

Embedding the dual complex of hyper-rectangular partitions

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May 2, 2014

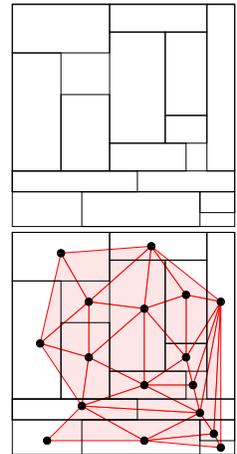
Abstract

A rectangular partition is the partition of an (axis-aligned) rectangle into interior-disjoint rectangles. We ask whether a rectangular partition permits a “nice” drawing of its dual, that is, a straight-line embedding of it such that each dual vertex is placed into the rectangle that it represents. We show that deciding whether such a drawing exists is NP-complete. Moreover, we consider the drawing where a vertex is placed in the center of the representing rectangle and consider sufficient conditions for this drawing to be nice. This question is studied both in the plane and for the higher-dimensional generalization of rectangular partitions.

1 Introduction

Problem statement and results We consider the partition of the d -dimensional cube into axis-parallel d -dimensional hyper-rectangles (or boxes) with disjoint interiors which we call *hyper-rectangular partitions*. The dual complex of such a partition is an abstract simplicial complex of dimension d and represents the connectivity of the boxes in the partition, possibly after a slight distortion if more than $d + 1$ boxes meet in a common point. Each vertex of the dual represents a box of the partition.

We pose the question of whether a partition permits a “nice” drawing of its dual in \mathbb{R}^d . We impose three conditions for niceness: First of all, the drawing must be straight, that is, a face must be drawn as the convex hull of the vertices that are on its boundary. That means that the drawing is completely determined by the positions of the vertices. Second, we disallow improper intersections of faces; for instance, edges are not allowed to cross except at their endpoints. The first two conditions can be summarized to that the drawing must yield a simplicial complex in \mathbb{R}^d . The final condition is that each vertex must be placed in the box that it represents; that guarantees some minimal amount



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of geometric closeness between a primary cell of the partition and its dual counterpart. If a drawing with these three conditions is possible, we call it an *embedding*. See the figure on the right for an illustration of a rectangular partition in \mathbb{R}^2 and an embedding of its dual.

We will see that not every hyper-rectangular partition permits an embedding of the dual, so a natural question to ask is how fast can the existence of an embedding be decided for a given partition. Our first main contribution is to show that this decision problem is NP-hard already in the planar case. More precisely, we restrict to the case that the boxes of a partition and the vertices of the dual are snapped to a uniform grid, and we prove that the problem is NP-complete under this constraint; see Section 3 for the precise statement.

The most natural choice for placing a vertex of the dual (for arbitrary dimension d) is the center of the corresponding box. We call a partition *center-embeddable* if this vertex placement extends to an embedding of the dual. For instance, if all boxes are d -dimensional cubes of same size, the partition is center-embeddable because its dual is simply a triangulation of the uniform grid defined by the box centers. We ask for simple, but less restrictive sufficient conditions on a partition to be center-embeddable. We call a partition β -balanced (with $\beta \geq 1$), if for any intersecting pair of boxes, the ratio of the longest side of the boxes divided by the shortest side of the boxes is at most β . As our second main contribution, we investigate the relationships between β -balanced and center-embeddable partitions: In the planar case, we show that center embeddability is guaranteed if the partition is $(3 - \epsilon)$ -balanced, for any $\epsilon > 0$. However, in \mathbb{R}^3 (and higher dimensions) we can construct a $(1 + \epsilon)$ -balanced partition that is not center-embeddable. The situation changes if we restrict to *cubical partitions*, consisting only of d -dimensional cubes: In this case, $(3 - \epsilon)$ -balanced partitions are center-embeddable in \mathbb{R}^3 , and we can construct a cubical β -balanced partition in \mathbb{R}^d that is not center-embeddable for any $\beta > d/(d - 2)$. It remains open whether cubical β -balanced partitions with $\beta < d/(d - 2)$ are center-embeddable in dimensions larger than 3.

Related work Rectangular partitions in \mathbb{R}^2 have a lot of applications, for instance in VLSI design [13], cartography [15], and database-related applications [14]; we refer to the survey by Eppstein [8] for more examples. In this context, rectangular partitions sometimes appear as the *rectangular dual* of a triangulated planar graph. Linear-time algorithms have been presented for the computation of such a rectangular dual [12]. Our situation, however, is different as the rectangular partition is the input object and we ask for a straight-line embedding of its dual where the vertices are constrained to lie inside rectangles. This can be seen as an instance of planar graph embedding with constraints; NP-hardness has been shown for other constraints, such as when fixing the length of each edge [5], and, more related to our approach, when restricting the placement of each vertex to a disc [10]. Furthermore, the problem of simultaneously embedding a planar graph and its dual on the integer grid has received some interest, e.g. [9]. Most related to our approach is the variant where the embedding of the primal graph is fixed and an embedding of the dual is sought for, such that only primal-dual pairs of edges intersect. Bern and Gilbert [3] show that the problem is linear-time solvable if all faces are convex and four-sided, and becomes NP-hard for convex five-sided faces. The latter is proven with a reduction from planar 3SAT and is similar to the proof presented in this work. However, although rectangular partitions can be seen as planar graphs with convex faces, there is no direct reduction from our problem because we

allow the dual graph to intersect the partition arbitrarily, not just at primal-dual pairs.

The higher dimensional equivalent of rectangular partitions (and their dual complexes) apparently has not been investigated from a theoretical point of view. Our motivation for this topic is originated in the approximation and simplification of d -dimensional image data [2]. The idea is to identify rectangular regions in which the image looks “similar”. In the simplest case, similarity means that the image values inside a region are similar, but different measures can be defined. The regions define a hyper-rectangular partition which can consist of substantially less elements than the original n^d image cells. In some situations, a piecewise linear approximation of the image is required, e.g., for computing a level set of the image. Standard techniques like bilinear interpolation cannot be applied because the non-uniformity of the partition leads to discontinuities; see [16] for a discussion with references. The standard approach is to triangulate the rectangular regions separately, such that triangulations of adjacent regions agree on their common boundary. This results in many simplices for regions with a large number of neighbors. Our dual complex construction constitutes an alternative to this standard approach; however, it does not necessarily embed in the ambient space nicely, which leads to the question considered in this work.

A special case of rectangular partitions are *hierarchical cubical subdivisions*; they arise from the initial box by a sequence of subdivisions, where a box in the partition is replaced with 2^d sub-boxes of half the side length. In 2 and 3 dimensions, these subdivisions are called *quad-trees* and *oct-trees*, respectively. A hierarchical cubical subdivisions is called *balanced* if adjacent boxes differ at most by a factor of 2 in side length. Note that balanced subdivisions are special cases of 2-balanced partitions in our notation. In [2], the dual complex of an oct-tree is used to approximate the *persistent homology* (see [6] for a definition) of the underlying image. It was also shown that the dual complex is center-embeddable, provided that the oct-tree is balanced. In [7], that result is generalized to general hierarchical cubical subdivisions; our results show that the hierarchical structure is crucial for obtaining this result, as for all $d \geq 5$, we can construct a non-hierarchical 2-balanced partition that is not center-embeddable.

Outline We introduce hyper-rectangular partitions and their dual complexes formally in Section 2. Section 3 is devoted to the NP-completeness proof of finding an embedding in the planar case. We study center-embeddability in Section 4. Section 5 concludes the paper.

2 Hyper-Rectangular Partitions and Dual Complexes

This section introduces the most important concepts needed for the results of this work.

We call a point set of the form $[a_1, b_1] \times \dots \times [a_d, b_d]$ with $a_i < b_i$ and $a_1, \dots, a_d, b_1, \dots, b_d \in \mathbb{Z}$ an *integral hyper-rectangle*, or just a *box*, with *lengths* $b_1 - a_1, \dots, b_d - a_d$. An *integral hyper-square*, or *square box* is a box where all lengths are equal. We can think of a box to be composed out of *hyper-pixels*, or *unit boxes* which are integer translates of $[0, 1]^d$. Let $B = [0, n]^d$ be a square box with an arbitrary $n > 0$. A *hyper-rectangular partition* $\mathcal{C} = (R_1, \dots, R_m)$ (with $m \leq n^d$) of B is a collection of boxes R_i such that their union equals B and their interiors are disjoint. If the union of the R_i is only a subset of B , we

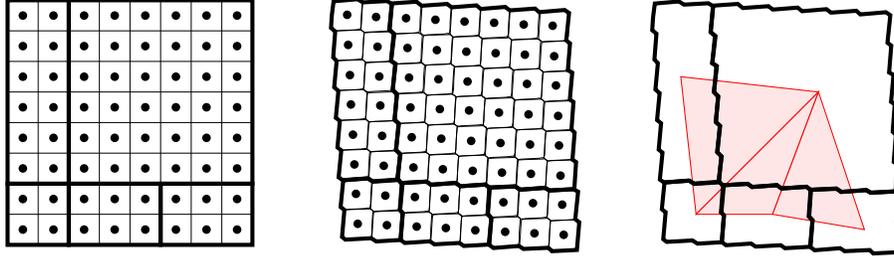


Figure 1: Left: A non-generic rectangular partition with 5 rectangles. Middle: The distorted pixels and rectangles. Right: The dual complex of the partition.

call the collection a *partial hyper-rectangular partition*. We usually omit the term “hyper-rectangular” and just talk about a *partition*. A partition is called *generic* if not more than $d + 1$ boxes intersect in a common point.

We associate an abstract d -dimensional *dual complex* $D(\mathcal{C})$ to a partition \mathcal{C} . For a generic partition, $D(\mathcal{C})$ is just the *nerve* of \mathcal{C} , that is, each box R_i is represented by a vertex v_i , and the simplices of $D(\mathcal{C})$ correspond to subsets of $\{R_1, \dots, R_m\}$ with non-empty intersection. This construction fails for non-generic partitions because intersections of $d + 2$ rectangles give rise to a $(d + 1)$ -simplex in the nerve. To circumvent this problem, we slightly perturb the boxes to obtain a generic situation, following the construction from [7]: The union of unit boxes can be seen as the Voronoi diagram of the pixel centers $P := \{(x_1 + 1/2, \dots, x_d + 1/2) \mid x_1, \dots, x_d \in \mathbb{Z}\}$. For an arbitrary $\epsilon \in (0, 1)$, we define the *distortion* of a point $p = (x_1, \dots, x_d) \in P$ as

$$T_\epsilon p := (x_1 - \epsilon \frac{\sum x_i}{2}, \dots, x_d - \epsilon \frac{\sum x_i}{2}).$$

We consider the Voronoi diagram of the distorted pixel centers. For a pixel U , we define its distortion \tilde{U} as the Voronoi cell of its distorted pixel center. For a box consisting of pixels U_1, \dots, U_k , we define its distortion as the union of the distorted pixels $\tilde{U}_1, \dots, \tilde{U}_k$. Finally, we define the dual complex $D(\mathcal{C})$ of a partition \mathcal{C} to be the nerve of the distorted boxes in the partition. See Figure 1 for an example in the plane. We remark that the dual complex does not depend on the choice of ϵ . Informally, the distortion is a way to remove high-dimensional simplices from the nerve in non-generic situations, thereby “preferring” connections between hyper-rectangle in the diagonal direction $(1, \dots, 1)$; we refer to [7] for more details.

For a hyper-rectangular partition $\mathcal{C} = (R_1, \dots, R_k)$, let v_i denote the vertex of the dual complex $D(\mathcal{C})$ that represents R_i . We call a mapping from $\{v_1, \dots, v_k\}$ to \mathbb{R}^d a *projection*. The *barycentric refinement* of the integer grid is the grid whose vertices are of the form $(a_1/2, \dots, a_d/2)$ with $a_1, \dots, a_d \in \mathbb{Z}$. A projection is *half-integral* if each v_i is mapped to a vertex of the barycentric refinement of the integer grid. A projection is *faithful* if each v_i is mapped in the interior of R_i . We assume in this work that projections are half-integral and faithful, unless otherwise stated. A projection π extends to the whole dual complex $D(\mathcal{C})$ by mapping a higher-dimensional simplex to the interior of the convex hull of the projected boundary vertices. Abusing notation, we let π also denote the extended mapping. We call

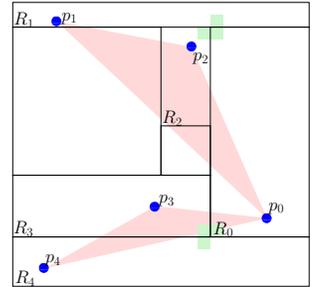
a projection an *embedding*, if this mapping is injective, or in other words, if the image is a simplicial complex in \mathbb{R}^d .

Seed configurations Consider a d -simplex σ of $D(\mathcal{C})$, dual to the intersection point w of the boxes R_0, \dots, R_d . By definition of $D(\mathcal{C})$, there is a *unique* collection of unit boxes U_0, \dots, U_d with $U_i \subseteq R_i$ such that the distorted unit boxes $\tilde{U}_0, \dots, \tilde{U}_d$ intersect in a *common point* as well. By definition, the U_i intersect in w . Let u_i be the center of U_i . We call (u_0, \dots, u_d) the *seed configuration* of σ .

In \mathbb{R}^d , a sequence of $d + 1$ points $p_0 = (x_{0,1}, \dots, x_{0,d}), \dots, p_d = (x_{d,1}, \dots, x_{d,d})$ has the *orientation*

$$O(p_0, \dots, p_d) = \text{sign det} \begin{pmatrix} 1 & x_{0,1} & \dots & x_{0,d} \\ \vdots & & \ddots & \\ 1 & x_{d,1} & \dots & x_{d,d} \end{pmatrix}.$$

We say that a projection π of $D(\mathcal{C})$ to \mathbb{R}^d *preserves the orientation of the d -simplex σ* , if $O(\pi(v_0), \dots, \pi(v_d)) = O(u_0, \dots, u_d)$, where u_0, \dots, u_d is the seed configuration of σ . For an equivalent formulation in the plane, assume that R_0, R_1, R_2 are arranged counterclockwisely around w . Then π preserves the orientation of σ , if the cycle $\pi(v_0), \pi(v_1), \pi(v_2)$ is counterclockwisely arranged, too. In the picture on the right, the seed configuration of the dual 2-simplices $\{R_0, R_1, R_2\}$ and $\{R_0, R_3, R_4\}$ is illustrated; note that the given projection does not preserve the orientation of the former 2-simplex, but does preserve the orientation of the latter.



The following result is a generalization of the “Geometric Realization Theorem” from [7]. In there, it was proven that for balanced hierarchical cubical subdivisions, the projection that maps a vertex to the center of the cube is always an embedding. The main property exploited in the proof is the “Orientation Lemma”, stating that the orientation of each d -simplex is preserved for such subdivisions. This is no longer true for arbitrary partitions, but the same proof idea can be used to show:

Theorem 1 (Embedding Theorem). *Let \mathcal{C} be a partition. Let π be a projection of $D(\mathcal{C})$ that preserves the orientation of at least one d -simplex. Then, π is an embedding if and only if it preserves the orientation of each d -simplex.*

The proof in [7] relies on some topological concepts; Appendix A repeats the argument for the convenience of the reader. The constraint that π preserves at least one d -simplex is required to rule out pathological cases such as a partition in \mathbb{R}^2 with only 3 rectangles, whose dual is always an embedding regardless of the orientation of the dual triangle (unless the triangles degenerate to a line). All our results are eventually reduced to investigate the orientation of projected d -simplices: Embedability results will be proved by showing that each d -simplex preserves orientation, non-embedability results by constructing examples where some d -simplex does not preserve orientation. In all these constructed examples, it will be easy to verify that at least one d -simplex preserves orientation; therefore we will tacitly ignore that assumption when applying Theorem 1.

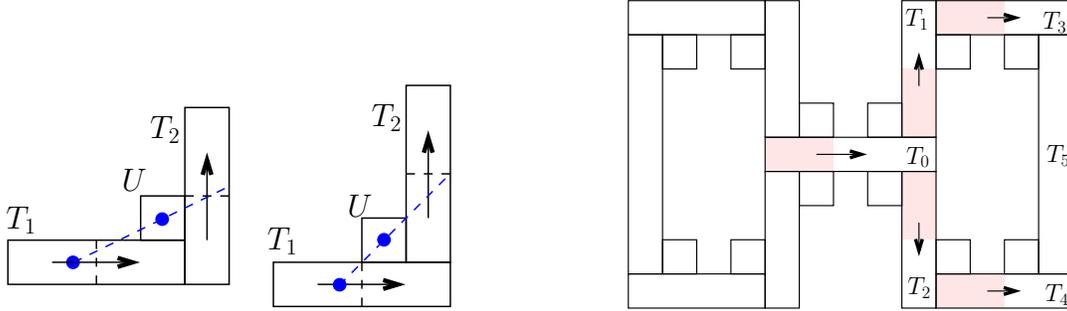


Figure 2: Left: The two possible configurations of L-joints (up to rotations and reflections). Right: A (partial) partition that does not permit an embedding.

3 NP-completeness

In this section, we concentrate on the planar case $d = 2$. We adapt our notation by leaving out the prefix “hyper-” on all defined terms.

Thin rectangles and L-joints We call an integral rectangle *thin* if one of its side has length 1, and the other side has length at least 4. The *length* ℓ of a thin rectangle is the length of its longer side. Clearly, any projection has to map the vertex of a thin rectangle to one of the ℓ pixel centers. Three intersecting rectangles are called an *L-joint* if two of the rectangles are thin and their union is L-shaped (up to rotation and reflection), and the third rectangle is the pixel in the bulge of the L-shape; see Figure 2 (left). If the L-joint belongs to a partition, it induces a 2-simplex in $D(\mathcal{C})$. An *L-path* is a sequence of thin rectangles such that two consecutive elements form an L-joint. Such an L-path can be cyclic, in which case we call it an *L-cycle*. It is convenient to think of the thin rectangles in an L-joint being directed consistently towards one end of the L-shape. If the direction is fixed, it makes sense to talk about the *front rectangle* and *back rectangle* of the L-joint, about the *front half* and *back half* of each thin rectangle, and about the *front pixel* and the *back pixel* which are the extremal pixels in a thin rectangle.

Lemma 2 (L-joint lemma). *Let (T_1, T_2, U) be a directed L-joint, with T_1 being the back rectangle, T_2 being the front rectangle, and U being the bulge pixel. Let π be a projection that preserves the orientation of the 2-simplex induced by the L-joint. Then, if π maps the vertex of T_1 to its back half, it also maps the vertex of T_2 to its back half.*

Proof. Figure 2 (left) displays the two possible connections of T_1 and T_2 , if both are of length 4. Recall that π is assumed to be half-integral and, therefore, has only 4 choices to place the vertex for both T_1 and T_2 . We can see that, if π preserves the orientation and maps the vertex of T_1 into its back half it must map the vertex of T_2 below the dashed line, so this vertex must go into the back half in both cases. The situation does not change if we extend T_1 to the left or T_2 to the top. \square

We can use L-joints to show that dual complexes cannot always be embedded: Consider the (partial) rectangular partition in Figure 2 (right). Let π be any projection of $D(\mathcal{C})$.

Assume for a contradiction that π preserves the orientation of all 2-simplices induced by the partial partition. Assume w.l.o.g. that the vertex of T_0 is projected to the left half of T_0 (otherwise, the symmetric argument applies). Direct the thin rectangles according to the arrows in the figure. By the L-joint lemma, the vertices of T_1 and T_2 must be placed in their back halves, and by repeating the argument, the same holds for the rectangles T_3 and T_4 . Using the L-joint Lemma again, the vertex of T_5 must be placed in the upper half of T_5 (caused by the L-joint with T_3) and also in the lower half of T_5 (caused by the L-joint with T_4), a contradiction. Filling out the partial partition with pixels, we obtain a full partition \mathcal{C} such that no projection preserves the orientation of all triangles. The Embedding Theorem (Theorem 1) asserts that there is no embedding.

The reduction We define the decision problem `Faithful_Embed` as follows: For a partition $\mathcal{C} = (R_0, \dots, R_m)$ of $[0, n] \times [0, n]$, is there an embedding of $D(\mathcal{C})$? Our goal is to show NP-completeness of this problem.

First of all, the problem is clearly in NP: Given a partition \mathcal{C} , we can compute $D(\mathcal{C})$ and check whether a specific projection causes an orientation switch for any triangle in polynomial time.

For the reduction, we define the `grid3sat` problem in the same way as in [10]:

grid3sat: Given some $N \times N$ grid (with N linear in n) in which some grid points are labeled as clauses and some as variables. Variables are connected with clauses by vertex-disjoint paths on the grid. A sign is associated to every such path indicating whether the corresponding variable is negated in the clause or not. Every clause is incident to exactly three paths. Is the formula described in this way satisfiable or not?

See Figure 3 (left) for an example. A consequence of the formulation is that every variable appears in at most four clauses. Therefore, the problem is a variant of the *planar 3,4-SAT* problem (see [10]) with the restriction that all paths follow grid edges.

As stated in [10], `grid3sat` is NP-complete. Consider an instance I of `grid3sat`, which is a graph on the integer grid representing a formula F (in 3-CNF form). Based on this graph, we will construct a rectangular partition \mathcal{C} such that the existence of a satisfying assignment for F is equivalent to the existence of an embedding of $D(\mathcal{C})$.

We barycentrically refine (see Section 2) the grid given by I a constant number of times (in fact 5 barycentric refinements suffice) and construct our partition inside this base grid. For that, we first replace vertices representing variables or clauses by boxes of sufficient size (which we fill later), and we replace paths connecting variables and clauses by disjoint L-paths. Each path inherits the sign of the corresponding path in I . We think of L-paths being directed from variables to clauses. See Figure 3 (right).

Inside each box representing a variable X_i , we place a **variable gadget** (Figure 4 (left)). In its center, it has an L-cycle with 4 thin rectangles of length 8, which we direct in clockwise direction. We call this cycle the *variable cycle* for X_i . Each L-path to a clause box that starts at a boundary side of the gadget is extended into the interior of the box and forms an L-joint with the corresponding side of the variable cycle. If the path is associated with a plus sign (that is, the variable appears in non-negated form in the corresponding clause),

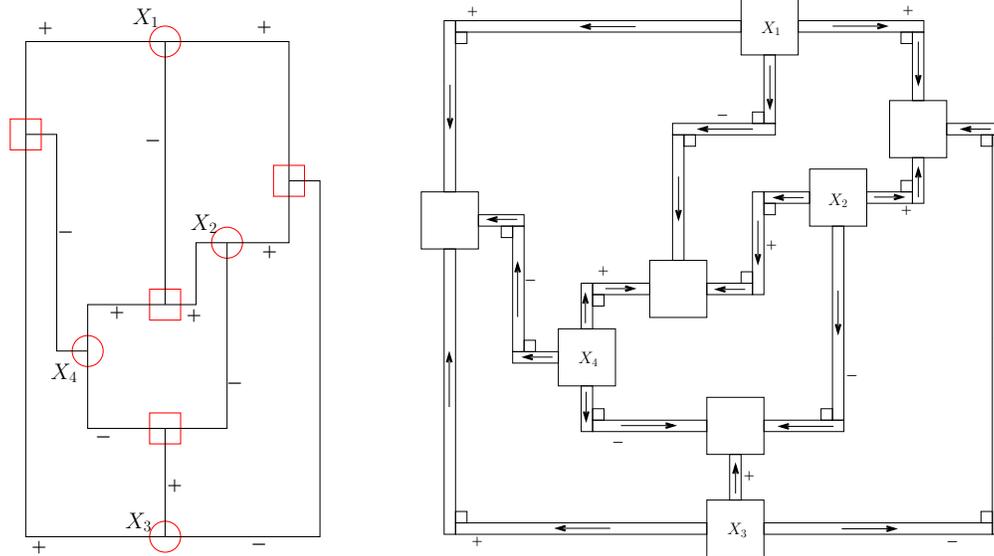


Figure 3: Left: A graph presenting the formula $(X_1 \vee X_2 \vee \bar{X}_3) \wedge (\bar{X}_1 \vee X_2 \vee X_4) \wedge (X_1 \vee \bar{X}_3 \vee \bar{X}_4) \wedge (\bar{X}_2 \vee X_3 \vee \bar{X}_4)$. Right: A corresponding representation as a partial integral rectangular partition, with L-path connections.

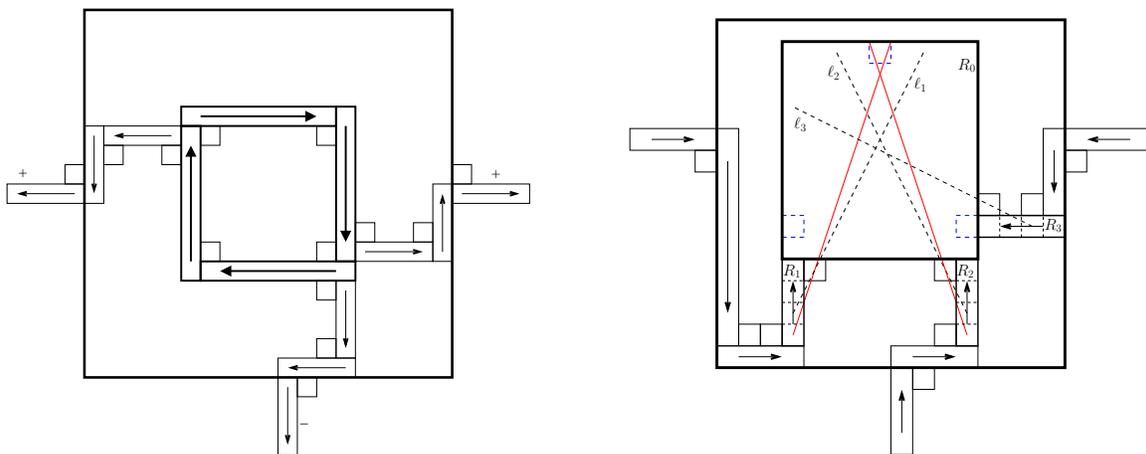


Figure 4: Left: The variable gadget for X_1 from Figure 3. Right: The clause gadget for $\bar{X}_2 \vee X_3 \vee \bar{X}_4$ from Figure 3. Note that R_1 , R_2 , R_3 are precisely of length 4; the dashed lines and the red lines are drawn for illustration purposes in Lemma 3.

the path is connected to the front pixel. If it is associated with a minus sign, it is connected to the back pixel.

Inside each box representing a clause, we place a **clause gadget** (Figure 4 (right)). The three L-paths arriving at the boundary of the gadget are extended to connect to the *clause rectangle* (rectangle R_0 in Figure 4 (right)), with the last rectangle being exactly 4 base squares long. If the L-paths arrive at different sides than displayed in Figure 4, we can just rotate the gadget by a multiple of 90° . The gadget is designed to satisfy the following two properties:

Lemma 3 (Clause Lemma). *Let v_0, v_1, v_2, v_3 be the vertices of R_0, R_1, R_2, R_3 , respectively, as in Figure 4 (right), and let π be a partial projection that maps v_i to a point p_i for $1 \leq i \leq 3$.*

1. *If p_1, p_2 , and p_3 are in the back half of R_1, R_2 , and R_3 , respectively, then π cannot be completed to an embedding.*
2. *If at least one of the p_i is in the front pixel of R_i , for $1 \leq i \leq 3$, then there exists an extension of π to v_0 such that π preserves the orientation of all triangles incident to v_0 .*

Proof. Consider Figure 4 (right); for the first part, let p_0 be the projection of v_0 in a completion of π . If π is an embedding, p_0 is left of ℓ_1 , right of ℓ_2 and below ℓ_3 , which is impossible. For the second part, if p_1 or p_2 are in the front pixel, we can set p_0 to be the center of the rightmost or leftmost dashed base square in Figure 4 (right), respectively. If p_3 is in the front pixel, we can choose the center of the topmost blue base square for p_0 . From the red line originating in R_1 in Figure 4 (right), it can be seen that the triangle induced by R_0, R_1 , and the pixel adjacent to both has the correct orientation. The same is true for the triangle spanned by R_0, R_2 and the pixel adjacent to both. \square

To summarize, we have constructed a partial integral rectangular partition, consisting of variable cycles, clause rectangles, and L-paths connecting them. We finally complete the partition by filling the empty spots with pixels. Let \mathcal{C} denote this partition. We can construct \mathcal{C} in polynomial time in n . The following lemma is sufficient to prove NP-hardness of `Faithful_Embed`.

Lemma 4. *There exists an embedding of $D(\mathcal{C})$ if and only if F has a satisfying assignment.*

Proof. “ \Rightarrow ”: Let π be an embedding. We define an assignment as follows: For every variable X_i , consider its variable cycle (which is clockwise arranged). Note that the L-joint Lemma implies that π either projects all vertices of the variable cycle to the front half, or it projects all vertices to the back half of the corresponding rectangles. We set X_i to **True** if π projects to the front half and to **False** otherwise. We show that this assignment satisfies F :

Let $C = L_i \vee L_j \vee L_k$ be a clause of F where $L_i = X_i$ or $L_i = \bar{X}_i$, and consider the corresponding clause gadget. Let R_i, R_j, R_k the last rectangles of the L-paths connecting the variable gadgets of X_i, X_j , and X_k , respectively, to the clause rectangle. Since π is an embedding, at least one of the vertices of R_i, R_j , or R_k is placed in the corresponding front half by the Clause Lemma. Assume w.l.o.g. that the vertex of R_i is placed in the front half. By iteratively applying the L-joint Lemma, all vertices in the L-path from X_i to C must

therefore be placed in the front half. If $L_i = X_i$, the path was associated with a plus sign, and therefore, the L-path is connected with the variable cycle of X_i at a front pixel (as for instance the left and right paths originating from the variable cycle in Figure 4 (left)). Since π preserves orientations, it follows again from the L-joint Lemma that the vertices in the variable cycle must also have been placed in the front half. Hence, our assignment sets X_i to **True** and therefore, the clause is satisfied. If $L_i = \bar{X}_i$, the argument is analogous.

“ \Leftarrow ”: We construct a projection based on a satisfying assignment. We start with the variable gadgets: If $X_i = 1$, we place the vertices in its variable cycle at the front pixel, otherwise at the back pixel. For every L-path that leaves a variable cycle, we distinguish two cases: If the path has a plus sign and $X_i = 1$, or if the path has a minus sign and $X_i = 0$, we place all vertices in the path in the front pixel; otherwise, we place all vertices in the back pixel. It is straight-forward to verify that this projection preserves the orientation for each 2-simplex induced by L-joints. Finally, let $C = L_i \vee L_j \vee L_k$ be a clause as above. Consider the corresponding clause gadget and let R_i, R_j, R_k denote the connecting (thin) L-path rectangles. By our construction, the vertex of R_i is in the front pixel if and only if $L_i = 1$, same for j and k . Since we have a satisfying assignment, at least one literal is satisfied, so one of the vertices is in the front pixel. By the Clause Lemma, we can therefore place the vertex of the clause rectangle such that all triangles preserve their orientation.

Our projection preserves all orientations of the constructed partial partition. It is not difficult to see that the additional 2-simplices caused by filling the empty spots with base squares preserve their orientation as well (in fact, any projection preserves their orientation). Therefore, the constructed projection is an embedding by the Embedding Theorem. \square

In conclusion, we can summarize

Theorem 5. *Faithful_Embed is NP-complete.*

A slight variation of our proof shows that the problem remains NP-hard when allowing projections that are not half-integral (see Appendix B). However, it is unclear whether this variant of the problem also remains in NP.

4 Center projections and β -balancing

We turn back to arbitrary dimensions. We define the *center* of a box $[a_1, b_1] \times \dots \times [a_d, b_d]$ to be the point $((a_1 + b_1)/2, \dots, (a_d + b_d)/2)$. The *center projection* is the projection of $D(\mathcal{C})$ that maps each vertex to the center of the corresponding box. Clearly, this projection is half-integral and faithful. We call \mathcal{C} *center-embeddable* if the center projection is an embedding.

We define the *balance* of a set of boxes $\{R_0, \dots, R_k\}$ to be the length of the longest side among all boxes R_0, \dots, R_k divided by the length of the shortest side among all boxes R_0, \dots, R_k . We define the balance of a k -simplex σ of $D(\mathcal{C})$ to be the balance of the set of dual boxes. The *aspect ratio* of a box R is the balance of $\{R\}$. Obviously R has aspect ratio 1 if and only if R is a square box. We call $D(\mathcal{C})$ β -*balanced* for some $\beta \geq 1$, if each simplex

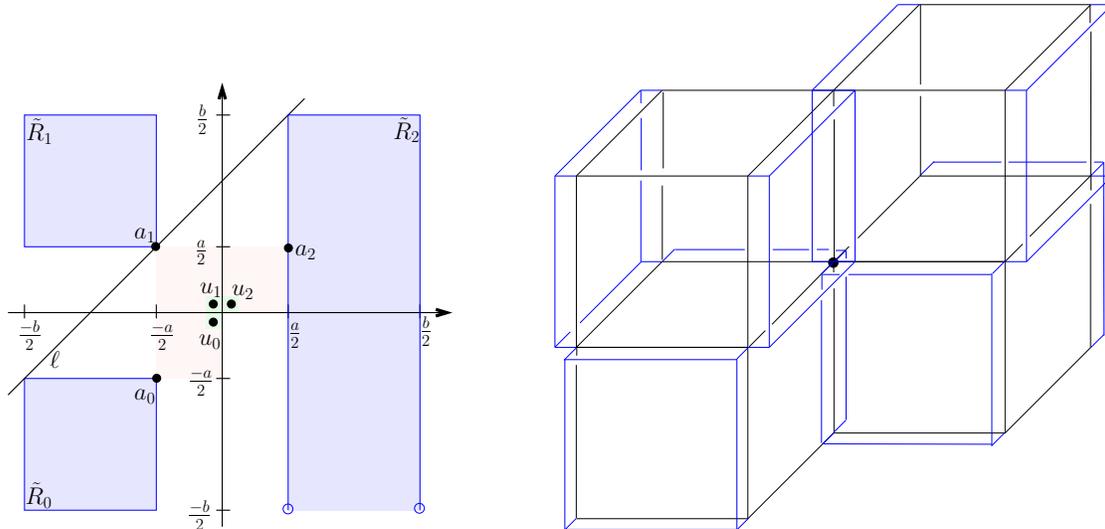


Figure 5: Left: Illustrations of the proof of Theorem 6. The drawing assumes that $\beta = 3$. In this case, we can find a line that touches C_0 , C_1 , and C_2 , leading to a counterexample for center-embeddability. Right: Illustration of the construction in the proof of Theorem 7.

of $D(\mathcal{C})$ has balance at most β .¹ Informally, β -balanced partitions have the property that boxes are not too skinny, but also neighboring boxes do not differ too much in side lengths.

The simplest sufficient condition for a partition to be center-embeddable is that all boxes are square boxes of same size. We investigate several generalizations of this trivial criterion.

Planar results It is not hard to prove that a partition in \mathbb{R}^2 consisting only of squares is always center-embeddable, because the edge connecting the centers of two squares does not leave the union of the squares. However, if we only bound the aspect ratio of any rectangle by $1 + \epsilon$ (with $\epsilon > 0$), this property does not hold in general, and we can construct a counterexample for center-embeddability. For β -balanced partitions, we prove a tight bound for center-embeddability.

Theorem 6 (Planar Balancing Theorem). *A β -balanced partition of \mathbb{R}^2 is center-embeddable for $\beta < 3$, and there exists a 3-balanced partition that is not center-embeddable.*

Proof. Consider a β -balanced partition. For the first part, it is enough to prove that the center projection preserves the orientation of each 2-simplex for $\beta < 3$ according to the Embedding Theorem. So, let σ be a 2-simplex, caused by the intersection of three rectangles R_0, R_1, R_2 . W.l.o.g., we assume that the intersection point is the origin, that R_0 is completely contained in the lower-left quadrant, R_1 is contained in the upper-left quadrant, and R_2 is contained in the halfplane $x \geq 0$; all other cases are obtained by suitable rotations and reflections. Let a denote the length of the shortest side among the R_i , and let b denote the length of the longest side; we have $\frac{b}{a} \leq \beta$. We let C_i denote the set of possible positions of the center of R_i . By the imposed constraints, it is straight-forward to verify that C_i are

¹This is equivalent to require that any edge in $D(\mathcal{C})$ has balance at most β , which is the definition given in the introduction.

rectangles as illustrated in Figure 5 (left). Note that the lower side of C_2 does not belong to the set of possible centers, while the other boundary sides do.

Recall that the orientation of σ is preserved if $O(p_0, p_1, p_2) = O(u_0, u_1, u_2)$, where the u_i define the seed configuration of σ ; in our case, $u_0 = (-1/2, -1/2)$, $u_1 = (-1/2, 1/2)$, and $u_2 = (1/2, 1/2)$. We scale each u_i by a , obtaining the points a_0, a_1, a_2 ; obviously, $O(u_0, u_1, u_2) = O(a_0, a_1, a_2)$. Assume that the orientation of σ is not preserved. Then, if we continuously move the points a_i to the points p_i along the connecting line segment (which lies in C_i), there must be a time where the three points become collinear. It follows that there must be a line in \mathbb{R}^2 which intersects each C_i . Such a line, however, exists if and only if $b \geq 3a$, as can be seen by elementary geometric arguments. It follows that the orientation of σ is preserved if $\beta < 3$.

The construction also yields the counterexample for $\beta = 3$: Consider the rectangles intersecting in the origin with centers $p_0 = (-3/2, -1/2)$, $p_1 = (-1/2, 1/2)$, $p_2 = (1/2, 3/2)$. We can fill the bounding box by pixels to obtain a 3-balanced partition. The 2-simplex dual to the origin is projected to a line segment because p_0, p_1, p_2 are collinear. Therefore, the partition is not center-embeddable. \square

The counterexample constructed in the proof of Theorem 6 yields a non-generic partition. If we assume genericity, the set C_2 defined in the proof loses the upper boundary side, and a line intersecting each C_i can only be found if $b > 3a$. It follows that generic 3-balanced partitions are center-embeddable. However, generic β -balanced partitions that are not center-embeddable can be constructed for any $\beta > 3$.

Higher dimensions Restricting to square boxes (that is, aspect ratio 1) does not guarantee center-embeddability for any $d > 2$: In \mathbb{R}^3 , the triangle spanned by the centers of three intersecting cubes might leave the union of the three cubes, and a counterexample can be constructed; see [2] and [7]. We show that also restricting to β -balanced partitions does not guarantee embeddability, unless $\beta = 1$:

Theorem 7. *For any $\beta > 1$, there exists a β -balanced partition in \mathbb{R}^d that is not center-embeddable.*

Proof. We restrict to $d = 3$ for the simplicity of presentation although our construction generalizes to any dimension: For simplicity, we talk about *cubes* and *voxels* instead of square boxes and unit boxes throughout the proof. Choose $b \in \mathbb{Z}$ such that $\beta \geq \frac{b+2}{b} > 1$. Consider four cubes of length b having the origin as a corner, such that the centers lie in a common plane (the black boxes in Figure 5 (right)). Then, extend the cubes by a layer of voxels at two opposite faces, such that all voxels around the origin are covered by the four boxes (the blue boxes in Figure 5 (right)). This ensures that the 3-simplex induced by the four boxes belongs to the dual complex of the partition. The extension does not change the centers, consequently, the center projection cannot preserve the orientation of that simplex. Moreover, all boxes have side length b and $b + 2$. It remains to show that the arrangement of the four boxes can be completed to a partition without destroying β -balancing. We give details of this step in Appendix C. \square

The cubical case We finally consider β -balanced partitions in \mathbb{R}^d where all boxes are square boxes. We call such partitions *cubical*. We first construct counterexamples in arbitrary dimension to show:

Theorem 8. *For $d \geq 3$ and $\beta > \frac{d}{d-2}$, there exists a β -balanced cubical partition in \mathbb{R}^d that is not center-embeddable.*

Proof. For $d \geq 3$, $a, b \in \mathbb{Z}$ positive, and $\delta \in \{0, 1\}$ consider the $(d+1) \times (d+1)$ -matrix

$$M_d^{(\delta)} := \begin{pmatrix} 1 & -\frac{a}{2} & & \cdots & & -\frac{a}{2} \\ 1 & \frac{b}{2} & -\frac{b}{2} & \cdots & \cdots & -\frac{b}{2} \\ 1 & -\frac{b}{2} + \delta & \frac{b}{2} & -\frac{b}{2} & \cdots & -\frac{b}{2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & -\frac{b}{2} + \delta & \cdots & -\frac{b}{2} + \delta & \frac{b}{2} & -\frac{b}{2} \\ 1 & -\frac{b}{2} + \delta & \cdots & \cdots & -\frac{b}{2} + \delta & \frac{b}{2} \end{pmatrix}.$$

It can be shown by elementary row operations (Appendix D) that

$$\det M_d^{(0)} = \frac{1}{2} b^{d-1} (da - (d-2)b).$$

In particular, note that the determinant is positive if $\frac{b}{a} < \frac{d}{d-2}$, vanishes if $\frac{b}{a} = \frac{d}{d-2}$, and is negative if $\frac{b}{a} > \frac{d}{d-2}$.

Let a and b be such that $\beta > \frac{b}{a} > \frac{d}{d-2}$, so that $\det M_d^{(0)} < 0$. If we choose a and b large enough, it follows that $\det M_d^{(1)} < 0$ as well, just because $M_d^{(1)}$ constitutes a small perturbation of $M_d^{(0)}$ for large a and b . The rows of $M_d^{(1)}$ define $d+1$ points p_0, \dots, p_d in \mathbb{R}^d (ignoring the first column). We define the square box R_i with center p_i and side length a for R_0 , and length b for R_1, \dots, R_d . By this choice, it can be seen easily that all 2^d unit boxes around the origin are covered by the R_i , so R_0, \dots, R_d form a d -simplex σ in the dual complex. Moreover, $u_0 := (-1/2, \dots, -1/2) \in R_0$, $u_1 := (1/2, -1/2, \dots, -1/2) \in R_1$, $u_2 := (1/2, 1/2, -1/2, \dots, -1/2) \in R_2$, and so on, thus u_0, \dots, u_d form a seed configuration and it is easily seen to that $O(u_0, \dots, u_d) > 0$. Because $O(p_0, \dots, p_d) < 0$ by construction, the center projection does not preserve the orientation of σ .

The last step of the proof is to show that we can complete the initial configuration R_0, \dots, R_d to a complete β -balanced cubical partition in \mathbb{R}^d . The details of this step are skipped for brevity. See Appendix C for details. \square

We are able to show that the constructed counterexample is the worst case in \mathbb{R}^3 . Precisely, we state:

Theorem 9. *Cubical β -balanced partitions in \mathbb{R}^3 are center-embeddable for $\beta < 3$.*

Proof. For simplicity, we restrict to generic partitions in the proof; the general case works with the same methodology, but requires closer investigation of the distortion defined for the dual complex construction. As in Theorem 7, we use the terms cube and voxel instead of square box and hyper-pixel.

Consider 4 cubes R_0, \dots, R_3 intersecting in a point p . Let a be the length of the shortest and b be the length of the longest side among the R_0, \dots, R_3 . Assume w.l.o.g. that $p = (0, 0, 0)$. By genericity, the union of the R_i covers all 8 voxels adjacent to p , and each cube covers at least one voxel. Represent these voxels by the 3-bit numbers $0, \dots, 7$ where the 1st/2nd/3rd bit of voxel j is set to 1 if and only if the x -/ y -/ z -coordinate of the voxel is positive. Assign to each R_i a subset of $\{0, \dots, 7\}$, denoting the voxels that it occupies. Only three cases are possible, namely that R_i covers a single voxel, that R_i covers two voxels which are face adjacent, or four voxels which lie in a common halfspace. We consider C_i , the set of possible centers for R_i . If R_i covers only one voxel adjacent to the origin, say voxel 0, then C_i is the line segment connecting $(-a/2, -a/2, -a/2)$ and $(-b/2, -b/2, -b/2)$. If R_i covers two voxels which are face adjacent, say voxels 0 and 1, C_i is the trapezoid spanned by the four points $(\pm a/2, -a/2, -a/2)$, $(\pm b/2, -b/2, -b/2)$; more precisely, the center can not lie on either of the two non-parallel lines because this would prevent the cube to span over both voxels, so C_i is the trapezoid with those two sides excluded. Finally, if R_i covers four voxels in a common hyperplane, say voxels 4, \dots , 7, C_i is the polytope spanned by the 8 points $(\pm a/2, \pm a/2, a/2)$, $(\pm b/2, \pm b/2, b/2)$; the polytope has 6 faces, two of them being parallel. More precisely, the center can not lie on a boundary face except for the two parallel sides, so C_i is that polytope with the other 4 faces excluded.

Assume for a contradiction that the center projection does not preserve the orientation of the simplex spanned by R_0, \dots, R_3 . Scaling the seed configuration of the simplex by a factor of a , we obtain a simplex spanned by a_0, \dots, a_3 with a_i in the closure of C_i with the same orientation as the seed configuration. Because this orientation is not preserved, there are points p_0, \dots, p_3 with $p_i \in C_i$ such that the orientation is different. It follows that there are points p'_0, \dots, p'_3 with $p'_i \in C_i$ whose orientation is zero. In other words, there exists a plane which intersects C_0, \dots, C_3 . However, C_0, \dots, C_3 are completely determined by the way of how the boxes R_i decompose the voxels $\{0, \dots, 7\}$ adjacent to the origin, and there are only two configurations possible, up to rotations and reflections: the *regular configuration*

$$\{0\}, \{1\}, \{2, 3\}, \{4, 5, 6, 7\},$$

and the *singular configuration*

$$\{0, 1\}, \{2, 3\}, \{4, 6\}, \{5, 7\}.$$

Consequently, there must be a plane intersecting the two line segments, the trapezoid and the polytope defined by the regular configuration, or a plane intersecting the four trapezoids defined by the singular configuration. It can be verified, however, that such a plane does not exist by formulating the statement in terms of a quantified system of inequalities and using a quantifier elimination program [4]² (see Appendix E). Hence, the center-projection preserves the orientation of every simplex and is an embedding. \square

The proof idea can in principle be extended to higher dimensions. However, there are two problems: First, the number of configurations to check increases; for instance, there are already 3 configurations to check in \mathbb{R}^4 . Second, and more seriously, the complexity of the

²We used qepcad: <http://www.usna.edu/cs/~qepcad/B/QEPCAD.html>

quantifier elimination increases dramatically for higher dimensions; in fact, we were not able to verify that β -balanced cubical partitions in \mathbb{R}^4 are center-embeddable for $\beta < \frac{d}{d-2} = 2$.

5 Conclusion

We have proven positive and negative results about partitions and their duals in this paper; it is NP-hard to decide whether an embedding exists already in the plane; if we project vertices to centers of rectangles, there are some simple balancing conditions that ensure embeddability in the plane, namely, if all rectangles are squares or if the partition is $(3 - \epsilon)$ -balanced with $\epsilon > 0$. In higher dimensions, however, we need to combine both properties, that means, consider cubical $(3 - \epsilon)$ -balanced partitions to get a guarantee that partitions can be embedded. We are posing the question whether another simple condition could guarantee center-embeddability as well.

Quad-tree decompositions and their higher-dimensional analogues have the nice property that they can be balanced with a greedy strategy, without granulating the partition too much. It seems unclear, however, how a planar partition can be turned into a 3-balanced partition efficiently by subdividing rectangles.

This work has considered hyper-rectangular partitions in general – can better embeddability results be derived for certain restrictions, in particular for partitions that arise from a sequence of hyperplane cuts (sometimes also called guillotine constructions)? We expect the answer to be negative in the plane; however, the situation might be different in higher dimensions.

On a technical side, a natural improvement would be to replace the computer-assisted proof in Theorem 9 with a geometric argument; this would hopefully extend into higher dimensions and prove our conjecture that β -balanced partitions in \mathbb{R}^d are center-embeddable for $\beta < \frac{d}{d-2}$. A second question is about a variant of `Faithful_embed`: while it is clear that deciding the existence of an embedding remains hard in three dimensions, it is not clear whether the restriction to cubical partitions simplifies the problem.

Acknowledgments

The author thanks Herbert Edelsbrunner for fruitful discussions, David Eppstein for pointing out related work, and the anonymous referees for their extremely valuable feedback.

A The Embedding Theorem

Theorem 1 (Embedding Theorem). *Let \mathcal{C} be a partition. Let π be a projection of $D(\mathcal{C})$ that preserves the orientation of at least one d -simplex. Then, π is an embedding if and only if it preserves the orientation of each d -simplex.*

We give some more details about its proof which is based on [7]. A set is called *contractible* if it can be reduced to a single point by a continuous deformation [11].

Lemma 10. *Any intersection of k distorted boxes of \mathcal{C} is either empty or contractible.*

Proof. The proof given in the “Fractal Distortion Lemma” in [7] extends to arbitrary partitions (it is neither exploited that the boxes are cubes, nor that they are arranged hierarchically). \square

We now give a proof of the Embedding Theorem which works similar to the proof of the “Geometric Realization Theorem” from [7].

Assume first that π preserves the orientation of all d -simplices of $D(\mathcal{C})$. Let $B \subset \mathbb{R}^d$ be the box that is partitioned by \mathcal{C} . We attach two layers of unit boxes at the boundary of B , obtaining an extended box \bar{B} . Note that this yields a partition $\bar{\mathcal{C}}$, which is an extension of \mathcal{C} . We extend the projection π of \mathcal{C} to $\bar{\pi}$ of $\bar{\mathcal{C}}$ by mapping vertices in the new layers to the corresponding box centers.

We compactify \mathbb{R}^d to the d -dimensional sphere \mathbb{S}^d by adding a vertex at infinity. Similarly, we compactify the dual complex $D(\bar{\mathcal{C}})$ by adding a new vertex at infinity and connecting it to every simplex of the boundary of $D(\bar{\mathcal{C}})$. Because of the Lemma 10, we can apply the Nerve Theorem [6] to $D(\bar{\mathcal{C}})$, which states that $D(\bar{\mathcal{C}})$ triangulates a d -dimensional ball, and therefore, its compactification triangulates \mathbb{S}^d . Hence, the extended projection $\bar{\pi}$ defines a continuous mapping $g : \mathbb{S}^d \rightarrow \mathbb{S}^d$. The *degree of g* at a point x not in any $(d - 1)$ -simplex is the number of d -simplices that contain $g^{-1}(x)$, counting a d -simplex positive or negative depending on the orientation of its image under g . This degree is 1 in between the two outside layers, because all boxes are of unit size and therefore, the orientation of all d -simplices can be chosen positive. However, the degree of a map is a global property that does not depend on the specific location of x . Hence, it is 1 for any x .

We can assume that the orientation of all seed configurations is positive; because all orientations are preserved by π , the orientation of all d -simplices is positive. Because the degree of g is 1, it follows that every point is in the interior of exactly one projected d -simplex. This implies injectivity, so π is an embedding.

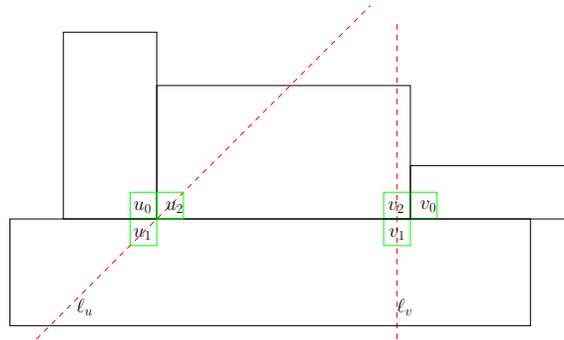


Figure 6: Illustration for the fact that the orientations of (u_0, u_1, u_2) and (v_0, v_1, v_2) are not the same; indeed, if we map u_i to v_i for $0 \leq i \leq 2$, the line l_u is mapped to the line l_v , but the halfspace left of l_u is mapped to the halfspace right of l_v . The same kind of argument can be applied in higher dimensions, replacing the dashed lines by hyperplanes and defining a notion of “left” and “right” by a common direction, for instance the vector from the center of v_0 to the center of u_0 .

For the opposite direction, assume that π does not preserve the orientation of some d -simplex of $D(\mathcal{C})$. Because there exists at least one d -simplex whose orientation is preserved,

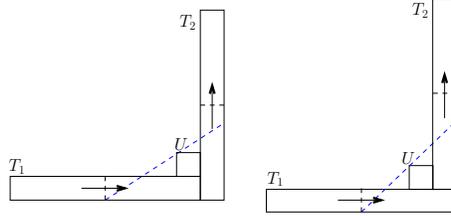


Figure 7: The two possible configurations of L-joints (up to rotations and reflections). If the vertex of T_1 is set in the back half, the pixel of T_2 must also be set in the back half, regardless of where the vertex of U is placed.

there exists some $(d - 1)$ -simplex τ in $D(\mathcal{C})$ incident to one simplex σ_1 whose orientation is preserved and to one simplex σ_2 whose orientation is not preserved. It suffices to prove that σ_1 and σ_2 are mapped to the same side of the supporting $(d - 1)$ -hyperplane of τ .

To see that, let u_0, \dots, u_n and v_0, \dots, v_n denote the seed configurations of σ_1 and σ_2 , respectively. W.l.o.g., we assume that the seed configurations are ordered such that u_1, \dots, u_n and v_1, \dots, v_n are the pixels that belong to boxes in τ , and furthermore, for all $1 \leq i \leq n$, u_i and v_i belong to the same box of the partition. By these choices, we have that

$$O(u_0, \dots, u_n) = -O(v_0, \dots, v_n).$$

See Figure 6 for an illustration in \mathbb{R}^2 ; the general case can be derived with an argument similar to [1, p.167]. Now, let w_1, \dots, w_n denote the projections of the boxes of τ , and let u and v denote the projections of the boxes corresponding to u_0 and v_0 , respectively. W.l.o.g., let σ_1 be spanned by u and w_1, \dots, w_n and σ_2 be spanned by v and w_1, \dots, w_n . Because the orientation of σ_1 is preserved, and the orientation of σ_2 is not preserved, we have that

$$O(u, w_1, \dots, w_n) = O(v, w_1, \dots, w_n).$$

It follows that u and v are mapped to side of the hyperplane spanned by w_1, \dots, w_n .

B Non-integral projections

In this section, we reconsider the decision problem `Faithful_Embed` without the restriction that our projections are half-integral. We show that the problem remains NP-hard in this relaxed setup.

The proof strategy follows the same steps as in Section 3. We require, however, slightly more skinny rectangles: We call a rectangle *thin* if one of its sides has length one, and the other has length at least 8 (as opposed to length 4 in Section 3). The definitions of L-path, L-joint, front pixel, front half etc. directly carry over. Also, the L-joint lemma remains true for arbitrary projections; see Figure 7.

The NP-hardness proof uses the same gadgets as in the half-integral case, with minor modifications: The variable gadget now consists of rectangles of length 8, and all paths to clauses are also formed by rectangles of at least this length. The clause gadget is modified as

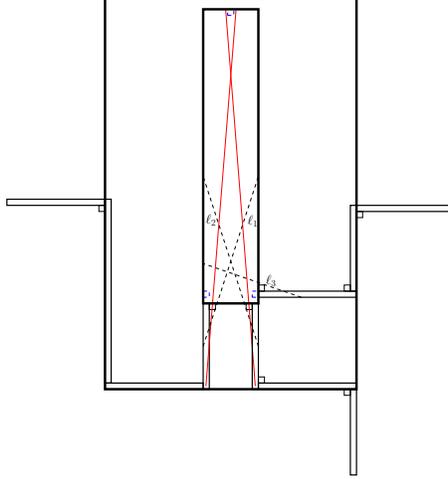


Figure 8: The modified clause gadget: If the vertices of all three incoming rectangles are placed in the back half, the projection of the clause triangle must be left of ℓ_1 , right of ℓ_2 and below ℓ_3 , which is impossible. If the projection is integral and at least one of the incoming rectangles has its vertex in the front pixel, we can choose the center of one of the dashed pixels as projection for the clause rectangle.

illustrated in Figure 8. It has three incoming paths, connecting variables to the clause, and the last rectangle is of length 14. The gadget is designed to satisfy the following properties similar to the Clause Lemma in Section 3:

1. If all three incoming thin rectangles have their vertices in the back half, the projection cannot be completed to an embedding.
2. If the projection is half-integral, and at least one of the incoming rectangles has its vertex in the front pixel, we can place a vertex in the clause rectangle such that the orientation of all incident triangles is preserved.

We can show that a formula encoded in a `grid3sat` instance is satisfiable if and only if the corresponding partition has an embedding. The proof is the same as for Lemma 4: Given an embedding, all vertices in a variable cycle must be either mapped to the front half or the back half, and we set the variable to 1 or 0 accordingly. This is a satisfying assignment, because for any clause, one of the incoming rectangles must have its vertex in the front half, and applying the L-joint lemma backwards ensures that the corresponding literal satisfies the clause. Vice versa, given a satisfying assignment, we can construct a half-integral projection in exactly the same way as in the proof of Lemma 4 which yields an embedding.

C Details of the counterexample constructions

We have created configurations in Theorems 7 and 8 which lead to a simplex whose orientation is not preserved under the center projection. We have to show that those initial

configurations can be completed to a β -balanced partition. For Theorem 7, this is relatively straight-forward. For the proof, we need the following definition: Let \mathcal{C} be a partition in \mathbb{R}^d , B a box in the partition with corners (q_1, \dots, q_D) (with $D = 2^d$), and p a point in the interior of B . *Splitting* B at p means to replace B in \mathcal{C} with D boxes defined by the corners p and q_i , with $1 \leq i \leq D$.

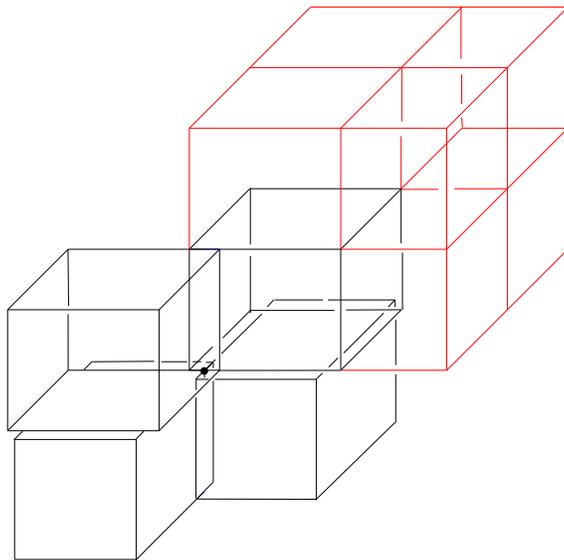


Figure 9: A part of the complete partition, after the split of one of the \tilde{B}_i .

Lemma 11. *The partial partition R_0, \dots, R_3 from Theorem 7 can be completed to a β -balanced partition in \mathbb{R}^3 .*

Proof. Let $b \in \mathbb{Z}$ as in Theorem 7, such that $\beta > \frac{b+2}{b-2}$. We will construct a partition of $B := [-2b, 2b]^3$, consisting of exactly $64 = 8 \cdot 8$ boxes, with R_0, \dots, R_3 among them. Fix a corner a_i of B (with $1 \leq i \leq 8$). It lies in an octant of \mathbb{R}^3 with respect to the origin, and there is exactly one box R_j (with $0 \leq j \leq 3$) that extends into this octant. Let c_j denote the corner of R_j that is closest to the origin (by construction, it is one unit length away from the origin). Note that c_j does not necessarily lie in the same octant as a_i . We consider the box \tilde{B}_i with (opposite) corners a_i and c_j . Observe that the side lengths of \tilde{B}_i are $2b - 1$, $2b$, or $2b + 1$. Also, B_1, \dots, B_8 form a partition of B , and each R_j (with $0 \leq j \leq 3$) lies in the corner of some B_i .

We split each \tilde{B}_i into 8 boxes: If \tilde{B}_i contains some R_j (with $0 \leq j \leq 3$), we let c_j^* denote the corner of R_j opposite to c_j , and split \tilde{B}_i at c_j^* ; clearly, one of the 8 created boxes is R_j . See Figure 9 for an illustration. If \tilde{B}_i does not contain any R_j , we simply split at an integer point closest to its center. It is simple to verify that all boxes created by these operations have side lengths of at least $b - 2$ and at most $b + 2$. Since $\frac{b+2}{b-2}$ by assumption, the rectangular partition is β -balanced, but not center-embeddable, because it contains R_0, \dots, R_3 . \square

For Theorem 8, the completion of the initial configuration is technically more challenging than in Theorem 7 because we can only use square boxes to fill the empty space. Indeed,

our construction needs much more filling space to “balance out” the differences between the square boxes, and it requires square boxes of many different size, all close to b .

The construction needs two preparatory lemmas. The first is a basic fact from elementary number theory

Lemma 12. *Let $z_0 := 1$ and z_1, \dots, z_k be the first k prime numbers (starting with $z_1 = 2$). Let $b \in \mathbb{Z}$ such that $b = \lambda \cdot (z_1 \cdots z_k) + 1$ with some $\lambda \in \mathbb{Z}$. Then, the $(k + 1)$ numbers $b + z_0 - 1, \dots, b + z_k - 1$ are pairwise coprime.*

Proof. Assume for a contradiction that $b + z_i - 1$ and $b + z_j - 1$ have a common prime factor p . Then $p = z_\ell$ for $1 \leq \ell \leq k$, because the two numbers differ by less than z_k . Moreover, $b - 1$ is a multiple of z_ℓ . It follows that z_ℓ divides both z_i and z_j , which is a contradiction. \square

The next lemma proves that we can fill arbitrary large boxes in \mathbb{R}^d just using square boxes with 2^d pairwise coprime side lengths.

Lemma 13. *Let $D := 2^d$ and let $b_1 < \dots < b_D$ be pairwise coprime integers. Then there exists some $L_0 \in \mathbb{Z}$ such that every box with smallest side length at least L_0 can be partitioned into square boxes of side lengths b_1, \dots, b_D . Moreover, the partition can be chosen such that a box of side length b_1 is in one of its corners.*

Proof. Set $B := \{b_1, \dots, b_D\}$. Choose L_0 such that for any integer $L \geq L_0$, and any two disjoint subsets B_1, B_2 of B , L can be represented as

$$L = \lambda_1 \prod_{b \in B_1} b + \lambda_2 \prod_{b \in B_2} b$$

with λ_1, λ_2 non-negative numbers. We prove inductively that any box with minimal side length at least L_0 can be partitioned. If $d = 1$, the statement is trivial. Let B be such a box in \mathbb{R}^d with $d > 1$ and let \tilde{B} denote the same box projected into \mathbb{R}^{d-1} . Using the induction hypothesis with the numbers $b_1, \dots, b_{D/2}$, we can partition \tilde{B} with square boxes in \mathbb{R}^{d-1} . If we carry this partition into \mathbb{R}^d with the lower side of every box being in the plane $x_d = 0$, this does not yield a box because the square boxes have different heights in x_d -direction; however, we can stack up boxes to match their heights. Formally, define $P_1 = b_1 \cdots b_{D/2}$ and place P_1/b_i vertical copies for a square box of length b_i . In this way, we construct a box whose projection to \mathbb{R}^{d-1} is \tilde{B} , and whose length in x_d -direction is P_1 . We can do exactly the same construction using the integers $b_{D/2+1}, \dots, b_D$, which yields a box of length $P_2 = b_{D/2+1} \cdots b_D$ in x_d -direction. Let L_d be the length of B in x_d -direction. By assumption, we can find non-negative integers λ_1, λ_2 with $\lambda_1 P_1 + \lambda_2 P_2 = L_d$. Thus, placing λ_1 copies of the box of first type and λ_2 copies of the box of second type gives the desired box. \square

Lemma 14. *The partial partition R_0, \dots, R_d from Theorem 8 can be completed to a β -balanced cubical partition in \mathbb{R}^3 .*

Proof. The first step is to choose a and b appropriately. Note that the proof of Theorem 6 requires that $\beta > \frac{b}{a} \geq \frac{d}{d-2}$, and such that $\det M_d^{(1)}$ is negative. We impose additional

conditions: Let $D := 2^d$ and z_1, \dots, z_{D-1} be the sequence of the first $D - 1$ prime numbers (note that the magnitude of z_{D-1} only depends on d). We choose a and b such that $b = \lambda z_1 \cdots z_{D-1} + 1$ for some $\lambda \in \mathbb{Z}$, such that

$$\beta > \frac{b + z_{D-1} - 1}{a} > \frac{b}{a} \geq \frac{d}{d-2},$$

and such that $\det M_d^{(1)}$ is negative. In particular, applying Lemma 12 yields D pairwise coprime integers $b := b_1 < \dots < b_D = b + z_{D-1} - 1$.

Lemma 13 applied on the sequence b_1, \dots, b_D asserts the existence of some L_0 such that any box with side lengths at least L_0 can be filled. We choose some $L \geq L_0 + 1$ that is a multiple of a , and set $B := [-L, L]^d$. We have to show that we can construct a partition of B containing R_0, \dots, R_d with square boxes of sizes a, b_1, b_2, \dots, b_D only. By our choice of a and b , the so-obtained cubical partition is β -balanced, and the claim is proven.

The construction works similar as in Lemma 11. We first partition B into D boxes \tilde{B}_j (which are not square boxes in general), each anchored at one corner of B , and with $d + 1$ of them containing one of the initial square boxes R_i . Each \tilde{B}_j has side lengths of at least $L - 1 \geq L_0$; therefore, it can be filled with square boxes of lengths b_1, \dots, b_D according to Lemma 13. If \tilde{B}_j contains some R_i , we can enforce that R_i is in the corner of the filling, as required. A special case is that \tilde{B}_j contains R_0 which is of length a . However, by construction, this \tilde{B}_j is a square box of length L , and we can simply fill it with square boxes of length a because L is a multiple of a . \square

D Determinant bound

We prove in this section that for the $(d + 1) \times (d + 1)$ matrix

$$M_d^{(0)} := \begin{pmatrix} 1 & -\frac{a}{2} & \cdots & \cdots & -\frac{a}{2} \\ 1 & \frac{b}{2} & -\frac{b}{2} & \cdots & \cdots & -\frac{b}{2} \\ 1 & -\frac{b}{2} & \frac{b}{2} & -\frac{b}{2} & \cdots & -\frac{b}{2} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & -\frac{b}{2} & \cdots & -\frac{b}{2} & \frac{b}{2} & -\frac{b}{2} \\ 1 & -\frac{b}{2} & \cdots & \cdots & -\frac{b}{2} & \frac{b}{2} \end{pmatrix},$$

it holds that

$$\det M_d^{(0)} = \frac{1}{2} b^{d-1} a - \frac{1}{2} (d-2) b^d = \frac{1}{2} b^{d-1} (da - (d-2)b).$$

Indeed, we subtract the $(i - 1)$ -st row from the i -th row for $i = n + 1, \dots, 2$ to obtain

$$\det M_d^{(0)} = \det \begin{pmatrix} 1 & -\frac{a}{2} & \cdots & \cdots & -\frac{a}{2} \\ 0 & \frac{a+b}{2} & \frac{a-b}{2} & \cdots & \cdots & \frac{a-b}{2} \\ 0 & -b & b & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -b & b & \cdots \end{pmatrix} = \det \begin{pmatrix} \frac{a+b}{2} & \frac{a-b}{2} & \cdots & \cdots & \frac{a-b}{2} \\ -b & b & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & -b & b \end{pmatrix},$$

where empty spots are zero. We simplify further by factoring out $1/2$ in the second row, factoring out b in rows $3, \dots, d+1$, and shifting the second row to the bottom, thereby changing the sign of every row that goes up. With these steps, we obtain

$$\det M_d^{(0)} = \det \begin{pmatrix} \frac{a+b}{2} & \frac{a-b}{2} & \cdots & \cdots & \frac{a-b}{2} \\ -b & b & & & \\ & -b & b & & \\ & & \ddots & \ddots & \\ & & & -b & b \end{pmatrix} = \frac{1}{2} b^{d-1} \det \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ a+b & a-b & \cdots & \cdots & a-b \end{pmatrix}.$$

It is straight-forward to verify that the determinant of the rightmost $(d \times d)$ -matrix equals the sum of its last row, which is $da - (d-2)b$.

E Details on quantifier elimination

The proof of Theorem 9 relies on the non-existence of planes that intersect certain point sets in \mathbb{R}^3 . We will give some details on how we prove this non-existence. Recall from the proof that we defined a and b to be the shortest and longest sides among the cubes R_0, \dots, R_3 . By scaling, we can just assume that $a = 1$. Because of the assumption, it follows that $b \leq \beta < 3$.

Regular case The regular configuration

$$\{0\}, \{1\}, \{2, 3\}, \{4, 5, 6, 7\},$$

induces the point sets

$$\begin{aligned} C_0^{(b)} &:= \text{hull}\{(-1, -1, -1), (-b, -b, -b)\} \\ C_1^{(b)} &:= \text{hull}\{(1, -1, -1), (b, -b, -b)\} \\ C_2^{(b)} &:= \text{hull}\{(-1, 1, -1), (-b, b, -b), (1, 1, -1), (b, b, -b)\} \\ &\quad - \text{hull}\{(-1, 1, -1), (-b, b, -b)\} - \text{hull}\{(1, 1, -1), (b, b, -b)\} \\ C_3^{(b)} &:= \text{hull}\{(-1, -1, 1), (-b, -b, b), (-1, 1, 1), (-b, b, b), (1, -1, 1), (b, -b, b), (1, 1, 1), (b, b, b)\} \\ &\quad - \text{hull}\{(-1, -1, 1), (-b, -b, b), (-1, 1, 1), (-b, b, b)\} \\ &\quad - \text{hull}\{(-1, -1, 1), (-b, -b, b), (1, -1, 1), (b, -b, b)\} \\ &\quad - \text{hull}\{(-1, 1, 1), (-b, b, b), (1, 1, 1), (b, b, b)\} \\ &\quad - \text{hull}\{(1, -1, 1), (b, -b, b), (1, 1, 1), (b, b, b)\}, \end{aligned}$$

where hull stands for the convex hull of a point set. Assume for a contradiction that e is a plane intersecting $C_0^{(b)}, \dots, C_3^{(b)}$. Since $C_i^{(b)} \subset C_i^{(3)}$, e intersects $C_0^{(3)}, \dots, C_3^{(3)}$ as well. Let e be defined by $t_0, \dots, t_3 \in \mathbb{R}$ via the equation

$$(t_0, t_1, t_2) \cdot (x, y, z) + t_3 = 0.$$

We first claim that $t_0 \neq 0$. Indeed, if $t_0 = 0$, the projection of e into the yz -plane is a line. This line has to intersect the projections of the $C_i^{(3)}$. However, these projections give the subsets of the rectangles defined in Figure 5 (left), and it can be seen with similar methods as in Theorem 6 that no such line exists for $\beta < 3$. Moreover, it can be easily verified that e is not parallel to any coordinate plane.

Since $t_0 \neq 0$, we can normalize such that $t_0 = 1$. We define the induced mapping

$$E : \mathbb{R}^3 \rightarrow \mathbb{R}, (x, y, z) \mapsto (1, t_1, t_2) \cdot (x, y, z) + t_3.$$

Clearly, $e = E^{-1}(0)$, and if for two points $p_1, p_2 \in \mathbb{R}^3$, $E(p_1) \cdot E(p_2) < 0$, then p_1 and p_2 are separated in different halfspaces by e . From this property, we can immediately deduce

Lemma 15. *e intersects each $C_i^{(3)}$, $1 \leq i \leq 4$ if and only if E satisfies the following four formulas*

$$\begin{aligned} E((-1, -1, -1)) \cdot E((-3, -3, -3)) &\leq 0 \\ E((1, -1, -1)) \cdot E((3, -3, -3)) &\leq 0 \\ E((-1, 1, -1)) \cdot E((3, 3, -3)) &< 0 \quad \vee \quad E((1, 1, -1)) \cdot E((-3, 3, -3)) < 0 \\ (E((-1, -1, 1)) \cdot E((3, 3, 3)) &< 0 \quad \vee \quad E((1, -1, 1)) \cdot E((-3, 3, 3)) < 0 \\ &\vee \quad E((-1, 1, 1)) \cdot E((3, -3, 3)) < 0 \\ &\vee \quad E((1, 1, 1)) \cdot E((-3, -3, 3)) < 0 \end{aligned}$$

Proof. The \Leftarrow -direction is simple to proof. For the \Rightarrow -direction, note that the first two statements are trivially satisfied. For the third statement, note first that e cannot contain the whole line segment from $(-1, 1, -1)$ to $(1, 1, -1)$, neither the line segment from $(-3, 3, -3)$ to $(3, 3, -3)$ (because this would contradict $t_0 \neq 0$). Let f be the plane that contains $C_2^{(3)}$. The intersection of e and f is a line on f that intersects the boundary of $C_2^{(3)}$ in exactly two points. It is therefore clear that this line separates two opposite vertices in two halfplanes. Thus, this pair is separated into two halfspaces by e and this is precisely what is checked by the third statement.

For the last statement, note that e cannot completely contain any boundary face of $C_3^{(3)}$, because this would mean that e is parallel to a coordinate plane which is easily checked to be impossible. Therefore, e intersects through the interior of $C_3^{(3)}$. Therefore e separates at least one of the four pairs of opposite vertices of the polytope, and this is checked in the fourth statement. \square

With Lemma 15, we can formulate a quantified formula in the real variables t_1, t_2, t_3 which is true if and only if a plane e as required exists. However, using a quantifier elimination program, we can easily compute that the formula is in fact false.

Singular case This case is completely analogous. For completeness, we write down the sets:

$$\begin{aligned}
C_0^{(b)} &:= \text{hull}\{(-1, -1, -1), (-b, -b, -b), (1, -1, -1), (b, -b, -b)\} \\
&\quad - \text{hull}\{(-1, -1, -1), (-b, -b, -b)\} - \text{hull}\{(1, -1, -1), (b, -b, -b)\} \\
C_1^{(b)} &:= \text{hull}\{(-1, 1, -1), (-b, b, -b), (1, 1, -1), (b, b, -b)\} \\
&\quad - \text{hull}\{(-1, 1, -1), (-b, b, -b)\} - \text{hull}\{(1, 1, -1), (b, b, -b)\} \\
C_2^{(b)} &:= \text{hull}\{(-1, -1, 1), (-b, -b, b), (-1, 1, 1), (-b, b, b)\} \\
&\quad - \text{hull}\{(-1, -1, 1), (-b, -b, b)\} - \text{hull}\{(-1, 1, 1), (-b, b, b)\} \\
C_3^{(b)} &:= \text{hull}\{(1, -1, 1), (b, -b, b), (1, 1, 1), (b, b, b)\} \\
&\quad - \text{hull}\{(1, -1, 1), (b, -b, b)\} - \text{hull}\{(1, 1, 1), (b, b, b)\}.
\end{aligned}$$

Defining e and E as above, we obtain:

Lemma 16. e intersects each $C_i^{(3)}$, $1 \leq i \leq 4$ if and only if E satisfies the following four formulas

$$\begin{aligned}
&E((-1, -1, -1)) \cdot E((3, -3, -3)) < 0 \quad \vee \quad E((1, -1, -1)) \cdot E((-3, -3, -3)) < 0 \\
&E((-1, 1, -1)) \cdot E((3, 3, -3)) < 0 \quad \vee \quad E((1, 1, -1)) \cdot E((-3, 3, -3)) < 0 \\
&E((-1, -1, 1)) \cdot E((-3, 3, 3)) < 0 \quad \vee \quad E((-1, 1, 1)) \cdot E((-3, -3, 3)) < 0 \\
&E((1, -1, 1)) \cdot E((3, 3, 3)) < 0 \quad \vee \quad E((1, 1, 1)) \cdot E((3, -3, 3)) < 0.
\end{aligned}$$

Again, it can be proven with a quantifier elimination program that the induced quantified formula is false.

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