Superconformally covariant operators and super-W-algebras

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Superdifferential operators of order $2n+1$ which are covariant with respect to superconformal changes of coordinates on a compact super-Riemann surface are studied herein. It is shown that all such operators arise from super-Möbius covariant ones. A canonical matrix representation is presented and applications to classical super-W-algebras are discussed.

I. INTRODUCTION

The study of linear nth order differential operators in one complex variable which are conformally covariant represents a classic subject in the mathematical literature.[1,2] In recent years, these topics have regained considerable interest and a variety of applications in mathematics and physics have been discussed (see Ref. 3 for a partial review and further references). In particular, it was realized[4-6] that these operators give rise to classical W-algebras.

The natural supersymmetric extension of this subject consists of the study of operators of the form $D^{2n+1}+\cdots$ (where $D\equiv\partial/\partial\theta+\theta\partial/\partial z$ and $n=0,1,2,\ldots,$) which are defined on compact super-Riemann surfaces. The subclass of these operators which only depends on the projective structure (and not on additional variables) has been investigated in detail in Ref. 3: these are the so-called super-Bol operators. In the present work, we will be concerned with the most general operators of order $2n+1$ which are superconformally covariant. Along the lines of Ref. 5, we will study their general structure, their classification, and discuss the applications to classical super-W-algebras.

This article is organized as follows. After introducing the necessary tools and notation in Sec. II, we discuss the general form of covariant operators of order $2n+1$. In the sequel, specific subclasses of these operators are constructed by starting from operators which are covariant with respect to superprojective changes of coordinates. The first examples are provided by the super-Bol operators $\mathscr{L}_n$ mentioned above and the second class is given by operators $M_{W_k}^{(n)}$ (with $1\leqslant k\leqslant2n+1$) which do not only depend on the superprojective structure, but also on some superconformal fields W_k. (In the applications to W-algebras, the projective structure is related to the superstress tensor while the conformal fields W_k correspond to the currents for the W symmetries.) By adding the operators $M_{W_k}^{(n)}$ to the super-Bol operator $\mathscr{L}_n$, we obtain again a covariant operator of order $2n+1$

$$\mathscr{L}=\mathscr{L}_n+M_{W_1}^{(n)}+\cdots+M_{W_{2n+1}}^{(n)}. \tag{1}$$

In Sec. VI we will show that this already encompasses the most general case. In other words, all operators of order $2n+1$ which are superconformally covariant can be cast into the form (1). Thereafter, it is shown that the operators $M_{W_k}^{(n)}$ represent special cases of operators which are bilinear and covariant and which are of independent interest. However, in the sequel, we return to the linear operators and we present a matrix representation for them, thereby elucidating the underlying algebraic structure. Section IX is devoted to the application of the previous results to the description of classical super-W-algebras. In fact, the aforementioned matrix representation provides a convenient set of generators for these algebras and the Poisson brackets between these generators involve the previously constructed covariant operators. While our main discussion is concerned with superfields, the derivation of component field results is addressed in Sec. X. We conclude with some remarks on topics which are closely related to our subject (covariant operators of even degree, in $N=2$ supersymmetry and in higher dimensions, singular vectors of the Neveu–Schwarz algebra). In an Appendix we collect some of the algebraic concepts which are referred to in the main text.

II. GENERAL FRAMEWORK

Let us first recall the notions which are needed in the sequel.[7] The arena we work on is a compact $N=1$ super-Riemann surface (SRS) parametrized by local coordinates $\mathbf{z}=(z,\theta)$. (Our discussion applies equally well to a real one-dimensional supermanifold for which case the changes of coordinates are superdiffeomorphisms.) The canonical derivatives are denoted by $\partial=\partial_z$ and $D=\partial_\theta+\theta\partial$. By definition, any two sets of local coordinates on the SRS, $\mathbf{z}$ and $\mathbf{z}'$, are related by a superconformal transformation $\mathbf{z}\to\mathbf{z}'(\mathbf{z})$, i.e., a transformation satisfying $Dz'=\theta'(D\theta')$. This condition implies $D=(D\theta')D'$ and $(D\theta')(D'\theta)=1$.

Throughout the text, the Jacobian of the superconformal change of coordinates $\mathbf{z}\to\mathbf{z}'(\mathbf{z})$ will be denoted by

$$e^{-w}\equiv D\theta'.$$

By $\mathscr{F}_n$ we denote the space of superconformal fields of weight $n/2$ on the SRS, i.e., superfields with transformation properties $C_n(\mathbf{z})\to C_n'(\mathbf{z}')=e^{nw}C_n(\mathbf{z})$. The field C_n is taken to have Grassmann parity $(-)^n$.

The super-Schwarzian derivative of the coordinate transformation $\mathbf{z}\to\mathbf{z}'(\mathbf{z})$ is defined by

$$\mathscr{S}(z',z)=-[D^3w+(\partial w)(Dw)]=\frac{\partial^2\theta'}{D\theta'}-2\frac{(\partial\theta')(D^3\theta')}{(D\theta')^2}. \tag{2}$$

Under the composition of superconformal transformations, $\mathbf{z}\to\mathbf{z}'\to\mathbf{z}''$, it transforms according to

$$\mathscr{S}(\mathbf{z}'',\mathbf{z})=e^{-3w}\mathscr{S}(\mathbf{z}'',\mathbf{z}')+\mathscr{S}(\mathbf{z}',\mathbf{z}), \tag{3}$$

which implies $\mathscr{S}(\mathbf{z}',\mathbf{z})=-e^{-3w}\mathscr{S}(\mathbf{z},\mathbf{z}')$.

Coordinates belonging to a superprojective atlas on the SRS will be denoted by capital letters, $\mathbf{Z}=(Z,\Theta)$. They are related to each other by superprojective (super-Möbius) transformations, i.e., superconformal changes of coordinates $\mathbf{Z}\to\mathbf{Z}'(\mathbf{Z})$ for which $\mathscr{S}(\mathbf{Z}',\mathbf{Z})=0$. Direct integration of this equation and of $D_\Theta Z'=\Theta'(D_\Theta\Theta')$ gives

$$Z' = \frac{aZ+b}{cZ+d} + \Theta \frac{\gamma Z+\delta}{(cZ+d)^2}, \quad (ad-bc=1),$$

$$\Theta' = \frac{\gamma Z+\delta}{cZ+d} + \Theta \frac{1+(1/2)\delta\gamma}{cZ+d}. \tag{4}$$

Here, a,b,c,d are even and γ,δ are odd constants; we redefined the parameters in such a way that the even part of the transformation for Z coincides with ordinary projective transformations. The associated Jacobian then reads

$$D_\Theta \Theta' = (\tilde{c}Z + \tilde{d} + \Theta\tilde{\gamma})^{-1},$$

with $\tilde{c} = c(1 - \frac{1}{2}\delta\gamma)$, $\tilde{d} = d(1 - \frac{1}{2}\delta\gamma)$, $\tilde{\gamma} = c\delta - d\gamma$.

III. THE MOST GENERAL COVARIANT OPERATORS

The most general superdifferential operator which is linear, superanalytic, and of order $2n+1$ has the local form

$$\mathscr{L} = a_0 D^{2n+1} + a_1 D^{2n} + a_2 D^{2n-1} + \cdots + a_{2n+1},$$

where the coefficients $a_p \equiv a_p^{(n)}(\mathbf{z})$ are analytic superfields. We will always take a_p to have Grassmann parity $(-)^p$. If the leading coefficient a_0 does not have any zeros, one can achieve $a_0 \equiv 1$ by dividing by this coefficient. In the sequel, we will make this choice and we will study this type of operator on the SRS from the point of view of superconformal changes of coordinates.

The requirement that $\mathscr{L}$ maps superconformal fields (of a generic weight $p/2$) again to superconformal fields, i.e.,

$$\mathscr{L}: \mathscr{F}_p \to \mathscr{F}_{p+2n+1},$$

determines the transformation laws of the coefficients $a_1,...,a_{2n+1}$ under a superconformal change of coordinates $\mathbf{z} \to \mathbf{z}'(\mathbf{z})$. For the first two coefficients, one finds that

$$a_1' = e^w \{a_1 - (p+n)(Dw)\},$$

$$a_2' = e^{2w}\{[a_2 - n(p+n)(\partial w)] + n(Dw)a_1\}. \tag{5}$$

From the first of these equations, we see that a_1 is a superconformal field of weight $\frac{1}{2}$ if $p = -n$. In that case, a_1 can be eliminated from $\mathscr{L}$ by performing the rescaling

$$\mathscr{L} \to g^{-1}\mathscr{L}g, \quad \text{with} \quad g(\mathbf{z}) = \exp\left\{-\int_{\mathbf{z}_0}^{\mathbf{z}} d\tilde{\mathbf{z}}\, a_1(\tilde{\mathbf{z}})\right\}, \tag{6}$$

where the integral is to be understood as an indefinite integral,[7] i.e., $D\int_{\mathbf{z}_0}^{\mathbf{z}} d\tilde{\mathbf{z}}\, a_1(\tilde{\mathbf{z}}) = a_1(\mathbf{z})$. As a result, one obtains an operator of the form

$$\mathscr{L}^{(n)} = D^{2n+1} + \sum_{p=2}^{2n+1} a_p D^{2n+1-p}. \tag{7}$$

From now on, we will consider this operator which will be referred to as *"normalized operator"* of order $2n+1$.

Lemma 3.1: There are transformation laws of the coefficients a_p under superconformal changes of coordinates $\mathbf{z} \to \mathbf{z}'(\mathbf{z})$ such that the operator $\mathscr{L}^{(n)}$ maps superconformal fields to superconformal fields; more precisely,

$$\mathscr{L}^{(n)} : \mathscr{F}_{-n} \to \mathscr{F}_{n+1}, \tag{8}$$

which entails that $\mathscr{L}^{(n)}$ transforms as

$$\mathscr{L}^{(n)} \to \mathscr{L}^{(n)'} = e^{(n+1)w} \mathscr{L}^{(n)} e^{nw}. \tag{9}$$

The transformation laws of the first few coefficients are explicitly given by

$$a_2' = e^{2w} a_2,$$

$$a_3' = e^{3w} \left\{ \left[a_3 - \binom{n+1}{2} \mathscr{S} \right] + (Dw) a_2 \right\},$$

$$a_4' = e^{4w} \left\{ \left[a_4 - \binom{n+1}{3} D\mathscr{S} \right] + (n-1)(Dw) \left[a_3 - \binom{n+1}{2} \mathscr{S} \right] + (n-1)(\partial w) a_2 \right\}, \tag{10}$$

$$a_5' = e^{5w} \left\{ \left[a_5 - 2\binom{n+1}{3} \partial\mathscr{S} \right] + 2(Dw) \left[a_4 - \binom{n+1}{3} D\mathscr{S} \right] + 2(n-1)(\partial w) \left[a_3 - \binom{n+1}{2} \mathscr{S} \right] \right.$$

$$\left. + \tfrac{1}{2}(n-1)[(n+2)(D^3 w) + (n+6)(\partial w)(Dw)] a_2 \right\},$$

$$a_6' = e^{6w} \left\{ \left[a_6 - 2\binom{n+1}{4} D^3\mathscr{S} \right] + (n-2)(Dw) \left[a_5 - 2\binom{n+1}{3} \partial\mathscr{S} \right] \right.$$

$$+ 2(n-2)\partial w \left[a_4 - \binom{n+1}{3} D\mathscr{S} \right] + \binom{n-1}{2}[(D^3 w) + 5(Dw)(\partial w)] \left[a_3 - \binom{n+1}{2} \mathscr{S} \right]$$

$$\left. + \tfrac{1}{6}(n-1)(n-2)[(3+2n)(Dw)(D^3 w) + (n+3)(\partial^2 w) + (n+9)(\partial w)^2] a_2 \right\},$$

where $a_p' = a_p'(\mathbf{z}')$, $a_p = a_p(\mathbf{z})$, and where $\mathscr{S} \equiv \mathscr{S}(\mathbf{z}', \mathbf{z})$ denotes the super-Schwarzian derivative.

Since the coefficient a_2 of $\mathscr{L}^{(n)}$ transforms homogeneously, it is possible to set it to zero in a consistent way. If we do so, we have $a_3(\mathbf{z}) = (1/2)n(n+1)\mathscr{R}_{z\theta}(\mathbf{z})$, where $\mathscr{R}$ represents a superprojective (or super-Schwarzian) connection on the SRS:[3] locally, the latter is given by a collection of odd superfields $\mathscr{R}_{z\theta}$ which are locally superanalytic and which transform under a superconformal change of coordinates according to

$$\mathscr{R}_{z'\theta'}(\mathbf{z}') = e^{3w} [\mathscr{R}_{z\theta}(\mathbf{z}) - \mathscr{S}(\mathbf{z}',\mathbf{z})]. \tag{11}$$

In the general case (a_2 not identically zero), we conclude from Eqs. (10) and (11) that

$$\tilde{a}_3 \equiv a_3 - \tfrac{1}{2} Da_2 = \tfrac{1}{2} n(n+1)\mathscr{R}. \tag{12}$$

On a compact SRS, there is a one-to-one correspondence between superprojective connections and superprojective structures (i.e., superprojective atlases). This relation is expressed by[3]

$$\mathscr{R}_{z\theta}(\mathbf{z}) = \mathscr{S}(\mathbf{Z},\mathbf{z}), \tag{13}$$

where $\mathbf{Z}$ belongs to a projective coordinate system and $\mathbf{z}$ to a generic one. Note that the quantity (13) transforms as in Eq. (11) with respect to a conformal change of $\mathbf{z}$ and that it is inert under a super-Möbius transformation of $\mathbf{Z}$.

Since $\tilde{a}_3(\mathbf{z}) \propto \mathscr{S}(\mathbf{Z},\mathbf{z})$ [or $a_3(\mathbf{z}) \propto \mathscr{S}(\mathbf{Z},\mathbf{z})$ for $a_2 \equiv 0$], this coefficient vanishes if $\mathbf{z}$ is chosen to belong to the same superprojective atlas as $\mathbf{Z}$. In the next section, we will start from such an atlas and define simple operators which are covariant with respect to super-Möbius transformations. Then, we go over to a generic coordinate system and recover conformally covariant operators.

IV. EXAMPLE 1: SUPER-BOL OPERATORS

In this section, we recall some results from Ref. 3. We start from a superprojective atlas with coordinates $\mathbf{Z},\mathbf{Z}'$ related by Eqs. (4). A field $\mathscr{C}_n(\mathbf{Z})$ transforming covariantly with respect to these changes of coordinates, $\mathscr{C}'_n(\mathbf{Z}') = (D_\Theta \Theta')^{-n} \mathscr{C}_n(\mathbf{Z})$, is called a quasiprimary field of weight $n/2$.

Obviously, the simplest normalized operator of order $2n+1$ is given by D_Θ^{2n+1}. For this operator, we have the following result which can be proven by induction:

Lemma 4.1 (Super-Bol Lemma): For quasiprimary superfields $\mathscr{C}_{-n}$ of weight $-n/2$, the field $D_\Theta^{2n+1}\mathscr{C}_{-n}$ is quasiprimary of weight $(n+1)/2$, i.e., it transforms under projective changes of coordinates according to

$$(D_\Theta^{2n+1}\mathscr{C}_{-n})' = (D_\Theta \Theta')^{-(n+1)} D_\Theta^{2n+1}\mathscr{C}_{-n}. \tag{14}$$

We now go over from the projective coordinates $\mathbf{Z}$ to generic coordinates $\mathbf{z}$ by a super-conformal transformation $\mathbf{z} \to \mathbf{Z}(\mathbf{z})$. Then, D_Θ^{2n+1} becomes the so-called super-Bol operator $\mathscr{L}_n$ acting on the conformal field $C_{-n}(\mathbf{z})$

$$D_\Theta^{2n+1}\mathscr{C}_{-n} = (D\Theta)^{-(n+1)} \mathscr{L}_n C_{-n}, \quad \text{with} \quad \mathscr{C}_{-n}(\mathbf{Z}) \equiv (D\Theta)^n C_{-n}(\mathbf{z}). \tag{15}$$

In operatorial form, this relation reads

$$\mathscr{L}_n = (D\Theta)^{n+1} \left(\frac{1}{D\Theta} D \right)^{2n+1} (D\Theta)^n$$

$$= [D - nB][D - (n-1)B] \cdots [D + nB], \tag{16}$$

where we introduced the quantity $B = D \ln D\Theta$.

Corollary 4.1: The super-Bol operator $\mathscr{L}_n$ as defined by Eq. (15) represents a normalized and conformally covariant operator of order $2n+1$. It depends only on the superprojective structure, i.e., it depends on B only through the superprojective connection

$$\mathscr{R}(\mathbf{z}) = \mathscr{S}(\mathbf{Z},\mathbf{z}) = \partial B - B D B. \tag{17}$$

The explicit expression for $\mathscr{L}_n$ has the form

$$\mathscr{L}_n = D^{2n+1} + \tfrac{1}{2} n(n+1) \mathscr{R} D^{2n-2} + \tfrac{1}{6} n(n^2-1)(D\mathscr{R}) D^{2n-3}$$

$$+ \tfrac{1}{3} n(n^2-1)(\partial \mathscr{R}) D^{2n-4} + \tfrac{1}{12} n(n^2-1)(n-2)(D^3\mathscr{R}) D^{2n-5} + \tfrac{1}{8} n(n^2-1)(n-2)$$

$$\times [(\partial^2 \mathscr{R}) + \tfrac{1}{3}(2n+3)\mathscr{R}(D\mathscr{R})] D^{2n-6} + \cdots. \tag{18}$$

For later reference, we display the first few super-Bol operators

$$\mathcal{L}_0 = D,$$

$$\mathcal{L}_1 = D^3 + \mathcal{R},$$

$$\mathcal{L}_2 = D^5 + 3\mathcal{R}\partial + (D\mathcal{R})D + 2(\partial\mathcal{R}), \tag{19}$$

$$\mathcal{L}_3 = D^7 + 6\mathcal{R}\partial^2 + 4(D\mathcal{R})D^3 + 8(\partial\mathcal{R})\partial + 2(D^3\mathcal{R})D + 3[\partial^2\mathcal{R} + 3\mathcal{R}\,D\mathcal{R}],$$

$$\mathcal{L}_4 = D^9 + 10\mathcal{R}\partial^3 + 10(D\mathcal{R})D^5 + 20(\partial\mathcal{R})\partial^2 + 10(D^3\mathcal{R})D^3 + 5[3\partial^2\mathcal{R} + 11\mathcal{R}\,D\mathcal{R}]\partial$$

$$+ [3D^5\mathcal{R} + 9(D\mathcal{R})^2 + (\partial\mathcal{R})\mathcal{R}]D + 4[\partial^3\mathcal{R} + 7\mathcal{R}\,D^3\mathcal{R} + 9(D\mathcal{R})(\partial\mathcal{R})].$$

Methods for constructing $\mathcal{L}_n$ and further properties of these operators are given in Ref. 3 and in Sec. VIII below. Here, we only note the following. Under a conformal change of **z**, the field B transforms like a superaffine connection

$$B_{\theta'}(\mathbf{z}') = e^w[B_\theta(\mathbf{z}) + Dw]. \tag{20}$$

Henceforth, $[D' - pB']C'_p = e^{(p+1)w}[D - pB]C_p$ where C_p has conformal weight $p/2$ and thereby we can locally define a supercovariant derivative

$$\nabla_{(p)} : \mathcal{F}_p \to \mathcal{F}_{p+1},$$

$$C_p \mapsto \nabla_{(p)}C_p = [D - pB]C_p. \tag{21}$$

Writing $\nabla^l_{(p)} \equiv \nabla_{(p+l-1)} \cdots \nabla_{(p+1)}\nabla_{(p)}$, the factorization equation (16) for the super-Bol operators reads

$$\mathcal{L}_n = \nabla^{2n+1}_{(-n)}. \tag{22}$$

It should be emphasized that superprojective structures exist on compact SRS's of any genus and that superprojective connections are globally defined on such spaces.[3] Thereby, the lemma and corollary stated above also hold there. By contrast, the covariant derivative defined by Eq. (21) only exists locally. In fact, the quantity B is not invariant under superprojective changes of the coordinate **Z**; under a conformal change of **z**, the field B transforms like a superaffine connection (these quantities only exist globally on SRS's of genus one). Nevertheless, the local definition (21) is useful at intermediate stages[5] and we will use it again in the next section.

V. EXAMPLE 2: OPERATORS PARAMETRIZED BY CONFORMAL FIELDS

Consider a fixed $n \in \mathbf{N}$ and some tensors $W_k \in \mathcal{F}_k$ (with $1 \leqslant k \leqslant 2n+1$). In analogy to the expression (22), we can locally introduce covariant operators $M^{(n)}_{W_k}$ of order $2n+1-k$ in terms of the fields W_k and the covariant derivative (21): we define them by

$$M^{(n)}_{W_k} : \mathcal{F}_{-n} \to \mathcal{F}_{n+1},$$

$$C_{-n} \mapsto \sum_{l=0}^{2n+1-k} \beta^{(n)}_{kl}(\nabla^l_{(k)}W_k)\nabla^{2n+1-k-l}_{(-n)}C_{-n}, \tag{23}$$

where $\beta_{kl}^{(n)}$ denote some complex numbers. These operators are linear in W_k and its derivatives up to order $2n+1-k$. Under a superconformal change of coordinates they transform as $M_{W_k}^{(n)} \to e^{(n+1)w} M_{W_k}^{(n)} e^{nw}$.

Proposition 5.1: There exist numerical coefficients $\beta_{kl}^{(n)}$, normalized to $\beta_{k0}^{(n)}=1$, such that the operators $M_{W_k}^{(n)}$ depend on B only through the superprojective connection $\mathcal{R} = \partial B - BDB$. The coefficients are explicitly given by the following expressions where we distinguish between k,l, even and odd:

$$\beta_{2k,2l}^{(n)} = \frac{\binom{n-k}{l}\binom{k+l-1}{l}}{\binom{2k+l-1}{l}}, \quad \beta_{2k,2l+1}^{(n)} = \frac{\binom{n-k}{l}\binom{k+l}{l+1}}{\binom{2k+l}{l+1}}, \quad (1 \leqslant k \leqslant n),$$

$$(24)$$

$$\beta_{2k+1,2l}^{(n)} = \frac{\binom{n-k}{l}\binom{k+l}{l}}{\binom{2k+l}{l}}, \quad \beta_{2k+1,2l+1}^{(n)} = \frac{\binom{n-k}{l+1}\binom{k+l}{l}}{\binom{2k+l+1}{l+1}}, \quad (0 \leqslant k \leqslant n).$$

The proof is by construction and uses a variational argument:[8,5] we vary B subject to the condition $\delta \mathcal{R} = 0$ and require $\delta M_{W_k}^{(n)} = 0$. The condition $\delta \mathcal{R} = 0$ implies

$$(\partial \delta B) = (\delta B)(DB) + B(D\delta B). \tag{25}$$

To proceed further, we need the following results. From Eq. (25) and the definition of the covariant derivative, one can derive the following operatorial relations:

$$\nabla_{(p)}(\delta B) = -(\delta B)\nabla_{(p-1)} + (\nabla_{(1)}\delta B),$$

$$\nabla_{(p)}(\nabla_{(1)}\delta B) = (\nabla_{(1)}\delta B)\nabla_{(p-2)}, \tag{26}$$

$$\nabla_{(p+1)}\nabla_{(p)}(\delta B) = (\delta B)\nabla_{(p)}\nabla_{(p-1)}.$$

Using these relations, one shows that

$$\delta\nabla_{(p)}^l = d_l^{(p)}(\delta B)\nabla_{(p)}^{l-1} + b_l^{(p)}(\nabla_{(1)}\delta B)\nabla_{(p)}^{l-2}, \tag{27}$$

with coefficients

$$d_{2n}^{(p)} = -n, \quad b_{2n}^{(p)} = -n(p+n-1),$$

$$(28)$$

$$d_{2n+1}^{(p)} = -(p+n), \quad b_{2n+1}^{(p)} = -n(p+n).$$

Indeed, one has $\delta\nabla_{(p)} = -p\delta B$ (i.e., $d_1^{(p)} = -p, b_1^{(p)} = 0$) and Eq. (27) is then proved by induction: one is led to the recursion relations $d_{l+1}^{(p)} = -(d_l^{(p)}+p+l)$ and $b_{l+1}^{(p)} = d_l^{(p)}+b_l^{(p)}$ with solutions given by Eqs. (28).

Now, we are ready to prove the proposition. Variation of the operator $M_{W_k}^{(n)}$ leads to the recursion relations

(a) $\beta_{k,l+1}^{(n)} d_{l+1}^{(k)} + (-)^{k+l} \beta_{k,l}^{(n)} d_{2n+1-k-l}^{(-n)} = 0$,   for $0 \leqslant l \leqslant 2n-k$,

(b) $\beta_{k,l+2}^{(n)} b_{l+2}^{(k)} + \beta_{k,l}^{(n)} b_{2n+1-k-l}^{(-n)} = 0$,   for $0 \leqslant l \leqslant 2n-k-1$,

with $\beta_{k,0}^{(n)} = 1$. One first shows that (b) follows from (a) by using the explicit form of the coefficients $d_l^{(p)}$ and $b_l^{(p)}$ given by Eqs. (28). Then, one solves (a) and finds the solution (24).

□

Corollary 5.1: Let $\mathscr{W}_k$ (with $1 \leqslant k \leqslant 2n+1$) be a quasiprimary field of weight k/2. Then, the operator $M_{\mathscr{W}_k}^{(n)}$, as defined by its action on a quasiprimary field, $\mathscr{C}_{-n}$,

$$M_{\mathscr{W}_k}^{(n)} \mathscr{C}_{-n} \equiv \sum_{l=0}^{2n+1-k} \beta_{kl}^{(n)} (D_\Theta^l \mathscr{W}_k) D_\Theta^{2n+1-k-l} \mathscr{C}_{-n}, \qquad (29)$$

with $\beta_{kl}^{(n)}$ given by Eq. (24), transforms linearly under a superprojective change of coordinates

$$(M_{\mathscr{W}_k}^{(n)} \mathscr{C}_{-n})' = (D_\Theta \Theta')^{-(n+1)} (M_{\mathscr{W}_k}^{(n)} \mathscr{C}_{-n}). \qquad (30)$$

This operator is related to the operator $M_{W_k}^{(n)}$ by a change of variables in analogy to Eq. (15)

$$M_{\mathscr{W}_k}^{(n)} \mathscr{C}_{-n} = (D\Theta)^{-(n+1)} M_{W_k}^{(n)} C_{-n}, \quad \text{with} \quad \mathscr{W}_k(\mathbf{Z}) \equiv (D\Theta)^{-k} W_k(\mathbf{z}).$$

As an illustration of our results, we give explicit expressions for the simplest $M_{W_k}^{(n)}$

$$M_{W_{2n+1}}^{(n)} = W_{2n+1}, \quad M_{W_{2n}}^{(n)} = W_{2n}D + \tfrac{1}{2}(DW_{2n}),$$

$$M_{W_{2n-1}}^{(n)} = W_{2n-1}\partial + \frac{1}{2n-1}(DW_{2n-1})D + \frac{n}{2n-1}(\partial W_{2n-1}), \qquad (31)$$

$$M_{W_{2n-2}}^{(n)} = W_{2n-2}D^3 + \tfrac{1}{2}(DW_{2n-2})\partial + \tfrac{1}{2}(\partial W_{2n-2})D + \frac{n}{2(2n-1)}(D^3 W_{2n-2}) + \frac{n^2}{2n-1} W_{2n-2}\mathscr{R}.$$

VI. CLASSIFICATION OF COVARIANT OPERATORS

By adding the covariant operators considered in the last two sections, we obtain the operator (1) which is again covariant and of order $2n+1$. In fact, the procedure leading to the expression (1) can be reversed to show that all normalized linear differential operators of order $2n+1$ which are superholomorphic and superconformally covariant can be cast into the form (1) with $W_1 = 0 = W_3$. In particular, this means that all normalized and conformally covariant operators come from Möbius covariant ones.

Theorem 6.1 (Classification Theorem): *Let Eq. (7) be the local form of a normalized differential operator of order $2n+1$. According to Lemma 3.1, there exist transformation laws for the coefficients a_k (under superconformal changes of coordinates) such that $\mathscr{L}^{(n)}: \mathscr{F}_{-n} \to \mathscr{F}_{n+1}$. If a_k are chosen to transform in this way, one can find a reparametrization of these coefficients in terms of superconformal fields $W_k \in \mathscr{F}_k$ (with $1 \leqslant k \leqslant 2n+1$) such that $\mathscr{L}^{(n)}$ is given by expression (1).*

W_k are polynomials in a_k and their derivatives. These relations are invertible and allow one to express a_k as differential polynomials in $\mathscr{R}$ and in W_k (with coefficients that are differential polynomials in $\mathscr{R}$).

To summarize, any holomorphic and covariant operator of the form (7) can be parametrized by a superprojective connection $\mathscr{R}$ and $2n+1$ superconformal fields $W_1,...,W_{2n+1}$ (which are differential polynomials in a_k).

The proof is by construction. In fact, with Eq. (12)

$$a_3 = \tfrac{1}{2} n(n+1)\mathscr{R} + \tfrac{1}{2} D a_2 \qquad (32)$$

the W_k are easily found in terms of a_k by setting to zero the coefficient of D^{2n+1-k} in $\mathscr{L} - \mathscr{L}^{(n)}$. For the first seven fields, we find

$$W_1 = 0, \quad W_2 = a_2, \quad W_3 = 0,$$

$$(33)$$

$$W_4 = a_4 - \tfrac{1}{6} n(n^2-1) D\mathscr{R} - \tfrac{1}{2}(n-1)\partial a_2,$$

$$W_5 = a_5 - \tfrac{1}{2} D a_4 - \tfrac{1}{12}(n-1) D^3 a_2 - \tfrac{1}{4} n(n^2-1)\partial\mathscr{R} - \tfrac{1}{6}(n-1)(3n+2)\mathscr{R} a_2,$$

$$W_6 = a_6 - \tfrac{1}{5}(n-2) D a_5 - \tfrac{2}{5}(n-2)\partial a_4 + \tfrac{1}{10}(n-1)(n-2)\partial^2 a_2 - \tfrac{1}{20}(n-1)(n-2)(2n+3)\mathscr{R} D a_2$$

$$- \tfrac{1}{30}(n-1)(n-2)(2n+3)(D\mathscr{R}) a_2 + \tfrac{1}{20} n(n-2)(n^2-1) D^3\mathscr{R},$$

$$W_7 = a_7 - \tfrac{1}{2} D a_6 - \tfrac{1}{2}(n-2)\partial a_5 + \tfrac{1}{5}(n-2) D^3 a_4 + \tfrac{1}{40}(n-1)(n-2) D^5 a_2 - \tfrac{1}{10}(n-2)(5n+3)\mathscr{R} a_4$$

$$+ \tfrac{1}{20}(n-1)(n-2)(5n+3)\mathscr{R}\partial a_2 + \tfrac{1}{20} n(n-2)(n^2-1)\partial^2\mathscr{R} - \tfrac{3}{40} n(n-2)(n^2-1)\mathscr{R} D\mathscr{R}.$$

These fields transform covariantly.

VII. COVARIANT BILINEAR OPERATORS

The expression $M_{W_k}^{(n)} C_{-n}$ can be viewed as the result of a bilinear and covariant map J, i.e., $M_{W_k}^{(n)} C_{-n} \propto J(W_k, C_{-n})$. The mapping J represents the graded extension of Gordan's transvectant.[1,2,5] It will reappear in the next sections in the context of the matrix representation for linear covariant operators and in the Poisson brackets of super-W-algebras.

To define this extension, we proceed as for the definition of $M_{W_k}^{(n)}$. First, we note that for any $\mu, \nu, m \in \mathbf{Z}$, the map

$$J_{\mu\nu}^m : \mathscr{F}_\mu \otimes \mathscr{F}_\nu \to \mathscr{F}_{\mu+\nu+m},$$

$$(34)$$

$$(F,G) \mapsto \sum_{l=0}^{m} \gamma_l^m(\mu,\nu)(\nabla_{(\mu)}^{m-l} F)(\nabla_{(\nu)}^l G)$$

[with $\gamma_l^m(\mu,\nu)$ denoting a numerical factor] is bilinear and covariant. We then have

Proposition 7.1: There exist numerical coefficients $\gamma_l^m(\mu,\nu)$ such that the operator $J_{\mu\nu}^m$ depends on B only through the superprojective connection $\mathscr{R} = \partial B - BDB$. These coefficients are explicitly given by

$$\gamma_{2l}^{2m}(\mu,\nu) = (-1)^l \binom{m}{l} \frac{1}{(\mu)_{m-l}(\nu)_l}, \quad \gamma_{2l+1}^{2m}(\mu,\nu) = (-1)^{l+\mu}\binom{m-1}{l}\frac{m}{(\mu)_{m-l}(\nu)_{l+1}},$$

$$\gamma_{2l}^{2m+1}(\mu,\nu) = (-1)^l\binom{m}{l}\frac{1}{(\mu)_{m+1-l}(\nu)_l}, \quad \gamma_{2l+1}^{2m+1}(\mu,\nu) = -(-1)^{l+\mu}\binom{m}{l}\frac{1}{(\mu)_{m-l}(\nu)_{l+1}},$$

$$(35)$$

where we made use of the Pochammer symbol

$$(r)_0 \equiv 1, \quad (r)_l = r(r+1)\cdots(r+l-1).$$

The proof proceeds along the lines of the proof of Proposition 5.1: the coefficients are determined by requiring that $\delta J_{\mu\nu}^m(F,G) = 0$ for variations δB satisfying $\delta\mathscr{R} = 0$. From Eq. (27), we get the following recursion relations for the coefficients $\gamma \equiv \gamma(\mu,\nu)$:

$$\gamma_{2l}^{2m} = -(-1)^\mu \frac{m+\mu-l}{l}\gamma_{2l-1}^{2m}, \quad \gamma_{2l+1}^{2m} = (-1)^\mu\frac{m-l}{l+\nu}\gamma_{2l}^{2m},$$

$$\gamma_{2l}^{2m+1} = (-1)^\mu\frac{m-l+1}{l}\gamma_{2l-1}^{2m+1}, \quad \gamma_{2l+1}^{2m+1} = -(-1)^\mu\frac{m+\mu-l}{\nu+l}\gamma_{2l}^{2m+1}.$$

Using a convenient choice for γ_0, one is led to the solution (35).   □

The last proposition can be reformulated by saying that

$$J_{\mu\nu}^m(\mathscr{F},\mathscr{G}) \equiv \sum_{l=0}^m \gamma_l^m(\mu,\nu)(D_{\ominus}^{m-l}\mathscr{F})(D_{\ominus}^l\mathscr{G}) \tag{36}$$

[with γ given by Eqs. (35)] is a quasiprimary superfield of weight $\frac{1}{2}(\mu+\nu+m)$ if $\mathscr{F}$ and $\mathscr{G}$ are quasiprimary of weight $\mu/2$ and $\nu/2$, respectively. In other words, the operator (36) is bicovariant with respect to super-Möbius transformations.

The super-Gordan transvectant [i.e., the mapping $J_{\mu\nu}^m$ with γ given by Eqs. (35)] has the symmetry properties

$$J_{\mu\nu}^{2m}(F,G) = (-1)^{m+\mu\nu}J_{\nu\mu}^{2m}(G,F),$$

$$J_{\mu\nu}^{2m+1}(F,G) = -(-1)^{m+\mu\nu}J_{\nu\mu}^{2m+1}(G,F). \tag{37}$$

As a simple example, we consider $\mu=2=\nu$, $m=3$, and $F=G\equiv V\in\mathscr{F}_2$ for which we get

$$6J_{2,2}^3(V,V) = 2V(\nabla_{(2)}^3 V) - 3(\nabla_{(2)}V)(\nabla_{(2)}^2 V)$$

$$= 2V(D^3 V) - 3(DV)(\partial V) - 4\mathscr{R}V^2. \tag{38}$$

As noted at the beginning of this section, the quantity $M_{W_k}^{(n)}C_{-n}$ is a special case of $J_{\mu\nu}^m(F,G)$: in fact, for

$$\mu=k, \quad \nu=-n, \quad m=2n+1-k, \quad F=W_k, \quad G=C_{-n}$$

we find

$$J_{\mu\nu}^m(F,G) = \begin{cases} \dfrac{(k-1)!}{n!} M_{W_{2k}}^{(n)} C_{-n}, & \text{for} \quad \mu = 2k \\[2ex] \dfrac{k!}{n!} M_{W_{2k+1}}^{(n)} C_{-n}, & \text{for} \quad \mu = 2k+1. \end{cases} \tag{39}$$

VIII. MATRIX REPRESENTATION

The differential equation $F_{2n+1} = \mathcal{L}^{(n)} f_{2n+1}$ which is of order $2n+1$ is equivalent to a system of $2n+1$ first-order equations. The latter system can be cast into matrix form which provides an elegant and efficient way for determining explicit expressions for the covariant operators. Moreover, the matrix representation exhibits most clearly the underlying algebraic structure which is due to the covariance with respect to super-Möbius transformations.

Let us first discuss the case of the super-Bol operator $\mathcal{L}_n$. We consider it in its factorized form [see Eq. (22)]

$$\mathcal{L}_n = (D - nB) \cdots (D + nB), \tag{40}$$

where B represents a superaffine connection. For $n = 0, 1, \ldots$, the scalar equation

$$F_{2n+1} = \mathcal{L}_n f_{2n+1}, \quad \text{with} \quad f_{2n+1} \in \mathcal{F}_{-n} \tag{41}$$

is equivalent to the $(2n+1) \times (2n+1)$ matrix equation

$$\mathbf{F} = \tilde{Q}_n \mathbf{f}, \tag{42}$$

with $\mathbf{F} = (F_{2n+1}, 0, \ldots, 0)^t$ and $\mathbf{f} = (f_1, \ldots, f_{2n+1})^t$. Here,

$$\tilde{Q}_n = -J_- + D\mathbf{1} - BH, \tag{43}$$

where $\mathbf{1}$ stands for the unit matrix and

$$J_- = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} n & 0 & \cdots & \cdots & 0 \\ 0 & n-1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -n \end{pmatrix}. \tag{44}$$

The matrices J_- and H satisfy

$$[H, J_-] = -J_-. \tag{45}$$

Together with the upper triangular matrix

$$J_+ = \begin{pmatrix} 0 & n & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & -1 & & & & \vdots \\ \vdots & & \ddots & n-1 & & & \vdots \\ \vdots & & & \ddots & -2 & & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & -n \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \tag{46}$$

which satisfies

$$[H, J_+] = J_+, \quad \{J_+, J_-\} = H, \tag{47}$$

they generate an $\mathrm{osp}(1|2)$ algebra. In fact,[9] these matrices represent the superprincipal embedding $\mathrm{osp}(1|2)_{\mathrm{pal}}$ of $\mathrm{osp}(1|2)$ into the superalgebra $\mathrm{sl}(\underline{n}+1|n)$. For an elaboration on the algebraic structure, we refer to the Appendix. The matrix $\widetilde{Q}_n$ can be cast into a canonical form Q_n by conjugation with a group element $N \in \mathrm{OSP}(1|2)_{\mathrm{pal}} \subset \mathrm{SL}(n+1|n)$. This makes the dependence on the superprojective structure manifest:

Theorem 8.1 (Matrix representation for the super-Bol operator): *The scalar operator $\mathscr{L}_n$ is equivalent to the matrix operator Q_n defined by*

$$Q_n \equiv \hat{N}^{-1} \widetilde{Q}_n N = -J_- + D\mathbf{1} - \mathscr{R} J_+^2. \tag{48}$$

Here, $\mathscr{R} = \partial B - B(DB)$ represents a superprojective connection and

$$N = \exp\{-BJ_+ - (DB)J_+^2\}, \quad \hat{N} = \exp\{+BJ_+ - (DB)J_+^2\}. \tag{49}$$

As for the proof, we first note that J_+ (J_+^2) is odd (even) with respect to the $\mathbf{Z}_2$-grading $i+j \pmod 2$ of matrix elements defined in the Appendix. In Eq. (49), these generators are multiplied with odd (even) parameters and therefore the corresponding expressions represent well-defined elements of the supergroup $\mathrm{SL}(n+1|n)$.

The element $\hat{N} \equiv \exp\{\hat{M}\}$ follows from $N \equiv \exp\{M\}$ by changing the sign of the anticommuting part of the algebra element M. (This operation represents an automorphism of the superalgebra.[9]) The consideration of $\hat{N}$ is necessary in the conjugation (48), because the operator $\widetilde{Q}_n$ has a grading different from the one considered for the superalgebra (e.g., $\widetilde{Q}_n$ contains odd elements on the diagonal).

The result (48) is a simple consequence of the relations

$$\hat{N}^{-1} J_- N = J_- - BH - (DB)J_+ + 2B(DB)J_+^2,$$

$$\hat{N}^{-1} D N = D\mathbf{1} - (DB)J_+ + [B(DB) - (\partial B)]J_+^2,$$

$$\hat{N}^{-1} BHN = B[H - BJ_+ - 2(DB)J_+^2],$$

which follow from Eqs. (45) and (47).

The matrix (48) still describes the operator $\mathscr{L}_n$.[10] In fact, one can transform an equation of the form (42) by upper triangular matrices

$$N(\mathbf{z}) = \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & * \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

which leads to the equation

$$\mathbf{F}' = \widetilde{Q}_n' \mathbf{f}', \tag{50}$$

with $\mathbf{F}' \equiv \hat{N} \mathbf{F}$, $\mathbf{f}' \equiv N\mathbf{f}$, and $\widetilde{Q}_n' \equiv \hat{N} \widetilde{Q}_n N^{-1}$. Since $F'_{2n+1} = F_{2n+1}$ and $f'_{2n+1} = f_{2n+1}$, the matrix equation (50) is equivalent to the scalar equation (41), i.e., Q_n' still describes the operator $\mathscr{L}_n$.
$\square$

To conclude our discussion of the super-Bol operator, we remark that the matrix representation (48) for $\mathscr{L}_n$ is equivalent to a representation in terms of an $(n+1)\times(n+1)$ matrix involving $D\mathscr{R}$ and the operators D, ∂ as given in Ref. 3.

Next, we consider the most general covariant operators $\mathscr{L}^{(n)}$ as discussed in Secs. III and VI.

Theorem 8.2 (Matrix representation for covariant operators): *For $n=0,1,...,$ the most general normalized and covariant operator of order $2n+1$ is given by the scalar equation*

$$F_{2n+1}=\mathscr{L}^{(n)}f_{2n+1}, \quad \text{with} \quad f_{2n+1}\in\mathscr{F}_{-n}, \tag{51}$$

which is obtained after elimination of $f_1,...,f_{2n}$ from the matrix equation

$$(F_{2n+1},0,...,0)^t=Q^{(n)}(f_1,...,f_{2n+1})^t. \tag{52}$$

Here, $Q^{(n)}$ is the $(2n+1)\times(2n+1)$ matrix

$$Q^{(n)}=-J_-+D\mathbf{1}+\sum_{k=1}^{2n}V_{k+1}M_k, \tag{53}$$

where $V_3\equiv\mathscr{R}$, $V_k\in\mathscr{F}_k$ for $k=2$ and for $4\leqslant k\leqslant 2n+1$ and $M_k=(M_1)^k$ with

$$M_1=\begin{pmatrix} 0 & n & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & 1 & & & & \vdots \\ \vdots & & \ddots & n-1 & & & \vdots \\ \vdots & & & \ddots & 2 & & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & n \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}. \tag{54}$$

A few comments concerning this result and its interpretation are in order. First, it should be noted that the representation (48) for the super-Bol operator is a special case of the representation (53), since $J_+^2=-(M_1)^2$.

The representation (53) obtained here as a generalization of Eq. (48) is analogous to the results based on the work of Drinfel'd and Sokolov and on constrained Wess–Zumino–Witten models (see Refs. 11, 5, 6, 9, and references therein). The superconformal fields V_k are introduced in the matrix representation along with the highest weight generators of the $osp(1|2)_{pal}$ subalgebra of $sl(n+1|n)$. These are the generators M_p which satisfy the graded commutation relation

$$[J_+,M_p\}=0,$$

where M_p is characterized by its H eigenvalue

$$[H,M_p]=pM_p.$$

The smallest value is $p=1$ and one easily finds for M_1 the matrix (54) as a solution of the previous equations. The integer powers of M_1 still belong to the superalgebra and one readily shows that

$$[H,(M_1)^p\}=p(M_1)^p, \quad [J_+,(M_1)^p\}=0 \tag{55}$$

from which we conclude that $M_p=(M_1)^p$.

In conclusion, we note that the superconformal fields V_k occurring in Eq. (53) represent a parametrization of the operator $\mathscr{L}^{(n)}$ which is equivalent to the one in terms of W_k discussed in Sec. VI: the two sets of superfields are related to each other by differential polynomials. For concreteness, we illustrate the situation for $n=3$. In this case, Eq. (53) yields

$$\mathscr{L}^{(3)} = \mathscr{L}_3 + M^{(3)}_{W_2} + \cdots + M^{(3)}_{W_7},$$

where $\mathscr{L}_3$ is the super-Bol operator and $M^{(3)}_{W_k}$ the operators of Sec. V with W_k depending on V_k and their derivatives according to

$$W_2 = 12V_2, \quad W_3 = 0, \quad W_4 = 10V_4 + 44(V_2)^2, \quad W_5 = 5V_5,$$

$$W_6 = 2V_6 + 48(V_2)^3 + 36V_2V_4, \tag{56}$$

$$W_7 = V_7 + 18V_2V_5 + \tfrac{9}{5}[2V_2(D^3V_2) - 3(DV_2)(\partial V_2) - 4\mathscr{R}(V_2)^2].$$

Up to an overall factor, the expression in brackets coincides with the super-Gordan transvectant $J^3_{2,2}$ applied to the pair of fields (V_2,V_2), see Eq. (38). Obviously, $V_k \in \mathscr{F}_k$ implies $W_k \in \mathscr{F}_k$ and the last set of relations is invertible since the leading term of W_k is always given by V_k.

For later reference, we also summarize the results for $n=1$ and $n=2$. In these cases, one finds, respectively, $\mathscr{L}^{(1)} = \mathscr{L}_1 + M^{(1)}_{W_2}$ with

$$W_2 = 2V_2 \tag{57}$$

and $\mathscr{L}^{(2)} = \mathscr{L}_2 + M^{(2)}_{W_2} + \cdots + M^{(2)}_{W_5}$ with

$$W_2 = 6V_2, \quad W_3 = 0, \quad W_4 = 4V_4 + 8(V_2)^2, \quad W_5 = 4V_5. \tag{58}$$

For any value of n, we have $W_2 = n(n+1)V_2$ where the coefficient $n(n+1)$ represents the sum of elements of the matrix M_1.

IX. CLASSICAL SUPER-W-ALGEBRAS

Classical super-W_n-algebras represent nonlinear extensions of the classical super-Virasoro algebra. We recall that the latter is generated by the superstress tensor $\mathscr{T}$ which transforms like a superprojective connection $\mathscr{R}$ or rather like the combination (12)

$$\tilde{a}^{(n)}_3 \equiv a^{(n)}_3 - \tfrac{1}{2}Da^{(n)}_2 = \iota_n\mathscr{R}, \quad \text{with} \quad \iota_n \equiv \tfrac{1}{2}n(n+1), \tag{59}$$

which involves the coefficients $a^{(n)}_2$ and $a^{(n)}_3$ of a normalized covariant operator of order $2n+1$. The Poisson bracket defining the super-Virasoro algebra is then given by[12]

$$\{\tilde{a}^{(n)}_3(z'), \tilde{a}^{(n)}_3(z)\} = \tfrac{1}{2}[\iota_n D^5 + 3\tilde{a}^{(n)}_3\partial + (D\tilde{a}^{(n)}_3)D + 2(\partial\tilde{a}^{(n)}_3)]\delta(z-z'), \tag{60}$$

where $\delta(z-z') = (\theta-\theta')\delta(z-z')$. By substituting $\tilde{a}^{(n)}_3 = \iota_n\mathscr{R}$ into this relation, we see that the operator on the rhs coincides with $\tfrac{1}{2}\iota_n\mathscr{L}_2$ where $\mathscr{L}_2$ is the super-Bol operator (19).

Arguing along the lines of the bosonic theory,[4-6] one can say that the super-W_n-algebra is generated by the superstress tensor $\tilde{a}_3^{(n)}$ and the superconformal fields W_k (with $2 \leqslant k \leqslant 2n+1$) which parametrize covariant operators of order $2n+1$ according to Sec. VI. It follows from the results of Gel'fand and Dickey and their generalization to odd superdifferential operators[13] that the Poisson brackets between these generators form a closed algebra. For the $n=1$ and $n=2$ cases, these brackets have recently been constructed[14] (see also Refs. 15, 16). Starting from normalized covariant operators $\mathscr{L}^{(n)}$, the authors of Refs. 14 and 15 found that these brackets take the simplest form if they are written in terms of superconformal fields V_k which are specific differential polynomials in the coefficients $a_k^{(n)}$ of $\mathscr{L}^{(n)}$. For the examples studied, these combinations are precisely the combinations V_k that we encountered for the matrix representation of covariant operators in Sec. VIII. Thus, these fields seem to be better suited for the parametrization of super-W-algebras than the combinations W_k discussed in Sec. VI for the classification of covariant operators. In any case, our results provide a general method for determining these combinations.

Our results admit a further application to the formulation of super-W-algebras. The $n=1$ and $n=2$ Poisson brackets derived in Ref. 14 by virtue of a long and tedious calculation involve a large number of terms which depend on the generators $\tilde{a}_3^{(n)}$ and V_k. All of these contributions can be rewritten in a compact way in terms of the covariant operators constructed in the present work (i.e., $\mathscr{L}_n, M_{W_k}^{(n)}, J_{\mu\nu}^m$) and of some covariant trilinear operators. In the following, we briefly summarize the results of Ref. 14 while emphasizing and elucidating the underlying algebraic structure.

For $n=1$, one starts from the covariant operator

$$\mathscr{L}^{(1)} = D^3 + a_2^{(1)} D + a_3^{(1)}$$

and Eqs. (59),(57),(33) then yield

$$\mathscr{R} = \tilde{a}_3^{(1)} = a_3^{(1)} - \tfrac{1}{2} D a_2^{(1)},$$

$$2V_2 = W_2 = a_2^{(1)}. \tag{61}$$

The superstress tensor $\mathscr{T} \equiv \tilde{a}_3^{(1)}$ and the superconformal field $\mathscr{I} \equiv 2V_2$ satisfy the algebra {in the $\{\mathscr{T}, \mathscr{I}\}$ bracket of Ref. 14, there is an obvious sign error which we have corrected here [cf. Eq. (82) below]}

$$\{\mathscr{I}(z'), \mathscr{I}(z)\} = 2[D^3 + \mathscr{T}] \delta(z-z'),$$

$$\{\mathscr{T}(z'), \mathscr{T}(z)\} = \tfrac{1}{2}[D^5 + 3\mathscr{T}\partial + (D\mathscr{T})D + 2(\partial\mathscr{T})] \delta(z-z'), \tag{62}$$

$$\{\mathscr{T}(z'), \mathscr{I}(z)\} = -[(\partial\mathscr{I}) + \mathscr{I}\partial - \tfrac{1}{2}(D\mathscr{I})D] \delta(z-z').$$

Noting that $\mathscr{T} = \mathscr{R}$ in the present case, we recognize the super-Bol operators $\mathscr{L}_1$ and $\mathscr{L}_2$ on the rhs of the first and second equations, respectively. The operator in the last relation is covariant, since it can be obtained from the super-Gordan transvectant $J_{\mu\nu}^m$ as a linear operator $-2J_{2,-2}^2(\mathscr{I}, \cdot)$. This relation represents the transformation law of the superconformal field $\mathscr{I}$ under a superconformal change of coordinates generated by the stress tensor $\mathscr{T}$. For a field $C_k \in \mathscr{F}_k$, it generalizes to

$$\{\mathscr{T}(z'), C_k(z)\} = \lambda_k J_{k,-2}^2(C_k, \cdot) \delta(z-z'), \quad \text{with} \quad \lambda_k = -(-1)^k k. \tag{63}$$

Component field expressions for $n=1$ follow from the θ-expansions

$$\mathscr{I}(\mathbf{z}) = J(z) + \theta G^2(z), \quad \mathscr{T}(\mathbf{z}) = G^1(z) + \theta T(z), \qquad (64)$$

where $T(z)$ represents the ordinary stress tensor. These component fields together with their commutation relations following from Eqs. (62) define a representation of the $N=2$ super-conformal algebra.

For $n=2$, one considers

$$\mathscr{L}^{(2)} = D^5 + a_2^{(2)} D^3 + a_3^{(2)} D^2 + a_4^{(2)} D + a_5^{(2)}$$

and Eqs. (59),(58),(33) now lead to

$$\mathscr{T} \equiv 3\mathscr{R} = \widetilde{a}_3 = a_3 - \tfrac{1}{2} D a_2,$$

$$\mathscr{I} \equiv 6V_2 \fallingdotseq W_2 = a_2,$$

$$\mathscr{U} \equiv 4V_4 = W_4 - 8(V_2)^2 = a_4 - \tfrac{1}{3} D a_3 - \tfrac{1}{3} \partial a_2 - \tfrac{2}{9}(a_2)^2, \qquad (65)$$

$$\mathscr{W} \equiv 4V_5 \fallingdotseq W_5 = a_5 - \tfrac{1}{2} D a_4 - \tfrac{1}{2} \partial a_3 + \tfrac{1}{6} D^3 a_2 - \tfrac{4}{9} a_2 \widetilde{a}_3,$$

where $a_k \equiv a_k^{(2)}$. (In order to avoid confusion with the Schwarzian derivative, the superfield S of Ref. 14 is being referred to as $\mathscr{U}$.) The $n=2$ super-Poisson brackets read

$$\{\mathscr{I}(\mathbf{z}'),\mathscr{I}(\mathbf{z})\} = 6\mathscr{L}_1 \delta(\mathbf{z}-\mathbf{z}'), \quad \{\mathscr{T}(\mathbf{z}'),\mathscr{T}(\mathbf{z})\} = \tfrac{3}{2}\mathscr{L}_2 \delta(\mathbf{z}-\mathbf{z}'),$$

$$\{\mathscr{T}(\mathbf{z}'),\mathscr{I}(\mathbf{z})\} = -2J_{2,-2}^2(\mathscr{I},\cdot)\delta(\mathbf{z}-\mathbf{z}'), \quad \{\mathscr{T}(\mathbf{z}'),\mathscr{U}(\mathbf{z})\} = -4J_{4,-2}^2(\mathscr{U},\cdot)\delta(\mathbf{z}-\mathbf{z}'),$$

$$\{\mathscr{T}(\mathbf{z}'),\mathscr{W}(\mathbf{z})\} = 5J_{5,-2}^2(\mathscr{W},\cdot)\delta(\mathbf{z}-\mathbf{z}'), \quad \{\mathscr{I}(\mathbf{z}'),\mathscr{W}(\mathbf{z})\} = 2J_{4,-1}^2(\mathscr{U},\cdot)\delta(\mathbf{z}-\mathbf{z}'), \qquad (66)$$

$$\{\mathscr{U}(\mathbf{z}'),\mathscr{U}(\mathbf{z})\} = -[\tfrac{2}{3}\mathscr{L}_3 + \tfrac{1}{27}M_{w_4}^{(3)} - \tfrac{1}{90}M_{w_7}^{(3)}]\delta(\mathbf{z}-\mathbf{z}'), \quad \{\mathscr{U}(\mathbf{z}'),\mathscr{I}(\mathbf{z})\} = 2\mathscr{W}\delta(\mathbf{z}-\mathbf{z}'),$$

$$\{\mathscr{W}(\mathbf{z}'),\mathscr{U}(\mathbf{z})\} = [\tfrac{16}{3}J_{2,-4}^6(\mathscr{I},\cdot) + 20J_{5,-4}^3(\mathscr{W},\cdot) - J_{6,-4}^2(w_6,\cdot) + \tfrac{5}{27}K_{2,4,-4}^2(\mathscr{I},\mathscr{U},\cdot)]$$

$$\times \delta(\mathbf{z}-\mathbf{z}'),$$

$$\{\mathscr{W}(\mathbf{z}'),\mathscr{W}(\mathbf{z})\} = -[\tfrac{1}{6}\mathscr{L}_4 + \tfrac{2}{9}J_{4,-4}^5(w_4,\cdot) + \tfrac{26}{9}J_{7,-4}^2(\mathscr{I}\mathscr{W},\cdot) - \tfrac{5}{126}K_{2,5,-4}^2(\mathscr{I},\mathscr{W},\cdot)$$

$$+ \tfrac{1}{540}K_{2,2,-4}^5(\mathscr{I},\mathscr{I},\cdot)]\delta(\mathbf{z}-\mathbf{z}'),$$

with

$$w_4 = 90\mathscr{U} - 2\mathscr{I}^2, \quad w_6 = \tfrac{8}{81}\mathscr{I}^3 - \tfrac{28}{9}\mathscr{I}\mathscr{U}, \quad w_7 = 20\mathscr{I}\mathscr{W} + 27J_{2,2}^3(\mathscr{I},\mathscr{I}) \qquad (67)$$

and

$$K^2_{2,4,-4}(\mathscr{I},\mathscr{U},\cdot)=\mathscr{I}(\nabla\mathscr{U})\nabla-2(\nabla\mathscr{I})\mathscr{U}\nabla+2(\nabla^2\mathscr{I})\mathscr{U}-\mathscr{I}(\nabla^2\mathscr{U})-\tfrac{3}{2}(\nabla\mathscr{I})(\nabla\mathscr{U}),$$

$$K^2_{2,5,-4}(\mathscr{I},\mathscr{W},\cdot)=4\mathscr{I}(\nabla\mathscr{W})\nabla-10(\nabla\mathscr{I})\mathscr{W}\nabla-5(\nabla^2\mathscr{I})\mathscr{W}+2\mathscr{I}(\nabla^2\mathscr{W})+7(\nabla\mathscr{I})(\nabla\mathscr{W}),$$

$$(68)$$

$$K^5_{2,2,-4}(\mathscr{I},\mathscr{I},\cdot)=47\mathscr{I}(\nabla^3\mathscr{I})\nabla^2-\tfrac{141}{2}(\nabla\mathscr{I})(\nabla^2\mathscr{I})\nabla^2+\mathscr{I}(\nabla^4\mathscr{I})\nabla-\tfrac{3}{2}(\nabla^2\mathscr{I})^2\nabla+\tfrac{45}{2}(\nabla\mathscr{I})$$

$$\times(\nabla^3\mathscr{I})\nabla+24\mathscr{I}(\nabla^5\mathscr{I})-2(\nabla^2\mathscr{I})(\nabla^3\mathscr{I})-46(\nabla\mathscr{I})(\nabla^4\mathscr{I}).$$

In Eqs. (66), the commutator of $\mathscr{T}$ with itself and the one of $\mathscr{T}$ with the other fields represent special cases of the relations (60) and (63), respectively. The operators (68) are examples of trilinear and covariant operators,

$$K^m_{\mu\nu\rho}:\mathscr{F}_\mu\otimes\mathscr{F}_\nu\otimes\mathscr{F}_\rho\to\mathscr{F}_{\mu+\nu+\rho+m}$$

$$(F,G,H)\mapsto K^m_{\mu\nu\rho}(F,G,H),$$

$$(69)$$

which only depend on superconformal fields and on the superprojective connection $\mathscr{R}$ (i.e., on the superprojective structure). Whereas the bilinear operators discussed in Sec. VII are unique (up to an overall normalization), this is in general not the case for multilinear generalizations. For any value of n, the Poisson brackets are at most cubic in the generators $W_k(2\leqslant k\leqslant 2n+1)$ (Ref. 5) and therefore they involve at most quadrilinear covariant operators.

X. PROJECTION TO COMPONENT FIELDS

All superfields $F(\mathbf{z})$ admit a θ-expansion $F(\mathbf{z})=f(z)+\theta\psi(z)$ where the component field f has the same Grassmann parity as F. In particular, the superprojective connection $\mathscr{R}$ can be written as[3]

$$\mathscr{R}_{z\theta}(\mathbf{z})=\frac{i}{2}\rho_{z\theta}(z)+\theta[\tfrac{1}{2}r_{zz}(z)],\qquad(70)$$

where r_{zz} corresponds to an ordinary projective connection on the Riemann surface underlying the SRS.

In order to project down the supercovariant derivative (21) to component field expressions, we note that the superconformal transformation $\mathbf{Z}=\mathbf{Z}(\mathbf{z})$ relating the generic coordinates $\mathbf{z}$ to the projective coordinates $\mathbf{Z}$ satisfies $DZ=\Theta D\Theta$. This equation implies $(D\Theta)^2=\partial Z+\Theta\partial\Theta$ and therefore

$$B\equiv D\ln D\Theta=\tfrac{1}{2}D\ln\partial Z+\tfrac{1}{2}D\left[\frac{\Theta\partial\Theta}{\partial Z}\right].\qquad(71)$$

In the following, we denote the lowest component of a generic superfield F by $F|$. Writing $B=(i/2)\beta+\theta[(1/2)b]$ and using Eq. (71), we get

$$b=\partial\ln\partial Z|+\text{SUSY},\qquad(72)$$

where SUSY stands for the contributions which are due to supersymmetry. By virtue of Eq. (21) we obtain the component field expression

$$(\nabla^2_{(p)}C_p)| = \left[\partial - \frac{p}{2}b\right]C_p| - \frac{i}{2}\beta(DC_p)|$$

$$= \mathscr{D}_{(p/2)}c_{p/2} + \text{SUSY}. \tag{73}$$

Here, $C_p| \equiv c_{p/2}$ represents an ordinary conformal field of weight $p/2$ and $\mathscr{D}_{(p/2)} \equiv \partial - (p/2)(\partial \ln \partial Z|)$ denotes the ordinary covariant derivative acting on such fields (see, e.g., Ref. 5). From the θ expansions of $\mathscr{R}$ and B and from Eqs. (17), (72), we conclude that b and β give rise to the following local form of the projective connection r:

$$r = \partial^2 \ln \partial Z| - \tfrac{1}{2}(\partial \ln \partial Z|)^2 + \text{SUSY}. \tag{74}$$

The projection of the super-Bol operator $\mathscr{L}_n$ has been discussed in Ref. 3 and it was shown that

$$(D\mathscr{L}_{n-1}C_{-(n-1)})| = L_n c_{-(n-1)/2} + \text{SUSY}, \tag{75}$$

where L_n denotes the ordinary Bol operator of order n. For instance, for $n = 3$

$$(D\mathscr{L}_2 C_{-2})| = L_3 c_{-1} + \text{SUSY}, \quad \text{with} \quad L_3 = \partial^3 + 2r\partial + (\partial r).$$

For the operator $M_{W_k}^{(n)}$ introduced in Eq. (23), we can extract the purely bosonic contribution in the same way

$$(DM_{W_{2k}}^{(n-1)}C_{-(n-1)})| = M_{w_k}^{(n)} c_{-(n-1)/2} + \text{SUSY}. \tag{76}$$

Here, $W_{2k}| \equiv w_k$ is an ordinary conformal field of weight k and $M_{w_k}^{(n)}$ represents the known bosonic result[5,2]

$$M_{w_k}^{(n)} = \sum_{l=0}^{n-k} \alpha_{kl}^{(n)} (\mathscr{D}_{(k)}^l w_k) \mathscr{D}_{(-(n-1)/2)}^{n-k-l}, \quad \text{with} \quad \alpha_{kl}^{(n)} = \frac{\binom{k+l-1}{l}\binom{n-k}{l}}{\binom{2k+l-1}{l}} \tag{77}$$

and with $\mathscr{D}$ denoting the covariant derivative introduced in Eq. (73). We note that the derivation of Eq. (76) makes use of the relation

$$\beta_{2k,2l}^{(n-1)} + \beta_{2k,2l-1}^{(n-1)} = \alpha_{kl}^{(n)}, \quad (0 \leqslant l \leqslant n-k-1). \tag{78}$$

Proceeding in the same way for the bilinear operator $J_{2\mu,2\nu}^m$, we find

$$DJ_{2\mu,2\nu}^{2m-1}(F_{2\mu}, G_{2\nu})| = j_{\mu\nu}^m(f_\mu, g_\nu) + \text{SUSY}. \tag{79}$$

Here, $F_{2\mu}| \equiv f_\mu$, $G_{2\nu}| \equiv g_\nu$ are ordinary conformal fields of weight μ and ν, respectively, and j denotes Gordan's transvectant[1,2,5]

$$j_{\mu\nu}^m(f_\mu, g_\nu) = \sum_{l=0}^m \delta_l^m(\mu,\nu)(\mathscr{D}_{(\mu)}^{m-l} f_\mu)(\mathscr{D}_{(\nu)}^l g_\nu), \quad \text{with} \quad \delta_l^m(\mu,\nu) = (-1)^l \binom{m}{l} \frac{1}{(\mu)_{m-l}(\nu)_l}. \tag{80}$$

The result (79) follows from the relation

$$\gamma_{2l}^{2m-1}(2\mu,2\nu) + \gamma_{2l-1}^{2m-1}(2\mu,2\nu) = \delta_l^m(\mu,\nu), \quad (1 \leqslant l \leqslant m-1). \tag{81}$$

By construction, the operators (75), (76), and (79) depend on the quantity $\partial \ln \partial Z |$ only through the projective connection (74).

Component field expressions for the super-Poisson brackets of Sec. IX immediately follow from the θ-expansions. Here, we only note that the transformation law of superconformal fields $C_k(\mathbf{z}) = c_{k/2}(z) + \theta c_{(k+1)/2}(z)$ as described by the superbracket (63) implies the usual transformation law of ordinary conformal fields under conformal changes of coordinates $z \rightarrow z'(z)$. In fact, from Eqs. (63) and (64), it follows that

$$\{T(z'), c_k(z)\} = [(\partial c_k) + k c_k \partial] \delta(z-z')$$

$$= k j_{k,-1}^1(c_k, \cdot) \delta(z-z'), \tag{82}$$

where j denotes Gordan's transvectant (80).

So far we have outlined the projection of our superfield results and we have verified that they encompass the known bosonic expressions. For the presentation of complete component field results, it may be convenient to interpret a superfield $F(\mathbf{z}) = f(z) + \theta \psi(z)$ as a doublet $(f(z), \psi(z))^t$ and to describe the action of a superdifferential operator $\mathscr{L}$ on it by a 2×2 matrix whose elements are ordinary differential operators. This approach was considered in Ref. 16 where some basic aspects of covariance and super-W-algebras have been discussed in terms of this formalism.

XI. CONCLUDING REMARKS

We have discussed superconformally covariant differential operators and their relevance for classical super-W-algebras. Our discussion was restricted to operators of odd degree in D, i.e., $\mathscr{L} = D^{2n+1} + \cdots$. Let us now comment on operators of even degree, i.e., operators of the form $\mathscr{L} = D^{2n} + a_1 D^{2n-1} + a_2 D^{2n-2} + \cdots = \partial^n + a_1 D^{2n-1} + a_2 \partial^{n-1} + \cdots$. One first notices that for these operators it is *not* possible to eliminate the coefficient a_1 of the subleading term in the way it was done for bosonic operators or for the supersymmetric operators discussed in this article. One can however eliminate a_2 and if one requires a_2 to stay zero under superconformal changes of coordinates, one finds that $\mathscr{L}$ has to map $\mathscr{F}_{1-n}$ to $\mathscr{F}_{1+n}$. Yet, if one tries to generalize the super-Bol operators and considers $\nabla_{(\rho)}^{2n}$, one finds for any choice of ρ that this expression depends on the connection B not only through the superprojective connection $\mathscr{R}$, but also through other differential polynomials in B. This means in particular that the super-Bol lemma does not hold in this case and that the analysis presented in this article does not carry over directly to normalized superdifferential operators of even degree. They do not seem to be relevant for classical super-W-algebras (see also Refs. 16, 17), but, interestingly enough, the even operator $\mathscr{L}^{(n)} D$ (where $\mathscr{L}^{(n)} = D^{2n+1} + \cdots$ is a normalized covariant operator) represents the Lax operator for the generalized $N=2$ super-Korteweg–de Vries (KdV) hierarchy.[15]

Our discussion of differential operators and classical W-algebras has been based on $N=1$ superconformal fields. The super-W-algebras considered in Sec. IX are however $N=2$ algebras. For instance, in the $n=1$ case, one has the $N=2$ super-Virasoro algebra involving the super-stress tensor and the superfield a_2 which contains the $U(1)$ current J and the second super-charge G^2 as component fields. As mentioned in Sec. III, we can consistently set a_2 to zero which reduces the symmetry algebra to $N=1$. Thus, the $N=1$ formalism used here encompasses both the $N=1$ and $N=2$ cases. To exhibit the $N=2$ structure more clearly for $a_2 \neq 0$, one could use an $N=2$ superfield formalism.[18] In that case, the osp(1|2) algebra should be replaced by an sl(2|1) algebra according to the discussion of the spin content of super-W-algebras in Ref. 9.

As we were finishing this work, we became aware of Ref. 19 where a general formula for the basic singular vectors of the Neveu–Schwarz (NS) algebra was presented. If written in terms of matrices, this formula uses the same representation of osp(1|2) as the one we encountered in Sec. VIII. In terms of our notation in Sec. VIII, the main result of Ref. 19 reads

$$\mathbf{F} = \left[-J_- + \sum_{p=0}^{n} \binom{2p}{p} G_{-(2p+1)/2} \left(\frac{t}{2} \right)^p J_+^{2p} \right] \mathbf{f}. \tag{83}$$

Here, $f_1, \dots, f_{2n+1}$ and F_{2n+1} belong to the Verma module $V_{(c,h)}^{NS}$ of the NS algebra; more specifically, $f_{2n+1} \equiv |h\rangle$ represents a highest weight vector and $F_{2n+1} \equiv |\psi_{1,q}\rangle$ a singular vector of the fundamental type (i.e., all the others can be obtained from these ones[19]). G_r denotes the fermionic generator of the NS algebra and t a complex parameter.

In conclusion, we mention that some conformally covariant operators on ordinary Riemann surfaces (like Bol operators) admit a natural generalization to higher dimensional Riemannian manifolds.[20] The corresponding covariant operators can be constructed by virtue of a matrix representation or by using homomorphisms of Verma modules. Superconformally covariant operators as discussed in Ref. 3 and in the present work should allow for an analogous generalization to higher dimensional supermanifolds and our results should provide the appropriate basis for their construction.

ACKNOWLEDGMENT

APPENDIX: SOME FACTS ABOUT SUPERALGEBRAS

In this appendix, we collect some general results on superalgebras relevant for our discussion of the algebraic structure of matrix operators in Sec. VIII.

By definition, the superalgebra $sl(n+1|n)$ is the graded algebra of $(2n+1) \times (2n+1)$ matrices M with vanishing supertrace

$$\text{str}_1 M \equiv \sum_{i=1}^{n+1} M_{ii} - \sum_{i=n+2}^{2n+1} M_{ii} = 0. \tag{A1}$$

The matrix M is made up of blocks

$$M = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \tag{A2}$$

with A, B even and C, D odd.

In Sec. VIII, we encountered another representation of $sl(n+1|n)$ which is based on a different definition of the grading and trace. In this representation, one associates a $\mathbf{Z}_2$-grading $i+j$ (mod 2) to the element M'_{ij} of the matrix $M' \in sl(n+1|n)$. Then, there are no more even and odd blocks as in Eq. (A2), rather the elements of a row or column of M' are alternatively odd and even. The supertrace is now given by an alternating sum over the diagonal elements of the matrix

$$\text{str}_2 M' \equiv \sum_{i=1}^{2n+1} (-1)^{i+1} M'_{ii}. \tag{A3}$$

Note that in both expressions (A1) and (A3) the number of plus (minus) signs is $n+1(n)$. The graded commutator is defined by

$$[M',N'\}_{ik} \equiv \sum_{j=1}^{2n+1} (M'_{ij}N'_{jk} - (-1)^{(i+j)(j+k)} N'_{ij}M'_{jk}) \tag{A4}$$

and one has

$$\mathrm{str}_2[M',N'\}=0.$$

The two different representations of $\mathrm{sl}(n+1|n)$ are related by a similarity transformation

$$M'=G^{-1}MG, \tag{A5}$$

where G is a permutation matrix, $G_{ij}=\delta_{iP(j)}$ with

$$P(2i+1)=i+1, \quad \text{for} \quad 0\leqslant i\leqslant n,$$

$$P(2i)=n+i+1, \quad \text{for} \quad 1\leqslant i\leqslant n.$$

In the following, we consider supertraceless matrices of the type M'. To simplify the notation, we will suppress the "prime."

Let E_{ij} denote the matrix

$$(E_{ij})_{kl}=\delta_{ik}\delta_{jl}. \tag{A6}$$

As a basis of the Cartan subalgebra of $\mathrm{sl}(n+1|n)$, we can take the matrices

$$h_i=E_{ii}+E_{i+1,i+1}, \quad \text{for} \quad 1\leqslant i\leqslant 2n,$$

while the simple roots can be chosen to be fermionic and represented by

$$e_i=E_{i,i+1} \quad \text{(positive roots)}, \quad f_i=E_{i+1,i} \quad \text{(negative roots)}.$$

Then, the superprincipal embedding of the graded algebra $\mathrm{osp}(1|2)$ in $\mathrm{sl}(n+1|n)$ is given by $J_- \equiv \Sigma_{i=1}^n f_i$ and H,J_+ [as given by Eqs. (44),(46)] supplemented by the bosonic generators $X_\pm = J_\pm^2$. These matrices satisfy the graded commutation relations (45) and (47) from which it follows that

$$[H,X_\pm]=\pm 2X_\pm, \quad [X_+,X_-]=-H, \quad [J_\mp,X_\pm]=\pm J_\pm. \tag{A7}$$

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