ .$$

Conversely, given a smooth curve $\eta_t \in \mathcal{D}$ with η_0 = identity, $\frac{d\eta_t}{dt} \circ \eta_t^{-1} = X_{\eta_t} \circ \eta_t^{-1} = X_t$ is a time-dependent vector field which generates η_t as its flow.

Thus in the Einstein system, if one gives the shift vector field X_t, then the actual shift of M is its flow $\eta_t \in \mathcal{D}$, a curve in $\mathcal{D}$. Equivalently one may specify the actual shift $\eta_t \in \mathcal{D}$ and compute the shift vector field as above. It is because of the presence of the shift vector field that the group must be included in the configuration space.

The relationship between the Lie derivative terms and the shift vector field can be explained geometrically as follows. Suppose that for $\bar{N}_t = 1$, $\bar{X}_t = 0$, $(\bar{g}_t, \bar{k}_t) \in \mathcal{M} \times S_2(M)$ is a solution to the Einstein system with initial conditions $(\bar{g}_0, \bar{k}_0)$; that is,

$$\frac{\partial \bar{g}_t}{\partial t} = \bar{k}_t \ ,$$

$$\frac{\partial \bar{k}_t}{\partial t} = S_{\bar{g}_t}(\bar{k}_t) - 2 \operatorname{Ric}(\bar{g}_t) \ .$$

Now let X_t be an arbitrary shift vector field with flow η_t, η_0 = identity. Then $(g_t, k_t) = ((\eta_t^{-1})^* \bar{g}_t, (\eta_t^{-1})^* \bar{k}_t)$ are solutions to the evolution equations with $N_t = 1$, X_t = given shift vector field, and the same initial data as before. This follows by a direct verification:

$$\frac{\partial g_t}{\partial t} = \frac{\partial (\eta_t^{-1})^* \bar{g}_t}{\partial t}$$

$$= (\eta_t^{-1})^* \frac{\partial \bar{g}_t}{\partial t} - L_{X_t}((\eta_t^{-1})^* \bar{g}_t)$$

$$= k_t - L_{X_t} g_t \ ,$$

where we have used the fact that

$$\frac{d}{dt}\,(\eta_t^{-1})^* g = -L_{X_t}\,(\eta_t^{-1})^* g \text{ ; see [7] , p. 32 .}$$

Similiarly,

$$\frac{\partial k_t}{\partial t} = \frac{\partial (\eta_t^{-1})^* \bar{k}_t}{\partial t}$$

$$= (\eta_t^{-1})^* \frac{\partial \bar{k}_t}{\partial t} - L_{X_t} (\eta_t^{-1})^* \bar{k}_t$$

$$= S_{g_t}(k_t) \quad -2\mathrm{Ric}\,(g_t) \quad - \quad L_{X_t} k_t$$

since $S_{\bar{g}}(\bar{k})$ and Ric $(\bar{g})$ are tensors and hence commute with $(\eta_t^{-1})^*$; that is, $(\eta_t^{-1})^*(\mathrm{Ric}\,(\bar{g})) = \mathrm{Ric}((\eta_t^{-1})^* \bar{g}) = \mathrm{Ric}(g)$.

The significance of this result may be clarified as follows: Besides the realization of $\mathcal{A}$ as $\mathcal{D} \times \mathcal{M}$ by "right translations," there is a realization of $\mathcal{A}$ as $\mathcal{D} \times \mathcal{M}$ by "left translations" defined as follows:

$$\Phi_L : \mathcal{A} \to \mathcal{D} \times \mathcal{M} \; ; \quad g_\eta \mapsto (\eta^{-1})^* (g_\eta \circ \eta^{-1}) \; .$$

These two realizations of $\mathcal{A}$ are entirely analogous to the two realizations of TSO(3) for the rigid body into body and space coordinates respectively; see Arnold [1] . Thus the introduction of a shift may be viewed merely as shifting from body to space coordinates by use of the coordinate change η_t .

4. The Lapse Function and the Intrinsic Shift Vector Field

To discuss the lapse we assume that the shift vector field $X_t = 0$. (They can be handled simultaneously by using the semi-direct product on $\mathcal{T} \times \mathcal{D}$.) If we choose the lapse $N_t = 1$, then the evolution of g is parameterized by a canonical evolution parameter, the proper time τ. But suppose that g is a solution of the Einstein system for an arbitrary lapse N. One constructs a space-time on $\mathbb{R} \times M$ in a tubular neighborhood of M by the Lorentz metric (in coordinates)

$$g_{\mu\nu}\,dx^{\mu}dx^{\nu} = -N^{2}dt^{2} + g_{ij}\,dx^{i}dx^{j}\,.$$

The proper time function $\tau(t,m) = \tau_t(m) = \tau(t,x^k)$ (in this tubular neighborhood of M) is then just the time coordinate in Gaussian normal coordinates $(\tau(t,x^k), \bar{x}^i(t,x^k))$, where $\bar{x}^i(t,x^k)$ is the space part of the Gaussian coordinates. To find the relation between the lapse N_t and τ_t, we consider the transformation of $g_{\mu\nu}$ to Gaussian normal coordinates; writing out $\bar{g}^{oo} = g^{\mu\nu}\frac{\partial\tau}{\partial x^{\mu}}\frac{\partial\tau}{\partial x^{\nu}}$ yields

$$-1 = -\frac{1}{N^2}\left(\frac{\partial\tau}{\partial t}\right)^2 + g^{k\ell}\frac{\partial\tau}{\partial x^k}\frac{\partial\tau}{\partial x^{\ell}}\,,$$

which is solved for N_t to give

$$N_t = \frac{d\tau_t}{dt}\,\frac{1}{\sqrt{1+\|\mathrm{grad}\ \tau_t\|^2}}\,,$$

where $\|\mathrm{grad}\ t\|^2 = g^{k\ell}\frac{d\tau}{dx^k}\frac{d\tau}{dx^{\ell}}$ is computed with respect to the inverse $g^{k\ell}$ of the time-dependent metric $g_{k\ell}$ ($= {}^4g^{ij}$ since the shift is zero). The factor $\frac{1}{\sqrt{1+\|\mathrm{grad}\,\tau\|^2}}$ takes into account the fact that in general the lapse depends on space coordinates and therefore pushes up the hypersurface M through $\mathbb{R}\times M$ unevenly.

The single first order partial differential equation for τ

$$\left(\frac{d\tau}{dt}\right)^2 - N^2 g^{k\ell}\frac{d\tau}{dx^k}\frac{d\tau}{dx^{\ell}} = N^2$$

can be reduced to a system of eight first-order ordinary differential equations by the Cauchy method of characteristics. Of course this system of ordinary differential equations is just the system of geodesic equations of the Lorentz metric $g_{\mu\nu}$ (for unit timelike geodesics) in Hamiltonian form. If we choose on the non-characteristeric hypersurface $t = 0$ the initial condition: $\tau(0,m) = 0$ (corresponding to geodesics normal to $t = 0$), then we are assured of a unique $\tau(t,m)$ that satisfies the above equation with the initial condition $\tau(0,m) = 0$. Note that $\frac{d\tau}{dt} = N$ on this initial hypersurface.

The condition

$$\bar{g}^{0i} = -\frac{1}{N^2}\frac{\partial \bar{x}^i}{\partial t}\frac{\partial \tau}{\partial t} + \frac{\partial \bar{x}^i}{\partial x^n}\left(g^{mn}\frac{\partial \tau}{\partial x^m}\right) = 0$$

gives an equation for the space part $\bar{x}^i(t,x^k)$ of the Gaussian normal coordinate system,

$$\frac{\partial \bar{x}^i}{\partial t}(t,x^k) = \frac{N(t,x^k)}{\sqrt{1+\|\mathrm{grad}\,\tau\|^2}}\frac{\partial \bar{x}^i}{\partial x^n}(t,x^k)\left(g^{mn}(t,x^k)\frac{\partial \tau}{\partial x^m}(t,x^k)\right).$$

Set $Y_t = -\dfrac{N(t,x^k)}{\sqrt{1+\|\mathrm{grad}\,\tau\|^2}}\,\mathrm{grad}\,\tau$; then the above equation can be written as

$$\frac{d\varphi_G}{dt} = -D\varphi_G\cdot(Y),$$

where φ_G is the spatial part of the Gaussian normal coordinates and $D\varphi_G$ is, in coordinates, the Jacobian matrix of φ_G. But the identity

$$\frac{d}{dt}\left(f_t^{-1}\circ f_t\right) = \frac{df_t^{-1}}{dt}\circ f_t + Df_t^{-1}\cdot\frac{df_t}{dt} = \frac{df_t^{-1}}{dt}\circ f_t + Df_t^{-1}\cdot Y_t\circ f_t = 0$$

then shows that this equation is solved by $\varphi_G = f_t^{-1}$ if f_t is the flow of Y_t. We call Y_t the <u>intrinsic shift of the lapse</u> since it describes the "tilting" of the Gaussian normal coordinates due to the space dependence of the lapse function. The above argument shows that the partial differential equation for the space part of the Gaussian normal coordinate system can be solved by an ordinary differential equation, namely finding the flow of the intrinsic shift. Finally, the inverse to the contravariant metric

$$\begin{aligned}\bar{g}^{ij}(\tau(t,x^k),\bar{x}^i(t,x^k)) &= \frac{\partial \bar{x}^i}{\partial x^m}(t,x^k)\frac{\partial \bar{x}^j}{\partial x^n}(t,x^k)\,g^{mn}(t,x^k) - \frac{1}{N^2}\frac{\partial \bar{x}^i}{\partial t}(t,x^k)\frac{\partial \bar{x}^j}{\partial t}(t,x^k)\\ &= \frac{\partial \bar{x}^i}{\partial x^m}(t,x^k)\frac{\partial \bar{x}^j}{\partial x^n}(t,x^k)\Bigg(g^{mn}(t,x^k) - \frac{1}{1+\|\mathrm{grad}\,\tau\|^2}\,g^{m\ell}(t,x^k)\frac{\partial \tau}{\partial x^\ell}(t,x^k)\,g^{nr}(t,x^k)\\ &\qquad\cdot\frac{\partial \tau}{\partial x^r}(t,x^k)\Bigg)\end{aligned}$$

solves the evolution equations with $N = 1$ (and the same initial data) if $g_{ij}(t, x^k)$ solves the Einstein equations with an arbitrary N. Writing g^{-1} for the contravariant components of g, the above equation can be written intrinsically as

$$\bar{g}^{-1}\left(\tau(t,m),\, \varphi_G(t,m)\right) = D\varphi_G(t,m) \otimes D\varphi_G(t,m)\left(g^{-1}(t,m) - \frac{\operatorname{grad}\tau(t,m)}{\sqrt{1+\|\operatorname{grad}\tau\|^2}} \otimes \frac{\operatorname{grad}\tau(t,m)}{\sqrt{1+\|\operatorname{grad}\tau\|^2}}\right).$$

Our prescription shows how, given a solution to the Einstein equation with an arbitrary N, to find the solution to the Einstein equations with $N = 1$ and the same initial data by solving ordinary differential equations only. A similiar prescription is available to go from solutions for $N = 1$ to solutions for arbitrary N; see [5]. To take into account the lapse function we introduce the relativistic time translation group $\mathcal{T} = C^\infty(M;\mathbb{R})$ (a group under pointwise addition of functions). As $\mathcal{T}$ is a vector space, $T\mathcal{T} = \mathcal{T} \times \mathcal{T}$. For a given lapse N_t and a solution g_t to Einstein's equations with this lapse, we construct a curve $\tau_t \in \mathcal{T}$ such that

$$\left(\frac{d\tau}{dt}\right)^2 - N^2\,\|\operatorname{grad}\tau\|^2 = N^2$$

and $\tau_0 = 0$. Thus to find the curve in $\mathcal{T}$ corresponding to a given lapse N we must first solve Einstein's equations with this particular lapse.

In the case that N depends only on the time coordinate, then τ_t and N_t are simply related by $\tau_t = \int_0^t N_\lambda d\lambda$. Moreover, if $(\bar{g}_t, \bar{k}_t)$ is a solution to the Einstein system with initial conditions $(\bar{g}_0, \bar{k}_0)$ and lapse $\bar{N}_t = 1$, then the solution with $N_t = f(t)$ (and $X_t = 0$) and the same initial conditions is just the reparameterized curve $(g_t, k_t) = (\bar{g}_{\tau(t)}, \bar{k}_{\tau(t)})$. This is easily seen, as

$$\frac{\partial g_t}{\partial t} = \frac{\partial \bar{g}_{\tau(t)}}{\partial t} = \frac{\partial \bar{g}_{\tau(t)}}{\partial \tau}\frac{d\tau(t)}{dt} = N_t\,\bar{k}_{\tau(t)} = N_t\,k_t$$

and

$$\frac{\partial k_t}{\partial t} = \frac{\partial \bar{k}_{\tau(t)}}{\partial t} = \frac{\partial \bar{k}_{\tau(t)}}{\partial \tau}\frac{d\tau(t)}{dt} = N_t\left(S_{\bar{g}_{\tau(t)}}(\bar{k}_{\tau(t)}) - 2\,\mathrm{Ric}(\bar{g}_{\tau(t)})\right)$$

$$= N_t\,S_{g_t}(k_t) - 2N_t\,\mathrm{Ric}(g_t)\ .$$

5. The Einstein Lagrangian on $\mathcal{T}\times\mathcal{Q}\approx\mathcal{T}\times\mathcal{D}\times\mathcal{M}$

Since $\mathcal{M}$ is an open convex cone in $S_2(M)$, $T\mathcal{M}=\mathcal{M}\times S_2(M)$. On $\mathcal{M}$ we define the DeWitt metric $\mathcal{G}$ (see DeWitt [3], and Fischer-Marsden [5]) by

$$\mathcal{G}_g : T_g\mathcal{M}\times T_g\mathcal{M} = S_2(M)\times S_2(M)\to\mathbb{R}$$

$$\mathcal{G}_g(h_1,h_2) = \int_M \left((\mathrm{Tr}\,h_1)(\mathrm{Tr}\,h_2) - h_1\cdot h_2\right)\mu_g ,$$

where μ_g is the volume element associated with the metric g (in coordinates $\mu_g = \sqrt{\det g}\ dx^1\wedge dx^2\wedge dx^3$). $\mathcal{G}$ is a non-degenerate but weak metric on $\mathcal{M}$; here weak means that the map $\mathcal{G}_g^{\#} : T_g\mathcal{M}\to T_g^{*}\mathcal{M}$, defined by $\mathcal{G}_g^{\#}(h_1)\cdot h_2 = \mathcal{G}_g(h_1,h_2)$ is an injection, by the non-degeneracy, but is not an isomorphism.

We now introduce a potential $V:\mathcal{M}\to\mathbb{R}$ defined by

$$V(g) = 2\int_M R(g)\,\mu_g$$

(twice the integrated scalar curvature). If on $T\mathcal{M}$ we consider the Lagrangian

$$L = T - V : T\mathcal{M} = \mathcal{M}\times S_2(M)\to\mathbb{R} ,$$

defined by $\qquad L(g,h) = \frac{1}{2}\mathcal{G}_g(h,h) - V(g)$,

then a computation shows that Lagrange's equations give the Einstein system with lapse $N_t = 1$ and shift $X_t = 0$.

The DeWitt metric $\mathcal{G}$ on $\mathcal{M}$ is extended to $\mathcal{D}\times\mathcal{M}\approx\mathcal{Q}$ by defining on each fiber $T_{(\eta,g)}(\mathcal{D}\times\mathcal{M}) = T_\eta\mathcal{D}\times S_2(M)$

$$\mathcal{G}(\eta,g) : (T_\eta\mathcal{D}\times S_2(M))\times(T_\eta\mathcal{D}\times S_2(M))\to\mathbb{R}$$

$$\mathcal{G}(\eta,g)\left((X_{\eta_1},h_1),(X_{\eta_2},h_2)\right) = \mathcal{G}_g(h_1+L_{X_{\eta_1}\circ\eta_1^{-1}}g,\ h_2+L_{X_{\eta_2}\circ\eta_2^{-1}}g).$$

The Lagrangian L on $T\mathcal{M}$ is now extended to a Lagrangian on $T(\mathcal{D}\times\mathcal{M})$ by

$$\bar{L} : T(\mathcal{D}\times\mathcal{M}) = T\mathcal{D}\times\mathcal{M}\times S_2(M)\to\mathbb{R}$$

$$\bar{L}(X_\eta,g,h) = L\left(g,\ h + L_{X_\eta\circ\eta^{-1}}g\right)$$

$$= \frac{1}{2} \mathcal{G}_g(h + L_{X_\eta \circ \eta^{-1}} g, \quad h + L_{X_\eta \circ \eta^{-1}} g) \quad - V(g) \; .$$

Note that the factor $\mathcal{D}$ is now essential as X_η is explicitly involved in L.

Now $\mathcal{G}$ is a degenerate metric on $\mathcal{D} \times \mathcal{M}$ since if

$$\mathcal{G}(\eta, g)\left(h + L_{X_\eta \circ \eta^{-1}} g, \; k + L_{Y_\eta \circ \eta^{-1}} g\right) = 0 \quad \text{for all} (Y_\eta, k) \in T_\eta \mathcal{D} \times S_2(M),$$

then

$$h + L_{X_\eta \circ \eta^{-1}} g = 0 \; ,$$

but h and X_η need not be zero independently. This degeneracy has the effect of introducing some ambiguity into the equations of motion. However, the degeneracy of $\mathcal{G}$ is such that we are free to specify a curve of diffeomorphisms $\eta_t \in \mathcal{D}$; thus the ambiguity in the equations of motion is completely removed by the specification of the shift vector field X_t.

Using $\bar{L} : T(\mathcal{D} \times \mathcal{M}) \to \mathbb{R}$, we construct on $T(\mathcal{T} \times \mathcal{D} \times \mathcal{M})$ the

Einstein Lagrangian

$$L_E : T(\mathcal{T} \times \mathcal{D} \times \mathcal{M})$$

defined by

$$L_E(\xi, N, X_\eta, g, h) = \int_M N\left\{\left(\frac{h + L_{X_\eta \circ \eta^{-1}} g}{N}\right) \cdot \left(\frac{h + L_{X_\eta \circ \eta^{-1}} g}{N}\right) - \left[\mathrm{Tr}\left(\frac{h + L_{X_\eta \circ \eta^{-1}} g}{N}\right)\right]^2\right\} \mu_g$$

$$- 2 \int_M N \; R(g) \, \mu_g \; .$$

L_E now picks up a degeneracy in the $\mathcal{T}$ direction, as well as in the $\mathcal{D}$ direction, allowing for the arbitrary specification of N_t as well as X_t. However, once N_t and X_t are specified, the degeneracy of L_E is completely removed and the evolution equations are well-defined. A computation then shows that Lagrange's equations in the "non-degenerate direction", together with the arbitrarily specified lapse function N_t and shift vector field X_t, are the Einstein equations of evolution (see [5] for details).

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