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String Field Theory Vertices, Integrability and Boundary States

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Abstract

We study Neumann coefficients of the various vertices in the Witten's open string field theory (SFT). We show that they are not independent, but satisfy an infinite set of algebraic relations. These relations are identified as so-called Hirota identities. Therefore, Neumann coefficients are equal to the second derivatives of tau-function of dispersionless Toda Lattice hierarchy (this tau-function is just a partition sum of normal matrix model). As a result, certain two-vertices of SFT are identified with the Neumann boundary states on an arbitrary curve.

We further analyze a class of SFT surface states, which can be re-written in the closed string language in terms of boundary states. This offers a new correspondence

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between open string states and closed string states (boundary states) in SFT. We conjecture that these special states can be considered as describing D-branes and other extended objects as "solitons" in SFT. We consider some explicit examples, one of them is a surface states corresponding to orientifold.

Contents

1 Introduction and summary of main results

A new structure seems to be underlying the open string field theory (OSFT) [1]. Namely, it becomes increasingly clear that a number of objects of SFT can be given an interpretation in terms of *integrable hierarchies*.

On the other hand, studies of tachyon dynamics and other related hypotheses (see e.g. [2]) within the scope of open SFT (see e.g. [3, 4]) have risen the question of description of closed string states in the open SFT. Surprisingly, traveling an unexploited path of integrability in SFT, we in fact will be able to address these questions as well.

It has been shown in [5] that various states in CFT can be associated with tau-functions of dispersionless KP and Toda Lattice hierarchies[1]. Times of integrable hierarchy parameterize the dependence of these states on an arbitrary conformal transformation. Vacuum state for the holomorphic scalar field in the plane ("open string picture") corresponds to dKP hierarchy[2]. Boundary states in case of arbitrary scalar field in the plane ("closed string picture") are described by the whole dToda hierarchy, with holomorphic and antiholomorphic sectors being mixed.

An important example of this construction is a surface state in open SFT [7]. Neumann coefficients of the surface state were shown in [5] to be not independent but to satisfy an infinite set of algebraic relations. These relations are nothing else but so-called *Hirota identities* for dKP hierarchy. It means that the Neumann coefficients are just second derivatives of (dispersionless) KP tau-function.

Hirota identities distinguish the tau-functions of dispersionless integrable hierarchies from any other function of infinite number of variables. They turned out to be equivalent to the condition for a state to be Bogolyubov transform of a vacuum in case of "open string picture". Namely, on the CFT side Hirota identities are the conditions which guarantee existence of some operators $b_k, \bar{b}_k$ annihilating the transformed state. These operators are linear combinations of original creation and annihilation operators $a_k, \bar{a}_k$.

For the "closed string picture" the Hirota identities of dispersionless Toda Lattice hierarchy mean that corresponding state is annihilated by combinations $b_k \pm \bar{b}_k$, being thus Bogolyubov transform of Neumann or Dirichlet boundary state. Such a state is exponential of combination quadratic in creation operators $a_k^+, \bar{a}_k^+$ with coefficients being second derivatives of dispersionless Toda Lattice tau-function with respect to all times t_k and $\bar{t}_k$. It is worth noting that this is the same tau-function that is equal to a partition sum of normal matrix model [8, 9].

Later, it was observed in [10] that Neumann coefficients of Witten's three-vertex in open SFT could also be expressed via derivatives of particular tau-function of dispersionless Toda Lattice hierarchy. Thus boundary states in "closed string picture" and three-vertex in open SFT are expressed in terms of the same data and should be somehow related. However, naively there is no immediate connection between these two objects.

Moreover, a three-vertex in SFT naturally depends on three independent sets of creation operators [7]. This gives nine infinite matrices of the corresponding Neumann coefficients N^{IJ} $(I, J = 1, 2, 3)$. On the other hand in dToda hierarchy there are only two infinite sets of times $t_k, \bar{t}_k$ and, consequently, three infinite matrices of second derivatives $F^{AB} \equiv \partial_{t^{(A)}}\partial_{t^{(B)}}F$, where $A, B = 1, 2$ and correspond to t_k and $\bar{t}_k$. This difficulty did not appear in [10] because for the particular case of Witten's vertex (corresponding to special choice of conformal maps which define gluing of three open string world sheets) all matrices of the Neumann coefficients are expressed via just two independent ones.

[1]Throughout this paper we will often use terms "dKP" and "dToda" when referring to the dispersionless KP and the dispersionless Toda Lattice hierarchies [6] correspondingly.

[2]Up to some subtleties, dKP hierarchy can be thought of as a holomorphic sector of dToda hierarchy.

Still, even for this case the three-vertex depends on three independent sets of creation operators, although some coefficients in front of them coincide. Because of this, the exact identification between combinations of Neumann coefficients and second derivatives of tau-functions were just guessed so that they would satisfy Hirota identities. The meaning of these formulae was unclear. Also, it was unclear if there was any relation between the three-vertex and boundary states.

In this paper we address these questions. We find explicit connection between the SFT two-vertices and boundary states. By its nature a two-vertex relates correlators in the tensor product of two open strings to correlators of one open string. On the other hand, a boundary state relates closed strings to open strings. In order for a two-vertex to be a boundary state it should combine two open strings into one closed. To achieve this, conformal transformations that define the two-vertex should map world sheets of two open strings into that of a closed string. We state a simple condition such transformations should obey. Using this connection, we derive in a systematic way the formulae relating Neumann coefficients for general SFT vertex and second derivatives of tau-function. This relation is a trivial consequence of the fact, that any n-vertex, contracted with vacua states in $n-2$ sectors is just a two-vertex in the remaining two sectors. In particular, this explains the results guessed in [10] for the case of Witten's three-vertex.

There is one more conceptual problem in [10] which we would like to address here. In the case of Witten's three-vertex that we considered till now all three conformal maps were fixed. It means that corresponding Neumann coefficients are just numbers, not functions of t_k, which are identified with second derivatives of tau function evaluated at some particular fixed values of t_k. However, the whole experience of integrable systems (in its application e.g. to matrix models, SYM theories, quantum Hall effect) teaches us that it is important and usually fruitful to introduce the dynamics w.r.t. these t_k even if we are interested at the end only in the results for some fixed times. (We will comment on the application of this ideology to the present case latter). This is why, once we have found a tau function (or its second derivatives) calculated at some particular values, it is interesting to try to understand what could this object mean at different, arbitrary values of t_k. Note, that for the "chiral" case of KP[3] we have already found an answer to this question. KP case was related to the surface states (or "conformally transformed vacuum" [5]). These states (one-vertices) $|\Sigma\rangle$ are defined for an arbitrary conformal map from some proper family. We can parameterize this family of states $|\Sigma(t_k)\rangle$ by times t_k and in such way describe dynamics of surface states. Thus the functional dependence of KP tau-function with respect to t_k (not just its value at fixed t_k) makes an appearance in SFT. We will discuss the consequences of this below.

However, we are interested in finding the full Toda dynamics in the SFT. We see from [10] that it indeed exists. Natural object for this dynamics (as this paper will show) is a two-vertex. Again, we want to make Neumann coefficients of the two-vertices varying with the t_k or (to put it differently) depending on the arbitrary conformal maps. What can be the meaning of such an object and where can it naturally appear?

Let us contract a surface state with Witten's three-vertex. According to "gluing

[3] See footnote 2 on the preceding page

theorem" [11] the result will be some two-vertex. This two-vertex can be considered as a *multiplication operator*. For a given surface state $|\Sigma\rangle$ this operator $\hat{\Sigma}$ is defined in the following way

$$\hat{\Sigma} : |X\rangle \to |X * \Sigma\rangle \quad \forall\ |X\rangle$$

and can be realized via two-vertex $|V_{12}(\Sigma)\rangle$. Again, this two-vertex can be viewed as a functional on the whole family of surface states. The resulting $|V_{12}(t_k)\rangle$ is Toda tau-function not for fixed, but for dynamical t_k.

This object, which was just introduced following the intuition led by integrable structure, has quite interesting interpretation by itself. It leads to a very intriguing possibility of describing solitonic objects of open SFT as some boundary states. Indeed, as we just said, some two-vertices can be interpreted as boundary states [4]. Thus multiplication operators $|V_{12}(\Sigma)\rangle$ will in fact be boundary states for some special surface states $|\Sigma\rangle$. A very interesting problem is to find a criteria the surface state should satisfy to give rise to a boundary state. We do not try to address this question in its full mathematical generality, but rather focus on various physically relevant examples.

It is a plausible assumption that some of these boundary states have relations to D-branes. Thus, as a first example of such surface states in SFT, we come to vacuum SFT (VSFT) (see, e.g. [4]) where D-branes as solitonic solutions of equations of motion were conjectured [3]. In the framework of VSFT these solutions are described via star-algebra projector states – they square to themselves under star-product [12]. Their identification as branes is based on the fact that they yield correct ratios of tensions of branes of different dimensions. We take one of such projectors, which evaded an interpretation so far – the identity state – and find that it does give rise to orientifold boundary state. We present some evidence that this should be also true in case of *sliver state*. In general we conjecture that the solutions identified in [3] with D-branes lead to Dirichlet and Neumann boundary states. If true, this would not only allow to test the conjecture of [3], but also provide a new candidate for description of closed strings in the open string field theory.

There is a variety of results which could be easily obtained once the integrability structure of the problem is identified. Because t_k's parameterize the whole family of surface states and surface states form a sub-algebra under star product it should be possible to re-write star multiplication analytically in terms of t_k. This would allow us, for example, use the well-known integrable reductions to identify the finite-dimensional sub-algebras of star-product. Another possible development can be made by giving sense to derivatives of Neumann coefficient w.r.t. t_k, i.e. the third derivatives of tau-function. In [13] it was shown that associativity (WDVV) equations [14] are solved by tau-function as a direct consequences of Hirota identities. The integrable structure of SFT, identified in this paper, thus strongly suggests the presence of such associativity algebra. The identification of the chiral rings of associative operators in SFT is just one of the numerous possibilities opened by integrability.

The paper has two logical parts. We start by introducing the setup and showing

[4]Again, this observation, which can be made directly, originated from integrable structure – from the fact that two-vertex and boundary state are expressed in terms of the same data, second derivatives of the same Toda tau-function

how conformally transformed vacuum of scalar CFT is related to the holomorphic (anti-holomorphic) part of dToda tau-function in Section 2. (The corresponding material about (dispersionless) Toda Lattice hierarchy is reviewed in Appendix A). This state has an interpretation as a surface state of OSFT. We show that Neumann coefficients of such surface state satisfy Hirota identities. In Section 3 we consider the free scalar CFT with imposed Neumann (Dirichlet) boundary conditions on an arbitrary analytic curve $\mathcal{C}$. We show that it is just a conformally transformed Neumann (Dirichlet) boundary state on a circle. We relate the boundary state on a curve $\mathcal{C}$ to the whole dToda tau-function (not only to the holomorphic part of it). We review the basic objects of SFT, two- and three-vertices in Section 4. The two-vertex gets identified with a Neumann boundary state on a curve. Having done that, we are able to relate the two-vertex to the same Toda tau-function, it is done in Section 4.1. In Section 5 we comment on the relation between three-vertex and dToda tau-function.

Sections 6–6.4 comprise the second part of the paper, addressing the issue which can be considered independently of the story with integrability. Although the motivation comes from the problem of finding dynamical interpretation for times t_k discussed above. In Section 6 we investigate the consequences of the correspondence between two-vertex and a boundary state. We conjecture a way to build boundary states that correspond to solitonic solutions of certain VSFT and explain how these boundary states would allow to test conjecture of [3]. We explore this correspondence first on the sub-algebra of *wedge states*. This is done in the Section 6.1. Section 6.2 contains an explicit example of the correspondence, we show in it that the identity state of the star-algebra can be identified with the orbifold boundary state. This triggers some additional comments, presented in the Section 6.3. We discuss all the open issues in the Section 6.4.

All technical details are given in the appendices. Appendix A contains a digest of various topics related to integrable hierarchies. It is intended to give a reader who is unfamiliar with the subject a quick overview of KP and Toda Lattice hierarchies as well as their dispersionless limits; introduces basic notions and approaches, and provide (non-exquisite) list of references. We give a proof of relation between Neumann boundary state on a circle and a two-vertex of SFT in Appendix B. In Appendix C two-vertex of SFT defined by arbitrary conformal transformations is related to the a boundary state on an analytic curve. Finally, in Appendix D we give a full list of identification between the two-vertex and tau-function of Toda Lattice hierarchy.

2 Vacuum state in CFT and dToda tau-function

Consider a scalar field defined on a whole complex plane. It is a sum of independent holomorphic and anti-holomorphic components[5]

$$\phi(w,\bar{w}) = \phi_0 + \phi(w) + \bar{\phi}(\bar{w}) = \phi_0 + b_0 \log w + \bar{b}_0 \log \bar{w} - \sum_{k=-\infty}^{\infty}{}' \left(\frac{b_k}{k\, w^k} + \frac{\bar{b}_k}{k\, \bar{w}^k} \right) \tag{2.1}$$

The operators $b_k, \bar{b}_k$ are harmonics of two independent currents $J(w) = \partial\phi(w,\bar{w})$, $\bar{J}(\bar{w}) = \bar{\partial}\phi(w,\bar{w})$

$$b_k = \oint_\infty \frac{dw}{2\pi i} w^k J(w) \qquad \bar{b}_k = \oint_\infty \frac{d\bar{w}}{2\pi i} \bar{w}^k \bar{J}(\bar{w}) \tag{2.2}$$

After quantization they obey the usual commutation relations

$$[b_k, b_n] = [\bar{b}_k, \bar{b}_n] = k\delta_{k,-n} \tag{2.3}$$

One can construct a Fock space which is a tensor product of Fock spaces for holomorphic and antiholomorphic sectors correspondingly, in which operators b_k and b_{-k} ($\bar{b}_k$ and $\bar{b}_{-k}$) act as annihilation and creation operators. The vacua in these Fock spaces are defined by

$$b_k\,|p_b\rangle = 0, \quad \forall\, k \geq 1 \tag{2.4}$$

$$\bar{b}_k\,|\bar{p}_b\rangle = 0, \quad \forall\, k \geq 1 \tag{2.5}$$

where p_b ($\bar{p}_b$) are eigenvalues of the operators b_0 ($\bar{b}_0$) (see the footnote 5). Since $\phi(w,\bar{w})$ is a scalar field, currents $J(w)$ and $\bar{J}(\bar{w})$ are primary fields with conformal dimensions $\Delta = (1,0)$, $\bar{\Delta} = (0,1)$ correspondingly. Thus under conformal transformation $(z,\bar{z}) \to (w(z), \bar{w}(\bar{z}))$ they change as

$$J(w) = \frac{dz}{dw} \mathcal{J}(z) \tag{2.6}$$

$$\bar{J}(w) = \frac{d\bar{z}}{d\bar{w}} \bar{\mathcal{J}}(\bar{z}) \tag{2.7}$$

These transformations can also be written in terms of operators U_w representing elements of Virasoro group

$$U_w = \exp(\sum_n v_n L_n) \tag{2.8}$$

where v_n's are harmonics of $v(z) = \sum v_n z^{n+1}$ and field $v(z)$ is defined by equation

$$e^{v(z)\partial_z} z = w(z) \tag{2.9}$$

[5]For field $\phi(w,\bar{w})$ to be single-valued in the complex plane one should impose the condition

$$b_0 = \bar{b}_0$$

However it is convenient for us to keep both of these operators for now.

For the purpose of this paper let us constrain ourselves to the univalent transformations, i.e. those that map $z = \infty$ into $w = \infty$ in one-to-one manner. Then $w(z)$ has the form

$$w(z) = \frac{z}{r} + \sum_{k \geq 0} \frac{p_k}{z^k} \tag{2.10}$$

Here we note that r as well as other parameters are complex. The b_k's can be expressed through the conformally transformed current $\mathcal{J}(z)$ as

$$b_k = U_w^{-1} a_k U_w = \oint \frac{dz}{2\pi i} \Big(w(z) \Big)^k \mathcal{J}(z) \tag{2.11}$$

and similarly for $\bar{b}_k$'s. Here a_k's are harmonics of the current $\mathcal{J}(z)$. For univalent transformations (2.10) this means for b_k's

$$\begin{aligned} b_n &= \sum_{k=0}^{n} C_{n,k} a_k + \sum_{k=1}^{\infty} C_{n,-k} a_{-k} \quad \forall n > 0 \\ b_{-n} &= \sum_{k=n}^{\infty} C_{-n,-k} a_{-k} \qquad\qquad\qquad\quad \forall n > 0 \end{aligned} \tag{2.12}$$

$$C_{n,k} = \oint \frac{dz}{2\pi i} z^{-k-1} \Big(w(z) \Big)^n \quad \forall k, n \tag{2.13}$$

By expressing b_k's via a_k's, we establish the map from the Fock space of the former operators to that of the latter. The natural question to ask is how the image of the vacuum (2.4–2.5) behaves under this map. Let $|p_a\rangle$ and $|\bar{p}_a\rangle$ be vacua in the Fock spaces of operators a_k ($\bar{a}_k$). They are defined via equations similar to (2.4–2.5) but with respect to operators a_k. Since $b_k(a) = U_w^{-1} a_k U_w$ the state

$$|w\rangle \otimes |\bar{w}\rangle = U_w^{-1} |p_a\rangle \otimes \bar{U}_{\bar{w}}^{-1} |\bar{p}_a\rangle \tag{2.14}$$

in the Fock space of operators a_k ($\bar{a}_k$) satisfies

$$b_k(a) |w\rangle = \bar{b}_k(\bar{a}) |\bar{w}\rangle = 0 \quad \forall k \geq 1 \tag{2.15}$$

We will call the state $|w\rangle \otimes |\bar{w}\rangle$ *conformally transformed vacuum.* Since the relation between b_k's and a_k's is linear we can look for the solution of eq. (2.15) in the form

$$|w\rangle \otimes |\bar{w}\rangle = \exp\left(\frac{1}{2} \sum_{k,m=0}^{\infty} a_{-k} a_{-m} N_{km} \right) |p_a\rangle \otimes \exp\left(\frac{1}{2} \sum_{k,m=0}^{\infty} \bar{a}_{-k} \bar{a}_{-m} \bar{N}_{km} \right) |\bar{p}_a\rangle \tag{2.16}$$

The coefficients N_{km} are subject to constraints, following from equations

$$b_n |w\rangle = \left[\sum_{k=0}^{n} C_{n,k} a_k + \sum_{k=0}^{\infty} C_{n,-k} a_{-k} \right] \exp\left(\frac{1}{2} \sum_{i,j=1}^{\infty} a_{-i} a_{-j} N_{ij} \right) |p_a\rangle = 0 \tag{2.17}$$

Thus

$$\sum_{k=1}^{n} kC_{n,k}N_{km} + C_{n,-m} = 0\ , \quad m \geq 0, n \geq 1 \tag{2.18}$$

One can show (see Appendix A in [5]) that these equations are solved by

$$\begin{aligned} N_{km} &= \frac{1}{km} \oint \frac{dz}{2\pi i} \oint \frac{d\zeta}{2\pi i} z^k \zeta^m \partial_z \partial_\zeta \log\Big(w(z) - w(\zeta)\Big) \\ N_{0m} &= \frac{1}{m} \oint \frac{dz}{2\pi i} z^m \partial_z \log\Big(w(z)\Big) \end{aligned} \tag{2.19}$$

Coefficients $\bar{N}_{km}$ depend on $\bar{w}$ in the same way. The above result can be checked by direct calculations.

Solution (2.19) requires some comments. For any function $w(z)$ regardless of its interpretation as a conformal map (2.10) coefficients $N_{kn},\ k,n \geq 0$ defined via (2.19) *are not independent*, but rather obey an infinite set of relations of which the first few are

$$\begin{aligned} N_{22} &= N_{13} - \frac{1}{2} N_{11}^2 \\ N_{23} &= N_{14} - N_{11} N_{12} \\ N_{33} &= \frac{1}{3} N_{11}^3 - N_{11} N_{13} - N_{12}^2 + N_{15} \\ &\dots \end{aligned} \tag{2.20}$$

It is, of course, possible (although not obvious) to see these relations directly from eq. (2.19). To appreciate how non-trivial they are one may want to take a look at the example of Neumann matrix in, say, [15], eq. (2.14).

Note that the answer (2.19) was obtained in [7, 16] from a different point of view. It was shown there that coefficients N_{km} are harmonics of conformally transformed scalar field propagator

$$\langle J(w(z)) J(w(\zeta)) \rangle = \partial_z \partial_\zeta \log\Big(w(z) - w(\zeta)\Big) \tag{2.21}$$

However, it was absolutely not obvious that one should search for the relations of the type (2.20) in the setup of [7]. And indeed, they were not found.

To see the nature of these relations one should recognize an underlying structure of conformally transformed vacuum. It was shown in [5] that one can introduce the generating function $F(t_0, t_k, \bar{t}_k)$ such that

$$N_{km} = \frac{1}{km} \frac{\partial^2 F}{\partial t_k \partial t_m}, \quad N_{k0} = \frac{1}{k} \frac{\partial^2 F}{\partial t_k \partial t_0}, \quad N_{\bar{k}\bar{m}} = \frac{1}{km} \frac{\partial^2 F}{\partial \bar{t}_k \partial \bar{t}_m}, \quad N_{\bar{k}0} = \frac{1}{k} \frac{\partial^2 F}{\partial \bar{t}_k \partial t_0} \tag{2.22}$$

By substituting (2.22) into (2.18) the latter becomes equivalent to the set of equations

$$(z - \zeta) e^{D(z) D(\zeta) F} = z e^{-\partial_{t_0} D(z) F} - \zeta e^{-\partial_{t_0} D(\zeta) F} \tag{2.23}$$

(for definition of operator $D(z)$ as well as all other notations and short review of Toda Lattice Hierarchy see Appendix A, particularly section A4). One recognizes in (2.23) *Hirota equations of dispersionless Toda Lattice Hierarchy.* This means, that function F defined in (2.22) is the (logarithm of) tau-function of this hierarchy. We may define function $w(z)$ as generating function for the second derivatives $\partial_{t_0}\partial_{t_k}F$

$$D(z)\partial_{t_0}F = -\log\frac{w(z)}{z/r} \tag{2.24}$$

Eqs. (2.22) and (2.24) are equivalent to the equations for N_{0m} in (2.19), if functions $w(z)$ in both of them are the same. Then we can rewrite (2.23) in the form

$$D(z)D(\zeta)F - \frac{1}{2}\partial_{t_0}^2 F = \log\frac{w(z)-w(\zeta)}{z-\zeta} \tag{2.25}$$

Expanding l.h.s. of eq. (2.25) in z^{-1}, ζ^{-1} we obtain eqs. (2.19) for N_{km}.

By expressing from (2.23) derivatives $\partial_{t_0}\partial_{t_k}F$ via $\partial_{t_1}\partial_{t_k}F$ and substituting it back, Hirota eqs. (2.23) can also be rewritten in the *pure holomorphic form*, which is also called *Hirota equations of dispersionless KP hierarchy* [17]

$$\exp(D(z)D(\zeta)F) = 1 - \frac{D(z)\partial_{t_1}F - D(\zeta)\partial_{t_1}F}{z-\zeta} \tag{2.26}$$

Expanding it in z^{-1}, ζ^{-1} and using identifications (2.22) we get equations on coefficients N_{km}. They are nothing else but equations (2.20).

3 Boundary states in CFT and dToda tau-function

In the previous Section we showed that interpretation for the purely (anti)holomorphic second derivatives of the dispersionless tau-function $F(t_0, t_k\bar{t}_k)$ of dToda hierarchy can be found in terms of the Neumann coefficients of the conformally transformed vacuum states. The natural question would be whether there is a similar interpretation for the mixed (holomorphic-antiholomorphic) derivatives of this function (see Appendix A4 for details). To find such interpretation we would have to find a state in the Fock space of scalar field, which mixes holomorphic and anti-holomorphic Fock spaces of its components.

Consider again the scalar field in the plane w (c.f. eq. (2.1))

$$\phi(w,\bar{w}) = \phi_0 + b_0\log w + \bar{b}_0\log\bar{w} - \sum_{k=-\infty}^{\infty}{}' \left(\frac{b_k}{k\,w^k} + \frac{\bar{b}_k}{k\,\bar{w}^k}\right) \tag{3.1}$$

and impose the Neumann

$$\partial_{\mathrm{n}}\phi(w,\bar{w})\Big|_{|w|=1} = 0 \tag{3.2}$$

or Dirichlet

$$\phi(w,\bar{w})\Big|_{|w|=1} = 0 \tag{3.3}$$

boundary conditions on the unit circle. Presence of the boundary makes holomorphic and anti-holomorphic modes dependent. As it is well known, there are two ways to realize it in quantum theory. One can either solve boundary conditions at the classical level: $\bar{b} = f(b)$ ("open string picture") and therefore express explicitly $\bar{b}$ in terms of b or quantize the theory first and then impose boundary conditions as constraints on quantum states ("closed string picture"). The latter leads to boundary state construction [18]. Boundary state $|B\rangle\rangle$ is defined by

$$\langle 0| V(b,\bar{b})\Big|_{\bar{b}=f(b)} |0\rangle_{open} = \langle 0| V(b,\bar{b}) |B\rangle\rangle_{closed} \tag{3.4}$$

The boundary conditions (3.3) or (3.2) imply constraints $b_n \pm \bar{b}_{-n} = 0$. In "closed string picture" they should be imposed on states. In the present simple case of the free scalar theory they define the boundary state uniquely via

$$(b_n \pm \bar{b}_{-n}) |B\rangle\rangle_{circle} = 0, \ \forall\, n \tag{3.5}$$

where upper sign (+) corresponds to Neumann boundary conditions and lower sign (−) to Dirichlet conditions. In what follows we are going to work mostly with the case of Neumann boundary state. We can solve (3.5) explicitly

$$|N_b\rangle\rangle_{circle} = \int dp d\bar{p}\, \delta(p+\bar{p}) \exp\left\{ -\sum_{k>0} \frac{b_{-k}\bar{b}_{-k}}{k} \right\} |p\rangle \otimes |\bar{p}\rangle \tag{3.6}$$

We would like to emphasize that for the case of Neumann boundary state single-valuedness condition $p = \bar{p}$ (see the footnote 5, p. 7) together with the delta-function in the above equation selects $p = \bar{p} = 0$ as the only choice. However, for the purpose of this paper, we will not impose single-valuedness here.

Now we would like to find Neumann boundary state $|N\rangle\rangle_{\mathcal{C}}$ for the boundary conditions imposed on an arbitrary analytic curve $\mathcal{C}$. This can be done applying conformal transformations in the way analogous to Section 2. In accordance to Riemann mapping theorem there exists a conformal transformation $w(z)$ that maps any analytic curve $\mathcal{C}$ to a unit circle and exterior of the curve into exterior of the unit circle. If a scalar field $\phi(w,\bar{w})$ satisfies Neumann (Dirichlet) boundary conditions on the circle, then conformally transformed field $\phi(w(z),\bar{w}(\bar{z}))$ satisfies the same condition on the curve $\mathcal{C}$. Since $b_n = U_w^{-1} a_n U_w$ (and analogous for antiholomorphic sector)[6] the boundary conditions (3.5) imply the following for $|N\rangle\rangle_{\mathcal{C}}$

$$\left(U_w^{-1} a_n U_w + \bar{U}_{\bar{w}}^{-1} \bar{a}_{-n} \bar{U}_{\bar{w}} \right) |N\rangle\rangle_{\mathcal{C}} = 0 \tag{3.7}$$

Here U_w and $\bar{U}_{\bar{w}}$ act on holomorphic/antiholomorphic sectors of $|N\rangle\rangle_{\mathcal{C}}$. It is clear that

$$|N\rangle\rangle_{\mathcal{C}} = U_w^{-1} \bar{U}_{\bar{w}}^{-1} |N_a\rangle\rangle_{circle} \tag{3.8}$$

[6] See eqs. (2.8–2.11) for details

is a solution of (3.7) (compare to conformally transformed vacuum in eq. (2.14)). Here $|N_a\rangle\rangle_{circle}$ is defined by equations similar to (3.5), but in the Fock space of operators a_k, $\bar{a}_k$.

We proceed to finding this solution. After plugging in the eqs. (3.5) $b_k(a)$ given by eqs. (2.12), one sees that the state $|N\rangle\rangle_{\mathcal{C}}$ obeys

$$\left[\sum_{k=0}^{n} C_{n,k} a_k + \sum_{k=1}^{\infty} C_{n,-k} a_{-k} + \sum_{k=n}^{\infty} \bar{C}_{-n,-k} \bar{a}_{-k}\right] |N\rangle\rangle_{\mathcal{C}} = 0 \tag{3.9}$$

where $C_{k,n}$ are given by eq. (2.13). As before we try to solve (3.9) as

$$\begin{aligned} |N\rangle\rangle_{\mathcal{C}} &= \int dp\, d\bar{p}\, \delta(p+\bar{p}) \times \\ & \exp\left\{\sum_{k,n\geq 0} B_{k\bar{n}} a_{-k}\bar{a}_{-n} + \frac{1}{2}\sum_{k,n\geq 0} B_{kn} a_{-k} a_{-n} + \frac{1}{2}\sum_{k,n\geq 0} B_{\bar{k}\bar{n}} \bar{a}_{-k}\bar{a}_{-n}\right\} |p\rangle \otimes |\bar{p}\rangle \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9) we find that the coefficients B_{kn} obey the same equations (2.18) as N_{kn} for conformally transformed vacuum state (and analogs thereof for $B_{\bar{k}\bar{n}}$ with coefficients being conjugated and a_k interchanged with $\bar{a}_k$). Thus the solutions for them are the Neumann coefficients (2.19). As for expressions $B_{k\bar{n}}$ with mixed indices, they can be extracted from a condition similar to (2.18)

$$\sum_{k=1}^{n} k C_{n,k} B_{k\bar{m}} + \bar{C}_{-n,-m} = 0 \tag{3.11}$$

Due to the fact that expressions for B_{kn} and $B_{\bar{k}\bar{n}}$ are the same as in Section 2 (and can therefore be identified with second derivatives of tau-function F), we may try the following identification for $B_{k\bar{n}}$

$$B_{k\bar{n}} = \frac{1}{kn}\frac{\partial^2 F}{\partial t_k \partial \bar{t}_n} \tag{3.12}$$

Again, one can show (see [5], Appendix B) that equation (3.11) together with identification (3.12) is equivalent to Hirota equations (A.30). Thus the identification gives us

$$B_{k\bar{n}} = -\frac{1}{kn}\oint \frac{dz}{2\pi i}\oint \frac{d\bar{z}}{2\pi i} z^k \bar{z}^n \partial_z \partial_{\bar{z}} \log\left(1 - \frac{1}{w(z)\bar{w}(\bar{z})}\right) \tag{3.13}$$

We present the final result for Neumann (upper sign) and Dirichlet (lower sign) boundary state written in terms of derivatives of tau-function

$$\begin{aligned} |B\rangle\rangle_{\mathcal{C}} &= \int dp \exp\left\{\frac{p^2}{2}\frac{\partial^2 F}{\partial t_0^2} + \frac{1}{2}\sum_{k,n=1}^{\infty}\left(a_{-k}a_{-n}\frac{\partial^2 F}{\partial t_k \partial t_n} + \bar{a}_{-k}\bar{a}_{-n}\frac{\partial^2 F}{\partial \bar{t}_k \partial \bar{t}_n}\right) + \right. \\ &+ \left. p\sum_{k=1}^{\infty}\left(a_{-k}\frac{\partial^2 F}{\partial t_0 \partial t_k} \mp \bar{a}_{-k}\frac{\partial^2 F}{\partial t_0 \partial \bar{t}_k}\right) \mp \sum_{k,n=1}^{\infty} a_{-k}\bar{a}_{-n}\frac{\partial^2 F}{\partial t_k \partial \bar{t}_n}\right\} |p\rangle \otimes |\mp p\rangle \end{aligned} \tag{3.14}$$

here we have fixed the normalization factor to be $\frac{p^2}{2}\frac{\partial^2 F}{\partial t_0^2}$, so that the state $|D\rangle\rangle$ would be the generating function for *all* second derivatives of dispersionless 2D Toda tau-function.

In the case of Dirichlet boundary state, eq. (3.14) has an interesting interpretation. If we realize operators a_{-k} and $\bar{a}_{-k}$ as operators of multiplications on s_k and $\bar{s}_k$, then the state $|D\rangle\rangle_{\mathcal{C}}$ as a function of s_k, $\bar{s}_k$ will itself be a tau-function of dispersionless Toda hierarchy (see [5] for details). Neumann boundary state, being quite similar in construction, lacks this interpretation to the best of our knowledge. Namely, the difference which prevents such interpretation is the following. First of all, relative minus sign in front of mixed derivative (compared to (anti)holomorphic sectors) appears in the case of Neumann boundary conditions. Secondly, in this case $p = 0$ if we require that scalar field (2.1) is single valued. Under such condition terms with derivatives with respect to t_0 would disappear.

Another comment which should be made about (3.14) is that we never used the fact that t_k and $\bar{t}_k$ are complex conjugated. This also implies that in general $w(z)$ and $\bar{w}(\zeta)$ in above formulae can be independent rather than complex conjugated. Eq. (3.14) will work formally for this general case as well, although the physical interpretation of $|B\rangle\rangle_{\mathcal{C}}$ is less clear. We will come back to this point at the end of Section 4.

4 Two-vertex of SFT as a boundary state

In Section 2 we have already mentioned that the ansatz (2.19) (without identification with Toda) was obtained in [7, 16] in the context of open SFT. At first glance there can not be any connection of $B_{k\bar{n}}$ (defined via (3.10)–(3.12)) with SFT, because these coefficients appeared in the expression for the boundary state, whereas in open string field theory one works in the "open string picture" and there is no place for boundary state as such. Nevertheless, recently it was noticed [10] that mixed derivatives of dispersionless Toda tau-function (at particular values of its arguments) do appear in the description of SFT three-vertex. In this Section we will show that it is not a coincidence. To wit, there is a close relation between boundary states and SFT vertices. Using this relation we will derive the identification between dispersionless Toda tau-function and Neumann coefficients in SFT in a systematic way.

Let us remind the basic facts about the *Cubic String Field Theory*. Action of this String Field Theory has the following schematic form [1]

$$S_{SFT} = \frac{1}{2}\int \Phi * Q_{BRST}\Phi + \frac{1}{3}\int \Phi * \Phi * \Phi \qquad (4.1)$$

We will not discuss the kinetic term here. The $*$-product in (4.1) is the main defining object of SFT. It encodes all the interactions by specifying how two of the incoming strings are glued into the resulting one. All strings can be off-shell. Thus star-product is a map from tensor product of two string Hilbert spaces into that of a third one

$$*: \ \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3 \qquad (4.2)$$

One way to specify $*$-product would be to introduce a *three-vertex* $|V_{123}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, such that

$$|A * B\rangle \equiv \langle A| \otimes \langle B| \, |V_{123}\rangle \tag{4.3}$$

where by $\langle A|$ we mean a *BPZ conjugation* of $|A\rangle$. Then interaction term in (4.1) can be written as

$$\int \Phi_A * \Phi_B * \Phi_C = \langle C|A * B\rangle \tag{4.4}$$

In order to define this three-vertex three in general independent conformal transformations can be used

$$\langle 0| \, h_1(V_A) h_2(V_B) h_3(V_C) \, |0\rangle = \langle A| \otimes \langle B| \otimes \langle C| \, |V_{123}\rangle \tag{4.5}$$

where V_A, V_B, V_C are vertex operators creating states $|A\rangle$, $|B\rangle$, and $|C\rangle$ correspondingly. It should be noted that all three vertex operators in the l.h.s. of (4.5) act *on the same* Hilbert space. Maps $h_I(z), I = 1, 2, 3$ are fixed. They contain in themselves all the information about $|V_{123}\rangle$. Different choices of $h_I(z)$ will lead to different off-shell interactions in String Field Theory. For example, the choice that leads to Witten's three-vertex is

$$h_1(z) = T^2 \circ h(z), \quad h_2(z) = T \circ h(z), \quad h_3(z) = h(z) \tag{4.6}$$

where $h(z)$ is a conformal map that carries a unit circle into a 120° wedge thereof and T is a 120° rotation

$$h(z) = \left(\frac{z+i}{z-i}\right)^{2/3}, \qquad T(z) = e^{2\pi i/3} z \tag{4.7}$$

The three-vertex $|V_{123}\rangle$ is defined as[7]

$$|V_{123}\rangle = \int \prod_{I=1}^{3} dp_I \, (2\pi) \, \delta\left(\sum_{I=1}^{3} p_I\right) \exp\left(\frac{1}{2} \sum_{I,J=1}^{3} \sum_{m,n=0}^{\infty} a^I_{-n} a^J_{-m} N^{IJ}_{mn}\right) |p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle \tag{4.8}$$

Here each operator a^I acts on its own vacuum $|p_I\rangle$, momenta p_I are eigen values of zero-modes a^I_0. The values of *Neumann coefficients* N^{IJ}_{nm} depend only on conformal maps h_I

$$N^{IJ}_{km} = \frac{1}{km} \oint \frac{dz}{2\pi i} \oint \frac{d\zeta}{2\pi i} z^k \zeta^m \partial_z \partial_\zeta \log\Big(h_I(z) - h_J(\zeta)\Big) \tag{4.9}$$

$$N^{IJ}_{0m} = -\frac{1}{m} \oint \frac{dz}{2\pi i} z^m \partial_z \log\Big(h_I(\infty) - h_J(z)\Big) \tag{4.10}$$

$$N^{IJ}_{00} = \begin{cases} \log([h'_I](\infty)) & I = J \\ \log(h_I(\infty) - h_J(\infty)) & I \neq J \end{cases} \tag{4.11}$$

[7]Depending on $h_I(z)$ it is more natural to work with *bra* or *ket* vertices. Throughout this paper we will be working with the ket ones, which are BPZ conjugated as compared to, say, [7]. It means that our $h_I(z)$ and $\tilde{h}_I(z)$ in [7] are related as: $\tilde{h}_I(z) = h_I(-\frac{1}{z})$.

where

$$[h'](z) \equiv -\frac{dh(z)}{d1/z} \tag{4.12}$$

In the above we assumed that we are going to work with the expansion of functions $h_I(z)$ around infinity[8]. Note that N^{II}_{kn} for $k,n>0$ are given by analogs of eq. (2.19).

Along with the three-vertex (4.3) one can in principle define n-vertex $|V_{1\ldots n}\rangle$ for arbitrary n as

$$\langle 0|\, h_1(V_1)\ldots h_n(V_n)\,|0\rangle = \langle V_1| \otimes \ldots \otimes \langle V_n|\ \ |V_{1\ldots n}\rangle \tag{4.13}$$

The simplest case is a one-vertex (also known as a *surface state* $|h_1\rangle$), defined by

$$|h_1\rangle = \exp\left(\frac{1}{2}\sum_{n,m=1}^{\infty} a_{-n}N^{11}_{nm}a_{-m}\right)|0\rangle \tag{4.14}$$

where N^{11}_{kn} as functions of some conformal transformation $h_1(z)$ are given by eq. (4.9) for $I = J = 1$. It is easy to see that this object coincides with conformally transformed vacuum state $|w\rangle$ (2.16) of Section 2 for $p = 0$ (thus the notation) and its Neumann coefficients N_{kn} are expressed via second derivatives of dKP tau-function. Like in the case of Dirichlet boundary state (see discussion after eq. (3.14)), surface state itself is a tau-function of dKP if one realized operators a_k as multiplication on variables s_k. For discussion see [5].

The next example is a two-vertex $|V_{12}\rangle$, defined by

$$\langle 0|\, h_1(V_1)h_2(V_2)\,|0\rangle = \langle V_1| \otimes \langle V_2|\ \ |V_{12}\rangle \tag{4.15}$$

It can be written again in terms of Neumann coefficients (4.9–4.11)

$$|V_{12}\rangle = \int dp_1\, dp_2\, 2\pi\delta(p_1+p_2)\exp\left(\frac{1}{2}\sum_{I,J=1}^{2}\sum_{n,m=0}^{\infty} a^I_{-n}N^{IJ}_{nm}a^J_{-m}\right)|p_1\rangle\otimes|p_2\rangle \tag{4.16}$$

We will also use a notation

$$|V_{12}\rangle \equiv |h_1, h_2\rangle \tag{4.17}$$

specifying explicitly on which conformal transformations the two-vertex $|V_{12}\rangle$ depends.

By its very definition (4.16) the two-vertex $|V_{12}\rangle$ is similar to the boundary state[9] (3.4). We would like to explore this similarity in details and find out what boundary theory the two-vertex (4.16) corresponds to.

It is not hard to find explicit conformal maps that define the two-vertex state which is Neumann boundary state on a unit circle. The required conformal maps are

$$h_1(z) = z, \quad h_2(z) = \frac{1}{z} \tag{4.18}$$

[8]Eqs. (4.9)–(4.11) may seem to be ill-defined if $h_I(z)$'s have poles at infinity. In Appendix B we show how to deal with such formulae in these cases.

[9]If one identifies a^1_k and a^2_k in (4.16) with a_k and $\bar{a}_k$ of Section 3. Note, however, that in case of two-vertex there is no additional condition requiring $p_1 = p_2$.

Indeed, the Neumann coefficients (4.9) for these maps are[10] (see Appendix B for details)

$$N^{12}_{nm} = N^{21}_{nm} = -\frac{1}{n}\delta_{n,m} \tag{4.19}$$

$$N^{11}_{nm} = N^{22}_{nm} = -\frac{1}{n}\delta_{n,-m} \tag{4.20}$$

Plugging (4.19–4.20) into definition of a vertex state (4.16) we get

$$|V_{12}\rangle = \int dp_1\, dp_2\, 2\pi\delta(p_1+p_2) \exp\left(-\sum_{n>0} \frac{a^1_{-n}a^2_{-n}}{n}\right) |p_1\rangle \otimes |p_2\rangle \tag{4.21}$$

Eq. (4.21) coincides with the boundary state on a unit circle (3.6) (see the footnote 9 on the page before). As it has already been mentioned there this would become the usual Neumann boundary state if we imposed single-valuedness condition. Note, that for two-vertex state there is no a priori reason to do it.

We can finally establish the relation between a two-vertex defined by arbitrary maps and Neumann boundary state on an arbitrary curve $|N\rangle\rangle_{\mathcal{C}}$. Let function $w(z)$ map exterior of this curve to the exterior of a unit circle. To differentiate between this conformal map and conformal maps defining two-vertex we still denote the latter by $h(z)$. In Appendix C we show that

$$|h_1, h_2\rangle = |N\rangle\rangle_{\mathcal{C}} \tag{4.22}$$

where

$$h_1(z) = w(z), \quad h_2(z) = \frac{1}{\bar{w}(z)} \tag{4.23}$$

or some $GL(2,\mathbb{C})$ transformation thereof[11]. In case of the circle $w(z) = \bar{w}(z) = z$ and (4.23) becomes (4.18) as it should. Note, that given any analytic curve one can construct a two-vertex $|w, \bar{w}^{-1}\rangle$ out of it. Inverse is not true. Given arbitrary h_1 and h_2 one can not always find an appropriate analytic curve (obtained via (4.23) $w(z)$ and $\bar{w}(z)$ will not be complex conjugated). Thus, two-vertex may be thought of as generalization of the boundary state.

To understand the meaning of eq. (4.23) let us come back to eq. (4.22). Recall, that by its definition (4.15) two-vertex $|h_1, h_2\rangle$ belongs to the tensor product of Hilbert spaces of two *open strings* and one can think of each open string as living in the exterior of a unit circle in its own complex plane[12]. At the same time, boundary state in the r.h.s.

[10] Neumann coefficients N^{IJ}_{mn} are invariant under $GL(2,\mathbb{C})$ transformations. Thus we could consider

$$h_1(z) = \frac{az+b}{cz+d}, \quad h_2(z) = \frac{(1/z)a+b}{(1/z)c+d}$$

with $ad-bc \neq 0$.

[11] Because of $GL(2,\mathbb{C})$ freedom in definition (4.18) mentioned before (the footnote 10 on this page) the same ambiguity is present in (4.23). However, requiring specific analytic properties of $w(z)$ and $h_I(z)$, this freedom can be completely fixed. For details see Appendix C.

[12] We are working with the *ket*-states, thus natural picture for the open string world sheets are exteriors of various curves, e.g. unit circles.

of (4.22) belongs to the Hilbert space of closed strings. In the simplest case (4.18) world sheets of the two open strings are represented by complex planes with removed unit disks. Then transformation $h_1(z) = z$ maps (identically) this world sheet onto the exterior of the unit circle, while $h_2(\bar{z}) = 1/\bar{z}$ maps world sheet of the second string onto the interior of the unit circle. The reason why we write here $h_2(\bar{z})$ instead of $h_2(z)$ is the following. Eq. (4.18) should be actually written as $h_1(z_1)$ and $h_2(z_2)$ to stress that functions h_1 and h_2 are defined on different spaces. When we glue two world sheets of the open strings into the one world sheet of the closed string we want maps h_1 and h_2 to match continuously across the boundary (unit circle). This means that we should identify z_1 with z and z_2 with $\bar{z}$ (then $h_1(z) = h_2(\bar{z})$ when $z\bar{z} = 1$)[13]. Under this identification two-vertex (4.21) is related to Neumann boundary state on the unit circle (3.6). This means that a correlator in the r.h.s. of (4.15) is computed in the closed string picture with Neumann boundary state on the unit circle.

More general situation (4.23) is obtained if one considers open strings with Neumann boundary conditions on an arbitrary analytic curve $\mathcal{C}$. Now the world sheet of the open string is a complex plane with the interior of the curve $\mathcal{C}$ removed. In order to combine two such open strings into a closed string as required by eq. (4.22) we should map the exterior of the curve $\mathcal{C}$ (world sheet of the first string) into the exterior of the unit circle on the world sheet of the closed string and the world sheet of the second string into the interior of the same unit circle. The transformations $h_1(z)$ and $h_2(\bar{z})$ of eq. (4.23) do precisely that.

4.1 The identification of two-vertex with dToda tau-function

In Sections 3 and 4 we were able to relate Neumann boundary state on an arbitrary analytic curve to two different objects - dispersionless Toda tau-function and two-vertex of SFT. That shows that there is a close connection between the two in their own right. Namely, one can express a two-vertex $|h_1, h_2\rangle$ in terms of second derivatives of Toda tau-function $\partial_{t_k}\partial_{t_n}F$ exactly as it was done in eq. (3.14). To see how it comes around let us integrate over zero-mode delta-function in the definition of two-vertex (4.16) and get rid of one of the zero modes a_0. The two-vertex now takes the form

$$|V_{12}\rangle = \int dp \, \exp\left\{ \begin{aligned} &\frac{1}{2}p^2(N_{00}^{11} + N_{00}^{22} - N_{00}^{12} - N_{00}^{21}) \\ &+\frac{1}{2}p\sum_{n=1}^{\infty}(N_{0n}^{11} - N_{0n}^{21})a_{-n}^1 \\ &+\frac{1}{2}p\sum_{n=1}^{\infty}(N_{0n}^{12} - N_{0n}^{22})a_{-n}^2 \\ &+\frac{1}{2}\sum_{I,J=1,2}\sum_{n,m=1}^{\infty} a_{-n}^I N_{nm}^{IJ} a_{-m}^J \end{aligned} \right\} |p\rangle \otimes |-p\rangle \tag{4.24}$$

[13]Compare this with the comment in the footnote 9 on page 15.

Here p and $-p$ are eigen-values of operators a_0^1 and a_0^2 correspondingly. If we identify a_k^1 with a_k and a_k^2 with $\bar{a}_k$ on (see the footnote 9 on page 15) and equate the above two-vertex to the boundary state of eq. (3.14) in accordance with eq. (4.23) we get the following relations between Neumann coefficients and second derivatives of Toda tau-function F

$$\frac{\partial^2 F}{\partial t_0^2} = N_{00}^{11} + N_{00}^{22} - N_{00}^{12} - N_{00}^{21} \tag{4.25}$$

$$\frac{1}{k}\frac{\partial^2 F}{\partial t_0 \partial t_k} = N_{0k}^{11} - N_{0k}^{21} \tag{4.26}$$

$$\frac{1}{k}\frac{\partial^2 F}{\partial t_0 \partial \bar{t}_k} = N_{0k}^{22} - N_{0k}^{12} \tag{4.27}$$

$$\frac{1}{nk}\frac{\partial^2 F}{\partial t_n \partial t_k} = N_{nk}^{11} \tag{4.28}$$

$$\frac{1}{nk}\frac{\partial^2 F}{\partial \bar{t}_n \partial \bar{t}_k} = N_{nk}^{22} \tag{4.29}$$

$$\frac{1}{nk}\frac{\partial^2 F}{\partial t_n \partial \bar{t}_k} = -N_{nk}^{12} \tag{4.30}$$

This is the identification we were looking for. Several comments are in order. First of all, let us stress one more time that derivatives of the tau-function F are not independent. They satisfy Hirota identities. The corresponding combinations of Neumann coefficients should satisfy them too. In more technical terms it means the following. The second derivatives of tau function $\partial_{t_n}\partial_{t_k}F$ are related by Hirota identities eq. (A.29)-(A.30) to maps $w(z)$ and $\bar{w}(z)$. If one replaces $w(z)$ and $\bar{w}(z)$ with $h_I(z)$ in accordance with relation found in the previous section, eq. (4.23), the result should be just a Neumann coefficients in eq. (4.28) - (4.30). This is indeed the case. Some subtlety arises in identification of a zero-mode sector due to different analytical properties of $w(z)$ and $h_I(z)$. We discuss the issue as well as give all details about the identification in Appendix D.

From the definition of two-vertex eq. (4.15) one can see that it is invariant under arbitrary $GL(2,\mathbb{C})$ transformation of defining conformal maps h. This requires Neumann coefficient N_{km} with $k, m > 0$ to be invariant under it as well. As for N_{0m} they are not necessarily invariant but must form an invariant combination after zero-mode δ-function in eq. (4.16) being integrated over. Not surprisingly, this is exactly the combination one sees in eqs. (4.26)-(4.27).

The identifications (4.25)-(4.30) were first found in [10]. There the correct $GL(2,\mathbb{C})$ invariant combinations eqs. (4.26)-(4.27) were guessed. Solving the Hirota identities gave a value of $\log 16/27$ for $\partial_{t_0^2}^2 F$. Our approach has several advantages. We were able to derive the identification for all derivatives of the tau-function including $\partial_{t_0}\partial_{t_k}F$. We derived the $\partial_{t_0^2}^2 F = \log 16/27$ as well. It is a value of Neumann coefficients in eq. (4.25) calculated for a particular choice of conformal maps, eq. (4.6). But perhaps more importantly, our approach makes it very clear that the appearance of the same tau-function in both objects, the two-vertex and conformally transformed boundary state, is not a coincidence. This fact will be crucial for us in Section 6.

5 Star algebra and dToda tau-function

Finally we can establish a relation between the three-vertex in eq. (4.8) and dToda tau-function. As before we start with integrating over momentum delta-function in the three-vertex. The result is eq. (4.8) with $a_0^3 = -a_0 - a_0'$, $a_0 = a_0^1$ and $a_0' = a_0^2$. We get

$$\begin{aligned} |V_{123}\rangle &= \int dp dp' \exp\Big(\frac{1}{2}\sum_{I,J=1}^{3} a_0^I a_0^J N_{00}^{IJ} + \\ &+ \sum_{J=1}^{3}\sum_{n>0} p(N_{0n}^{1J} - N_{0n}^{3J})a_{-n}^J + \sum_{J=1}^{3}\sum_{n>0} p'(N_{0n}^{2J} - N_{0n}^{3J})a_{-n}^J + \\ &+ \frac{1}{2}\sum_{I,J=1}^{3}\sum_{m,n>0} a_{-n}^I a_{-m}^J N_{mn}^{IJ}\Big) |p\rangle_1 \otimes |p'\rangle_2 \otimes |-p-p'\rangle_3 \end{aligned} \tag{5.1}$$

Now we can make the following observation. Since a three-vertex belongs to the tensor product of Hilbert spaces of three independent open strings one can multiply it by bra-vacuum belonging to any of the three sectors. If we choose the vacuum to have zero momentum the result will be just a two-vertex. For example, let us take a vacuum of the third string then

$$\langle 0_3|h_1, h_2, h_3\rangle = |h_1, h_2\rangle \tag{5.2}$$

and similarly for vacua of the first and second string. All resulting two-vertices can be identified with Toda tau-function independently. It sets the correspondence between the three-vertex (in particular the one that defines star-product) and Toda tau function.

We would like to note that this is what was done in effect in [10]. There, one worked with a specific choice of the conformal maps $h_I(z)$, namely the maps that correspond to Witten's three vertex eq. (4.6). In that case there are only two independent sets of Neumann coefficients, N^{11} and N^{12}. The identification was made for them only. Therefore, it was actually the identification with a single two-vertex sector. For such specific choice all other sectors were equal to this one.

In SFT the choice of the three-vertex is fixed. It means that its defining conformal maps correspond to tau-function evaluated at some particular values of times $t_k, \bar{t}_k$ and other values of times never come into play. There are examples though, when the dependence on conformal maps is important. For example, the so-called *surface states* are defined with respect to arbitrary Riemann surfaces (hence their name). Surface states are conformally transformed vacua and they are related to tau-function of dispersionless KP hierarchy [5]. By varying t_k we are able to change from one surface state to the other. Thus dKP tau-function describes an infinite sequence of surface states, related to each other by conformal transformations. The similar statement is true for the dToda tau-function as we showed in this paper. Indeed, dToda tau-function parameterizes a set of boundary states on arbitrary curves and thus a set of two-vertices of SFT. For a given star-product (i.e. for the three-vertex $|V_{123}(h_1, h_2, h_3)\rangle$ with some particular $h_I(z)$'s) and any surface state $\langle g|$ defined by a conformal map $g(z)$ we can define the two-vertex

$$|V_{12}(g_1, g_2)\rangle \equiv \langle g\,|V_{123}\rangle \tag{5.3}$$

Resulting conformal maps $g_1(z)$ and $g_2(z)$ are now some functionals of $g(z)$. Their exact form is determined by three conformal maps $h_I(z)$ of the three-vertex $|V_{123}\rangle$. We can repeat this procedure and fuse the resulting two-vertex with one more surface state, $|S\rangle$. In accordance to eq. (4.3) the result will be star-product of $|S\rangle$ with the initial surface state $|g\rangle$

$$\langle S\,|V_{12}(g)\rangle = |S * g\rangle \tag{5.4}$$

Thus one can view $|g_1, g_2\rangle = |V_{12}(g)\rangle$ as a *multiplication operator* on a surface state $|g\rangle$. Hence, the dToda tau-function parameterizes the space of such multiplication operators.

6 Surface states and boundary states

As we have seen in the previous section, one can associate a multiplication operator with any state. For a given state $|\Sigma\rangle$ such operator $\hat{\Sigma}$ is defined in the following way

$$\hat{\Sigma} : |X\rangle \to |X * \Sigma\rangle \quad \forall\ |X\rangle \tag{6.1}$$

For a surface state multiplication operator $\hat{\Sigma}$ is nothing else but a two-vertex $|V_{12}(\Sigma)\rangle$.

As we have shown in Section 4, some of the two-vertices can be interpreted as boundary states. The condition two-vertex should satisfy to be a boundary state has very simple geometric meaning. Two conformal maps that define the two-vertex should map world-sheets of two open strings into surfaces that can be glued into one world-sheet of a closed string (see discussion in Section 4 after eq. (4.23)). Namely, world-sheet of the one of the strings should be mapped to the *interior* of a unit disk, the world-sheet of the other to the *exterior* of the unit disk. Together they can be combined into the whole complex plane – world-sheet of the closed string.

We see that although all surface states lead to multiplication operators, which are some two-vertices, not all of these operators are boundary states (functions $w(z)$ and $\bar{w}(z)$ in eq. (4.23) in general are *not* complex conjugated). We come to a very interesting problem of finding a criteria that the surface state should satisfy to give rise to a boundary state. Moreover, the identification with boundary states would present these surface states in absolutely different light giving them closed string interpretation. To address the problem we will use the result of Generalized Gluing Theorem [11].

Precise formulation of our problem is the following. Let the three-vertex be defined by three conformal maps $h_I(z)$ (say, given by (4.6)), and the surface state is defined by some conformal map $f(z)$. The contraction of the tree-vertex and a surface state will give rise to a two-vertex defined by maps $g_1(z)$ and $g_2(z)$:

$$|g_1, g_2\rangle \equiv \langle f\,|h_1, h_2, h_3\rangle \tag{6.2}$$

We want to find the condition which state $|f\rangle$ should satisfy to give rise to the two-vertex $|g_1, g_2\rangle$ which can be interpreted as a boundary state. Eq. (6.2) also means, that

$$\langle f\,|A * B\rangle = \langle A| \otimes \langle B|\,|g_1, g_2\rangle\,, \quad \forall\, A, B \tag{6.3}$$

If we rewrite (6.3) as

$$\sum_{\Phi_r} \langle h_1 \circ A\, h_2 \circ B\, h_3 \circ \Phi_r \rangle \langle f \circ \Phi_r \rangle = \langle F_1 \circ h_1 \circ A\, F_1 \circ h_2 \circ B \rangle \tag{6.4}$$

then Generalized Gluing Theorem [11] gives the following answer

$$g_1 = F_1 \circ h_1(z) \qquad g_2 = F_1 \circ h_2(z) \tag{6.5}$$

where map F_1 is defined by some requirements of analyticity that we outline below [14] and by

$$F_1 \circ h_3(z) = F_2 \circ I \circ f \circ I(z) \equiv w(z) \tag{6.6}$$

where I is BPZ inversion $I(z) = -1/z$. The geometric meaning of eq. (6.6) is the following [11]. Transformation $h_3(z)$ maps the interior of the unit disk to interior of some curve $\mathcal{C}_1$ (wedge with the angle $2\pi/3$). Correspondingly $f \circ I$ maps the exterior of the unit disk into the exterior of another curve $\mathcal{C}_2$. Let us denote the compliments of these regions by D_1 and D_2. The F_1 and $F_2 \circ I$ should be such that the images of D_1 under F_1 and D_2 under $F_2 \circ I$ are compliment of each other as well:

$$F_2 \circ I(D_2) = \mathbb{C}^1 \backslash F_1(D_1) \tag{6.7}$$

This procedure can be visualized as follows. Any vertex can be represented by a sphere (that is a full complex plane) with holes. The boundaries of these holes are images of a unit circle under defining maps h_I. The contraction of two vertexes with maps h_I and f corresponds to gluing two holes on two spheres together to form a new sphere with other holes left unchanged. Let z_1 be a coordinate on a unit disk to be mapped by h_3 and z_2 is coordinate on a unit disk to be mapped by f. The two curves are glued by identification in z plane

$$z_1 = -\frac{1}{z_2} \tag{6.8}$$

To construct a global coordinate w on resulting surface, further maps F_1 and F_2 are needed.

Before starting the search for the surface states that give rise to the boundary states let us consider a simple example of so-called *wedge state* [19], which will turn out to be useful in the future.

6.1 Contraction of the wedge states

Wedge states are the family of surface states, which form a subalgebra under star-product. General wedge state $|f_n\rangle$ is defined by means of conformal map

$$f_n(z) = h^{-1}\left(h^{\frac{2}{n}}(z)\right), \quad n > 0 \tag{6.9}$$

[14]for exact formulation see [11]

Here $h(z)$ is a famous $SL(2,\mathbb{C})$ map, which maps upper half-plane into the interior of the unit circle and which in particular translates upper half-disk $\{|z| \le 1, \Im z \ge 0\}$ onto the vertical half-disk $\{|\xi| \le 1, \Re\xi \ge 0\}$:

$$h(z) = \frac{1+iz}{1-iz} \tag{6.10}$$

Note, that identity state, vacuum state and sliver state are examples of wedge states for $n = 1, 2, \infty$ correspondingly. The transformation $F_{1,2}$ for a given wedge state, satisfying properties discussed above (eqs. (6.6)–(6.7)) can be easily found[15] to be

$$F_1(u) = u^{\frac{3}{n+1}}, \quad F_2(v) = N\left(e^{i\varphi} v^{\frac{n}{n+1}}\right) \tag{6.11}$$

(where $N(z)$ is some $SL(2,\mathbb{C})$ transformation which will not be important for us). Eq. (6.11) means that according to (6.5) conformal maps for two-vertex, associated with the wedge state (6.9) via (6.2) are given by

$$\begin{aligned} g_1(z) &\equiv F_1 \circ h_1(z) = \left(h^{\frac{2}{3}}(z)\right)^{\frac{3}{n+1}} = h^{\frac{2}{n+1}}(z) \\ g_2(z) &\equiv F_1 \circ h_2(z) = \left(e^{\frac{2\pi i}{3}} h^{\frac{2}{3}}(z)\right)^{\frac{3}{n+1}} = e^{\frac{2\pi i}{n+1}} h^{\frac{2}{n+1}}(z) \end{aligned} \tag{6.12}$$

Another way to understand eq. (6.12) is by using the approach of [20][16]. Wedge states can be thought of as canonical half disks with added to them wedges of an angle $\pi(n-1)$ [20, 21]. Thus, the procedure of contracting a wedge state with the three-vertex is equivalent to the gluing two canonical half-disks (corresponding to the transformations $h_{1,2}$ of three-vertex) to the wedge of the angle $\pi(n-1)$. All together this gives rise to the cone with the "angle deficit" of $\pi(n+1)$. To smooth this conic singularity into the plane one needs to apply the transformation $u^{2/(n+1)}$ to the original half-disks. This gives transformations g_1, g_2 of (6.12).

In the next Section we will see that this exercise actually allowed us to find an example of the boundary state.

6.2 Example: identity state and orientifold boundary state

Let us now come back to the problem stated at the beginning of the Section 6, the problem of finding surface states whose multiplication operators are some boundary states.

We can approach it from the other end. Namely, we can first try to find the smoothing transformations F_1 and F_2 in eq. 6.6 and only then find the map f that defines the corresponding surface state. As we mentioned above (see Sec. 4) in order for two-vertex $|g_1, g_2\rangle$ to be a boundary state one of the defining maps (say, $g_1(z)$) should map the region D on which surface state is defined to a unit disk, while $g_2(z)$ would map the same region D into a compliment of the unit disk. Represented as a sphere with holes, such

[15] Compare with similar computations in [11], Section 3.D.

[16] We are indebted to M. Schnabl for teaching us this method.

two-vertex will look like a sphere with two holes that have a common boundary but cover a complimenting regions. This should be achieved by conformal map F_1 applied to two Witten's maps

$$h_1(z) = h^{2/3}(z) \quad h_2(z) = e^{2\pi i/3} h^{2/3}(z) \tag{6.13}$$

where $h(z)$ is given by (6.10). The simplest solution is $F_1(z) = z^{3/2}$. Indeed, resulting $F_1 \circ h_1$ and $F_1 \circ h_2$ will map two unit disks into the whole complex plane. We see that such F_1 was found for the case of the wedge state with $n = 1$ (eq. (6.11)). This wedge state is an identity element of the star-product [12, 21]

$$|\mathcal{I}\rangle * |X\rangle = |X\rangle\,, \qquad \forall\, X \tag{6.14}$$

The identity state was originally constructed in [22]. As a consequence of being an identity element it has an obvious property – under star-product it squares to itself. One can try to interpret it in the framework of vacuum SFT [3]. VSFT equations of motion for brane ansatz requires the brane solutions to square under star-product to themselves. Denote the multiplication operator corresponding to identity state by $|R\rangle \equiv |V_{12}(\mathcal{I})\rangle$. This $|R\rangle$ is easy to find [22]. It should act on any state $|X\rangle$ like

$$\langle X\,|R\rangle = |X * \mathcal{I}\rangle = |X\rangle \tag{6.15}$$

That is $|R\rangle$ is a *reflector state* – two-vertex that realizes BPZ conjugation, i.e. maps any state $\langle X|$ into its BPZ conjugated $|X\rangle$. From (6.12) it is obvious that $|R\rangle$ corresponds to two-vertex with conformal maps

$$g_1(z) = z, \quad g_2(z) = -\frac{1}{z} \tag{6.16}$$

The explicit form is given by (c.f. Appendix. B)

$$|R\rangle = \int dp\, \exp\left(-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} a^1_{-n} a^2_{-n} \right) |p\rangle_1 \otimes |-p\rangle_2 \tag{6.17}$$

The reflector state is indeed a boundary state. Namely, it is an orientifold boundary state that evaded an interpretation in VSFT so far, although it is also a legitimate solution of equations of motion of VSFT. It corresponds to boundary conditions on a cross-cap [18]

$$X(\sigma+\pi,\tau) = X(\sigma,\tau) \tag{6.18}$$

$$\partial_\tau X(\sigma+\pi,\tau) = -\partial_\tau X(\sigma,\tau) \tag{6.19}$$

The presence of the orientifold would fit very well in the physical picture of VSFT. Orientifolds are not dynamical. Unlike D25 branes in bosonic theory they do not have tachyonic modes and will not decay when tachyon condenses. On the other hand, orientifolds do couple to closed strings and should be present in a true closed string vacuum.

Another interesting question comes from the following observation. Identity state (as majority of the wedge states) is singular from the point of view of defining it conformal transformation. However, contracted with the three-vertex it became a reflector state, defined by the very simple, $GL(2,\mathbb{C})$ transformation. It is possible that this example can help to find a more general set of surface states that corresponds to some geometrically simple two-vertices. Many other states of interest in VSFT correspond to singular transformations as well. There is a possibility that their multiplication operators will be two-vertices built from non-singular transformations as well. If true it would make it possible among other things to describe them in terms of dToda tau-function. We will return to this observation in Section 6.4.

6.3 Involution

In the previous section we found one example of a surface state that leads to an orientifold. It is natural to ask if this solutions is unique or not. To answer this, let us come back to the case of three-vertex, contracted with the wedge state and compare the results (6.12) with the results of Section 4 eq. (4.23). As we just saw (Sec. 6.2) in case of identity state (i.e. wedge state with the $n = 1$), the results of contraction is

$$g_1(z) = z; \quad g_2(z) = I \circ g_1(z) = -\frac{1}{z} \tag{6.20}$$

(with $I(z)$ being BPZ inversion) or $GL(2,\mathbb{C})$ transformation thereof.

Apparently, eq. (6.20) is not compatible with the eq. (4.23) – the condition for the two-vertex $|g_1, g_2\rangle$ to be a boundary state:

$$g_1(z) = w(z) \quad g_2(z) = \frac{1}{\bar{w}(z)} \tag{6.21}$$

The reason for that is that the above condition is a condition specifically on a Neumann boundary state on a curve. Not just a boundary state. It is easy to follow the derivation of eq. (6.21) to see that it is a consequence of the equation of a unit circle $z\bar{z} = 1$ (see discussion at the end of the Section 4 on page 13). Repeating this derivation we can see that formally eq. (6.20) is compatible with another curve described by $\bar{z} = -1/z$ which can be though as a circle with the *imaginary radius* $r = i$. This should be compared with the statement in [18], where it was observed, that orientifold boundary state looks like Neumann boundary state on the circle of imaginary radius.

This observation suggests the new interpretation to those boundary states which are *not* the (Neumann) states on the curve. May it have something to do with the fact that we have a BPZ conjugation in place of the ordinary (Hermitian) conjugation? If yes, then the question is – how should we modify the relation (6.21) to accommodate for this change.

Recall (see e.g. book [23], Chapter 6.1) that usually in CFT Hermitian conjugation is defined via $z \to 1/\bar{z}$. That is for the primary field with the conformal dimensions $(\Delta, \bar{\Delta})$ the operation of Hermitian conjugation is defined as

$$[\mathcal{O}_{\Delta,\bar{\Delta}}(z,\bar{z})]^{+} = \bar{z}^{-2\Delta} z^{-2\bar{\Delta}} \mathcal{O}_{\Delta,\bar{\Delta}}(1/\bar{z}, 1/z) \tag{6.22}$$

Comparing (6.22) with (6.21) we come to the conclusion that relation between g_1 and g_2 in (6.21) is the relation between the variable and its Hermitian conjugated. Then the natural idea would be to put $I(z) = -1/z$ in place of Hermitian conjugation eq. (6.21) to get the new condition

$$\tilde{g}_1(z) = w(z) \quad \tilde{g}_2(z) = I \circ w(z) = -\frac{1}{w(z)} \tag{6.23}$$

Eq. (6.23) would mean

$$\tilde{g}_2(z) = I \circ \tilde{g}_1(z) = -\frac{1}{\tilde{g}_1(z)} \tag{6.24}$$

We will call such two-vertices *BPZ boundary states*[17] .

From eq. (6.12) we see that for the two-vertex, obtained from an arbitrary wedge state $|n\rangle$ one has

$$g_2(z) = g_1 \circ I(z) = g_1(I(z)) \tag{6.25}$$

Comparing (6.25) with (6.24) one can see, that only those wedge states would give rise to the BPZ boundary states, which have the following *necessary* condition

$$g_1(z) = I \circ g_1 \circ I(z) \tag{6.26}$$

It is also immediately clear how to deal with the "general case", when surface state is *not* the wedge state. Indeed, in that case eq. (6.25) still holds (see (6.5) together with (6.13)):

$$g_1(z) = F_1 \circ h^{2/3}(z) \quad g_2(z) = F_1 \circ h^{2/3}(I(z)) \tag{6.27}$$

(we used the property $h(I(z)) = -h(z)$ for (6.10)). Hence, condition (6.26) is the criterion for the two-vertices to be BPZ boundary states.

This gives some new ideas of how to look for boundary states. For example, we may start by solving the equation (6.26):

$$g(z)g(-1/z) = -1 \tag{6.28}$$

Obvious solution of eq. (6.28) is

$$g(z) = z^{2k+1}, k \in \mathbb{N} \tag{6.29}$$

Another solution is given by the appropriate regular branch of the multi-valued function[18]

$$g(z) = z^{\frac{1}{2k+1}}, k \in \mathbb{N}, \quad \text{such branch that } g(-1) = -1 \tag{6.30}$$

The solution in (6.30) should be compared with eq. (6.12) to see that it describes wedge state $|4k+1\rangle$ $k \geq 0$. Thus, such family of wedge states describes BPZ boundary states. Note, that multiplication rule of wedge states is [21]

$$|r\rangle * |s\rangle = |r + s - 1\rangle \tag{6.31}$$

[17]Another suggestion would be to call such states *cross-cap boundary states*

[18]This equation was studied in different context in [24].

and as a result these surface states form a subalgebra:

$$|4k+1\rangle * |4m+1\rangle = |4(k+m)+1\rangle \tag{6.32}$$

On the other hand, solution (6.29) means that function $F_1(z)$ of (6.5) is such that

$$z^{2k+1} = F_1(h^{2/3}(z)) \Rightarrow F_1(u) = \Big(h^{-1}(u^{3/2})\Big)^{2k+1} \tag{6.33}$$

Geometric meaning of this function is not clear to us at the moment, but it is obvious, that using it and eq. (6.6) one can very simply obtain the function $f(z)$ (in r.h.s. of (6.6)), and thus the corresponding surface state.

Whether there are more solutions of (6.28) is not clear to us at the moment. Note, that they are *necessary* but not *sufficient* conditions! For each of them, the procedure of finding $f(z)$ via eq. (6.6) should still be realized. It would be very interesting to check these two solution and see the class of $f(z)$, which correspond to them.

6.4 Discussion

As we mentioned in the introduction the D-branes as a solitons in open strings arise in the context of Vacuum SFT [3]. It would be interesting to build the multiplication operator (6.1) for projectors, corresponding to the D-branes in VSFT, to see whether they correspond to boundary states, In first place we would like to do it for the sliver state. This is a state conjectured to correspond to space-filling D-brane in VSFT. It would be interesting to see explicitly that it is a Neumann boundary state (that of a space-filling D-brane) of some curve and find out the geometric meaning of this curve. This was partially fulfilled in the Section 6.1, eq. (6.12). However it is hard to analyze it fully in case of sliver, i.e. the singular limit $n \to \infty$ in eq. (6.9). Right now we would just like to stress one existing connection. Sliver state is a surface state that corresponds to the conformal map on infinitely-sheeted logarithm-like Riemann surface [12]. On the other hand, the two-vertex (4.21), corresponding to the Neumann boundary state, has logarithmic multi-valuedness as well. Indeed, recall (Section 2) that the scalar field we considered had a term $b_0 \log w + \bar{b}_0 \log \bar{w}$. In order for it to be single-valued on the complex plane w, b_0 should be equal to $\bar{b}_0$ (compare footnote 5 on page 7). As we had mentioned already in Section 3 we did not impose this condition, but instead required Neumann boundary conditions $b_0 = -\bar{b}_0$. As a result the scalar CFT and Neumann boundary state should be defined on multi-sheet logarithm Riemann surface as well. It will be very interesting to see how this Riemann surface and Riemann surface of a sliver state are related.

We would like also to note, that for the subalgebra of states, described in 6.3 (namely, states (6.30)) we can formally take limit $k \to \infty$ and thus it may be that sliver shares some of their properties. We leave this question to the future investigation.

To see, whether the conjectured relation between projectors and boundary states is true, one would have to perform several checks. One of them would be to compute one-point function of closed string vertex operators *in the background of given solution of*

SFT. For that one would have to act as follows. First, compute the two-vertex $|V_{12}(\Sigma)\rangle$, corresponding to the given surface state $|\Sigma\rangle$. For example this state can be projector, conjectured to correspond to a D-brane. Then, interpret this vertex as a boundary state $|\Sigma\rangle\rangle$ (as discussed in details in Section 4). Finally, compute the one-point function of any closed string vertex operator $V_{closed}(z,\bar{z})$:

$$\langle 0|\, V_{closed}(z,\bar{z})\, |\Sigma\rangle\rangle \stackrel{?}{=} \langle V\rangle_{disk,\, Dirichlet} \tag{6.34}$$

where correlator $\langle V_{closed}\rangle$ in the r.h.s. of (6.34) is computed on the disk with appropriate (Dirichlet) boundary conditions.

Another interesting check would be to calculate ratios of tensions corresponding to branes of different dimensions. In VSFT the tension is just a norm of a projector state $|\Sigma\rangle$. On the other hand the tension of a brane is extracted from the amplitude of closed string exchange between the branes

$$\langle\langle V_{12}(\Sigma)|\, \exp^{-(L_0^1+L_0^2)\pi/Y}\, |V_{12}(\Sigma)\rangle\rangle \;=\; \langle\Sigma|\, \langle V_{1^*23}|\, \exp^{-(L_0^1+L_0^2)\pi/Y}\, |V_{231^*}\rangle\, |\Sigma\rangle \tag{6.35}$$

where star in $\langle V_{1^*23}|$ and $|V_{231^*}\rangle$ means that corresponding state is BPZ conjugated. The parameter Y is a separation between branes. We do not that hope this norm will be equal to the norm of $|\Sigma\rangle$ itself. But it is possible that the ratio of tension of different branes will stay the same.

The interpretation of two-vertices as boundary states is useful realization of a star-product in closed SFT. To wit, it was suggested recently in [25] to describe D-branes as boundary states in closed SFT. It was also shown there that by taking so called HIKKO formulation of closed SFT [26] boundary states $|B\rangle\rangle$ corresponding to various D-brane solutions indeed obey the equation

$$|B\rangle\rangle \star |B\rangle\rangle = |B\rangle\rangle \tag{6.36}$$

That is, boundary states formally satisfy the same equations that are imposed on brane solutions in VSFT. The star-products in both theories are very different, of course. This prompts the possibility of a more general correspondence. Let us define the star-product of boundary states as usual Witten's star-product of corresponding surface states. Namely, if open string surface states $|f_1\rangle$, $|f_2\rangle$ and $|f_3\rangle$ give rise in a way suggested in our paper to boundary states and

$$|f_1\rangle * |f_2\rangle = |f_3\rangle \tag{6.37}$$

let us define the result of a star-product of $|B_1\rangle\rangle\star|B_2\rangle\rangle$ as a state $|B_3\rangle\rangle$ that corresponds to $|f_3\rangle$. Note, that in this case eq. (6.36) becomes the natural consequence of the hypothesis of [3] that surface states, corresponding to D-branes square to themselves under Witten's star product [1]. We do not know if such star-product is indeed a HIKKO star product or if it is a consistent star product at all. But if correct, this will be a very interesting observation. At this moment, though, this is just a conjecture and we leave it for future works. Let us just mention immediate consequence of this conjecture:

– boundary state (6.17), corresponding to orientifold, should serve as an *identity element* of the $\star$-algebra of closed SFT.

One of the main points of the paper is introduction of full tau-function dynamics in SFT. The SFT vertices we considered depended on a family of conformal maps. In terms of tau-function times t_k it means that Neumann coefficients of corresponding vertices become functions of t_k as well. This result opens the whole field of problems that can be pursued.

- Since surface states are just one-vertices and surface states form a sub-algebra under star product it should be possible to re-write star multiplication analytically in terms of t_k. This would allow us , in particular, to use the well-known integrable reductions to identify the finite-dimensional sub-algebras of star-product.

- In [13] it was shown that associativity (WDVV) equations [14] are solved by tau-function as a direct consequences of Hirota identities. The integrable structure of SFT, identified in this paper, thus strongly suggests the presence of such associativity algebra. Technically, it will involve the derivatives of Neumann coefficient w.r.t. t_k, i.e. the third derivatives of tau-function. The identification of the chiral rings of associative operators in SFT is just one of the numerous possibilities opened by integrability.

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A Toda Lattice Hierarchy and its dispersionless limit

In this Appendix we will review some facts about Toda Lattice hierarchy.

A1 Toda Lattice Equation

Equation of (dispersionful) Toda Lattice is given by

$$\partial_{t_1}\partial_{\bar{t}_1}\phi_n = e^{\phi_n-\phi_{n-1}} - e^{\phi_{n+1}-\phi_n} \tag{A.1}$$

It is convenient to introduce a new variable t_0, function $\phi(t_0)$ and *lattice spacing* $\hbar$ such that

$$\phi_n = \phi(t_0),\ \phi_{n\pm1} = \phi(t_0 \pm \hbar),\ \text{etc.} \tag{A.2}$$

Eq. (A.1) is known to be *integrable* (see e.g. [27, 6, 8] and refs. therein). In particular it has the so called *Zakharov-Shabat* or *zero curvature* representation

$$\partial_{t_1}\bar{H}_1 - \partial_{\bar{t}_1}H_1 + [\bar{H}_1,\, H_1] = 0 \tag{A.3}$$

where infinite matrices H_1 and $\bar{H}_1$ are defined [28] as

$$H_1 = \left(\begin{array}{cc|cccc} \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \; \vdots \\ \cdots & \partial_{\bar{t}_1}\phi(t_0-\hbar) & 1 & 0 & \cdots & \\ \hline \cdots & 0 & \partial_{\bar{t}_1}\phi(t_0) & 1 & \cdots & \\ \cdots & 0 & 0 & \partial_{\bar{t}_1}\phi(t_0+\hbar) & \ddots & \\ \cdots & \vdots & \vdots & \vdots & \ddots & \end{array}\right) \tag{A.4}$$

and

$$\bar{H}_1 = \left(\begin{array}{cc|ccc} \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \hline \cdots & e^{\phi(t_0)-\phi(t_0-\hbar)} & 0 & 0 & \cdots \\ \cdots & 0 & e^{\phi(t_0+\hbar)-\phi(t_0)} & 0 & \cdots \\ \cdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) \tag{A.5}$$

Toda Lattice equation can also be written in the form which introduces a *tau-function* $\tau(t_0, t_1, \bar{t}_1)$. Namely, if one defines

$$\phi(t_0) = \log\frac{\tau(t_0+\hbar)}{\tau(t_0)},\;\; e^{\phi(t_0)-\phi(t_0+\hbar)} = \frac{\tau(t_0+\hbar)\tau(t_0-\hbar)}{\tau(t_0)^2} \tag{A.6}$$

then equation (A.1) can be rewritten in the *Hirota form*

$$\frac{1}{2}D_{t_1}D_{\bar{t}_1}\tau(t_0)\tau(t_0) + \tau(t_0+\hbar)\tau(t_0-\hbar) = 0 \tag{A.7}$$

where *Hirota derivative* $D_xD_yf(x,y)f(x,y)$ is defined via

$$D_xD_yf(x,y)f(x,y) \equiv 2\left(\frac{\partial^2 f(x,y)}{\partial x\partial y}f(x,y) - \frac{\partial f(x,y)}{\partial x}\frac{\partial f(x,y)}{\partial y}\right) \tag{A.8}$$

We will see below that the tau-function will play an important role in describing the hierarchy.

A2 Toda Lattice Hierarchy

Eq. (A.3) is just a first equation of an infinite series of zero curvature equations that described the whole *Toda Lattice Hierarchy*. This hierarchy can be represented in the following form. Let us define two *Lax operators*

$$L = r(t_0)e^{\hbar\partial_{t_0}} + \sum_{k=0}^{\infty}u_k(t_0)e^{-k\hbar\partial_{t_0}} \tag{A.9}$$

and

$$\bar{L} = e^{-\hbar\partial_{t_0}} r(t_0) + \sum_{k=0}^{\infty} e^{k\hbar\partial_{t_0}} \bar{u}_k(t_0) \tag{A.10}$$

Then one can build (double infinite) series of *Lax-Sato equations*

$$\frac{\partial L}{\partial t_k} = [H_k,\, L] \tag{A.11}$$

$$\frac{\partial L}{\partial \bar{t}_k} = [\bar{H}_k,\, L] \tag{A.12}$$

defining the evolution with respect to all times t_k, $\bar{t}_k$. Here[19]

$$H_k = \left(L^k\right)_+ + \frac{1}{2}\left(L^k\right)_0; \quad \bar{H}_k = \left(\bar{L}^k\right)_- + \frac{1}{2}\left(\bar{L}^k\right)_0 \tag{A.13}$$

Compatibility of the system of equations (A.11–A.12) leads to Zakharov-Shabat equations

$$\frac{\partial H_n}{\partial t_k} - \frac{\partial H_k}{\partial t_n} + [H_n,\, H_k] = 0 \tag{A.14}$$

$$\frac{\partial \bar{H}_n}{\partial t_k} - \frac{\partial H_k}{\partial \bar{t}_n} + [\bar{H}_n,\, H_k] = 0 \tag{A.15}$$

(there is also an equation analogous to (A.14) for Hamiltonians $\bar{H}_k$ and variables $\bar{t}_n$). Eq. (A.3) was the first in series (A.14). The identification between eq. (A.3) and (A.14) is established by

$$r^2(t_0) = e^{\phi(t_0+\hbar)-\phi(t_0)}; \quad u_0(t_0) = \frac{\partial \phi(t_0)}{\partial t_1}; \quad \text{etc} \tag{A.16}$$

A3 Tau-function of Toda Lattice Hierarchy

There are many (to some extent) equivalent ways to describe integrable hierarchies. In various applications various of them are useful. For example, above we have shown two such formalisms: "Lax formalism", described by eqs. (A.9)–(A.13) and Zakharov-Shabat formalism, described by eqs. (A.14)–(A.15). Below, we are going to introduce one more way to describe the hierarchy, which is going to be extremely useful for us in what follows - the so called *tau-function approach.*

One can introduce a tau-function for the whole Toda Lattice Hierarchy in the following way. Consider eqs. (A.11), (A.12) as a compatibility condition for the auxiliary linear problem for function $\Psi(z|t) \equiv \Psi(z|t_0, t_k, \bar{t}_k)$

$$L\Psi(z|t) = z\Psi(z|t), \; \frac{\partial \Psi(z|t)}{\partial t_k} = H_k \Psi(z|t), \; k > 0 \tag{A.17}$$

Function $\Psi(z|t)$ are called *Backer-Akhiezer functions.*

[19] Sometimes eq. (A.13) is written in the form $H_k = \left(L^k\right)_+$, etc. This is the gauge choice, depending on the form of the Lax operators (A.9)

The solution of all the equations can be incorporated into a function of infinite variables $\tau(t_0, t_k, \bar{t}_k)$ such that

$$\Psi(z|t) = \frac{\tau\left(t_0; \boldsymbol{t} - \hbar\left[z^{-1}\right]; \bar{\boldsymbol{t}}\right)}{\tau(t_0; \boldsymbol{t}; \bar{\boldsymbol{t}})} e^{\frac{\xi(z)}{\hbar}} z^{\frac{t_0}{\hbar}} \tag{A.18}$$

where $\xi(z) \equiv \sum_{k=1}^{\infty} t_k z^k$ and notation $[z^{-1}]$ means

$$f\left(\boldsymbol{t} - \left[z^{-1}\right]\right) = f\left(t_1 - \frac{1}{z}, t_2 - \frac{1}{2z^2}, \ldots\right) \tag{A.19}$$

Similar equations can be written for the conjugated function $\Psi^*(z)$

$$\Psi^*(z|t_0, t_k, \bar{t}_k) \equiv \Psi^*(z|t) = \frac{\tau\left(t_0; \boldsymbol{t} + \hbar\left[z^{-1}\right]; \bar{\boldsymbol{t}}\right)}{\tau(t_0; \boldsymbol{t}; \bar{\boldsymbol{t}})} e^{-\frac{\xi(z)}{\hbar}} z^{-\frac{t_0}{\hbar}} \tag{A.20}$$

In terms of Backer-Akhiezer functions the whole Toda Lattice hierarchy can be compactly encoded in the form

$$\oint_{\infty} \frac{dz}{2\pi i} \Psi(z|t_0, t_k, \bar{t}_k) \Psi^*(z|t'_0, t'_k, \bar{t}'_k) = 0 \tag{A.21}$$

That is, one can show that condition (A.21) is equivalent to the system of equations (A.9)–(A.13) and thus equivalently describes Toda Lattice hierarchy.

Eq. (A.21) can be re-expressed in terms of infinite set of differential equations on tau-function, called *Hirota equations*[20]. This form of Hirota identities is similar to the one appearing in the case of KP (see [17]). For a general form, see also [27].

Introduce the operator

$$D(z) = \sum_{k=1}^{\infty} \frac{z^{-k}}{k} \partial_{t_k} \tag{A.22}$$

Then eq. (A.21) can be rewritten in the following form

$$\begin{aligned} &z\left(e^{\hbar(\partial_{t_0} - D(z))}\tau\right)\left(e^{-\hbar D(\zeta)}\tau\right) - \zeta\left(e^{\hbar(\partial_{t_0} - D(\zeta))}\tau\right)\left(e^{-\hbar D(z)}\tau\right) \\ &= (z - \zeta)\left(e^{-\hbar(D(z) + D(\zeta))}\tau\right)\left(e^{-\hbar\partial_{t_0}}\tau\right) \end{aligned} \tag{A.23}$$

$$\begin{aligned} &\left(e^{-\hbar D(z)}\tau\right)\left(e^{-\hbar \bar{D}(\bar{\zeta})}\tau\right) - \tau\left(e^{\hbar(\bar{D}(\bar{\zeta}) - D(z))}\tau\right) \\ &= \frac{1}{z\bar{\zeta}}\left(e^{\hbar(\partial_{t_0} + D(z))}\tau\right)\left(e^{\hbar(\partial_{t_0} + \bar{D}(\bar{\zeta}))}\tau\right) \end{aligned} \tag{A.24}$$

There are also other Hirota equations, which we do not provide here.

[20]The following result for Toda Lattice hierarchy Hirota identities was shown to us by A.Zabrodin. We are grateful to him for sharing with us this information.

A4 Dispersionless limit of Toda Lattice Hierarchy

Many integrable hierarchies (Toda Lattice being one of them) admit the so called *dispersionless limit*. New hierarchies are also integrable, according to the general theory of Witham hierarchies and hydro-dynamical brackets [29, 30]. These hierarchies also appear in many applications.

In the case of Toda, to obtain the dispersionless limit one takes formal limit $\hbar \to 0$ in eq. (A.1), in identification (A.2), (A.9), etc. In this limit, for example, the equation (A.1) takes the form (see e.g. [6])

$$\partial_{t_1}\partial_{\bar{t}_1}\phi(t_0) = \partial_{t_0} e^{\partial_{t_0}\phi(t_0)} \tag{A.25}$$

This equation is also integrable and is a part of dispersionless Toda lattice hierarchy. Again, all equations of this hierarchy can be encoded in the Hirota identities, which can be obtained by taking the same limit $\hbar \to 0$ in (A.24) (writing $\tau_\hbar = \exp(F/\hbar^2 + \mathcal{O}(1))$, etc.)

$$(z-\zeta)e^{D(z)D(\zeta)F} = ze^{-\partial_{t_0}D(z)F} - \zeta e^{-\partial_{t_0}D(\zeta)F} \tag{A.26}$$

$$1 - \exp(-D(z)\bar{D}(\bar{\zeta})F) = \frac{1}{z\bar{\zeta}}\exp(\partial_{t_0}(\partial_{t_0} + D(z) + \bar{D}(\bar{\zeta}))F), \tag{A.27}$$

where $D(z)$ is defined by (A.22). There is also Hirota identity in purely anti-holomorphic sector, it is identical to the holomorphic one (A.26).

The map $w(z)$ understood as a function of t_k is given by

$$\log\frac{w(z)}{z/r} = -\partial_{t_0}D(z)F(\mathbf{t}) \tag{A.28}$$

and $\log r^2 = \partial_{t_0}^2 F(\mathbf{t})$. This allows to rewrite Hirota equations (A.26), (A.27)

$$D(z)D(\zeta)F(\mathbf{t}) - \frac{1}{2}\partial_{t_0}^2 F(\mathbf{t}) = \log\frac{w(z)-w(\zeta)}{z-\zeta} \tag{A.29}$$

and

$$D(z)\bar{D}(\zeta)F = -\log\left(1 - \frac{1}{w(z)\bar{w}(\zeta)}\right) \tag{A.30}$$

Coupled with eq. (A.28) one can see that second derivatives $\partial_{t_k}\partial_{t_n}F$ for $k, n > 0$ as well as $\partial_{\bar{t}_k}\partial_{\bar{t}_n}F$ are completely determined by $\partial_{t_0}\partial_{t_k}F$.

By expressing from (A.26) derivatives $\partial_{t_0}\partial_{t_k}F$ via $\partial_{t_1}\partial_{t_k}F$, Hirota eqs. (A.26)(A.29) can also be rewritten in the *pure holomorphic form*, which is also called *Hirota equations of dispersionless KP hierarchy* [17]

$$\exp(D(z)D(\zeta)F) = 1 - \frac{D(z)\partial_{t_1}F - D(\zeta)\partial_{t_1}F}{z-\zeta} \tag{A.31}$$

B Two-vertex corresponding to Neumann boundary state on a circle

In this Appendix we want to make an explicit calculation for the two-vertex corresponding to conformal transformations, $h_1(z) = z$ and $h_2(z) = 1/z$. The Neumann coefficients are

$$N^{12}_{nm} = N^{21}_{mn} = \oint\oint \frac{dzdw}{(2\pi i)^2}\frac{z^n w^m}{nm}\frac{-1}{z^2}\frac{1}{(1/z-w)^2} = -\frac{1}{n}\delta_{n,m} \tag{B.1}$$

$$N^{11}_{nm} = N^{22}_{nm} = \oint\oint \frac{dzdw}{(2\pi i)^2}\frac{z^n w^m}{nm}\frac{1}{z^2}\frac{1}{w^2}\frac{1}{(1/z-1/w)^2} = -\frac{1}{m}\delta_{-n,m} \tag{B.2}$$

In the definition of the vertex state eq. (4.16) the sum is over positive n, thus only N^{12} contributes. Others are contracted with lowering operators and they annihilate the vacuum. Now we can bring this together to get

$$|V_{12}\rangle = \int dp \exp\left\{-\sum_{n>0}\frac{a^1_{-n}a^2_{-n}}{n}\right\}|p\rangle_1 \otimes |-p\rangle_2 \tag{B.3}$$

We assumed that zero-mode Neumann coefficients that stand in front of p and p^2 terms are zero (see eq.(4.24)). In dealing with these Neumann coefficients N^{IJ}_{0k}, $k \geq 0$ one has to manipulate with formal objects like $h_I(\infty)$, $[h'](\infty)$. Let us demonstrate how to do it. We make use of the following observation. The choice $h_1(z) = z$ and $h_2(z) = 1/z$ is not unique. One can take any $GL(2,\mathbb{C})$ transformations of $h_I(z)$. It is easy to see that Neumann coefficients are left unchanged. Thus a general choice for conformal transformations that leads to Neumann boundary state is given by (see the footnote 11, p. 16)

$$h_1(z) = \frac{az+b}{cz+d}, \quad h_2(z) = \frac{(1/z)a+b}{(1/z)c+d} \tag{B.4}$$

If any of the h's are not regular at infinity we can build new h's which are regular by applying any $GL(2,\mathbb{C})$ transformation. In this way we can "regularize" $h_I(\infty)$. Let us take this $GL(2,\mathbb{C})$ transformation to be small. For example, to deal with current $h_{1,2}(z)$ we choose the following $GL(2,\mathbb{C})$ matrix

$$\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \tag{B.5}$$

Then

$$h_1(z) = \frac{z}{1+\varepsilon z} \quad h_2(z) = \frac{1}{z+\varepsilon} \tag{B.6}$$

Now $h_{1,2}(\infty)$ are well defined. We can make the calculations and take the limit $\varepsilon \to 0$ at the end. As an example

$$N^{11}_{0n} = \frac{1}{n}\oint \frac{dz}{2\pi i} z^n \partial_z \log\left(\frac{1}{\varepsilon} - \frac{z}{1+\varepsilon z}\right) = 0 \quad \forall\, n > 0 \tag{B.7}$$

The coefficient of the p^2 term is[21]

$$\log\left\{-\frac{[h_1'](\infty)[h_2'](\infty)}{(h_1(\infty)-h_2(\infty))^2}\right\} = \log\frac{\varepsilon^2}{\varepsilon^2} = 0 \tag{B.8}$$

Thus the eq. (B.3) is indeed Neumann boundary state on a circle as one can see from eq. (3.6).

C Conformal transformations of a two-vertex state

Below, we give a proof that a two-vertex defined by arbitrary conformal maps corresponds to a Neumann boundary state defined on conformally transformed unit circle. Consider a curve $\mathcal{C}$ which is mapped to a unit circle by conformal transformations $w(z)$. Let us define the action of conformal transformations on operators and states. Under transformation $g(z)$

$$g(V) \equiv U_g V U_g^{-1} \tag{C.1}$$

$$g\,|\psi\rangle \equiv U_g\,|\psi\rangle \tag{C.2}$$

$$\text{So that} \quad g\Big(V\,|\psi\rangle\Big) = g(V)g\,|\psi\rangle \tag{C.3}$$

Here U_g are the elements of Virasoro group given by

$$U_g = \exp(\sum_n v_n L_n) \tag{C.4}$$

where v_n's are harmonics of $v(z) = \sum v_n z^{n+1}$ and field $v(z)$ is defined by equation

$$e^{v(z)\partial_z} z = g(z) \tag{C.5}$$

If the conformal transformation $g(z)$ is regular at infinity the expansion of the field $v(z)$ has only modes with $n \leq 1$. As a result we have

$$\langle 0|\,U_g = \langle 0|\,U_g^{-1} = \langle 0| \tag{C.6}$$

Again, denote the two-vertex (4.15) defined by maps h_1 and h_2 via $|h_1,h_2\rangle$. Therefore what we showed in Appendix B can be written as $|z,z^{-1}\rangle = |N\rangle\rangle_{circle}$, with $|N\rangle\rangle_{circle}$ being Neumann boundary state on a circle. The task is now to find which conformal transformations $w(z)$ and $\bar{w}(\bar{z})$ correspond to such $h_1(z)$ and $h_2(z)$ in definition of two-vertex that the resulting two-vertex is conformally transformed Neumann boundary state

$$|h_1,h_2\rangle = U_w^{-1}\bar{U}_{\bar{w}}^{-1}\,|N\rangle\rangle_{circle} \tag{C.7}$$

where U_w acts on a holomorphic part of $|N\rangle\rangle$ and $\bar{U}_{\bar{w}}$ acts on anti-holomorphic part. U_w^{-1} comes from the fact, that $a_k = U_w b_k U_w^{-1}$ and $b_k = U_w^{-1} a_k U_w$.

[21] Recall, that $[h'](z)$ is defined as $[h'](z) \equiv -\dfrac{dh(z)}{d1/z}$

To do this, let us calculate the two-vertex corresponding to composite conformal transformation $f_1 \circ w$ and $f_2 \circ \bar{w}$. We can do it in two steps. We can either apply the whole $f \circ w$ at once or take f first and w later. For any vertex operators V one can write using definition of two-vertex eq. (4.15)

$$\begin{aligned} \langle 0|_1 \otimes \langle 0|_2 \, V_1(a^1)V_2(a^2) \, |f_1 \circ w_1, f_2 \circ w_2\rangle &= \\ \langle 0| \, f_1 \circ w_1(V_1) f_2 \circ w_2(V_2) \, |0\rangle &= \\ \langle 0|_1 \otimes \langle 0|_2 \, w_1(V_1) w_2(V_2) \, |f_1, f_2\rangle & \end{aligned} \tag{C.8}$$

Now, we can use the explicit form $f(V)$ from eq. (C.1) to get

$$\begin{aligned} \langle 0|_1 \otimes \langle 0|_2 \, w_1(V_1) w_2(V_2) \, |f_1, f_2\rangle &= \\ \langle 0|_1 \otimes \langle 0|_2 \, U_{w_1} V_1(a_1) U_{w_1}^{-1} U_{w_2} V_2(a_2) U_{w_2}^{-1} \, |f_1, f_2\rangle &= \\ \langle 0|_1 \otimes \langle 0|_2 \, V_1(a_1) V_2(a_2) U_{w_1}^{-1} U_{w_2}^{-1} \, |f_1, f_2\rangle & \end{aligned} \tag{C.9}$$

In the last line we assumed that w_1 and w_2 leave the vacuum states invariant eq. (C.6). Comparing eq. (C.9) and eq. (C.8) we see that for any V_1 and V_2

$$\langle 0|_1 \otimes \langle 0|_2 \, V_1(a^1)V_2(a^2) \, |f_1 \circ w_1, f_2 \circ w_2\rangle = \tag{C.10}$$
$$\langle 0|_1 \otimes \langle 0|_2 \, V_1(a^1)V_2(a^2) U_{w_1}^{-1} U_{w_2}^{-1} \, |f_1, f_2\rangle \tag{C.11}$$

Thus we can say that corresponding states are equal

$$|f_1 \circ w_1, f_2 \circ w_2\rangle = U_{w_1}^{-1} U_{w_2}^{-1} \, |f_1, f_2\rangle \tag{C.12}$$

For the fixed choice of $f_1(z) = z$ and $f_2(z) = \frac{1}{z}$, $\left|z, \frac{1}{z}\right\rangle$ is just a Neumann boundary state $|N\rangle\rangle_{circle}$ on a unit circle, defined by eq. (3.6) (see also Appendix B). If $w_1(z) = w(z)$ and $w_2(z) = \bar{w}(z)$ then

$$\left|w, \frac{1}{\bar{w}}\right\rangle = U_w^{-1} \bar{U}_{\bar{w}}^{-1} \, |N\rangle\rangle_{circle} \tag{C.13}$$

It should be compared to eq. (C.7). Thus we achieved the result (4.23) — the two-vertex with conformal transformations

$$h_1 = w(z), \quad h_2(z) = \frac{1}{\bar{w}(z)} \tag{C.14}$$

is a Neumann boundary state $|N\rangle\rangle_{\mathcal{C}}$ on a curve $\mathcal{C}$.

Now we come back to the question, raised in the footnote 11, p. 16. There we discussed the problem that there is a freedom in relating a two-vertex to Neumann boundary state on a circle. Indeed, instead of $f_1(z) = z$ and $f_2(z) = \frac{1}{z}$ we could use general $GL(2, \mathbb{C})$ transformation thereof. Then we would get

$$h_1(z) = \frac{w(z)\, a + b}{w(z)\, c + d}, \quad h_2(z) = \frac{\frac{a}{\bar{w}(z)} + b}{\frac{c}{\bar{w}(z)} + d} \tag{C.15}$$

instead of (C.14). Or inverse

$$w(z) = \left(-\frac{d}{c}\right)\frac{h_1(z) - b/d}{h_1(z) - a/c}, \quad \bar{w}(z) = \left(-\frac{c}{d}\right)\frac{h_2(z) - a/c}{h_2(z) - b/d} \tag{C.16}$$

Up until now, we did not specify analytic properties of $h_I(z)$. It turns out that by comparing them with analytic properties of $w(z)$ one can fix this $GL(2,\mathbb{C})$ freedom. For the purpose of identification between Toda tau-function and Neumann boundary state on a curve we considered only univalent w and $\bar{w}$ (2.10), i.e.

$$w(z) = \frac{z}{r} + p_0 + \frac{p_1}{z} + \mathcal{O}(\frac{1}{z^2}) \tag{C.17}$$

Now eq. (C.14) assumes that h_1 is univalent too but expansion of h_2 starts with $1/z$. On the other hand in future applications h_1 and h_2 are taken to be regular at infinity

$$h_I(z) = h_I(\infty) - \frac{[h'_I](\infty)}{z} + \mathcal{O}(\frac{1}{z^2}),\ I = 1,2 \tag{C.18}$$

Insisting on both of the conditions (C.17), (C.18) as well as on eq. (C.16) will fix coefficients of $GL(2,\mathbb{C})$ transformation. It is easy to see that the first of the equations in (C.16) requires[22] $a/c = h_1(\infty)$ and the second gives $b/d = h_2(\infty)$. The scaling coefficient c/d is left unfixed for the moment. We can calculate how r and $\bar{r}$ in the expansion of w and $\bar{w}$ depend on corresponding coefficients in h. It is

$$\frac{1}{r} = \left(-\frac{d}{c}\right)\frac{h_1(\infty) - h_2(\infty)}{[h'_1](\infty)}, \quad \frac{1}{\bar{r}} = \left(-\frac{c}{d}\right)\frac{h_2(\infty) - h_1(\infty)}{[h'_2](\infty)} \tag{C.19}$$

If we require $\bar{r}$ to be complex conjugate of r then the factor d/c is determined. For the case when all $h_I(\infty)$ and $[h'_I](\infty)$ are real we have

$$\frac{d}{c} = -\sqrt{-\frac{[h'_1](\infty)}{[h'_2](\infty)}} \tag{C.20}$$

Let us give for completeness the final result

$$\begin{gathered} w(z) = \left(-\frac{d}{c}\right)\frac{h_1(z) - b/d}{h_1(z) - a/c}, \quad \bar{w}(z) = \left(-\frac{c}{d}\right)\frac{h_2(z) - a/c}{h_2(z) - b/d} \\ \frac{a}{c} = h_1(\infty) \qquad \frac{b}{d} = h_2(\infty) \qquad \frac{d}{c} = -\sqrt{-\frac{[h'_1](\infty)}{[h'_2](\infty)}} \end{gathered} \tag{C.21}$$

D The identification between dToda and a two-vertex

In this Appendix we check explicitly if identification between Neumann coefficients of two-vertex and second derivatives of Toda tau-function F is consistent. The formulation

[22] We do not consider the case when $h_1(\infty) = h_2(\infty)$.

of the problem is the following. The second derivatives of F as second derivatives of any Toda tau-function satisfy Hirota equations. We want to plug Neumann coefficients in the corresponding equations and see that they are still satisfied. For convenience let us list Hirota equations, definitions of Neumann coefficients, and the identification between them and second derivatives of F. Hirota equations are

$$\begin{aligned}
\partial_{t_0}^2 F &= \log|r|^2 && \text{(D.1)}\\
\partial_{t_0}\partial_{t_n} F &= \oint \frac{dz}{2\pi i} z^n \partial_z \log\left(w(z)r\right) \quad \text{for} \quad n>0 && \text{(D.2)}\\
\partial_{t_n}\partial_{t_k} F &= \oint\oint \frac{dz d\zeta}{(2\pi i)^2} z^n \zeta^k \partial_z \partial_\zeta \log \frac{w(z)-w(\zeta)}{z-\zeta} && \text{(D.3)}\\
\partial_{t_n}\partial_{\bar{t}_k} F &= -\oint\oint \frac{dz d\zeta}{(2\pi i)^2} z^n \zeta^k \partial_z \partial_\zeta \log\left(1-\frac{1}{w(z)\bar{w}(\zeta)}\frac{e^{\partial_{t_0}^2 F}}{|r|^2}\right) && \text{(D.4)}
\end{aligned}$$

Equations (D.2) completely determine the conformal transformation $w(z)$ and plugging it in eq. (D.3) and eq. (D.4) one can calculate the rest of the derivatives of F. The analytic properties of $w(z)$ are

$$w(z) = \frac{z}{r} + \sum_{k=0}^{\infty} \frac{p_k}{z^k} \tag{D.5}$$

Now we repeat definition of Neumann coefficients. They depend on conformal maps h_1, h_2 as

$$\begin{aligned}
N_{km}^{IJ} &= \frac{1}{km}\oint \frac{dz}{2\pi i}\oint \frac{d\zeta}{2\pi i} z^k \zeta^m \partial_z \partial_\zeta \log\Big(h_I(z)-h_J(\zeta)\Big) && \text{(D.6)}\\
N_{0m}^{IJ} &= -\frac{1}{m}\oint \frac{dz}{2\pi i} z^m \partial_z \log\Big(h_I(\infty)-h_J(z)\Big) && \text{(D.7)}\\
N_{00}^{IJ} &= \begin{cases} \log([h_I'](\infty)) & I=J \\ \log(h_I(\infty)-h_J(\infty)) & I\neq J \end{cases} && \text{(D.8)}
\end{aligned}$$

where $[h'](z) \equiv -\dfrac{dh(z)}{d1/z}$. And finally we give the full list of identifications

$$\begin{aligned}
\partial_{t_0}^2 F &= N_{00}^{11}+N_{00}^{22}-N_{00}^{12}-N_{00}^{21} && \text{(D.9)}\\
\frac{1}{k}\partial_{t_0}\partial_{t_k} F &= N_{0k}^{11}-N_{0k}^{21} && \text{(D.10)}\\
\frac{1}{k}\partial_{t_0}\partial_{\bar{t}_k} F &= N_{0k}^{22}-N_{0k}^{12} && \text{(D.11)}\\
\frac{1}{nk}\partial_{t_n}\partial_{t_k} F &= N_{nk}^{11} && \text{(D.12)}\\
\frac{1}{nk}\partial_{\bar{t}_n}\partial_{\bar{t}_k} F &= N_{nk}^{22} && \text{(D.13)}\\
\frac{1}{nk}\partial_{t_n}\partial_{\bar{t}_k} F &= -N_{nk}^{12} && \text{(D.14)}
\end{aligned}$$

Our strategy is the following now. From identifications in zero-mode sector, eqs. (D.9)-(D.11) we find the relation between conformal maps h and w.

$$|r|^2 = -\frac{[h_1'](\infty)[h_2'](\infty)}{(h_1(\infty) - h_2(\infty))^2} \tag{D.15}$$

$$w(z) = -\frac{dh_1(z) - b}{ch_1(z) - a} \tag{D.16}$$

$$\bar{w}(z) = -\frac{ch_2(z) - a}{dh_2(z) - b} \tag{D.17}$$

where coefficients a, b, c, d are the same as in eq. (C.21). They were obtain there by relating the two-vertex to Neumann boundary state and requiring both $h(z)$ to be regular at infinity.

If we plug this in place of w in eq. (D.4) for $\partial_{t_k}\partial_{\bar{t}_n}F$ we should get corresponding Neumann coefficients N^{12}_{kn} as predicted in eq. (D.14) That is we should prove that

$$\begin{aligned}
\partial_{t_n}\partial_{\bar{t}_k}F &= -\oint\oint \frac{dz d\zeta}{(2\pi i)^2} z^n \zeta^k \partial_z \partial_\zeta \log\left(1 - \frac{1}{w(h_1(z))\bar{w}(h_2(\zeta))}\frac{e^{\partial^2_{t_0}F}}{r\bar{r}}\right) \\
&= -\oint \frac{dz}{2\pi i} z^n \zeta^k \partial_z \partial_\zeta \log\Big(h_1(z) - h_2(\zeta)\Big) \\
&= -nkN^{12}_{nk}
\end{aligned} \tag{D.18}$$

Let us plug maps w and $\bar{w}$ found in eq. (D.16)-eq. (D.17), into the above equation.

$$\begin{aligned}
\partial_{t_n}\partial_{\bar{t}_k}F &= \\
&- \oint\oint \frac{dz d\zeta}{(2\pi i)^2} z^n \zeta^k \partial_z \partial_\zeta \log\left(1 - \frac{h_1(z) - h_2(\infty)}{h_1(z) - h_1(\infty)}\frac{h_2(\zeta) - h_1(\infty)}{h_2(\zeta) - h_2(\infty)}\right) = \\
&- \oint\oint \frac{dz d\zeta}{(2\pi i)^2} z^n \zeta^k \partial_z \partial_\zeta \log\Big(h_2(\zeta) - h_1(z)\Big)
\end{aligned} \tag{D.19}$$

It is indeed equal to $-nkN^{12}_{nk}$. Similar checks can be performed for identifications in eq. (D.12) and eq. (D.13). The checks are satisfied.

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