

Linear Integer Arithmetic Revisited

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Abstract. We consider feasibility of linear integer programs in the context of verification systems such as SMT solvers or theorem provers. Although satisfiability of linear integer programs is decidable, many state-of-the-art solvers neglect termination in favor of efficiency. It is challenging to design a solver that is both terminating and practically efficient. Recent work by Jovanović and de Moura constitutes an important step into this direction. Their algorithm CUTSAT is sound, but does not terminate, in general. In this paper we extend their CUTSAT algorithm by refined inference rules, a new type of conflicting core, and a dedicated rule application strategy. This leads to our algorithm CUTSAT++, which guarantees termination.

Keywords: Linear arithmetic, SMT, SAT, DPLL, Linear programming, Integer arithmetic

1 Introduction

Historically, feasibility of linear integer problems is a classical problem, which has been addressed and thoroughly investigated by at least two independent research lines: (i) integer and mixed real integer linear programming for optimization [15], (ii) first-order quantifier elimination and decision procedures for Presburger Arithmetic and corresponding complexity results [19,4,9,17,6,7,20,2,3,10,11,21,16]. We are interested in feasibility of linear integer problems, which we call simply *problems*, in the context of the combination of theories, as they occur, e.g., in the context of SMT solving or theorem proving. From this perspective, both these research lines address problems that are too general for our purposes: with the former, the optimization aspects go considerably beyond pure feasibility. The latter considers arbitrary Boolean combinations of constraints and quantifier alternation or even parametric problems.

Consequently, the SMT community has developed several interesting approaches on their own [5,12,1]. These solvers typically neglect termination and completeness in favour of efficiency. More precisely, these approaches are based on a branch-and-bound strategy, where the rational relaxation of an integer problem is used to cut off and branch on integer solutions. Together with the known a priori integer bounds [18] for a problem this yields a terminating and complete algorithm. However, these bounds are so large that for many practical problems

the resulting branch-and-bound search space cannot be explored in reasonable time. Hence, the a priori bounds are not integrated in the implementations of the approaches.

On these grounds, the recent work by Jovanović and de Moura [13,14] constitutes an important step towards an algorithm that is both efficient and terminating. The termination does no longer rely on bounds that are a priori exponentially large in the occurring parameters. Instead, it relies on structural properties of the problem, which are explored by their CUTSAT algorithm. The price for this result is an algorithm that is by far more complicated than the above-mentioned branch-and-bound approach. In particular, it has to consider divisibility constraints in addition to inequalities.

Our interest in an algorithm for integer constraints originates from a possible combination with superposition, e.g., see [8]. In the superposition context integer constraints are part of the first-order clauses. Variables in constraints are typically unguarded, so that an efficient decision procedure for this case is a prerequisite for an efficient combined procedure.

Our contribution is an extension and refinement of the CUTSAT algorithm, which we call CUTSAT++. In contrast to CUTSAT, our CUTSAT++ generally terminates. The basic idea of both algorithms is to reduce a problem containing unguarded integer variables to a problem containing only guarded variables. These unguarded variables are not eliminated. Instead, one explores the unguarded variables by adding constraints on smaller variables to the problem, with respect to a strict total ordering where all unguarded variables are larger than all guarded variables. After adding sufficiently many constraints, feasibility of the problem only depends on guarded variables. Then a CDCL style algorithm tests for feasibility by employing exhaustive propagation. The most sophisticated part is to turn an unguarded variable into a guarded variable. Quantifier elimination techniques, such as Cooper elimination [4], do so by removing the unguarded variable. In case of Cooper elimination, the price to pay is an exponentially growing Boolean structure and exponentially growing coefficients (see Section 3). Since integer linear programming is NP-complete, all algorithms known today can not prevent such a kind of behavior, in general. Since Cooper elimination does not care about the concrete structure of a given problem, the exponential behavior is almost guaranteed. The idea of both CUTSAT and CUTSAT++ is therefore to simulate a lazy variation of Cooper elimination. This leaves space for model assumptions and simplification rules in order for the algorithm to adapt to the specific structure of a problem and hence to systematically avoid certain cases of the worst-case exponential behavior observed with Cooper elimination.

The paper is organized as follows. After fixing some notation, in Section 2 we present three examples for problems where CUTSAT diverges. The divergence of CUTSAT can be fixed by respective refinements on the original CUTSAT rules. However, in a fourth examples the combination of our refinements results in a frozen state. Our conclusion is that CUTSAT lacks, in addition to our rule refinements, a third type of conflicting cores, which we call *diophantine conflicting core*. Theorem 9 in Section 3 actually implies that any procedure

that is based on what we call *weak Cooper elimination* needs to consider this type of conflicting core for completeness. In Sections 4-5 we refine the inference rules for the elimination of unguarded variables on the basis of our results from Section 3 and show their soundness, completeness and termination. We finally give conclusions and point at possible directions for future research.

2 Motivation

We use *variables* x, y, z, k , possibly with indices. Furthermore, we use *integer constants* a, b, c, d, e, l, v, u , and *linear polynomials* p, q, r, s , possibly with indices. As input *problems*, we consider finite sets of constraints C corresponding to and some times used as conjunction over their elements. Each constraint I is either an inequality $a_n x_n + \dots + a_1 x_1 + c \leq 0$ or a divisibility constraint $d \mid a_n x_n + \dots + a_1 x_1 + c$. We denote $\text{coeff}(I, x_i) = a_i \in \mathbb{Z} \setminus \{0\}$. $\text{vars}(C)$ denotes the set of variables occurring in C . We sometimes write $C(x)$ in order to emphasise that $x \in \text{vars}(C(x))$. A problem C is satisfiable if $\exists X : C$ holds, where $X = \text{vars}(C)$. For true we denote \top and for false we denote \perp . Since $d \mid cx + s \equiv d \mid -cx + -s$, we may assume that $c > 0$ for all $d \mid cx + s \in C$. A variable x is *guarded* in a problem C if C contains constraints of the form $x - u_x \leq 0$ and $-x + l_x \leq 0$. Otherwise, x is *unguarded* in C . Note that in [14] guarded variables are called *bounded*. A constraint is *guarded* if it contains only guarded variables. Otherwise, it is *unguarded*.

Our algorithm CUTSAT++ aims at deciding whether or not a given problem C is satisfiable. It either ends in the state *unsat*, or in a state $\langle v, \text{sat} \rangle$ where v is a satisfiable assignment for C . In order to reach one of those two final states, the algorithm produces *lower bounds* $x \geq b$ and *upper bounds* $x \leq b$ for the variables in C . The produced bounds are stored in a sequence $M = \llbracket \gamma_1, \dots, \gamma_n \rrbracket$, which describes a partial model. The empty sequence is denoted by $\llbracket \rrbracket$. We use $\llbracket M, \gamma \rrbracket$ and $\llbracket M_1, M_2 \rrbracket$ to denote the concatenation of a bound γ at the end of M , and M_2 at the end of M_1 , respectively.

By $\text{lower}(x, M) = b$ and $\text{upper}(x, M) = b$ we denote the value b of the smallest lower bound $x \geq b$ and largest upper bound $x \leq b$ for a variable x in M , respectively [14]. If there is no lower (upper) bound for x in M , then $\text{lower}(x, M) = -\infty$ ($\text{upper}(x, M) = \infty$). We canonically extend the definitions of upper and lower to polynomials.

A state in CUTSAT++ is of the form $S = \langle M, C \rangle$ or $S = \langle M, C \rangle \vdash I$, or one of the two *final states* $\langle v, \text{sat} \rangle$, *unsat* [14]. The *initial-state* for a problem C is $\langle \llbracket \rrbracket, C \rangle$. The corresponding partial model M is complete if all variables x in C are *fixed*, in the sense that $\text{upper}(x, M) = \text{lower}(x, M)$. In this case, we define $v[M]$ as the assignment that assigns to every variable x the value $\text{lower}(x, M)$. With $\text{val}(p, M) = \text{lower}(p, M)$ we denote the value assigned to a fixed polynomial p , i.e., $\text{val}(p, M)$ is only defined if all variables occurring in p are fixed in M . A state is *frozen* if it is no final state and no rule is applicable.

For a state $S = \langle M, C \rangle \vdash I$, inequality $p \leq 0$ is a *conflict* if $\text{lower}(p, M) > 0$. For a state $S = \langle M, C \rangle \vdash I$, divisibility constraint $d \mid ax + p$ is a *conflict* if all

variables in p are fixed, and $d \nmid ab + \text{lower}(p, M)$ for all b with $\text{lower}(x, M) \leq b \leq \text{upper}(x, M)$. In a state $S = \langle M, C \rangle \vdash I$, the constraint I is always a conflict.

Our CUTSAT++ algorithm is defined as a transition system consisting of the following rules:

Decide

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle \llbracket M, x \geq b \rrbracket, C \rangle \quad \text{if} \begin{cases} \text{upper}(x, M) \neq +\infty, \\ \text{lower}(x, M) < b = \text{upper}(x, M) \end{cases}$$

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle \llbracket M, x \leq b \rrbracket, C \rangle \quad \text{if} \begin{cases} \text{lower}(x, M) \neq -\infty, \\ \text{lower}(x, M) = b < \text{upper}(x, M) \end{cases}$$

Propagate

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle \llbracket M, x \geq_I b \rrbracket, C \rangle \quad \text{if} \begin{cases} J \in C \text{ is an inequality} \\ \text{coeff}(J, x) < 0, \\ \text{improves}(J, x, M), \\ b = \text{bound}(J, x, M), \\ I = \text{tight}(J, x, M) \end{cases}$$

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle \llbracket M, x \leq_I b \rrbracket, C \rangle \quad \text{if} \begin{cases} J \in C \text{ is an inequality} \\ \text{coeff}(J, x) > 0, \\ \text{improves}(J, x, M), \\ b = \text{bound}(J, x, M), \\ I = \text{tight}(J, x, M) \end{cases}$$

Propagate-Div

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle \llbracket M, x \geq_I c \rrbracket, C \rangle \quad \text{if} \begin{cases} D = (d \mid ax + p) \in C, \text{val}(p, M) = k \\ b = \text{lower}(x, M), d \nmid ab + k \\ c = \text{bound}(D, x, M), c \leq \text{upper}(x, M) \\ I = \text{div-derive}(D, x, M) \end{cases}$$

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle \llbracket M, x \leq_I c \rrbracket, C \rangle \quad \text{if} \begin{cases} D = (d \mid ax + p) \in C, \text{val}(p, M) = k \\ b = \text{upper}(x, M), d \nmid ab + k \\ c = \text{bound}(D, x, M), c \geq \text{lower}(x, M) \\ I = \text{div-derive}(D, x, M) \end{cases}$$

Conflict

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle M, C \rangle \vdash p \leq 0 \quad \text{if } p \leq 0 \in C, \text{lower}(p, M) > 0$$

Conflict-Div

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle M, C \rangle \vdash I \quad \text{if} \begin{cases} J = (d \mid ax + p) \in C, \text{val}(p, M) = k \\ b = \text{lower}(x, M), d \nmid ab + k \\ \text{bound}(J, x, M) > \text{upper}(x, M) \\ I = \text{div-derive}(J, x, M) \end{cases}$$

$$\langle M, C \rangle \Longrightarrow_{\text{CS}} \langle M, C \rangle \vdash I \quad \text{if} \begin{cases} J = (d \mid ax + p) \in C, \text{val}(p, M) = k \\ b = \text{upper}(x, M), d \nmid ab + k \\ \text{bound}(J, x, M) < \text{lower}(x, M) \\ I = \text{div-derive}(J, x, M) \end{cases}$$

Unsat-Div		
$\langle M, C \rangle$	$\Rightarrow_{\text{CS}} \text{unsat}$	if $\begin{cases} d \mid a_1x_1 + \dots + a_nx_n + c \in C \\ \text{gcd}(d, a_1, \dots, a_n) \nmid c \end{cases}$
Sat		
$\langle M, C \rangle$	$\Rightarrow_{\text{CS}} \langle v[M], \text{sat} \rangle$	if $v[M]$ satisfies C
Forget		
$\langle M, C \cup \{J\} \rangle$	$\Rightarrow_{\text{CS}} \langle M, C \rangle$	if $C \vdash_{\mathbb{Z}} J$, and $J \notin C$
Slack-Intro		
$\langle M, C \rangle$	$\Rightarrow_{\text{CS}} \langle M, C \cup C_s \rangle$	if $\begin{cases} \langle M, C \rangle \text{ is stuck} \\ x \text{ is stuck} \\ x_S \text{ is the slack-variable} \\ C_s = \{-x_S \leq 0, x - x_S \leq 0, \\ \quad -x - x_S \leq 0\} \end{cases}$
Resolve		
$\langle \llbracket M, \gamma \rrbracket, C \rangle \vdash I$	$\Rightarrow_{\text{CS}} \langle M, C \rangle \vdash \text{resolve}(\gamma, I)$	if γ is an implied bound
Skip-Decision		
$\langle \llbracket M, \gamma \rrbracket, C \rangle \vdash p \leq 0$	$\Rightarrow_{\text{CS}} \langle M, C \rangle \vdash p \leq 0$	if $\begin{cases} \gamma \text{ is a decided bound,} \\ \text{lower}(p, M) > 0 \end{cases}$
Unsat		
$\langle M, C \rangle \vdash b \leq 0$ $b > 0$	$\Rightarrow_{\text{CS}} \text{unsat}$	if γ is a decided bound,
Backjump		
$\langle \llbracket M, \gamma, M' \rrbracket, C \rangle \vdash J$	$\Rightarrow_{\text{CS}} \langle \llbracket M, x \geq_I b \rrbracket, C \rangle$	if $\begin{cases} \gamma \text{ is a decided bound,} \\ \text{coeff}(J, x) < 0, \\ \text{improves}(J, x, M), \\ I = \text{tight}(J, x, M), \\ b = \text{bound}(J, x, M) \end{cases}$
$\langle \llbracket M, \gamma, M' \rrbracket, C \rangle \vdash J$	$\Rightarrow_{\text{CS}} \langle \llbracket M, x \leq_I b \rrbracket, C \rangle$	if $\begin{cases} \gamma \text{ is a decided bound,} \\ \text{coeff}(J, x) > 0, \\ \text{improves}(J, x, M), \\ I = \text{tight}(J, x, M), \\ b = \text{bound}(J, x, M) \end{cases}$
Learn		
$\langle M, C \rangle \vdash I$	$\Rightarrow_{\text{CS}} \langle M, C \cup \{I\} \rangle \vdash I$	if $I \notin C$

Solve-Div-Left

$$\langle M, C \rangle \Longrightarrow_{\text{cs}} \langle M, C' \rangle \quad \text{if} \quad \begin{cases} \text{divisibility constraints } I_1, I_2 \in C \\ x \text{ is top in } I_1 \text{ and } I_2 \\ \text{all other vars. in } I_1, I_2 \text{ are fixed} \\ (I'_1, I'_2) = \text{div-solve}(x, I_1, I_2), \\ C' = C \setminus \{I_1, I_2\} \cup \{I'_1, I'_2\} \\ I'_2 \text{ is not a conflict} \end{cases}$$

Solve-Div-Right

$$\langle M, C \rangle \Longrightarrow_{\text{cs}} \langle M', C' \rangle \quad \text{if} \quad \begin{cases} \text{divisibility constraints } I_1, I_2 \in C \\ x \text{ is top in } I_1 \text{ and } I_2 \\ \text{all other vars. in } I_1, I_2 \text{ are fixed} \\ (I'_1, I'_2) = \text{div-solve}(x, I_1, I_2), \\ C' = C \setminus \{I_1, I_2\} \cup \{I'_1, I'_2\} \\ I'_2 \text{ is a conflict} \\ y = \text{top}(I'_2) \\ M' = \text{prefix}(M, y) \end{cases}$$

Resolve-Cooper

$$\langle M, C \rangle \Longrightarrow_{\text{cs}} \langle M', C \cup R_k \cup R_c \rangle \quad \text{if} \quad \begin{cases} (x, C') \text{ is a conflicting core} \\ x \in U, \text{ all } y \prec x \text{ are fixed and } C' \subseteq C \\ \text{if } J \in C \text{ is a conflict then } \text{top}(J) \not\prec x \\ \text{cooper}(x, C') = (R_k, R_c) \\ y = \min_{I \in R_c} \{\text{top}(I)\} \\ M' = \text{prefix}(M, y) \end{cases}$$

CUTSAT [14] includes all of the rules of CUTSAT++, except for the rules Solve-Div-Left, Solve-Div-Right, and Resolve-Cooper, which are explained in more detail in Section 4.

Via applications of the rule Decide, CUTSAT++ adds *decided bounds* $x \leq b$ or $x \geq b$ to the sequence M in search-state S [14]. A decided bound generally assigns a variable x to $\text{lower}(x, M)$ or $\text{upper}(x, M)$. Via applications of the propagation rules, CUTSAT++ adds *propagated bounds* $x \geq_I b$ or $x \leq_I b$ to the sequence M , where I is a generated constraint propagating the bound. To this end the function $\text{bound}(J, x, M)$ computes the strictest bound value b and the function $\text{tight}(J, x, M)$ computes the corresponding justification I for constraint J under the partial model M [14]. For an inequality J , $\text{bound}(J, x, M)$ is defined as follows:

$$\text{bound}(ax + p \leq 0, x, M) = \begin{cases} - \left\lceil \frac{\text{lower}(p, M)}{a} \right\rceil & \text{if } a > 0, \\ - \left\lfloor \frac{\text{lower}(p, M)}{a} \right\rfloor & \text{if } a < 0. \end{cases}$$

Whenever $a > 0$, J propagates only upper bounds for x . Whenever $a < 0$, J propagates only lower bounds for x . For a divisibility constraint D , $\text{bound}(D, x, M)$ is defined as follows:

$$\text{bound}(d \mid ax + p, x, M) = \begin{cases} \left\lceil \frac{d \left\lceil \frac{ab+k}{d} \right\rceil - k}{a} \right\rceil & \text{if } b = \text{lower}(x, M), \\ \left\lfloor \frac{d \left\lfloor \frac{ab+k}{d} \right\rfloor - k}{a} \right\rfloor & \text{if } b = \text{upper}(x, M), \end{cases}$$

where $a > 0$, $d > 0$, all variables in p are fixed, and $\text{lower}(p) = k$. Whenever we choose $b = \text{lower}(x, M)$ in the above function, D propagates a lower bound. Whenever we choose $b = \text{upper}(x, M)$ in the above function, D propagates an upper bound. The bound value $\text{bound}(D, x, M)$ for divisibility constraint D is computed in such a way that CUTSAT++ never skips a satisfiable solution for D :

Lemma 1. *Let $D = d \mid ax + p$ be a divisibility constraint with $a > 0$. Let $\langle M, C \rangle$ be a state where the polynomial p is fixed. Let $\text{bound}(d \mid ax + p, x, M)$ denote a lower bound value. (1) Then it holds for all $e \in \{\text{lower}(x, M), \dots, \text{bound}(d \mid ax + p, x, M) - 1\}$ that $d \nmid ae + \text{lower}(p, M)$. Otherwise, $\text{bound}(d \mid ax + p, x, M)$ denotes an upper bound value: (2) Then it holds for all $e \in \{\text{bound}(d \mid ax + p, x, M, \leq) + 1, \dots, \text{upper}(x, M)\}$ that $d \nmid ae + \text{lower}(p, M)$.*

Proof. We only prove the case that $\text{bound}(d \mid ax + p, x, M)$ denotes a lower bound. The proof for the second case is analogous. Assume for a contradiction that there exists an $e \in \{\text{lower}(x, M), \dots, \text{bound}(d \mid ax + p, x, M) - 1\}$ such that $d \mid ae + \text{lower}(p, M)$ holds. Since $d \mid ae + \text{lower}(p, M)$, it holds that $\lceil \frac{ae+k}{d} \rceil = \frac{ae+k}{d}$ and $\left\lceil \frac{d \lceil \frac{ae+k}{d} \rceil - k}{a} \right\rceil = \left\lceil \frac{d \frac{ae+k}{d} - k}{a} \right\rceil = e$. Since $\text{lower}(x, M) \leq e$, it holds that $\left\lceil \frac{a \text{lower}(x, M) + k}{d} \right\rceil \leq \left\lceil \frac{ae+k}{d} \right\rceil$. Thence,

$$\text{bound}(d \mid ax + p, x, M, \geq) = \left\lceil \frac{d \lceil \frac{ae+k}{d} \rceil - k}{a} \right\rceil \leq \left\lceil \frac{d \lceil \frac{ae+k}{d} \rceil - k}{a} \right\rceil = e.$$

Therefore, $e \notin \{\text{lower}(x, M), \dots, \text{bound}(d \mid ax + p, x, M) - 1\}$, which contradicts our initial assumption. \square

The rules of the CUTSAT++ calculus are restricted in such a way that M stays *consistent*, i.e. $\text{lower}(x, M) \leq \text{upper}(x, M)$ for all variables $x \in X$. CUTSAT++ also propagates only bounds that are more *strict* than the current bound for the variable x , e.g. only propagate lower bound $x \geq b$ if $b > \text{lower}(x, M)$. This behaviour is expressed by the following predicate for inequalities $J = ax + p \leq 0$: $\text{improves}(J, x, M) = \begin{cases} \text{lower}(x, M) < \text{bound}(J, x, M) \leq \text{upper}(x, M), & \text{if } a < 0, \\ \text{lower}(x, M) \leq \text{bound}(J, x, M) < \text{upper}(x, M), & \text{if } a > 0. \end{cases}$

The justifications annotated to the propagated bounds are necessary for a CDCL-like conflict resolution. In CDCL, boolean resolution is used to combine the current conflict $C \vee l$ with a clause used for unit propagation $C' \vee \bar{l}$ to receive a new conflict $C \vee C'$ without literals l or \bar{l} . For CUTSAT++ the function resolve fulfils a similar purpose:

$$\text{resolve}(x \bowtie_{cx+q \leq 0} b, ax + p \leq 0) = \begin{cases} |a|q + |c|p \leq 0, & \text{if } a \cdot c < 0, \\ ax + p \leq 0, & \text{otherwise.} \end{cases}$$

Whenever $J = ax + p \leq 0$ is a conflict in state $\langle \llbracket M, \gamma \rrbracket, C \rangle \vdash J$, where $C \vdash_{\mathbb{Z}} J$ and $\gamma = x \bowtie_{cx+q \leq 0} b$, $J' = \text{resolve}(x \bowtie_{cx+q \leq 0} b, ax + p \leq 0)$ is also a conflict in state $\langle M, C \rangle \vdash J'$ and $C \vdash_{\mathbb{Z}} J'$ [14]. The function resolve only results in a new conflict because CUTSAT++ requires that the justification I of bound $x \bowtie_I b$, with $\bowtie \in \{\leq, \geq\}$, in state $\langle M, C \rangle (\vdash J)$ fulfils the following conditions: Firstly, I

Consume

$$\langle M, \pm ax + as \oplus ak y + r \rangle \quad \Longrightarrow_{\text{tight}} \langle M, \pm ax + as + ak y \oplus r \rangle,$$

where $x \neq y$.

Resolve-Implied

$$\langle \llbracket M, \gamma \rrbracket, \pm ax + as \oplus p \rangle \quad \Longrightarrow_{\text{tight}} \langle M, \pm ax + as \oplus q \rangle,$$

where γ is a propagated bound and $q \leq 0 = \text{resolve}(\gamma, p \leq 0)$.

Decide-Lower

$$\langle \llbracket M, y \geq b \rrbracket, \pm ax + as \oplus cy + r \rangle \quad \Longrightarrow_{\text{tight}} \langle M, \pm ax + as + ak y \oplus r + (ak - c)q \rangle,$$

where $y \leq_I b$ in M , with $I = y + q \leq 0$, and $k = \lfloor \frac{c}{a} \rfloor$.

Decide-Lower-Neg

$$\langle \llbracket M, y \geq b \rrbracket, \pm ax + as \oplus cy + r \rangle \quad \Longrightarrow_{\text{tight}} \langle M, \pm ax + as \oplus cq + r \rangle,$$

where $y \leq_I b$ in M , with $I = y - q \leq 0$, and $c < 0$.

Decide-Upper

$$\langle \llbracket M, y \leq b \rrbracket, \pm ax + as \oplus cy + r \rangle \quad \Longrightarrow_{\text{tight}} \langle M, \pm ax + as + ak y \oplus r + (c - ak)q \rangle,$$

where $y \geq_I b$ in M , with $I = -y + q \leq 0$, and $k = \lfloor \frac{c}{a} \rfloor$.

Decide-Upper-Pos

$$\langle \llbracket M, y \leq b \rrbracket, \pm ax + as \oplus cy + r \rangle \quad \Longrightarrow_{\text{tight}} \langle M, \pm ax + as \oplus cq + r \rangle,$$

where $y \geq_I b$ in M , with $I = -y + q \leq 0$, and $c > 0$.

Round (and terminate)

$$\langle M, \pm ax + as \oplus b \rangle \quad \Longrightarrow_{\text{tight}} \pm x + s + \left\lfloor \frac{b}{a} \right\rfloor \leq 0$$

Fig. 1. Rule system that derives tightly propagating inequalities [14]

is an inequality and $C \vdash_{\mathbb{Z}} I$. Secondly, if $\bowtie = \leq$, then $\text{coeff}(I, x) = 1$. If $\bowtie = \geq$, then $\text{coeff}(I, x) = -1$. Finally, $\text{bound}(I, x, M) \bowtie b$, i.e. the justification I implies at least a bound as strong as $x \bowtie_I b$.

The function $\text{tight}(J, x, M) = I$ defined by the set of rules in Figure 2 calculates a justification I in variable x for the bound $x \bowtie_I b$ propagated from inequality $J = \pm ax + p \leq 0$, where $b = \text{bound}(J, x, M)$. A state in this rule system is a pair

$$\langle M', \pm ax + as \oplus r \rangle,$$

where $a > 0$, s and r are polynomials, and M' is a prefix of the initial M , i.e. $M = \llbracket M', M'' \rrbracket$. The initial state for $\text{tight}(\pm ax + p \leq 0, x, M)$ is $\langle M, \pm ax \oplus p \rangle$. The first goal is to produce an inequality where all coefficients are divisible by $a = \text{coeff}(J, x)$. To this end, we apply the rules (Fig. 2) until the polynomial on the right side of \oplus becomes empty. In this case, all coefficients are divisible by $a = \text{coeff}(J, x)$ and we derive the justification I with the final rule Round.

The function $\text{div-derive}(D, x, M)$ calculates a justification I in variable x for the bound $x \bowtie_I b$ propagated from the divisibility constraint $D = d \mid ax + p$,

where $b = \text{bound}(D, x, M)$. The justification computed by div-derive is derived from a set of inequalities describing the propagated bound value. For instance, let us look at the lower bound value:

$$\text{bound}(D, x, M) = \left\lceil \frac{d \lceil \frac{ab+k}{d} \rceil - k}{a} \right\rceil,$$

where $D = d \mid ax + p$, $a > 0$, $d > 0$, p is fixed, $k = \text{lower}(p, M)$, and $b = \text{lower}(x, M)$. The sub-term $c = \lceil \frac{ab+k}{d} \rceil$ is equal to the bound value computable from the *diophantine representation* $dz = ax + p$ of the divisibility constraint D :

$$\text{bound}(-dz + ax + p \leq 0, z, M) = \lceil \frac{ab+k}{d} \rceil.$$

Notice that z is a variable not occurring in the problem, and only introduced for the above calculation. The fitting tightly propagating inequality for the sub-terms are abbreviated with $\text{div-part}(D, x, M)$:

$$\text{div-part}(D, x, M) = \begin{cases} \text{tight}(-dz + ax + p \leq 0, x, M), & \text{if } b = \text{lower}(x, M), \\ \text{tight}(dz - ax - p \leq 0, x, M) & \text{if } b = \text{upper}(x, M). \end{cases}$$

For div-part , we forbid tight to apply the Consume rule to the variables x and z . The restriction to Consume guarantees that the inequality $\pm z + r \leq 0 = \text{div-part}(D, x, M)$ does not contain x . Given $I_2 = -z + r \leq 0 = \text{div-part}(D, x, M)$ and $I_1 = dz - ax - p \leq 0$, we use $\text{resolve}(x \geq_{I_2} c, I_1) = -ax + dr - p \leq 0 = I_3$ to receive the inequality that computes the complete lower bound:

$$\text{bound}(I_3, x, M) = \left\lceil \frac{d \text{lower}(r, M) - k}{a} \right\rceil \geq \left\lceil \frac{d \lceil \frac{ab+k}{d} \rceil - k}{a} \right\rceil.$$

Finally, we compute the tightly propagating inequality for a divisibility constraint $D = d \mid ax + p$ with the function $\text{div-derive}(D, x, M)$:

$$\text{div-derive}(D, x, M) = \begin{cases} I'_3 & \text{if } \begin{cases} b = \text{lower}(x, M), \\ I_2 = -z + r \leq 0 = \text{div-part}(D, x, M), \\ I'_3 = \text{tight}(-ax + dr - p \leq 0, x, M), \end{cases} \\ I'_3 & \text{if } \begin{cases} b = \text{upper}(x, M), \\ I_2 = z + r \leq 0 = \text{div-part}(D, x, M), \\ I'_3 = \text{tight}(ax + dr + p \leq 0, x, M). \end{cases} \end{cases}$$

The rule Slack-Intro is necessary to prevent a special type of frozen state called stuck state. A variable x is called *stuck* in state $S = \langle M, C \rangle$ if M contains no bounds for x and there is no inequality $I = ax + p \leq 0 \in C$ that propagates a bound for x [14]. Variables x with a constraint of the form $\pm x - b \leq 0 \in C$ are never stuck, as CUTSAT++ is able to propagate at least one bound for x , i.e., either $x \geq -b$ or $x \leq b$. A state S is a *stuck state* if all unfixed variables x are stuck and if the rules Sat, Unsat-Div, Conflict, and Conflict-Div are not applicable [14]. In a stuck state, Slack-Intro is applicable and one of the previously stuck variables x is turned unstuck by the constraints added to the problem. As recommended in [14], CUTSAT++ uses the same slack variable for all Slack-Intro applications.

We are now going to discuss three examples where CUTSAT diverges. The first one shows that, CUTSAT can apply Conflict and Conflict-Div infinitely often to constraints containing unguarded variables.

Example 2. Let

$$C := \underbrace{\{-x \leq 0\}}_{I_x} \underbrace{\{-y \leq 0\}}_{I_y} \underbrace{\{-z \leq 0\}}_{I_{z1}} \underbrace{\{z \leq 0\}}_{I_{z2}} \underbrace{\{z + 1 \leq 0\}}_{I_{z3}} \underbrace{\{1 - x + y \leq 0\}}_{J_1} \underbrace{\{x - y \leq 0\}}_{J_2}$$

be a problem. Let $S_i = \langle M_i, C \rangle$ for $i \in \mathbb{N}$ be a series of states with:

$$\begin{aligned} M_0 &:= \llbracket x \geq_{I_x} 0, y \geq_{I_y} 0, z \geq_{I_{z1}} 0, z \leq_{I_{z2}} 0 \rrbracket, \\ M_{i+1} &:= \llbracket M_i, x \geq_{J_1} i + 1, y \geq_{J_2} i + 1 \rrbracket. \end{aligned}$$

Let the variable order be given by $z \prec y \prec x$. CUTSAT with a two-layered strategy [14] has diverging runs starting in state $S'_0 = \langle \llbracket \rrbracket, C \rangle$. Let CUTSAT traverse the states $S'_0, S_0, S_1, S_2, \dots$ in the following fashion: CUTSAT reaches state S_0 from state S'_0 after propagating the constraints I_x, I_y, I_{z1} , and I_{z2} . CUTSAT reaches state S_{i+1} from state S_i after:

- fixing x to i with a Decision $\gamma_d^x := x \leq i$; $M_i^1 := \llbracket M_i, \gamma_d^x \rrbracket$ and $S_i^1 := \langle M_i^1, C \rangle$
- applying Conflict to Constraint J_1 because $\text{lower}(1 - x + y, M_i^1) > 0$; $M_i^2 := M_i^1$ and $S_i^2 := \langle M_i^2, C \rangle \vdash J_1$
- undoing the decided bound γ_d^x by applying Backjump because the predicate $\text{improves}(J_1, x, M_i, \geq)$ evaluates to true. The result is the exchange of γ_d^x with the bound $\gamma^x = x \geq_{J_1} i + 1$; $M_i^3 := \llbracket M_i^2, \gamma^x \rrbracket$ and $S_i^3 := \langle M_i^3, C \rangle$
- fixing y to i with a Decision $\gamma_d^y := y \leq i$; $M_i^4 := \llbracket M_i^3, \gamma_d^y \rrbracket$ and $S_i^4 := \langle M_i^4, C \rangle$
- applying Conflict to Constraint J_2 because $\text{lower}(x - y, M_i^4) > 0$; $M_i^5 := M_i^4$ and $S_i^5 := \langle M_i^5, C \rangle \vdash J_1$
- undoing the decided bound γ_d^y by applying Backjump because the predicate $\text{improves}(J_2, y, M_i^5, \geq)$ evaluates to true. The result is the exchange of γ_d^y with the bound $\gamma^y = y \geq_{J_2} i + 1$; $M_i^6 := \llbracket M_i^5, \gamma^y \rrbracket$ and $S_{i+1} = S_i^6 := \langle M_i^6, C \rangle$

Since $\{I_{z1}, I_{z3}\}$ is a conflicting core, the variable z is the minimal conflicting variable in the states S_i, S_i^1 and S_i^4 . Since I_{z1} and I_{z2} bound z , the conflicting core is also guarded. Therefore, Resolve-Cooper as defined in [14] is not applicable, which in turn implies that Conflict is applicable.

A straightforward fix to example 2 is to limit the application of the Conflict and Conflict-Div rules to guarded constraints. Our second example shows, that CUTSAT can still diverge by infinitely many applications of the Solve-Div rule [14].

Example 3. Let d_i be the sequence with $d_0 = 2$ and $d_{k+1} := d_k^2$ for $k \in \mathbb{N}$, let $C_0 = \{4 \mid 2x + 2y, 2 \mid x + z\}$ be a problem, and let $S_0 = \langle \llbracket \rrbracket, C_0 \rangle$ be the initial CUTSAT state. Let the variable order be given by $x \prec y \prec z$. Then CUTSAT has divergent runs $S_0 \Rightarrow_{\text{CS}} S_1 \Rightarrow_{\text{CS}} S_2 \Rightarrow_{\text{CS}} \dots$. For instance, let CUTSAT apply the Solve-Div rule whenever applicable. By an inductive argument, Solve-Div is applicable in every state $S_n = \langle \llbracket \rrbracket, C_n \rangle$, and the constraint set C_n has the following form:

$$C_n = \begin{cases} \{2d_n \mid d_n x + d_n y, d_n \mid \frac{d_n}{2} y - \frac{d_n}{2} z\} & \text{if } n \text{ is odd,} \\ \{2d_n \mid d_n x + d_n y, d_n \mid \frac{d_n}{2} x + \frac{d_n}{2} z\} & \text{if } n \text{ is even.} \end{cases}$$

Therefore, CUTSAT applies Solve-Div infinitely often and diverges.

A straightforward fix to example 3 is to limit the application of Solve-Div to maximal variables in the variable order \prec . Our third example shows, that

CUTSAT can apply Conflict and Conflict-Div [14] infinitely often. The example 4 differs from example 2 in that the conflicting core contains also unguarded variables.

Example 4. Let

$$C := \left\{ \underbrace{-x \leq 0}_{I_x}, \underbrace{-y \leq 0}_{I_y}, \underbrace{-z \leq 0}_{I_{z1}}, \underbrace{z \leq 0}_{I_{z2}}, \underbrace{1 - x + y + z \leq 0}_{J_1}, \underbrace{x - y - z \leq 0}_{J_2} \right\}$$

be a problem. Let $S_i = \langle M_i, C \rangle$ for $i \in \mathbb{N}$ be a series of states with:

$$M_0 := \llbracket x \geq_{I_x} 0, y \geq_{I_y} 0, z \geq_{I_{z1}} 0, z \leq_{I_{z2}} 0 \rrbracket,$$

$$M_{i+1} := \llbracket M_i, x \geq_{J_1} i + 1, y \geq_{J_2} i + 1 \rrbracket.$$

Let the variable order be given by $z \prec x \prec y$. CUTSAT has diverging runs starting in state $S'_0 = \langle \llbracket \rrbracket, C \rangle$. For instance, let CUTSAT traverse the states $S'_0, S_0, S_1, S_2, \dots$ in the following fashion: CUTSAT reaches state S_0 from state S'_0 after propagating the constraints I_x, I_y, I_{z2} and I_{z1} . CUTSAT reaches state S_{i+1} from state S_i in the same way as in example 2. Notice that the conflicting core $\{J_1, J_2\}$ is bounded in [14], which admits the application of Conflict.

For example 4, applying the fix suggested for example 2 results in a frozen state. Here, a straightforward fix is to change the definition of conflicting cores to cover only those cores where the conflicting variable is the maximal variable.¹

The fixes for our examples suggested above are restrictions of CUTSAT which have the consequence that Conflict(-Div) cannot be applied to unguarded constraints, Solve-Div is only applicable for the elimination of the maximal variable, and the conflicting variable x is the maximal variable in the associated conflicting core C' . However, our next and final example shows that these restrictions lead to frozen states.

Example 5. Let CUTSAT include restrictions to maximal variables in the definition of conflicting cores, and in the Solve-Div rule as described above. Let there be additional restrictions in CUTSAT to the rules Conflict and Conflict-Div such that these rules are only applicable to conflict constraints I where I contains no unguarded variable. Let

$$C := \left\{ \underbrace{-x \leq 0}_{I_{x1}}, \underbrace{x - 1 \leq 0}_{I_{x2}}, \underbrace{-y \leq 0}_{I_y}, \underbrace{6 \mid 4y + x}_J \right\}$$

be a problem. Let $M := \llbracket x \geq_{I_{x1}} 0, x \leq_{I_{x2}} 1, y \geq_{I_y} 0, x \geq 1, y \leq 0 \rrbracket$ be a bound sequence. Let the variable order be given by $x \prec y$. CUTSAT has a run starting in state $S'_0 = \langle \llbracket \rrbracket, C \rangle$ that ends in the frozen state $S = \langle M, C \rangle$. Let CUTSAT propagate I_{x1}, I_{x2}, I_y , and fix x to 1 and y to 0 with two Decisions. Through these Decisions, the constraint J is a conflict. Since y is unguarded, CUTSAT cannot apply the rule Conflict-Div. Furthermore, [14] has defined conflicting cores as either interval or divisibility conflicting cores. The state S contains neither an interval or a divisibility conflicting core. Therefore, CUTSAT cannot apply the rule Resolve-Cooper. The remaining rules are also not applicable because

¹ The restrictions to maximal variables in the definition of the conflicting core and to the Solve-Div rule were both confirmed as missing but necessary in a private communication with Jovanović.

all variables are fixed and there is only one divisibility constraint. Without the before introduced restrictions to the rules Conflict(-Div), Solve-Div, CUTSAT diverges on the example.

3 Weak Cooper Elimination

In order to fix the frozen state of Example 5 in the previous section, we are going to introduce in Section 4 a new conflicting core, which we call *diophantine conflicting core*. For understanding diophantine conflicting cores, as well as further modifications to be made, it is helpful to understand the connection between CUTSAT and a variant of Cooper's quantifier elimination procedure [4].

The original *Cooper elimination* takes a variable x , a problem $C(x)$, and produces a disjunction of problems, equivalent to $\exists x : C(x)$:

$$\exists x : C(x) \equiv \bigvee_{0 \leq k < m} C_{-\infty}(k) \vee \bigvee_{-ax+p \leq 0 \in C} \bigvee_{0 \leq k < a \cdot m} \left[a \mid p + k \wedge C\left(\frac{p+k}{a}\right) \right],$$

where $a > 0$, $m = \text{lcm}\{d \in \mathbb{Z} \mid (d \mid ax + p) \in C\}$. If there exists no constraint of the form $-ax + p \leq 0 \in C$, then $C_{-\infty}(x) = \{d \mid ax + p \in C\}$. Otherwise, $C_{-\infty}(x) = \perp$. One application of Cooper elimination results in a disjunction of quadratically many problems out of a single problem. Iteration causes an exponential increase in the coefficients due to the multiplication with a because division is not part of the language.

Weak Cooper elimination is a variant of Cooper elimination, which is very helpful to understand problems around CUTSAT. The idea is, instead of building a disjunction over all potential solutions for x , to add additional guarded variables and constraints without x that guarantee the existence of a solution for x . We assume here that $C(x)$ contains only one divisibility constraint for x . If not, exhaustive application of div-solve to divisibility constraints for x removes all constraints except one: $\text{div-solve}(x, d_1 \mid a_1x + p_1, d_2 \mid a_2x + p_2) = (d_1d_2 \mid dx + c_1d_2p_1 + c_2d_1p_2, d \mid -a_1p_2 + a_2p_1)$, where $d = \text{gcd}(a_1d_2, a_2d_1)$, and c_1 and c_2 are integers such that $c_1a_1d_2 + c_2a_2d_1 = d$ [4,14]. Now weak Cooper elimination takes a variable x , a problem $C(x)$ and produces a new problem by replacing $\exists x : C(x)$ with:

$$\exists K : \left(\{I \in C(x) \mid \text{coeff}(I, x) = 0\} \cup \{\text{gcd}(c, d) \mid s\} \cup \bigcup_{k \in K} R_k \right),$$

where $d \mid cx + s \in C(x)$, $k \in K$ is a newly introduced variable for every pair of constraints $-ax + p \leq 0 \in C(x)$ and $bx - q \leq 0 \in C(x)$,

$$R_k = \{-k \leq 0, k - m \leq 0, bp - aq + bk \leq 0, a \mid k + p, ad \mid cp + as + ck\}$$

is a resolvent for the same inequalities [14], where $m := \text{lcm}\left(a, \frac{ad}{\text{gcd}(ad, c)}\right) - 1$. Note that, there is still an existential quantifier $\exists K$ but all variables $k \in K$ are guarded by their respective R_k .

Let ν be a satisfiable assignment for the formula after one weak Cooper elimination step on $C(x)$. Then we compute a strictest lower bound $x \geq l_x$ and a strictest upper bound $x \leq u_x$ from $C(x)$ for the variable x under the assignment ν . We now argue that there is a value v_x for x such that $x \geq l_x$, $x \leq u_x$, and $d \mid cv_x + s$ are satisfied. Whenever $l_x \neq -\infty$ and $u_x \neq \infty$, the bounds $x \geq l_x$,

$x \leq u_x$ are given by respective constraints of the form $-ax + p \leq 0 \in C(x)$ and $bx - q \leq 0 \in C(x)$ such that $l_x = \lceil \frac{\nu(p)}{a} \rceil$ and $u_x = \lfloor \frac{\nu(q)}{b} \rfloor$. In this case the extension of ν with $\nu(x) = \frac{\nu(k+p)}{a}$ satisfies $C(x)$ because the constraint $a \mid k + p \in R_k$ guarantees that $\nu(x) \in \mathbb{Z}$, the constraint $bp - aq + bk \leq 0 \in R_k$ guarantees that $l_x \leq \nu(x) \leq u_x$, and the constraint $ad \mid cp + as + ck \in R_k$ guarantees that ν satisfies $d \mid cx + s \in C(x)$. Whenever $l_x = -\infty$ ($u_x = \infty$) we extend ν by an arbitrary small (large) value for x that satisfies $d \mid cx + s \in C(x)$. There exist arbitrarily small (large) solutions for x and $d \mid cx + \nu(s)$ because $\gcd(c, d) \mid s$ is satisfied by ν .

The advantage of weak Cooper elimination compared to Cooper elimination is that the output is again one conjunctive problem in contrast to a disjunction of problems. Our CUTSAT++ performs weak Cooper elimination not in one step but subsequently adds to the states the constraints from the R_k as well as the divisibility constraint $\gcd(c, d) \mid s$ with respect to a strict ordering on the unguarded variables.

The following equivalence, for which we have just outlined the proof, states the correctness of weak Cooper elimination:

$$\exists x : C(x) \equiv \exists K : \left(\{I \in C(x) \mid \text{coeff}(I, x) = 0\} \cup \{\gcd(c, d) \mid s\} \cup \bigcup_{k \in K} R_k \right).$$

The extra divisibility constraint $\gcd(c, d) \mid s$ in weak Cooper elimination is necessary whenever the problem $C(x)$ has no constraint of the form $-ax + p \leq 0 \in C(x)$ or $bx - q \leq 0 \in C(x)$. For example, let $C(x) = \{y - 1 \leq 0, -y + 1 \leq 0, 6 \mid 2x + y\}$ be a problem and x be the unguarded variable we want to eliminate. As there are no inequalities containing x , weak Cooper elimination without the extra divisibility constraint returns $C' = \{y - 1 \leq 0, -y + 1 \leq 0\}$. While C' has a satisfiable assignment $\nu(y) = 1$, $C(x)$ has not since $2x + 1$ is never divisible by 2 or 6.

Note that for any R_k introduced by weak Cooper elimination we can also show the following Lemma:

Lemma 6. *Let k be a new variable. Let $a, b, c > 0$. Then,*

$$\begin{aligned} & (\exists x : \{-ax + p \leq 0, bx - q \leq 0, d \mid cx + s\}) \\ \equiv & (\exists k : \{-k \leq 0, k - m \leq 0, bp - aq + bk \leq 0, a \mid k + p, ad \mid cp + as + ck\}). \end{aligned}$$

Proof. See [14] pp. 101-102 Lemma 4. □

That means satisfiability of the respective R_k guarantees a solution for the triple of constraints it is derived from. An analogous Lemma holds for the divisibility constraint $\gcd(c, d) \mid s$ introduced by weak Cooper elimination:

Lemma 7. $(\exists x : d \mid cx + s) \equiv \gcd(c, d) \mid s$.

Proof. We equivalently rewrite the two divisibility constraints into diophantine equations, viz. $\exists y : dy - cx = s$, and $\exists k : \gcd(c, d)k = s$ for $d \mid cx + s$, and $\gcd(c, d) \mid s$, respectively. We choose $d', c' \in \mathbb{Z}$ such that $d' \cdot \gcd(c, d) = d$ and

$c' \cdot \gcd(c, d) = c$. Assume that ν is a variable assignment such that $d\nu(y) - c\nu(x) = \nu(s)$ and therefore also $d \mid c\nu(x) + \nu(s)$. Thence $\nu(s) = d\nu(y) - c\nu(x) = \gcd(c, d) \cdot (d'\nu(y) - c'\nu(x))$. After extending ν with $\nu(k) = (d'\nu(y) - c'\nu(x))$, ν satisfies $\gcd(c, d)k = s$.

Assume, that ν is a variable assignment such that $\gcd(c, d)\nu(k) = \nu(s)$ holds and therefore also $\gcd(c, d) \mid \nu(s)$. By Bežout's Lemma there exist $a', b' \in \mathbb{Z}$, such that $a'd - b'c = \gcd(c, d)$. Thence $a'd\nu(k) - b'c\nu(k) = (a'd - b'c)\nu(k) = \gcd(c, d)\nu(k) = \nu(s)$. After extending ν with $\nu(y) = a'\nu(k)$ and $\nu(x) = b'\nu(k)$ the assignment ν satisfies $dy - cx = s$. \square

That means satisfiability of $\gcd(c, d) \mid s$ guarantees a solution for the divisibility constraint $d \mid cx + s$. The rule Resolve-Cooper (Fig. 4) in our CUTSAT++ exploits these properties by generating the R_k and constraint $\gcd(c, d) \mid s$ in the form of strong resolvents in a lazy way. Furthermore, it is not necessary for the divisibility constraints to be a priori reduced to one, as done for weak Cooper elimination. Instead, the rules Solve-Div-Left and Solve-Div-Right (Fig. 4) perform lazy reduction.

The *solution set* for variable x , assignment ν , and problem $C(x)$, is the set of values $S \subseteq \mathbb{Z}$ such that $v \in S$ if $C(v)$ is satisfied by ν . The solution set S_d of a divisibility constraint $d \mid cx + s$, variable x , and assignment ν is either empty or unbounded from above and below.

Lemma 8. *Let ν be an assignment for all variables except x . Let S_d be the solution set for variable x , assignment ν , and constraint $d \mid cx + s$. Then,*

$$S_d = \emptyset \text{ or } S_d = \{v_0 + ev' \mid e \in \mathbb{Z}\} \text{ for some } v_0, v' \in \mathbb{Z}.$$

Proof. In case $S_d \neq \emptyset$, there exists a value $v_0 \in S_d$ such that $d \mid cv_0 + \nu(s)$. We first prove that there exists a $v' \in \mathbb{Z}$ such that $d \mid c(v_0 + ev') + \nu(s)$ for all $e \in \mathbb{Z}$ and therefore $v_0 + ev' \in S_d$. We choose $v', e' \in \mathbb{Z}$ such that $c = e' \gcd(c, d)$ and $d = v' \gcd(c, d)$. Then we deduce for any $e \in \mathbb{Z}$:

$$\begin{aligned} d \mid c(v_0 + ev') + \nu(s) &\equiv d \mid cv_0 + \nu(s) + cev' \equiv \\ &d \mid cv_0 + \nu(s) + de'v' \equiv d \mid cv_0 + \nu(s) \end{aligned}$$

It remains to show that for every $v_k \in S_d$ there exists an $e \in \mathbb{Z}$ such that $v_0 + ev' = v_k$. As S_d is the solution set we know that $d \mid cv_0 + \nu(s)$ and $d \mid cv_k + \nu(s)$ are true. Thence $d \mid c(v_0 - v_k) \equiv d \mid cv_0 + \nu(s) - (cv_k + \nu(s))$. As $d = v' \gcd(c, d)$, the term $c(v_0 - v_k)$ is only divisible by d if $v_0 - v_k$ is divisible by v' . Therefore, $\exists e \in \mathbb{Z} : v_0 - v_k = ev'$. \square

This property allows us to choose an arbitrary small or large solution for x to satisfies $d \mid cx + \nu(s)$ in the correctness proof of weak Cooper Elimination. As mentioned in the outline of the proof, the ability to choose arbitrary small and large solutions for x is necessary when $C(x)$ contains no constraints of the form $-ax + p \leq 0$ or $bx - q \leq 0$.

Theorem 9.

$$\exists x : C(x) \equiv \exists K : \left(\{I \in C(x) \mid \text{coeff}(I, x) = 0\} \cup \{\gcd(c, d) \mid s\} \cup \bigcup_{k \in K} R_k \right)$$

Proof. First, we partition the problem $C(x)$ as follows:

$$\begin{aligned} C_l &= \{-ax + p \leq 0 \in C \mid a > 0\}, & C_u &= \{bx - q \leq 0 \in C \mid b > 0\}, \\ I_d &= d \mid cx + s \in C, & C_r &= \{I \in C \mid \text{coeff}(I, x) = 0\}. \end{aligned}$$

By Lemma 6, it holds for all $-ax + p \leq 0, bx - q \leq 0 \in C'$ with $a, b > 0$ that:

$$\begin{aligned} &(\exists x : C) \rightarrow (\exists x : \{-ax + p \leq 0, bx - q \leq 0, d \mid cx + s\}) \\ &\rightarrow (\exists k : \underbrace{\{-k \leq 0, k - m \leq 0, bp - aq + bk \leq 0, a \mid k + p, ad \mid cp + as + ck\}}_{R_k}). \end{aligned}$$

By Lemma 7, it holds that: $(\exists x : C) \rightarrow (\exists x : d \mid cx + s) \rightarrow \text{gcd}(c, d) \mid s$. As $C_r \subseteq C$ it also holds that: $(\exists x : C) \rightarrow C_r$. As all new variables $k \in K$ appear only in one resolvent R_k , the above implications prove

$$\exists x : C(x) \rightarrow \exists K : \left(\{I \in C \mid \text{coeff}(I, x) = 0\} \cup \{\text{gcd}(c, d) \mid s\} \cup \bigcup_{k \in K} R_k \right).$$

Assume, vice versa, that ν is a satisfiable assignment for the formula after one step of weak Cooper elimination. Then it is easy to deduce the following facts:

- Let S_l be the solution set for x, ν , and $I_l = -ax + p \leq 0 \in C(x)$ with $a > 0$. Then $S_l = \left\{ \lceil \frac{\nu(p)}{a} \rceil, \lceil \frac{\nu(p)}{a} \rceil + 1, \dots \right\}$.
- Let S_u be the solution set for x, ν , and $I_u = bx - q \leq 0 \in C(x)$ with $b > 0$. Then $S_u = \left\{ \dots, \lfloor \frac{\nu(q)}{b} \rfloor - 1, \lfloor \frac{\nu(q)}{b} \rfloor \right\}$.
- Let S_I be the solution set for x, ν , and $C_l \cup C_u$. Then $S_I = \bigcap_{I_l \in C_l} S_l \cap \bigcap_{I_u \in C_u} S_u$.
- Let the set S_I be bounded from below, i.e., $S_I = \{l, l + 1, \dots\}$ or $S_I = \{l, \dots, u\}$. Then $l = \max_{I \in C_l} \left\{ \lceil \frac{\nu(p')}{a'} \rceil \mid I = -a'x + p' \leq 0 \right\}$.
- Let the set S_I be bounded from above, i.e., $S_I = \{\dots, u - 1, u\}$ or $S_I = \{l, \dots, u\}$. Then $u = \min_{I \in C_u} \left\{ \lfloor \frac{\nu(q)}{b} \rfloor \mid I = b'x - q' \leq 0 \right\}$.
- By Lemma 7 $d \mid cx + \nu(s)$ is satisfiable because $\text{gcd}(c, d) \mid s$ is contained in the result formula of weak Cooper elimination. By Lemma 8 the set of solutions for x, ν , and $d \mid cx + s$ has the form $S_d = \{v_0 + v'\mu \mid v' \in \mathbb{Z}\}$.
- The solution set S for x, ν , and C' is $S = S_d \cap S_I$.

Next, we do a case distinction on the structure of $C(x)$:

- Let $C_l = \emptyset$, then S_I is unbounded from below. We choose a small enough $v \in S_d$, i.e., small enough $v' \in \mathbb{Z}$ such that $v = v_0 + v'\mu$. Then the assignment $x \mapsto v$ and $y \mapsto \nu(y)$ (if $y \neq x$) satisfies C' .
- Let $C_u = \emptyset$, then S_I is unbounded from above. We choose a large enough $v \in S_d$, i.e., large enough $v' \in \mathbb{Z}$ such that $v = v_0 + v'\mu$. Then the assignment $x \mapsto v$ and $y \mapsto \nu(y)$ for all $y \neq x$ satisfies C' .
- Let $|C_l|, |C_u| > 0$. We select $I_l = -ax + p \leq 0$ such that

$$\lceil \frac{\nu(p)}{a} \rceil = \max_{I \in C_l} \left\{ \lceil \frac{\nu(p')}{a'} \rceil \mid I = -a'x + p' \leq 0 \right\}$$
 and $I_u = bx - q \leq 0$ such that

$$\lfloor \frac{\nu(q)}{b} \rfloor = \min_{I \in C_u} \left\{ \lfloor \frac{\nu(q')}{b'} \rfloor \mid I = b'x - q' \leq 0 \right\}.$$

The resolvent for the two constraints I_l and I_u is

Algorithm 1: CombDivs($x, C'(x)$)

Input : The variable x and a set of LIA constraints $C'(x)$
Output : A set of LIA constraints $C(x)$ such that $C(x) \equiv C'(x)$ and there exists one divisibility constraint $d \mid cx + s \in C(x)$ such that $c > 0$

- 1 $C_d := \{d \mid cx + s \in C'(x) \mid c > 0\}$
- 2 $C(x) := C'(x) \setminus C_d$;
- 3 **if** ($C_d = \emptyset$) **then**
- 4 | **return** $C(x) \cup \{1 \mid x\}$;
- 5 **while** ($|C_d| > 1$) **do**
- 6 | Select $d_1 \mid a_1x + p_1, d_2 \mid a_2x + p_2 \in C_d$;
- 7 | $C_d := C_d \setminus \{d_1 \mid a_1x + p_1, d_2 \mid a_2x + p_2\}$;
- 8 | $d = \gcd(a_1d_2, a_2d_1)$;
- 9 | Choose c_1 and c_2 such that $c_1a_1d_2 + c_2a_2d_1 = d$;
- 10 | $C_d := C_d \cup \{d_1d_2 \mid dx + c_1d_2p_1 + c_2d_1p_2\}$;
- 11 | $C(x) := C(x) \cup \{d \mid -a_1p_2 + a_2p_1\}$;
- 12 **end**
- 13 **return** $C(x) \cup C_d$;

Fig. 2. An algorithm that combines constraints $C_d = \{d \mid cx + s \in C'(x) \mid c > 0\}$ until only one divisibility constraint in x remains

$R_k = \{-k \leq 0, k - m \leq 0, bp - aq + bk \leq 0, a \mid k + p, ad \mid cp + as + ck\}$. We will now show that $\frac{\nu(p+k)}{a}$ is in the set of solutions S of $C(x)$. All of the remaining deductions stem from the evaluation of the resolvent under ν . Since $a \mid \nu(p+k)$, $\frac{\nu(p+k)}{a} \in \mathbb{Z}$. Furthermore, since $\frac{\nu(p+k)}{a} \in \mathbb{Z}$ and $\nu(bp - aq + bk) \leq 0$, $\frac{\nu(p+k)}{a} \in S_I = \left\{ \lceil \frac{\nu(p)}{a} \rceil, \dots, \lfloor \frac{\nu(q)}{b} \rfloor \right\}$. Finally, since $ad \mid \nu(cp + as + ck) = ad \mid ac\nu(x) + a\nu(s) = d \mid c\nu(x) + \nu(s)$, $\frac{\nu(p+k)}{a} \in S_d$. We choose the assignment ν' with $x \mapsto \frac{\nu(p+k)}{a}$ and $y \mapsto \nu(y)$ for all $y \neq x$. Hence, ν' satisfies C' .

□

We stated that weak Cooper elimination can only be applied to those problems where $C(x)$ contains one divisibility constraint $d \mid ax + p$ in x . To expand weak Cooper elimination to any set of constraints $C'(x)$ we briefly explained how to exhaustively apply div-solve to eliminate all but one constraint $d \mid ax + p$ in x . The algorithm CombDivs(x, C) (Fig. 3) is a more detailed version of this procedure.

Lemma 10. *Let $C'(x)$ be a set of LIA constraints. Let $C(x)$ be the output of CombDivs($x, C'(x)$). Then $C(x) \equiv C'(x)$.*

Proof. Follows directly from the proof of equivalence [14] of the div-solve transformation. □

Since the output $C(x)$ of $\text{CombDivs}(x, C'(x))$ is equivalent to $C'(x)$ and fulfils the conditions of weak Cooper elimination, we conclude the following equivalence for the output of weak Cooper elimination applied to $C(x)$:

$$\exists x : C'(x) \equiv \exists K : \left(\{I \in C(x) \mid \text{coeff}(I, x) = 0\} \cup \{\text{gcd}(c, d) \mid s\} \cup \bigcup_{k \in K} R_k \right)$$

4 Strong Conflict Resolution Revisited

Weak Cooper elimination is capable of exploring all unguarded variables to eventually create a problem where feasibility only depends on guarded variables. It is simulated in a lazy manner through an additional set of CUTSAT++ rules (Fig. 4). Instead of eliminating all unguarded variables before the application of CUTSAT++, the rules perform the same intermediate steps as weak Cooper elimination, viz. the combination of divisibility constraints via div-solve and the construction of resolvents, to resolve and block conflicts in unguarded constraints. As a result, CUTSAT++ can avoid some of the intermediate steps of weak Cooper elimination. Furthermore, CUTSAT++ is not required to apply the intermediate steps of weak Cooper elimination one variable at a time. The lazy approach of CUTSAT++ does not eliminate unguarded variables. In the worst case CUTSAT++ has to perform all of weak Cooper elimination's intermediate steps. Then a strategy (Def. 17) guarantees that CUTSAT++ recognizes that all unguarded conflict constraints have been blocked by guarded constraints.

The eventual result is the complete algorithm CUTSAT++, which is a combination of the rules Resolve-Cooper, Solve-Div-Left, Solve-Div-Right (Fig. 4), a strictly-two-layered strategy (Def. 17), and the CUTSAT rules: Propagate, Propagate-Div, Decide, Conflict, Conflict-Div, Sat, Unsat-Div, Forget, Slack-Intro, Resolve, Skip-Decision, Backjump, Unsat, and Learn [14].

The advantage of the lazy approach is that CUTSAT++ might find a satisfiable assignment or detect unsatisfiability without encountering and resolving a large number of unguarded conflicts. This means the number of divisibility constraint combinations and introduced resolvents might be much smaller in the lazy approach of CUTSAT++ than during the elimination with weak Cooper elimination.

In order to simulate weak Cooper elimination, CUTSAT++ uses a total order \prec over all variables such that $y \prec x$ for all guarded variables y and unguarded variables x [14]. While the order needs to be fixed for all unguarded variables, the ordering among the guarded variables can be dynamically changed. In relation to weak Cooper elimination, the order \prec describes the elimination order for the unguarded variables, viz. $x_i \prec x_j$ if x_j is eliminated before x_i . A variable x is called *maximal* in a constraint I if x is contained in I and all other variables in I are smaller with respect to \prec . The maximal variable in I is also called its *top variable* ($x = \text{top}(I)$).

Definition 11. Let $S = \langle M, C \rangle$ be a state, $C' \subseteq C$, x the top variable in C' and let all other variables in C' be fixed. The pair (x, C') is a conflicting core if

- (1) $C' = \{-ax + p \leq 0, bx - q \leq 0\}$ and the lower bound from $-ax + p \leq 0$ contradicts the upper bound from $bx - q \leq 0$, i.e., $\text{bound}(-ax + p \leq 0, x, M) > \text{bound}(bx - q \leq 0, x, M)$ [14]; in this case (x, C') is called an interval conflicting core and its strong resolvent is $(\{-k \leq 0, k - a + 1 \leq 0\}, \{bp - aq + bk \leq 0, a \mid k + p\})$ [14]
- (2) $C' = \{-ax + p \leq 0, bx - q \leq 0, d \mid cx + s\}$ and $b_l = \text{bound}(-ax + p \leq 0, x, M)$, $b_u = \text{bound}(bx - q \leq 0, x, M)$, $b_l \leq b_u$ and for all $b_d \in [b_l, b_u]$ we have $d \nmid cb_d + \text{lower}(s, M)$ [14]; in this case (x, C') is called a divisibility conflicting core and its strong resolvent is $(\{-k \leq 0, k - m \leq 0\}, \{bp - aq + bk \leq 0, a \mid k + p, ad \mid cp + as + ck\})$ [14]
- (3) $C' = \{d \mid cx + s\}$ and for all $b_d \in \mathbb{Z}$ we have $d \nmid cb_d + \text{lower}(s, M)$; in this case (x, C') is called a diophantine conflicting core and its strong resolvent is $(\emptyset, \{\text{gcd}(c, d) \mid s\})$ where k is a fresh variable and $m = \text{lcm}\left(a, \frac{ad}{\text{gcd}(ad, c)}\right) - 1$.

We refer to the respective strong resolvents for a conflicting core (x, C') by the function $\text{cooper}(x, C')$ which returns a pair (R_k, R_c) as defined above. Note that the newly introduced variable k is guarded by the constraints in R_k . If there is a conflicting core (x, C') in state S , then x is called a *conflicting variable*. A *potential conflicting core* is a pair (x, C') if there exists a state S where (x, C') is a conflicting core.

Next we define a semantic generalization of strong resolvents. Since the strong resolvents generated out of conflicting cores will be further processed by CUT-SAT++, we must guarantee that any set of constraints implying the feasibility of the conflicting core constraints prevents a second application of Resolve-Cooper to the same conflicting core. All strong resolvents of Definition 11 are also strong resolvents in the sense of the below definition (see also end of Section 3).

Definition 12. A set of constraints R is a strong resolvent for the pair (x, C') if it holds that $R \rightarrow \exists x : C'$ and for all $J \in R : \text{top}(J) \prec x$.

Lemma 13. Let $C' \subseteq C$. Let $\text{cooper}(x, C') = (R_k, R_c)$. Let $R = R_k \cup R_c$. Then $\exists k : C \cup R \equiv C$. Furthermore, R is a strong resolvent for (x, C') .

Proof. Follows directly from the Lemmas 6 and 7. The interval conflicting core is the only new case. However, $\text{cooper}(x, \{-ax + p \leq 0, bx - q \leq 0\})$ is equivalent to $\text{cooper}(x, \{-ax + p \leq 0, bx - q \leq 0, 1 \mid x\})$. By Lemmas 6 and 7, $R \rightarrow \exists x : C'$. Finally, since k is the minimal element of \prec and all other variables in R appear in C' , where x is maximal, it holds that $J \in R : \text{top}(J) \prec x$. \square

The rule Resolve-Cooper (Fig. 4) requires that the conflicting variable x of the conflicting core (x, C') is the top-variable in the constraints of C' . This simulates a setting where all variables y with $x \prec y$ are already eliminated. We restrict Resolve-Cooper to unguarded constraints, because weak Cooper elimination modifies only unguarded constraints.

Lemma 14. Let $S = \langle M, C \rangle$ be a CUTSAT++ state. Let $C' \subseteq C$ and x be an unguarded variable. Let $R, R \subseteq C$, be a strong resolvent for (x, C') . Then Resolve-Cooper is not applicable to (x, C') . \square

Proof. Assume for contradiction that $D = (x, C')$ is a conflicting core, $R \in C$ is a strong resolvent for D in state S and Resolve-Cooper is applicable to D in state S . Resolve-Cooper requires that all variables $y \prec x$ are fixed (Fig. 4). This holds especially for all variables in R (Def. 12). Due to the restriction that every conflict $J \in C$ has $\text{top}(J) \not\prec \text{top}(I)$ in Resolve-Cooper, there is no conflict in R . Furthermore, since all variables $y \prec x$ are fixed, R is satisfied by the partial assignment defined by M . By Def. 11, all conflicting cores have no satisfiable solution for x under partial model M . However, by Def. 12, R satisfiable implies that there exists an x such that C' is satisfiable under M . This contradicts the assumption that (x, C') is a conflicting core! \square

For the resolvent R to block Resolve-Cooper from being applied to the conflicting core (x, C') , CUTSAT++ has to detect all conflicts in R . Detecting all conflicts in R is only possible if CUTSAT++ fixes all variables y with $y \prec x$ and if Resolve-Cooper is only applicable if there exists no conflict I with $\text{top}(I) \prec x$. Therefore, the remaining restrictions of Resolve-Cooper justify the above Lemma. If we add strong resolvents again and again, then CUTSAT++ will reach a state after which every encounter of a conflicting core guarantees a conflict in a guarded constraint. From this point forward CUTSAT++ won't apply Resolve-Cooper anymore. The remaining guarded conflicts are resolved with the rules Conflict and Conflict-Div [14].

The rules Div-Solve-Left and Div-Solve-Right (Fig. 4) combine divisibility constraints as it is done a priori to weak Cooper elimination. In these rules we restrict the application of $\text{div-solve}(x, I_1, I_2)$ to constraints where x is the top variable and where all variables y in I_1 and I_2 , with $y \neq x$, are fixed. The ordering restriction simulates the order of elimination, i.e., we apply $\text{div-solve}(x, I_1, I_2)$ in a setting where all variables y with $x \prec y$ appear to be eliminated in I_1 and I_2 . Otherwise, divergence would be possible (see example 3). Requiring smaller variables to be fixed prevents the accidental generation of a conflict for an unguarded variable x_i by $\text{div-solve}(x, I_1, I_2)$.

Thanks to an eager top-level propagating strategy, defined below, any unguarded conflict in CUTSAT++ is either resolved with Div-Solve-Right (Fig. 4) or CUTSAT++ constructs a conflicting core that is resolved with Resolve-Cooper. Both cases may require multiple applications of the Div-Solve-Left rule (Fig. 4). We define the following further restrictions on the CUTSAT++ rules that will eventually generate the above described behavior.

Definition 15. Let $\bowtie \in \{\leq, \geq\}$. We call a strategy for CUTSAT++ eager top-level propagating if we restrict propagations and decisions for every state $\langle M, C \rangle$ in the following way:

1. Let x be an unguarded variable. Then we only allow to propagate bounds $x \bowtie \text{bound}(I, x, M)$ if x is the top variable in I . Furthermore, if I is a divisibility constraint $d \mid ax + p$, then we only propagate $d \mid ax + p$ if:
 - (a) Either $\text{lower}(x, M) \neq -\infty$ and $\text{upper}(x, M) \neq \infty$
 - (b) Or if $\text{gcd}(a, d) \mid \text{lower}(p, M)$, and $d \mid ax + p$ is the only divisibility constraint in C with x as top variable.

Solve-Div-Left	$\langle M, C \rangle \Rightarrow_{\text{cs}} \langle M, C' \rangle$	if $\left\{ \begin{array}{l} \text{divisibility constraints } I_1, I_2 \in C \\ x \text{ is top in } I_1 \text{ and } I_2 \\ \text{all other vars. in } I_1, I_2 \text{ are fixed} \\ (I'_1, I'_2) = \text{div-solve}(x, I_1, I_2), \\ C' = C \setminus \{I_1, I_2\} \cup \{I'_1, I'_2\} \\ I'_2 \text{ is not a conflict} \end{array} \right.$
Solve-Div-Right	$\langle M, C \rangle \Rightarrow_{\text{cs}} \langle M', C' \rangle$	if $\left\{ \begin{array}{l} \text{divisibility constraints } I_1, I_2 \in C \\ x \text{ is top in } I_1 \text{ and } I_2 \\ \text{all other vars. in } I_1, I_2 \text{ are fixed} \\ (I'_1, I'_2) = \text{div-solve}(x, I_1, I_2), \\ C' = C \setminus \{I_1, I_2\} \cup \{I'_1, I'_2\} \\ I'_2 \text{ is a conflict} \\ y = \text{top}(I'_2) \\ M' = \text{prefix}(M, y) \end{array} \right.$
Resolve-Cooper	$\langle M, C \rangle \Rightarrow_{\text{cs}} \langle M', C \cup R_k \cup R_c \rangle$	if $\left\{ \begin{array}{l} (x, C') \text{ is a conflicting core} \\ x \in U, \text{ all } y \prec x \text{ are fixed and } C' \subseteq C \\ \text{if } J \in C \text{ is a conflict then } \text{top}(J) \not\prec x \\ \text{cooper}(x, C') = (R_k, R_c) \\ y = \min_{I \in R_c} \{\text{top}(I)\} \\ M' = \text{prefix}(M, y) \end{array} \right.$

In the above rules, $M' = \text{prefix}(M, y)$ defines the largest prefix of M that contains only decided bounds for variables x with $x \prec y$.

Fig. 3. Our strong conflict resolution rules

2. Let x be an unguarded variable. Then we only allow decisions $\gamma = x \bowtie b$ if:
 - (a) For every constraint $I \in C$ with $x = \text{top}(I)$ all occurring variables $y \neq x$ are fixed
 - (b) There exists no $I \in C$ where $x = \text{top}(I)$ and I is a conflict in $\llbracket M, \gamma \rrbracket$
 - (c) Either $\text{lower}(x, M) \neq -\infty$ and $\text{upper}(x, M) \neq \infty$ or there exists at most one divisibility constraint in C with x as top variable.

An eager top-level propagating strategy has two advantages. First, the strategy dictates an order of influence over the variables, i.e., a bound for unguarded variable x is only influenced by previously propagated bounds for variable y with $y \prec x$. Furthermore, the strategy makes only decisions for unguarded variable x when all constraints with $x = \text{top}(I)$ are fixed and satisfied by the decision. This means any conflict $I \in C$ with $x = \text{top}(I)$ is impossible as long as the decision for x remains on the bound sequence. For the same purpose, i.e., avoiding conflicts I where $x = \text{top}(I)$ is fixed by a decision, CUTSAT++ backjumps in the rules Resolve-Cooper and Solve-Div-Right to a state where this is not the case. To avoid frozen states resulting from the eager top-level propagating strategy, the slack variable x_S has to be the smallest unguarded variable in \prec . Otherwise, the constraints $x - x_S \leq 0$, $-x - x_S \leq 0$ introduced by Slack-Intro cannot be used to propagate bounds for variable x , and x would remain stuck.

Definition 16. A strategy is reasonable if it prefers the propagation of constraints of the form $\pm x - b \leq 0$ and applies the Forget Rule only finitely often [14].

Definition 17. A strategy is strictly-two-layered if:
(1) it is reasonable, (2) it is eager top-level propagating (Def. 15), (3) the Forget, Conflict, Conflict-Div rules only apply to guarded constraints, (4) Forget cannot be applied to a divisibility constraint or a constraint contained in a strong resolvent, and (5) only guarded constraints are used to propagate guarded variables.

A strictly-two-layered strategy is the final restriction to CUTSAT++. With the condition 17-(3) it partitions conflict resolution into two layers: Every unguarded conflict is handled with the rules Resolve-Cooper, Solve-Div-Left, and Solve-Div-Right (Fig. 4), every guarded conflict with the rules Conflict(-Div). The conditions 17-(1) and 17-(5) make the guarded variables independent from the unguarded variables. The conditions 17-(2) and 17-(4) give a guarantee that the rules Resolve-Cooper, Solve-Div-Left, and Solve-Div-Right are applied at most finitely often. We assume for the remainder of the paper that all runs of CUTSAT++ follow a strictly-two-layered strategy.

5 Termination and Completeness

The CUTSAT++ rules are Propagate, Propagate-Div, Decide, Conflict, Conflict-Div, Sat, Unsat-Div, Forget, Slack-Intro, Resolve, Skip-Decision, Backjump, Unsat, and Learn from [14], as well as Resolve-Cooper, Solve-Div-Left, and Solve-Div-Right (Fig. 4). Before we prove termination and completeness for CUTSAT++, we have to prove another property over strong resolvents. We have proven in Section 4 that Resolve-Cooper applied to conflicting core (x, C') adds a strong resolvent R , which blocks another application of Resolve-Cooper to (x, C') . However, CUTSAT++ is able to remove constraints from R with the rules Solve-Div-Left and Solve-Div-Right. This removes the original conflicting core R from our state. Nonetheless, CUTSAT++ is still unable to apply Resolve-Cooper to conflicting core (x, C') because the rules Solve-Div-Left and Solve-Div-Right guarantee that a new strong resolvent R' for conflicting core (x, C') is introduced:

Lemma 18. Let $S = \langle M, C \rangle$ be a state reachable by CUTSAT++ from the initial state $\langle \square, C_0 \rangle$ and let $S' = \langle M', C' \rangle$ be a state reachable by CUTSAT++ from S . Let C contain a strong resolvent R for (x, C'') . Then C' contains also a strong resolvent R' for (x, C'') .

Proof. Assume for a contradiction that S contains a strong resolvent R for (x, C'') and S' contains no strong resolvent $R' \in C'$ for (x, C'') . W.l.o.g. we assume that S' is the first state after S where $R \not\subseteq C'$. By Def. 17-(4), CUTSAT++ with a strictly-two-layered strategy cannot remove constraints from a strong resolvent R except with the rules Solve-Div-Right and Solve-Div-Left. Through

the equivalence proven for $\text{div-solve}(x, I_1, I_2)$ in [14], we know that there exist $I'_1, I'_2 \in C'$ such that $\{I'_1, I'_2\} \equiv \{I_1, I_2\}$ and $R \subseteq C' \setminus \{I'_1, I'_2\} \cup \{I_1, I_2\}$. Thus $R' = R \setminus \{I_1, I_2\} \cup \{I'_1, I'_2\}$ is a strong resolvent of (x, C'') such that

$$R' \rightarrow R \rightarrow \exists x : C''.$$

Furthermore, R' is a subset of C' , which contradicts our initial assumption! \square

Together with Lemma 14 this property implies that Resolve-Cooper is applied at most once to every conflicting core encountered by CUTSAT++. This is essential for our termination proof.

5.1 Proof for Termination

For the termination proof of CUTSAT++, we consider a (possibly infinite) sequence of rule applications $\langle \square, C_0 \rangle = S_0 \Rightarrow_{\text{CS}} S_1 \Rightarrow_{\text{CS}} \dots$ on a problem C_0 , following the strictly-two-layered strategy.

First, this sequence reaches a state S_s ($s \in \mathbb{N}_0^+$) after a finite derivation of rule applications $S_0 \Rightarrow_{\text{CS}} \dots \Rightarrow_{\text{CS}} S_s$ such that there is no further application of the rules Slack-Intro and Forget after state S_s :

Lemma 19. *Let $\langle \square, C_0 \rangle = S_0 \Rightarrow_{\text{CS}} S_1 \Rightarrow_{\text{CS}} \dots$ be a sequence of rule applications applied to a problem C_0 , following the strictly-two-layered strategy. Then the sequence reaches a state S_s ($s \in \mathbb{N}_0^+$) after at most finitely many rule applications $S_0 \Rightarrow_{\text{CS}} \dots \Rightarrow_{\text{CS}} S_s$ such that there is no further application of the rules Slack-Intro and Forget after state S_s .*

Proof. Such a state S_s exists for two reasons: Firstly, the strictly-propagating-strategy employed by CUTSAT++ is also reasonable. The reasonable strategy explicitly forbids infinite applications of the rule Forget. Secondly, the Slack-Intro rule is applicable only to stuck variables and only once to each stuck variable. Only the initial set of variables can be stuck because all variables x introduced during the considered derivation are introduced with at least one constraint $x - b \leq 0$ that allows at least one propagation for the variable. Therefore, the rules Slack-Intro and Forget are at most finitely often applicable. \square

Next, the sequence reaches a state S_w ($w \geq s$) after a finite derivation of rule applications $S_s \Rightarrow_{\text{CS}} \dots \Rightarrow_{\text{CS}} S_w$ such that there is no further application of the rules Resolve-Cooper, Solve-Div-Left or Solve-Div-Right after state S_w : The strictly-two-layered strategy guarantees that CUTSAT++ only learns unguarded constraints with the rules Resolve-Cooper, Solve-Div-Left, Solve-Div-Right, and Slack-Intro. We have shown in the previous paragraph that $S_s \Rightarrow_{\text{CS}} \dots \Rightarrow_{\text{CS}} S_w$ contains no application of the rule Slack-Intro. By Lemma 14, an application of Resolve-Cooper to the conflicting core (x, C') prevents any further applications of Resolve-Cooper to the same core. By Lemma 13, the constraints learned through an application of Resolve-Cooper contain only variables y such that $y \prec x$. Therefore, an application of Resolve-Cooper blocks a conflicting core (x, C') and introduces potential conflicting cores only for smaller variables than x . This strict decrease in the conflicting variables guarantees that

we encounter only finitely many conflicting cores in unguarded variables. Therefore, Resolve-Cooper is at most finitely often applicable. An analogous argument applies to the rules Solve-Div-Left and Solve-Div-Right. Thus the rules Resolve-Cooper, Solve-Div-Left and Solve-Div-Right are at most finitely often applicable.

Lemma 20. *Let $\langle \square, C_0 \rangle = S_0 \Longrightarrow_{CS} S_1 \Longrightarrow_{CS} \dots$ be a sequence of rule applications applied to a problem C_0 , following the strictly-two-layered strategy. Then the sequence reaches a state S_w after finitely many rule applications $S_0 \Longrightarrow_{CS} \dots \Longrightarrow_{CS} S_w$ such that there is no further application of the rules Resolve-Cooper, Solve-Div-Left, and Solve-Div-Right after state S_w .*

Proof. By Lemma 19, we assume w.l.o.g. that the sequence continues from a state S_s such that S_s is reached by the sequence after at most finitely many rule applications $S_0 \Longrightarrow_{CS} \dots \Longrightarrow_{CS} S_s$, and there is no further application of the rules Slack-Intro and Forget after state S_s . Let $x_1 \prec \dots \prec x_n$ be the order of variables for all unguarded variables x_i . We consider a weight vector that strictly decreases after every call to Resolve-Cooper, Solve-Div-Left and Solve-Div-Right. For this weight vector, we define $\text{cores}(x_i, C)$ as the set of potential conflicting cores in the problem C with conflicting variable x_i . Its subset $\text{woSR}(x_i, C) \subseteq \text{cores}(x_i, C)$ is defined so it contains all its potential conflicting cores without a strong resolvent $R \subseteq C$. It is easy to see that $|\text{cores}(x_i, C)| \leq 2^{|C|}$ and therefore both functions define finite sets. Now we define the weight vector $\text{weight}_c(S)$ for every state $S = \langle M, C \rangle (\vdash I)$:
 $\text{weight}_c(S) = (|\text{cores}(x_n, C)|, |\text{woSR}(x_n, C)|, \dots, |\text{cores}(x_1, C)|, |\text{woSR}(x_1, C)|)$
By Definition 17, Learn is only applicable to guarded constraints. Therefore, Resolve-Cooper, Solve-Div-Left, and Solve-Div-Right are the only rules learning potentially unguarded constraints and thereby the only rules that can increase $|\text{cores}(x_i, C)|$ and $|\text{woSR}(x_i, C)|$ between two subsequent states $S_i \Longrightarrow_{CS} S_{i+1}$. After all other transitions $S_i \Longrightarrow_{CS} S_{i+1}$, it holds that $\text{weight}_c(S_i) \geq_{\text{lex}} \text{weight}_c(S_{i+1})$. Whenever CUTSAT++ applies Solve-Div-Left, Solve-Div-Right or Resolve-Cooper, the weight strictly decreases, i.e., $\text{weight}_c(S') >_{\text{lex}} \text{weight}_c(S)$:

1. By Lemma 14, an application of Resolve-Cooper to conflicting core (x_i, C^*) implies that there is no strong resolvent $R' \subseteq C'$ for (x_i, C^*) . By Lemma 13, the new problem $C = C' \cup R$ contains a strong resolvent R for (x_i, C^*) . Therefore, $|\text{woSR}(x_i, C)| < |\text{woSR}(x_i, C')|$. By Definition 12, it holds for all $y \in \text{vars}(R)$ that $y \prec x$. Thence, Resolve-Cooper has not introduced new potential conflicting cores (x_j, C^{**}) with $j \geq i$ and $|\text{cores}(x_j, C)| \leq |\text{cores}(x_j, C')|$ for all $j \geq i$. By Lemma 18, $|\text{woSR}(x_j, C)| \leq |\text{woSR}(x_j, C')|$ for all $j > i$. Therefore, the weight decreases after an application of Resolve-Cooper, i.e., $\text{weight}_c(S') >_{\text{lex}} \text{weight}_c(S)$.
2. Let Solve-Div-Left (Solve-Div-Right) be applied to the pair of divisibility constraints (I_1, I_2) such that $\text{top}(I_1) = x_i$ and $\text{div-solve}(x_i, I_1, I_2) = (I'_1, I'_2)$. The new constraint set is $C = C' \setminus \{I_1, I_2\} \cup \{I'_1, I'_2\}$. The number of potential conflicting cores containing the same divisibility constraint $I = d \mid ax + p \in C''$ in problem $\hat{C} \uplus \{I\}$ is the same for all divisibility constraints with $\text{top}(I) = x$. This means, removing I_1 and replacing it with

I'_1 doesn't increase the number of cores, i.e., $|\text{cores}(x_i, C' \setminus \{I_1\} \cup \{I'_1\})| = |\text{cores}(x_i, C')|$. However, since $I_2 \in \text{cores}(x_i, C')$ and we replace I_2 with I'_2 where $\text{top}(I'_2) \prec x_i$, we will decrease the number of conflicting cores in x_i . It is easy to see, that we do not introduce any new conflicting cores for x_j with $j > i$. Thus $|\text{cores}(x_j, C)| = |\text{cores}(x_j, C')|$. Finally, Lemma 18 implies that $|\text{woSR}(x_j, C)| \leq |\text{woSR}(x_j, C')|$ for $j > i$. Therefore, $\text{weight}_c(C') >_{\text{lex}} \text{weight}_c(C)$.

We deduce that the weight_c vector monotonically decreases if we continue from the before mentioned state S_s . Since $>$ is a well-founded order, the lexicographic order $>_{\text{lex}}$ is also well-founded. The minimum of the weight order is $(\dots, 0, \dots)$. As $>_{\text{lex}}$ is well-founded, there exists no way to decrease the weight $\text{weight}_c(C_s)$ without reaching the minimum $(\dots, 0, \dots)$ after finitely many applications of the rules Solve-Div-Left, Solve-Div-Right, or Resolve-Cooper. Finally, the weight_c vector cannot decrease below $(\dots, 0, \dots)$ so CUTSAT++ is not able to apply Solve-Div-Left, Solve-Div-Right, or Resolve-Cooper after we reach a state S with $\text{weight}_c(S) = (\dots, 0, \dots)$. We conclude that the rules Solve-Div-Left, Solve-Div-Right, and Resolve-Cooper are at most finitely often applicable. \square

Next, the sequence reaches a state S_b ($b \geq w$) after a finite derivation of rule applications $S_w \Longrightarrow_{\text{CS}} \dots \Longrightarrow_{\text{CS}} S_b$ such that for every guarded variable x the bounds remain invariant, i.e., $\text{lower}(x, M_b) = \text{lower}(x, M_j)$ and $\text{upper}(x, M_b) = \text{upper}(x, M_j)$ for every state $S_j = \langle M_j, C_j \rangle (\vdash I_j)$ after $S_b = \langle M_b, C_b \rangle (\vdash I_b)$ ($j \geq b$): The strictly-two-layered strategy guarantees that only bounds of guarded variables influence the propagation of further bounds for guarded variables. Any rule application involving unguarded variables does not influence the bounds for guarded variables. A proof for the termination of the solely guarded case was already provided in [14]. At this point we know that the sequence after S_b contains no further propagations, decisions, or conflict analysis for the guarded variables.

Lemma 21. *Let $\langle \square, C_0 \rangle = S_0 \Longrightarrow_{\text{CS}} S_1 \Longrightarrow_{\text{CS}} \dots$ be a sequence of rule applications applied to a problem C_0 , following the strictly-two-layered strategy. Then the sequence reaches a state S_b after finitely many rule applications $S_0 \Longrightarrow_{\text{CS}} \dots \Longrightarrow_{\text{CS}} S_b$ such that for every guarded variable x the bounds remain invariant.*

Proof. This proof is based on the termination proof for CUTSAT on finite problems, i.e., problems without unguarded variables [14]. It uses a weight function that strictly decreases whenever CUTSAT++ changes a bound for a guarded variable and otherwise stays the same. By Lemmas 19 and 20, we assume w.l.o.g. that the sequence continues from a state S_w such that S_w is reached by the sequence after at most finitely many rule applications $S_0 \Longrightarrow_{\text{CS}} \dots \Longrightarrow_{\text{CS}} S_w$, and there is no further application of the rules Slack-Intro, Forget, Resolve-Cooper, Solve-Div-Left, and Solve-Div-Right after state S_w . The level_B of a state $S = \langle M, C \rangle$ is the number of decisions for guarded variables in M . The maximal prefix of M containing only j decisions for guarded variables is denoted

by $\text{B-subseq}_j(M) = M_j$ the j -th guarded subsequence. Since CUTSAT++ is relevant, it prefers to propagate simple constraints. This allows us to assume w.l.o.g. that M_0 contains both a lower and upper bound for all guarded variables x . The *guarded weight* of the j -th level $_B$ is defined by the function $w_B(M_j)$:

$$w_B(M_j) = \sum_{x \text{ is guarded}} (\text{upper}(x, M) - \text{lower}(x, M)).$$

The *guarded weight* of a state is the vector:

$$\text{weight}_B(\langle M, C \rangle) = \langle w_B(\text{B-subseq}_0(M)), \dots, w_B(\text{B-subseq}_n(M)) \rangle,$$

where n is the number of guarded variables. We order the two weight_B -vectors of two subsequent search-states with the well-founded lexicographic order $>_{\text{lex}}$ based on the well-founded order $>$. It is easy to see that the minimum of weight_B is $(\dots, 0, \dots)$ and that any change to a bound of a guarded variable changes the guarded weight weight_B . Furthermore, by the definition of the strictly-two-layered strategy, we see that we only propagate guarded variables with guarded constraints. Thus the strategy also implies that the conflict rules, Conflict, Conflict-Div, Backjump, Resolve, Skip-Decision, Unsat, and Learn, only handle guarded constraints. Given the proof for Theorem 2 in [14], we see that every application of Propagate, Propagate-Div and Decide applied to a guarded variable decreases weight_B strictly. We see in the same proof [14] that weight_B strictly decreases between one application of Conflict(-Div) and Backjump as long as the conflict rules handle only guarded constraints - as is the case for CUTSAT++. Since the bound sequence M is finite, the conflict rules are at most $|M|$ times applicable between one application of Conflict(-Div), and Backjump or Unsat. The remaining rules, Propagate, Propagate-Div and Decide applied to unguarded variables, have no influence on weight_B or the bounds of guarded variables. Since the guarded weight weight_B cannot decrease below $(\dots, 0, \dots)$, we conclude that CUTSAT++ is not able to change the bounds for guarded variables infinitely often. \square

Next, the sequence reaches a state S_u ($u \geq b$) after a finite derivation of rule applications $S_b \Rightarrow_{\text{CS}} \dots \Rightarrow_{\text{CS}} S_u$ such that also for every unguarded variable x the bounds remain invariant, i.e. $\text{lower}(x, M_b) = \text{lower}(x, M_j)$ and $\text{upper}(x, M_b) = \text{upper}(x, M_j)$ for every state $S_j = \langle M_j, C_j \rangle (\vdash I_j)$ after $S_u = \langle M_u, C_b \rangle (\vdash I_u)$ ($j \geq u$). After S_b , CUTSAT++ propagates and decides only unguarded variables or ends with an application of Sat or Unsat(-Div). CUTSAT++ employs the strictly-two-layered strategy which is also an eager top-level propagating strategy. The latter induces a strict order of propagation over the unguarded variables through the top-variable restriction for propagating constraints. Therefore, any bound for unguarded variable x is influenced only by bounds for variables $y \prec x$. This strict variable order guarantees that unguarded variables are propagated and decided only finitely often.

Lemma 22. *Let $\langle \square, C_0 \rangle = S_0 \Rightarrow_{\text{CS}} S_1 \Rightarrow_{\text{CS}} \dots$ be a sequence of rule applications applied to a problem C_0 , following the strictly-two-layered strategy. Then the sequence reaches a state S_u after finitely many rule applications $S_0 \Rightarrow_{\text{CS}} \dots \Rightarrow_{\text{CS}} S_u$ such that for every unguarded variable x the bounds remain invariant.*

Proof. By Lemmas 19, 20, and 21, we assume w.l.o.g. that the sequence continues from a state $S_b = \langle M_b, C_b \rangle (\vdash I_b)$ such that S_b is reached by the sequence after at most finitely many rule applications $S_0 \Longrightarrow_{CS} \dots \Longrightarrow_{CS} S_b$ and only the rules Sat, Unsat-Div, Propagate, Propagate-Div, and Decide are applied after S_b , whereas the last three only for unguarded variables. Assume for a contradiction that there exists an infinite CUTSAT++ run starting in S_b . This is easy to see, since there is only a finite number of unguarded variables and no rule to undo a decision, the Decide rule is applied at most finitely often. Furthermore, any application of Sat or Unsat-Div ends a run making it finite. This allows us to assume w.l.o.g. that there is no application to the rules Sat, Unsat-Div, and Decide in the infinite run starting in the state S_b .

Since there are at most finitely many variables in state S_b , and no rule to introduce further variables after S_b , there exists a smallest unguarded variable x that is propagated infinitely often. We assume w.l.o.g. that the run starting in S_b propagates only variables y bigger than or equal to x . Therefore, the bounds of all smaller variables y , remain invariant in all subsequent states $S_j = \langle M_j, C_j \rangle (\vdash I_j)$ of S_b , i.e., $\text{lower}(y, M_j) = \text{lower}(y, M_c)$ and $\text{upper}(y, M_j) = \text{upper}(y, M_c)$. Since there exists no applicable rule that changes the constraint set, we notice that the constraint set C_b also remains invariant for all states after S_b . Thus we find all constraints C^* used to propagate x in the set C_b . Since CUTSAT++ is eager top-level propagating, any constraint $I \in C^*$ has x as their top variable. This leads us to the deduction, that $\text{bound}(I, x, M_j) = \text{bound}(I, x, M_b)$ for all subsequent states $S_j = \langle M_j, C_j \rangle$ and inequalities $I \in C^*$. Since the bounds defined by the inequalities in C^* remain invariant after state S_b , CUTSAT++ propagates x at most finitely often with inequalities.

Therefore, there only exists an infinite CUTSAT++ run if x is propagated infinitely often with Propagate-Div. This allows us to assume w.l.o.g. that the run starting in S_b propagates x only with Propagate-Div. Next, we deduce that variable x stays unbounded in the remaining states of the derivation sequence. Otherwise there exists a finite set $x \in \{l_x, \dots, u_x\}$ bounding x , therefore allowing only finitely many propagations. In the case that x stays unbounded, the definition of the eager top-level propagating strategy states that Propagate-Div is only applicable to x if $I_d = d \mid ax + p \in C^*$ is the only divisibility constraint in C_b with x as their top variable. Furthermore, we know, because of Definition 15 and Lemma 7, that there must exist $v \in \mathbb{Z}$ such that $d \mid av + \text{lower}(p, M_k)$ is satisfied. Now if we consider Lemma 1 and the definition of the predicate $\text{improves}(I, x, M_c)$ then we see that Propagate-Div propagates x at most finitely often. More specifically, if the lower bound of x is $\text{lower}(x, M_c) = l_x \neq -\infty$ then Propagate-Div propagates for x at most $v - l_x$ lower bounds. If the upper bound of x is $\text{upper}(x, M_c) = u_x \neq \infty$ then Propagate-Div propagates for x at most $u_x - v$ upper bounds. This contradicts the assumption that x is the smallest variable propagated infinitely often, which in turn contradicts our initial assumption! \square

After state S_u , only the rules Sat, Unsat, and Unsat-Div are applicable, which lead to a final state. Hence, the sequence $S_0 \Longrightarrow_{\text{CS}} S_1 \Longrightarrow_{\text{CS}} \dots$ is finite. We conclude that CUTSAT++ always terminates:

Theorem 23. *If CUTSAT++ starts from an initial state $\langle \llbracket \cdot \rrbracket, C_0 \rangle$, then there is no infinite derivation sequence.*

Proof. By Lemmas 19, 20, 21, and 22, CUTSAT++ reaches a state S_u after which only the rules Sat, Unsat, and Unsat-Div are applicable, which lead to a final state. Therefore, CUTSAT++ does not diverge. \square

5.2 Frozen States

Our CUTSAT++ algorithm never reaches a frozen state. Let x be the smallest unfixed variable with respect to \prec . Whenever x is guarded we can propagate a constraint $\pm x - b \leq 0 \in C$ and then fix x with by introducing a decision. If we cannot propagate any bound for x , then x is unguarded and stuck, and therefore Slack-Intro is applicable. If we cannot fix x by introducing a decision, then x is unguarded and there is a conflict constraint. Guarded conflict constraints are resolved via the Conflict(-Div) rules. Unguarded conflict constraints are resolved via the strong conflict resolution rules. Unless a final state is reached CUTSAT++ has always a rule applicable.

The proof outlined above works because CUTSAT++ encounters only unguarded conflict constraints that are either the result of multiple contradicting divisibility constraints resolvable by Solve-Div-Left and Solve-Div-Right, or expressible via a conflicting core. Since conflicting cores are only defined over constraints and propagated bounds, we have to guarantee that CUTSAT++ never encounters an unguarded conflict constraint I where $x = \text{top}(I)$ is fixed with a Decision. We express this property with the following invariant fulfilled by every state visited by CUTSAT++:

Definition 24. *A state $S = \langle M, C \rangle (\vdash I)$ is called eager top-level propagated if for all unguarded variables x , all decisions $\gamma = x \bowtie b$ in $M = \llbracket M', \gamma, M'' \rrbracket$, and all constraints $I \in C$ with $\text{top}(I) = x$ the constraint I is no conflict in S and all other variables contained in I are fixed in M' .*

Lemma 25. *If S' is an eager top-level propagated state (Def. 24), then any successor state $S = \langle M, C \rangle (\vdash I)$ reachable by CUTSAT++ is eager top-level propagated.*

Proof. Let S' be an eager top-level propagated state and S its successor, i.e., $S' \Longrightarrow_{\text{CS}} S$. We prove this Lemma with a case distinction on the rule leading to the above transition:

1. Let the applied rule be Propagate(-Div). Then $S' = \langle M', C' \rangle$ and $S = \langle \llbracket M', x \bowtie_J b \rrbracket, C' \rangle$. Let $J' \in C'$ be the constraint used for propagation, i.e., J' fulfils the properties $\text{improves}(J', x, M')$, $\text{bound}(J', x, M') = b$ and $J =$

- tight(J', x, M') (or $J = \text{div-derive}(J', x, M')$). Let the unguarded variable y be fixed by a decided bound γ , i.e., $M' = \llbracket M'', \gamma, M''' \rrbracket$. Let $I \in C'$ be a constraint with $\text{top}(I) = y$. Since S' is eager top-level propagated, all variables in I are fixed in M' and M'' . The variable x is not fixed in M' , because the predicate $\text{improves}(J', x, M')$ must be true for $\text{Propagate}(-\text{Div})$ to be applicable. Therefore, x is not contained in I (or $\text{top}(I) = y \prec x$) and I is still no conflict in S . Furthermore, all variables in I are still fixed in $\llbracket M', x \bowtie_J b \rrbracket$. We conclude that S is eager top-level propagated.
2. Let the applied rule be **Decide**. Then $S' = \langle M', C' \rangle$ and $S = \langle \llbracket M', x \bowtie b \rrbracket, C' \rangle$. We will use the eager top-level propagating strategy (Def. 15) to prove that S is an eager top-level propagated successor state. We consider all unguarded variables y decided in S' by a decided bound γ . Let $I \in C'$ be a constraint with $\text{top}(I) = y$. The bound γ is part of M' , i.e., $M' = \llbracket M'', \gamma, M''' \rrbracket$. As S' is eager top-level propagated, all other variables contained in I are fixed in M' and M'' . Since $\text{lower}(x, M') < \text{upper}(x, M')$ is a condition of the **Decide** rule, the variable x is not fixed in M' . Therefore, x is not contained in I ($\text{top}(I) = y \prec x$), and I is still no conflict in S . Furthermore, all variables in I are still fixed in $\llbracket M', x \bowtie_J b \rrbracket$. Next, we prove that the newly decided variable x does not violate that S is eager top-level propagated. Considering Def. 15.2(a) we see that Def. 24.1 is fulfilled. Similarly, Def. 15.2(b) enforces Def. 24.2. We conclude that S is eager top-level propagated.
 3. Let the applied rule be **Unsat(-Div)** or **Sat**. Then the successor state S is neither a search- or conflict-state. The Lemma is thereby trivially fulfilled.
 4. Let the applied rule be **Forget**. Then $S' = \langle M', C' \cup \{J\} \rangle$ and $S = \langle M', C' \rangle$. Therefore, any conflict $I \in C'$ and any decision in S is also contained in S' . We conclude that S is eager top-level propagated.
 5. Let the applied rule be **Slack-Intro**. Then $S' = \langle M', C' \rangle$, x is a stuck variable in S' and $S = \langle M', C' \cup \{-x_S \leq 0, x - x_S \leq 0, -x - x_S \leq 0\} \rangle$. Either **Slack-Intro** was applied before and $-x_S \leq 0 \in C'$ or x_S has $\text{upper}(x_S, M) = \infty$, and $-x_S \leq 0$ is not a conflict in S . Since x was stuck in S' , it is unfixed, and the top variable in the new constraints $\{x - x_S \leq 0, -x - x_S \leq 0\}$. We conclude that S is eager top-level propagated.
 6. Let the applied rule be **Resolve-Cooper**. Then $S' = \langle M', C' \rangle$ and $S = \langle M, C' \cup R_c \cup R_k \rangle$. Notice that $M = \text{prefix}(M', y)$ with $y = \min_{I \in R_c} \{\text{top}(I)\}$. Therefore, M is the prefix of M' without decisions in variables greater or equal to y . Since $y \preceq x$ for all $x = \text{top}(I)$ for $I \in R$, we deduce that any $I \in R$ that is a conflict has no decision for its top variable x in S . Since M is a prefix of M' , every conflict $I \in C'$ appearing in state S also appears in state S' . Now it is easy to see that S is eager top-level propagated because S' was eager top-level propagated.
 7. Let the applied rule be **Solve-Div-Right**. Then $S' = \langle M', C' \cup \{I_1, I_2\} \rangle$ and $S = \langle M, C' \cup \{I'_1, I'_2\} \rangle$. We notice that $M = \text{prefix}(M', y)$ with $y = \text{top}(I'_2)$. Therefore, M is the prefix of M' without decisions in variables greater or equal to y , which includes especially the variable $x = \text{top}(I_1)$. We deduce that neither the top variable of I'_1 or I'_2 is fixed by a decision. Since M is a prefix of M' , every conflict $I \in C'$ appearing in state S also appears in state

S' . Now it is easy to see that S is eager top-level propagated because S' was eager top-level propagated.

8. Let the applied rule be Solve-Div-Left. Then $S' = \langle M', C' \cup \{I_1, I_2\} \rangle$ and $S = \langle M', C' \cup \{I'_1, I'_2\} \rangle$. Since the bound sequence is the same in both states, every conflict $I \in C'$ appearing in state S also appears in state S' . By the definition of the Solve-Div-Left rule, I'_2 is no conflict in state S . Note that div-solve is an equivalence preserving transformation. Thus if I'_1 were a conflict in S , and $\text{top}(I'_1) = x$ fixed by a Decision, then I_1 or I_2 is a conflict in S' . Therefore, I'_1 is no conflict or $\text{top}(I'_1) = x$ is not decided by a Decision. Now it is easy to see that S is eager top-level propagated because S' was eager top-level propagated.
9. Let the applied rule be Conflict or Conflict-Div. Then $S' = \langle M', C' \rangle$ and $S = \langle M', C' \rangle \vdash I$. It is easy to see that S is eager top-level propagated because S' is eager top-level propagated.
10. Let the applied rule be Resolve or Skip-Decision. Then $S' = \langle \llbracket M, \gamma \rrbracket, C' \rangle \vdash J'$ and $S = \langle M, C' \rangle \vdash J$. Since M is a prefix of M' , every conflict $I \in C'$ appearing in state S also appears in state S' . Now it is easy to see that S is eager top-level propagated because S' was eager top-level propagated.
11. Let the applied rule be Learn. Then $S' = \langle \llbracket M', \gamma \rrbracket, C' \rangle \vdash I$ and $S = \langle M', C' \cup I \rangle \vdash I$. Since CUTSAT++ uses a two-layered strategy (Def. 17), I is a guarded constraint. Now it is easy to see that S is eager top-level propagated because S' was eager top-level propagated.
12. Let the applied rule be Backjump. Then $S' = \langle \llbracket M', \gamma, M'' \rrbracket, C' \rangle \vdash I$ and $S = \langle \llbracket M', \gamma' \rrbracket, C' \rangle$. Since CUTSAT++ uses a two-layered strategy (Def. 17), I is a guarded constraint. Now it is easy to see that S is eager top-level propagated because S' was eager top-level propagated.

□

Since the initial state $\langle \llbracket \rrbracket, C_0 \rangle$ fulfils the eager top-level propagated properties trivially, it is clear that CUTSAT++ produces only eager top-level states, except for the final states. The eager top-level propagated property is so important, because it allows us to show that CUTSAT++ resolves any conflict I it encounters. In case the conflict is a guarded constraint this is done with the conflict rules. Otherwise, the conflict I is an unguarded constraint and CUTSAT++ simulates weak Cooper elimination with the strong conflict resolution rules. First, we use Solve-Div-Left to simulate Phase I. This either ends with a call to Solve-Div-Right resolving the conflict or CUTSAT++ finds a conflicting core. Then the conflicting core is resolved with the rules Resolve-Cooper.

Lemma 26. *Let $S = \langle M, C \rangle$ be a state reachable by CUTSAT++. Let $I \in C$ be a conflict in state S . Then state S is not frozen.*

Proof. Assume for a contradiction that state S is frozen. W.l.o.g. we assume that $x = \text{top}(I)$ is the smallest variable in our order that is top variable in a conflict $I' \in C$. If x is a guarded variable then Conflict or Conflict-Div is applicable, which contradicts our initial assumption! Therefore, x is an unguarded variable.

Furthermore, all variables y smaller than x are fixed. Otherwise, we deduce for the smallest unfixed variable y that either

- y is stuck and Slack-Intro is applicable
- Propagate is applicable to a constraint I' where $\text{top}(I') = y$
- C contains at least two divisibility constraints I_1, I_2 that have y as their top variable and Solve-Div-Left or Solve-Div-Right is applicable
- S contains a diophantine conflicting core (y, I_d) and Resolve-Cooper is applicable
- Decision is applicable to y because all conditions in Def. 15.2 are fulfilled

Since S is eager top-level propagated and I is a conflict with top variable x , we know that state S contains no decision for x (Def. 24 and Lemma 25). W.l.o.g. we assume that C contains at most one divisibility constraint I_d with x as its top variable. Otherwise, Solve-Div-Left or Solve-Div-Right are applicable, which contradicts our initial assumption! Let $x \geq b_l$ be the strictest lower bound $b_l = \text{bound}(x, I_l, M, \geq)$ for an inequality $I_l \in C$ with top variable x or $-\infty$ if there is no inequality propagating a lower bound. Let $x \leq b_u$ be the strictest upper bound $b_u = \text{bound}(x, I_u, M, \leq)$ for an inequality $I_u \in C$ with top variable x or ∞ if there is no inequality propagating an upper bound. Since the strictly-two-layered strategy forbids the application of Forget to unguarded constraints, CUTSAT++ never removes an unguarded inequality. Furthermore, any bound $x \bowtie b$ propagated from a divisibility constraint requires another bound $x \bowtie b'$ propagated from an inequality. We deduce that $b_u \neq \infty$ if $\text{upper}(x, M) \neq \infty$ and $b_l \neq -\infty$ if $\text{lower}(x, M) \neq -\infty$. Next, we do a case distinction on whether the bounds b_u and b_l are finite:

- Let $b_u = \infty$ and $b_l = -\infty$. Then it holds for all inequalities $ax + p \leq 0$ that $\text{lower}(ax + p) = -\infty$. Thus I is no inequality. A divisibility constraint is only a conflict if $\text{lower}(x, M) \neq -\infty$ and $\text{upper}(x, M) \neq \infty$. This contradicts the assumption that I is a conflict.
- Let $b_u = \infty$ and $b_l \in \mathbb{Z}$. Then it holds for all inequalities $ax + p \leq 0$ with $a < 0$ that $\text{lower}(ax + p) = -\infty$. Thus I is no inequality. A divisibility constraint is only a conflict if $\text{lower}(x, M) \neq -\infty$ and $\text{upper}(x, M) \neq \infty$. This contradicts the assumption that I is a conflict.
- Let $b_l = -\infty$ and $b_u \in \mathbb{Z}$. Then it holds for all inequalities $ax + p \leq 0$ with $a > 0$ that $\text{lower}(ax + p) = -\infty$. Thus I is no inequality. A divisibility constraint is only a conflict if $\text{lower}(x, M) \neq -\infty$ and $\text{upper}(x, M) \neq \infty$. This contradicts the assumption that I is a conflict.
- Let $b_u < b_l$. Then $(x, \{I_l, I_u\})$ is a conflicting core, and Resolve-Cooper is applicable. This contradicts the assumption that no rule is applicable.
- Let $\{b_l, \dots, b_u\} \neq \emptyset$. Then I is not an inequality. If $(x, \{I_l, I_u, I_d\})$ is a conflicting core, then Resolve-Cooper is applicable contradicting our initial assumption. Therefore, there exists a solution $b_d \in \{b_l, \dots, b_u\}$ for x satisfying I_d . Let D be the set of divisibility constraints used to propagate a bound for x in M . All constraints $D' \subseteq D$ not contained in C , i.e., $D' = D \setminus C = D \setminus \{I_d\}$ were eliminated with div-solve. It is easy to see that there exists a set of constraints $D^* = D^{**} \cup \{I_d\}$ contained in C that implies satisfiability of D :

$$D^* = D^{**} \cup \{I_d\} \rightarrow D,$$

and D^{**} contains only variables y smaller than x (Proof the same as for Lemma 18). In state S the set of divisibility constraints D^* is fixed, and satisfied under the partial assignment of M . Otherwise S would contain a conflict $I' \in D^* \subseteq C$ with $\text{top}(I') \prec x$. This implies that the solution $b_d \in \{b_l, \dots, b_u\}$ for x that satisfies I_d in M , also satisfies $D \cup \{I_d\}$. Furthermore, all propagated constraints are satisfied if x is set to b_d :

$$\text{lower}(x, M) \leq b_d \leq \text{upper}(x, M).$$

This contradicts the assumption that there exists a conflict I with $\text{top}(I) = x$.

□

The remainder of the proof follows directly the proof outline from above:

Theorem 27. *Let $S = \langle M, C \rangle$ be a state reachable by CUTSAT++. Then S is not frozen.*

Proof. Assume for a contradiction that $S = \langle M, C \rangle$ is a frozen state. It is easy to see that CUTSAT++ can propagate at least two bounds for every guarded variable and afterwards use a Decision to fix them. Therefore, we assume that all guarded variables are fixed. By Lemma 26, there is no conflict in state S . Since there is no conflict, at least one variable is unfixed or rule Sat [14] would be applicable. Therefore, there must exist a smallest unfixed and unguarded variable x . With the Slack-Intro-rules CUTSAT++ introduces for all variables at least one lower or upper bound. Therefore, there exists a violation to the conditions in Def. 15.2 or Decide would be applicable to x . Since x is the smallest unfixed variable, condition Def. 15-(2a) holds. Def. 15.2 c) is also easy to satisfy by applications of Solve-Div-Left or Solve-Div-Right. Therefore, Def. 15.2 b) is violated. This implies that there exists a constraint $I \in C$ that is a conflict in $S' = \langle \llbracket M, \gamma \rrbracket, C \rangle$, where γ is a decision in x and $x = \text{top}(I)$. However, by Lemma 26, it is not possible that $I \in C$ is a conflict in S or S would not be frozen. Finally, I is a conflict only in S' and not S if Propagate(-Div) is applicable to I . With Solve-Div-Left and Solve-Div-Right it is relatively easy to fulfil the conditions for Def. 15.1 and therefore Propagate(-Div) is applicable. We conclude that CUTSAT++ has always one applicable rule, which is a contradiction to our assumption! □

5.3 Proof for Completeness

All CUTSAT++ rules are sound, i.e., if $\langle M_i, C_i \rangle (\vdash I_i) \Longrightarrow_{\text{cs}} \langle M_j, C_j \rangle (\vdash I_j)$ then any satisfiable assignment v for C_j is a satisfiable assignment also for C_i . The rule Resolve-Cooper is sound because of the Lemmas 6 and 7. The soundness of Solve-Div-Left and Solve-Div-Right follows from the fact that div-solve is an equivalence preserving transformation. The soundness proofs for all other rules are either trivial or given in [14].

Summarizing, CUTSAT++ is terminating, sound, and never reaches a frozen state. This in combination with obvious properties of the rules Sat, Unsat, and Unsat-Div implies completeness:

Theorem 28. *If CUTSAT++ starts from an initial state $\langle \square, C_0 \rangle$ then it either terminates in the unsat state and C_0 is unsatisfiable, or it terminates with $\langle v, \text{sat} \rangle$ where v is a satisfiable assignment for C_0 .*

Proof. By Theorem 23, CUTSAT++ is terminating. By [14], and the Lemmas 6 and 7, CUTSAT++ is sound. By Theorem 27, CUTSAT++ never reaches a frozen state. Since CUTSAT++ is terminating and never reaches a frozen state, every application of CUTSAT++ ends via the rules Sat, Unsat, or Unsat-Div in one of the final states. The rule Sat is only applicable in a state $\langle M, C \rangle$ where $v[M]$ satisfies C and because of soundness also C_0 . The rules Unsat and Unsat-Div are only applicable to states $\langle M, C \rangle (\vdash I)$ where the constraint set C contains a trivially unsatisfiable constraint. In this case, it follows from the soundness of the CUTSAT++ rules that C_0 is unsatisfiable. \square

6 Conclusion and Future Work

The starting point of our work was an implementation of CUTSAT [14] as a theory solver for hierarchic superposition [8]. In that course, we observed divergence for some of our problems. The analysis of those divergences led to the development of the CUTSAT++ algorithm presented in this paper, which is a substantial extension of CUTSAT by means of the weak Cooper elimination describe in Section 3.

As a next step, we plan to develop a prototypical implementation of CUTSAT++, to test its efficiency on benchmark problems. Depending on the outcome, we consider integrating CUTSAT++ as a theory solver for hierarchic superposition modulo linear integer arithmetic [8].

Finally, we point at some possible improvements of CUTSAT++. We see great potential in the development of constraint reduction techniques from (weak) Cooper elimination [4]. For practical applicability such reduction techniques might be crucial. The choice of the variable order \prec has considerable impact on the efficiency of CUTSAT++. It might be possibly to derive suitable orders via the analysis of the problem structure. We might benefit from results and experiences of research in quantifier elimination with variable elimination orders.

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