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Average Distance in a General Class of Scale-Free Networks with Underlying Geometry

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Abstract

In Chung-Lu random graphs, a classic model for real-world networks, each vertex is equipped with a weight drawn from a power-law distribution (for which we fix an exponent $2 < \beta < 3$), and two vertices form an edge independently with probability proportional to the product of their weights. Modern, more realistic variants of this model also equip each vertex with a random position in a specific underlying geometry, which is typically Euclidean, such as the unit square, circle, or torus. The edge probability of two vertices then depends, say, inversely polynomial on their distance.

We show that specific choices, such as the underlying geometry being Euclidean or the dependence on the distance being inversely polynomial, do not significantly influence the average distance, by studying a generic augmented version of Chung-Lu random graphs. Specifically, we analyze a model where the edge probability of two vertices can depend arbitrarily on their positions, as long as the marginal probability of forming an edge (for two vertices with fixed weights, one fixed position, and one random position) is as in Chung-Lu random graphs, i.e., proportional to the product of their weights. The resulting class contains Chung-Lu random graphs, hyperbolic random graphs, and geometric inhomogeneous random graphs as special cases. Our main result is that this general model has the same average distance as Chung-Lu random graphs, up to a factor $1 + o(1)$. The proof also yields that our model has a giant component and polylogarithmic diameter with high probability.

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1 Introduction

Large real-world networks, like social networks or the internet infrastructure, are almost always *scale-free*, i.e., their degree distribution follows a power law. Such networks have been studied in detail since the 60s. One of the key findings is the *small-world phenomenon*, which is the observation that two nodes in a network typically have very small graph-theoretic distance. Theoretical models of random graphs that explain this phenomenon have been proposed since the 90s. In this line of research, one studies the diameter of a graph, i.e., the largest distance between any pair of vertices (in the largest component), and its average distance, i.e., the expected distance between two nodes chosen independently and uniformly at random (from the largest component). A random graph model is said to be a *small world* if its diameter is bounded by $(\log n)^{O(1)}$ or even $O(\log n)$, and an *ultra-small world* if its average distance is $O(\log \log n)$.

Chung-Lu random graphs are a prominent model of scale-free networks [11, 12]. In this model, every vertex v is equipped with a weight w_v, and two vertices u, v are connected independently with probability $\min\{1, \mathsf{w}_u \mathsf{w}_v/\mathsf{W}\}$, where W is the sum over all weights w_v. The weights are typically assumed to follow a power-law distribution. Chung-Lu random graphs have the ultra-small world property, since the average distance is $(2 \pm o(1)) \frac{\log \log(n)}{|\log(\beta-2)|}$, if the power-law exponent is $2 < \beta < 3$ [11, 12].

However, Chung-Lu random graphs fail to capture other important features of real-world networks, such as their high clustering coefficient. This is why dozens of papers study more realistic models, many of which combine Chung-Lu random graphs (or other classic models such as preferential attachment [3]) with an underlying geometry, see, e.g., hyperbolic random graphs [5, 24], geometric inhomogeneous random graphs [9], and many others [2, 6, 7, 8, 15, 21]. In these models, each vertex is additionally equipped with a random position in some underlying geometric space, and the edge probability of two vertices depends on their weights as well as the geometric distance of their positions. Typical choices for the geometric space are the unit square, circle, or torus, and for the dependence on the distance are inverse polynomial, exponential, or threshold functions. Such models can naturally yield a large clustering coefficient, since there are many edges among geometrically close vertices. For some of these models the average distance has been studied and shown to be the same as in Chung-Lu graphs, up to a factor $1 + o(1)$, see, e.g., [1, 6, 15].

For these results, it is unclear how much they depend on the particular choice of the underlying geometry. In particular, it is not known whether any of the important properties of Chung-Lu random graphs transfer to versions with a *non-metric* underlying space. Such spaces are well-motivated in the context of social networks, where two persons are likely to know each other if share a feature (e.g., they are in the same sports club) regardless of their differences in other features, which gives rise to a non-metric distance (see Section 7).

Our contribution In this paper we prove that all geometric variants of Chung-Lu random graphs have the same average distance $(2 \pm o(1)) \frac{\log \log(n)}{|\log(\beta-2)|}$ for $2 < \beta < 3$, showing universality of the ultra-small world property. We do this by analyzing a generic augmented version of Chung-Lu random graphs. Here, each vertex is equipped with a power-law weight w_v and a random position x_v in some ground space $\mathcal{X}$. Two vertices u, v form an edge independently with probability p_{uv} that only depends on the positions $\mathsf{x}_u, \mathsf{x}_v$ (and u, v and the weight sequence). The dependence on $\mathsf{x}_u, \mathsf{x}_v$ may be arbitrary, as long as the edge probability has the same marginal probabilities as in Chung-Lu random graphs. Specifically, for fixed x_u and random x_v we require that the marginal edge probability $\mathbb{E}_{\mathsf{x}_v}[p_{uv} | \mathsf{x}_u]$ is within constant factors of the Chung-Lu edge probability $\min\{1, \mathsf{w}_u \mathsf{w}_v / W\}$. This is a natural property for any augmented version of Chung-Lu random graphs. Note that our model is stripped off any geometric specifics, in fact, the ground space is not even required to be metric. We retain only the most important features, namely power-law weights and the right marginal edge probabilities. See Section 2 for details of our model.

It it quite surprising that the average distance can be computed so precisely in this generality. In particular, in another regime (for $\beta > 3$) our model is too general to obtain any meaningful results: There are instantiations that do not have a giant component, but the largest component is of polynomial size $n^{1-\Omega(1)}$ [4]. In such graphs it makes much less sense to analyze diameter and average distance, so the model is not useful in this regime – which makes it even more surprising that our model allows to precisely determine the average distance in the regime $2 < \beta < 3$.

Beyond the average distance, we establish that our model is scale-free and has a giant component and polylogarithmic diameter. This shows that all instantiations of augmented Chung-Lu random graphs are reasonable models for real-world networks. We remark that the clustering coefficient varies drastically between different instantiations of our model, as it encompasses the classic Chung-Lu random graphs that have clustering coefficient $n^{-\Omega(1)}$, as well as geometric variants that have constant clustering coefficient [9].

From a technical perspective, for our analysis of the average distance we can only borrow one step from previous proofs for Chung-Lu graphs and its variants, namely the "greedy path" argument (Lemma 5.2). The remainder of the proof is a delicate analysis of the k-neighborhood of a vertex restricted to small-weight vertices, and the probability that any node in this k-neighborhood is connected to a high-weight vertex (Lemma 5.5), from which we can then apply the "greedy path" argument (Theorem 5.9).

In Section 2 we present the details of our model and results. After preliminary and basic results (Sections 3 and 4), we determine the connectivity properties in Section 5 and the degree distribution in Section 6. We discuss special cases of our model in Section 7 and close in Section 8.

2 Model and Results

2.1 Definition of the Model

Power law weights For $n \in \mathbb{N}$ let $\mathsf{w} = (\mathsf{w}_1, \dots, \mathsf{w}_n)$ be a non-increasing sequence of positive weights. We call $\mathsf{W} := \sum_{v=1}^{n} \mathsf{w}_v$ the *total weight.* Throughout this paper we will assume that the weights follow a *power law.* More precisely, we assume that for some $2 < \beta < 3$ (the *power-law exponent* of w) and some $\overline{w} = \overline{w}(n)$ with $n^{\omega(1/\log\log n)} \leq \overline{w} \leq n^{(1-\Omega(1))/(\beta-1)}$, the sequence w satisfies the following conditions:

(PL1) $\mathsf{w}_{\min} := \min\{\mathsf{w}_v \mid 1 \leq v \leq n\} = \Omega(1)$,

(PL2) for all $\eta > 0$ there are $c_1, c_2 > 0$ with

$$c_1 \frac{n}{w^{\beta-1+\eta}} \leq \#\{1 \leq v \leq n \mid \mathsf{w}_v \geq w\} \leq c_2 \frac{n}{w^{\beta-1-\eta}},$$

where the first inequality holds for all $\mathsf{w}_{\min} \leq w \leq \overline{w}$ and the second for all $w \geq \mathsf{w}_{\min}$.

We remark that these are standard assumptions for power-law graphs with average degree $\Theta(1)$. Note that since $\overline{w} \leq n^{(1-\Omega(1))/(\beta-1)}$, there are $n^{\Omega(1)}$ vertices with weight at least $\overline{w}$. On the other hand, no vertex has weight larger than $(c_2 n)^{1/(\beta-1-\eta)}$.

Random graph model Let $\mathcal{X}$ be a non-empty set, and assume we have a measure μ on $\mathcal{X}$ that allows to sample elements from $\mathcal{X}$. We call $\mathcal{X}$ the *ground space* of the model and the elements in $\mathcal{X}$ *positions*. The random graph $\mathcal{G}(n, \mathcal{X}, \mathsf{w}, p)$ has vertex set $V = [n] = \{1, \dots, n\}$. For any vertex v we independently draw a position $\mathsf{x}_v \in \mathcal{X}$ according to measure μ. We connect any two vertices $u \neq v$ independently with probability $p_{uv} := p_{uv}(\mathsf{x}_u, \mathsf{x}_v) := p_{uv}(\mathsf{x}_u, \mathsf{x}_v; n, \mathcal{X}, \mathsf{w})$, where p is a (symmetric in u, v and measurable) function mapping to $[0, 1]$ and satisfying the following condition:

(EP1) there are constants $0 < c_1 \leq c_2$ such that for any u, v, if we fix position $\mathsf{x}_u \in \mathcal{X}$ and draw position x_v from $\mathcal{X}$ according to μ, then the marginal edge probability is

$$\mathbb{E}_{\mathsf{x}_v}[p_{uv}(\mathsf{x}_u, \mathsf{x}_v) \mid \mathsf{x}_u] = \Theta\left(\min\left\{1, \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}}\right\}\right).$$

For most results we also need the following condition, to ensure a unique giant component:

(EP2) for all $\eta > 0$, any u, v with $\mathsf{w}_u, \mathsf{w}_v \geq \overline{w}$, and any fixed positions $\mathsf{x}_u, \mathsf{x}_v \in \mathcal{X}$ we have

$$p_{uv}(\mathsf{x}_u, \mathsf{x}_v) \geq \left(\frac{n}{\overline{w}^{\beta-1-\eta}}\right)^{-1+\omega(1/\log\log n)}.$$

Discussion of the model Let us first argue why condition (EP2) is necessary to obtain a unique giant component. Suppose we have an instantiation of our model G on a space $\mathcal{X}$. We will see in this paper that with high probability G has a giant component that contains all high-degree vertices. Now make a copy $\mathcal{X}'$ of $\mathcal{X}$, and consider a graph where all vertices draw geometric positions from $\mathcal{X} \cup \mathcal{X}'$. Vertices in $\mathcal{X}$ are never connected to vertices in $\mathcal{X}'$, but within $\mathcal{X}$ and $\mathcal{X}'$ we use the same connection probabilities as for G. Then the resulting graph will satisfy all properties of our model except for (EP2), but it will have two giant components, one in $\mathcal{X}$ and one in $\mathcal{X}'$. As we will see, (EP2) ensures that the high-weight vertices form a single dense network, and that the graph has a unique giant component. However, for our results on the degree sequence (EP2) is not necessary.

Since the right hand side of (EP1) is the edge probability of Chung-Lu graphs, this is a natural condition for any augmented version of Chung-Lu graphs. For similar reasons as discussed for (EP2), we cannot further relax (EP1) to a condition on the marginal probability over random positions x_u and x_v, i.e, a condition like $\mathbb{E}_{\mathsf{x}_u, \mathsf{x}_v}[p_{uv}(\mathsf{x}_u, \mathsf{x}_v)] = \Theta\left(\min\left\{1, \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}}\right\}\right)$. Indeed, consider the same setup as above, with G, $\mathcal{X}$, and copy $\mathcal{X}'$. For two vertices of weight at most $\bar{w}$, connect them only if they are in the same copy of $\mathcal{X}$. For two vertices of weight larger than $\bar{w}$, always treat them as if they would come from the same copy (then condition (EP2) is satisfied). For a vertex u of weight at most $\bar{w}$ and v of weight larger than $\bar{w}$, connect them only if u is in $\mathcal{X}'$. Then the high-weight vertices form a unique component, but it is only connected to vertices in $\mathcal{X}'$, while the low-weight vertices in $\mathcal{X}$ may form a second giant component. Thus, in (EP1) it is necessary to allow any fixed x_u.

Sampling the weights In our definition we assume that the weight sequence w is fixed. However, if we sample the weights according to an appropriate distribution, then the sampled weights will follow a power law with probability $1 - n^{-o(1)}$, so that a model with sampled weights is almost surely included in our model. For the precise statement, see Lemma 4.6.

Examples We regain the Chung-Lu model by setting $\mathcal{X} = \{x\}$ (the trivial ground space) and $p_{uv} = \min\left\{1, \frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right\}$, since then (EP1) is trivially satisfied and (EP2) is satisfied for $2 < \beta < 3$.

We discuss more special cases in Sections 7. Among our examples are *geometric inhomogeneous random graphs* (GIRGs) that where introduced in [9]. Consider the d-dimensional ground space $\mathcal{X} = [0, 1]^d$ with the standard (Lebesgue) measure, where $d \geq 1$ is a (constant) parameter of the model. Let $\alpha \neq 1$ be a second parameter that determines how strongly the geometry influences edge probabilities. Finally, let $\|\cdot\|$ be the Euclidean distance on $[0, 1]^d$, where we identify 0 and 1 in each coordinate (i.e., we take the distance on the torus). We show in Theorem 7.3 that every edge probability function p satisfying

$$p_{uv} = \Theta\left(\min\left\{1, (||\mathsf{x}_u - \mathsf{x}_v||)^{-d\alpha} \cdot \left(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right)^{\max\{\alpha,1\}}\right\}\right) \tag{1}$$

follows (EP1) and (EP2), so it is a special case of our model. As was shown in [9], an instance of hyperbolic random graphs [5, 26, 22] satisfies (1) asymptotically almost surely (over the choice of random weights w), so this class also is a special case of our model.

In Section 7 we will see that GIRGs can be varied as follows. As before, let $\mathcal{X} = [0, 1]^d$. For $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d) \in \mathcal{X}$, we define the *minimum component distance* $\|x - y\|_{\min} := \min\{|x_i - y_i| \mid 1 \leq i \leq d\}$, where the differences $x_i - y_i \in [-1/2, 1/2)$ are computed modulo 1, or, equivalently, on the circle. This distance reflects the property of social networks that two individuals may know each other because they are similar in only one feature (e.g., they share a hobby), regardless of the differences in other features. Note that the minimum component distance is not a metric, since there are $x, y, z \in \mathcal{X}$ such that x and y are close in one component, y and z are close in one (different) component, but x and z are not close in any component. Let $V(r)$ be the volume of the ball $B_r(0) := \{x \in \mathcal{X} \mid \|x\|_{\min} \leq r\}$. Then any p satisfying

$$p_{uv} = \Theta\left(\min\left\{1, V(\|\mathsf{x}_u - \mathsf{x}_v\|)^{-\alpha} \cdot \left(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right)^{\max\{\alpha,1\}}\right\}\right)$$

satisfies conditions (EP1) and (EP2), so it is a special case of our model.[1]

2.2 Our Results

Our results generalize and improve the understanding of Chung-Lu random graphs, hyperbolic random graphs, and other models, as they are special cases of our fairly general model. We study the following fundamental structural questions.

Scale-free Since we plug in power-law weights w, we expect our model to be scale-free.

Theorem 2.1 (Section 6)**.** *Whp*[2] *the degree sequence of our random graphs, not necessarily fulfilling (EP2), follows a power law with exponent β and average degree $\Theta(1)$.*

Giant component and diameter The connectivity properties of our model for $\beta > 3$ are not very well-behaved, in particular since in this case even threshold hyperbolic random graphs do not possess a giant component of linear size [4]. Hence, for connectivity properties we restrict our attention to the regime $2 < \beta < 3$, which holds for most real-world networks [17].

[1] These examples also show that our model is incomparable to the (also very general) model of inhomogeneous random graphs studied by Bollobás, Janson, and Riordan [6]. Their model requires sufficiently many long-range edges, so that setting $\alpha > 1$ in (1) yields an edge probability that is not supported by their model, and it requires metric distances, so that the minimum component distance is also not supported by their model.

[2] We say that an event holds *with high probability* (whp) if it holds with probability $1 - n^{-\omega(1)}$.

Theorem 2.2 (Section 5)**.** *Let $2 < \beta < 3$. Whp the largest component of our random graph model has linear size, while all other components have size at most $\log^{O(1)} n$. Moreover, whp the diameter is at most $\log^{O(1)} n$.*

A better bound of $\Theta(\log n)$ holds for the diameter of Chung-Lu graphs [13]. It remains an open problem whether the upper bound $O(\log n)$ holds in general for our model.

Average distance As our main result, we determine the average distance between two randomly chosen nodes in the giant component to be the same as in Chung-Lu random graphs up to a factor $1+o(1)$, showing that the underlying geometry is negligible for this graph parameter.

Theorem 2.3 (Section 5)**.** *The average distance of our random graph model is $(2\pm o(1))\frac{\log\log n}{|\log(\beta-2)|}$ in expectation and with probability $1-o(1)$ for any $2 < \beta < 3$.*

3 Preliminaries and Notation

3.1 Notation

For $w \in \mathbb{R}_{\geq 0}$, we use the notation $V_{\geq w} := \{v \in V \mid \mathsf{w}_v \geq w\}$ and $V_{\leq w} := \{v \in V \mid \mathsf{w}_v \leq w\}$, as well as $\mathsf{W}_{\geq w} := \sum_{v \in V_{\geq w}} \mathsf{w}_v$ and $\mathsf{W}_{\leq w} := \sum_{v \in V_{\geq w}} \mathsf{w}_v$. For $u, v \in V$ we write $u \sim v$ if u and v are adjacent, and for $A, B \subseteq V$ we write $A \sim v$ if there exists $u \in A$ such that $u \sim v$, and we write $A \sim B$ if there exists $v \in B$ such that $A \sim v$. For a vertex $v \in V$, we denote its neighborhood by $\Gamma(v)$, i.e. $\Gamma(v) := \{u \in V \mid u \sim v\}$. We say that an event holds *with high probability* (whp) if it holds with probability $1 - n^{-\omega(1)}$.

3.2 Tools

In the proofs we will use the following concentration inequalities.

Theorem 3.1 (Chernoff-Hoeffding bound, Theorem 1.1 in [19])**.** *Let $X := \sum_{i\in[n]} X_i$ where for all $i \in [n]$, the random variables X_i are independently distributed in $[0, 1]$. Then*

(i) $\Pr[X > (1+\varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}[X]\right)$ *for all $0 < \varepsilon < 1$,*

(ii) $\Pr[X < (1-\varepsilon)\mathbb{E}[X]] \leq \exp\left(-\frac{\varepsilon^2}{2}\mathbb{E}[X]\right)$ *for all $0 < \varepsilon < 1$, and*

(iii) $\Pr[X > t] \leq 2^{-t}$ *for all $t > 2e\mathbb{E}[X]$.*

We will need a concentration inequality which bounds large deviations taking into account some bad event $\mathcal{B}$. We start with the following variant of McDiarmid's inequality as given in [23] (slightly simplified).

Theorem 3.2 (Theorem 3.6 in [23])**.** *Let $X_1, \ldots, X_m$ be independent random variables over $\Omega_1, \ldots, \Omega_m$. Let $X = (X_1, \ldots, X_m)$, $\Omega = \prod_{k=1}^m \Omega_k$ and let $f\colon \Omega \to \mathbb{R}$ be measurable with $0 \leq f(\omega) \leq M$ for all $\omega \in \Omega$. Let $\mathcal{B} \subseteq \Omega$ such that for some $c > 0$ and for all $\omega \in \overline{\mathcal{B}}, \omega' \in \Omega$ that differ in only one component we have*

$$|f(\omega) - f(\omega')| \leq c.$$

Then for all $t > 0$

$$\Pr[|f(X) - \mathbb{E}[f(X)]| \geq t] \leq 2e^{-\frac{t^2}{8mc^2}} + 2\frac{mM}{c}\Pr[\mathcal{B}]. \tag{2}$$

Our improved version of this theorem is the following, where in the Lipschitz condition both ω and ω' come from the good set $\overline{\mathcal{B}}$, but we have to consider changes of two components at once. Recently, a similar inequality has been proven by Combes [14].

Theorem 3.3. *Let $X_1, \dots, X_m$ be independent random variables over $\Omega_1, \dots, \Omega_m$. Let $X = (X_1, \dots, X_m)$, $\Omega = \prod_{k=1}^m \Omega_k$ and let $f \colon \Omega \to \mathbb{R}$ be measurable with $0 \le f(\omega) \le M$ for all $\omega \in \Omega$. Let $\mathcal{B} \subseteq \Omega$ such that for some $c > 0$ and for all $\omega \in \overline{\mathcal{B}}, \omega' \in \overline{\mathcal{B}}$ that differ in at most* two *components we have*

$$|f(\omega) - f(\omega')| \le c. \tag{3}$$

Then for all $t \ge 2M \Pr[\mathcal{B}]$

$$\Pr\left[|f(X) - \mathbb{E}[f(X)]| \ge t\right] \le 2e^{-\frac{t^2}{32mc^2}} + (2\tfrac{mM}{c} + 1)\Pr[\mathcal{B}].$$

Proof. We say that $\omega, \omega' \in \Omega$ are *neighbors* if they differ in exactly one component. Given a function f as in the statement, we define a function f' as follows. On $\overline{\mathcal{B}}$ the functions f and f' coincide. Let $\omega \in \mathcal{B}$. If ω has a neighbor $\omega' \in \overline{\mathcal{B}}$, then choose any such ω' and set $f'(\omega) := f(\omega')$. Otherwise set $f'(\omega) := f(\omega)$.

The constructed function f' satisfies the precondition of Theorem 3.2. Indeed, let $\omega \in \overline{\mathcal{B}}$ and $\omega' \in \Omega$ differ in only one position. If $\omega' \in \overline{\mathcal{B}}$, then since $f'(\omega) = f(\omega)$ and $f'(\omega') = f(\omega')$, and by the assumption on f, we obtain $|f'(\omega) - f'(\omega')| \le c$. Otherwise we have $\omega' \in \mathcal{B}$, and since ω' has at least one neighbor in $\overline{\mathcal{B}}$, namely ω, we have $f'(\omega') = f(\omega'')$ for some neighbor $\omega'' \in \overline{\mathcal{B}}$ of ω'. Note that both ω and ω'' are in $\overline{\mathcal{B}}$, and as they are both neighbors of ω' they differ in at most two components. Thus, by the assumption on f we have

$$|f'(\omega) - f'(\omega')| = |f(\omega) - f(\omega'')| \le c.$$

Hence, we can use Theorem 3.2 on f' and obtain concentration of $f'(X)$. Specifically, since $\Pr[X \neq X'] \le \Pr[\mathcal{B}]$, and thus $|\mathbb{E}[f(X)] - \mathbb{E}[f'(X)]| \le M \Pr[\mathcal{B}]$, we obtain

$$\begin{aligned}\Pr[|f(X) - \mathbb{E}[f(X)]| \ge t] &\le \Pr[\mathcal{B}] + \Pr[|f'(X) - \mathbb{E}[f'(X)]| \ge t - M\Pr[\mathcal{B}]] \\ &\le \Pr[\mathcal{B}] + \Pr[|f'(X) - \mathbb{E}[f'(X)]| \ge t/2],\end{aligned}$$

since $t \ge 2M \Pr[\mathcal{B}]$, which together with Theorem 3.2 proves the claim. □

4 Basic Properties

In this section, we prove some basic properties which repeatedly occur in our proofs. In particular we calculate the expected degree of a vertex and the marginal probability that an edge between two vertices with given weights is present. Let us start with the following abstract statement.

Lemma 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Then for any weights $0 \le w_0 \le w_1$,*

$$\sum_{v \in V, w_0 \le \mathsf{w}_v \le w_1} f(\mathsf{w}_v) \;=\; f(w_0) \cdot |V_{\ge w_0}| \;-\; f(w_1) \cdot |V_{> w_1}| \;+\; \int_{w_0}^{w_1} f'(w) \cdot |V_{\ge w}| dw.$$

Note in particular that if $f(0) = 0$, then, by using $w_0 = 0$ and $w_1 > \mathsf{w}_{\max}$, we have

$$\sum_{v \in V} f(\mathsf{w}_v) = \int_0^{w_1} |V_{\ge w}| f'(w) dw = \int_0^{\infty} |V_{\ge w}| f'(w) dw.$$

Proof. We start by defining a measure ν on $\mathbb{R}$ as follows: For every set $A \subseteq \mathbb{R}$ we set $\nu(A) = |\{v \in V : \mathsf{w}_v \in A, w_0 \le \mathsf{w}_v \le w_1\}|$. In other words, ν is the sum of all Dirac measures given by the vertex weights between w_0 and w_1. Then

$$\begin{aligned}\sum_{v\in V, w_0\le \mathsf{w}_v\le w_1} f(\mathsf{w}_v) \;&=\; \int_0^{\mathsf{w}_{\max}} f(w)d\nu(w) \;=\; \int_0^{\mathsf{w}_{\max}}\int_0^{w} f'(x)dxd\nu(w) \;+\; \int_0^{\mathsf{w}_{\max}} f(0)d\nu(w)\\ &=\; \int_0^{\mathsf{w}_{\max}}\int_0^{\infty} f'(x)\cdot \mathbb{1}_{\{x\le w\}}dxd\nu(w) \;+\; f(0)\cdot|V_{\ge w_0}\setminus V_{>w_1}|.\end{aligned}$$

Notice that $[0,\mathsf{w}_{\max}]$ is a compact set and $f'(x)$ is continuous by assumption. Hence $|f'(x)\cdot \mathbb{1}_{\{x\le w\}}|$ is globally bounded on $[0,\mathsf{w}_{\max}]$ and always zero for $x > \mathsf{w}_{\max}$. Thus, $f'(x)\cdot\mathbb{1}_{\{x\le w\}}$ is integrable and we can apply Fubini's theorem (see, e.g., [20]), which yields

$$\begin{aligned}\sum_{v\in V, w_0\le \mathsf{w}_v\le w_1} f(\mathsf{w}_v) \;&=\; \int_0^{\infty} f'(x)\int_0^{\mathsf{w}_{\max}} \mathbb{1}_{\{w\ge x\}}d\nu(w)dx \;+\; f(0)\cdot|V_{\ge w_0}\setminus V_{>w_1}|\\ &=\; \int_0^{\infty} f'(x)\cdot|V_{\ge \max\{x,w_0\}}\setminus V_{>w_1}|dx \;+\; f(0)\cdot|V_{\ge w_0}\setminus V_{>w_1}|\\ &=\; \int_0^{w_0} f'(x)\cdot|V_{\ge w_0}\setminus V_{>w_1}|dx\\ &\qquad +\; \int_{w_0}^{w_1} f'(x)\cdot|V_{\ge x}\setminus V_{>w_1}|dx \;+\; f(0)\cdot|V_{\ge w_0}\setminus V_{>w_1}|\\ &=\; f(w_0)\cdot|V_{\ge w_0}| \;-\; f(w_1)\cdot|V_{>w_1}| \;+\; \int_{w_0}^{w_1} f'(w)\cdot|V_{\ge w}|dw.\end{aligned}$$

□

Recall the assumptions on power-law weights in Section 2.1. In the next lemma we calculate the partial weight sums $\mathsf{W}_{\le w}$ and $\mathsf{W}_{\ge w}$.

Lemma 4.2. *The total weight satisfies* $\mathsf{W} = \Theta(n)$. *Moreover, for all sufficiently small* $\eta > 0$,

(i) $\mathsf{W}_{\ge w} = O(nw^{2-\beta+\eta})$ *for all* $w \ge \mathsf{w}_{\min}$,

(ii) $\mathsf{W}_{\ge w} = \Omega(nw^{2-\beta-\eta})$ *for all* $\mathsf{w}_{\min} \le w \le \overline{w}$,

(iii) $\mathsf{W}_{\le w} = O(n)$ *for all* w, *and*

(iv) $\mathsf{W}_{\le w} = \Omega(n)$ *for all* $w = \omega(1)$.

Proof. We start with (i) and use Lemma 4.1 with $w_0 = w$, $w_1 = \infty$ and $f(w) = w$. Our assumption (PL2) on the weights implies

$$\mathsf{W}_{\ge w} = |V_{\ge w}|\cdot w + \int_w^{\infty} |V_{\ge x}|dx = O\left(nw^{2-\beta+\eta} + \int_w^{\infty} nx^{1-\beta+\eta}dx\right) = O\left(nw^{2-\beta+\eta}\right).$$

For (ii) we similarly obtain

$$\mathsf{W}_{\ge w} = \Omega\left(nw^{2-\beta-\eta} + \int_w^{\mathsf{w}_{\max}} nx^{1-\beta-\eta}dx\right) = \Omega\left(nw^{2-\beta-\eta}\right).$$

For (iii), we see that if $w < \mathsf{w}_{\min}$, then clearly $\mathsf{W}_{\le w} = 0$. Otherwise, Lemma 4.1 with $w_0 = \mathsf{w}_{\min}$ and $w_1 = w$ implies

$$\mathsf{W}_{\le w} = |V_{\ge \mathsf{w}_{\min}}| \cdot \mathsf{w}_{\min} - |V_{>w}| \cdot w + \int_{\mathsf{w}_{\min}}^{w} |V_{\ge x}| dx \le n\mathsf{w}_{\min} + O\left(\int_{\mathsf{w}_{\min}}^{w} nx^{1-\beta+\eta} dx \right) = O(n),$$

and for (iv) we obtain

$$\mathsf{W}_{\le w} \ge \int_{\mathsf{w}_{\min}}^{w} |V_{\ge x}| dx - |V_{>w}| \cdot w = \Omega\left(\int_{\mathsf{w}_{\min}}^{w} nx^{1-\beta-\eta} dx \right) - O\left(nw^{2-\beta+\eta} \right) = \Omega(n) - o(n) = \Omega(n).$$

□

Next we consider the marginal edge probability of two vertices u, v with weights w_u, w_v. For a fixed position $\mathsf{x}_u \in \mathcal{X}$, we already know this probability by Equation (EP1) of our definition.

Lemma 4.3. *Fix $u \in [n]$ and $\mathsf{x}_u \in \mathcal{X}$. All edges $\{u, v\}$, $u \neq v$, are independently present with probability*

$$\Pr[u \sim v \mid \mathsf{x}_u] = \Theta(\Pr[u \sim v]) = \Theta\left(\min\left\{1, \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}}\right\}\right).$$

Proof. Let $u, v \in [n]$. Then by (EP1), it follows directly

$$\Pr[u \sim v] = \mathbb{E}_{\mathsf{x}_u}\left[\Pr_{\mathsf{x}_v}[u \sim v \mid \mathsf{x}_u] \right] = \mathbb{E}_{\mathsf{x}_u}\left[\Theta\left(\min\left\{1, \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}}\right\}\right)\right] = \Theta\left(\min\left\{1, \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}}\right\}\right).$$

Furthermore, for every fixed $\mathsf{x}_u \in \mathcal{X}$ the edges incident to u are independently present with probability $\Pr_{\mathsf{x}_v}[u \sim v \mid \mathsf{x}_u]$, as the event "$u \sim v$" only depends on x_v, and an independent random choice for the edge $\{u, v\}$ (after fixing x_u). □

The following lemma shows that the expected degree of a vertex is of the same order as the weight of the vertex, thus we can interpret a given weight sequence w as a sequence of expected degrees.

Lemma 4.4. *For any $v \in [n]$ we have $\mathbb{E}[\deg(v)] = \Theta(\mathsf{w}_v)$.*

Proof. Let v be any vertex. We estimate the expected degree both from below and above. By Lemma 4.3, the expected degree of v is at most

$$\sum_{u \neq v} \Pr[u \sim v] = \Theta\left(\sum_{u \neq v} \min\left\{1, \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}}\right\}\right) = O\left(\sum_{u \in V} \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}} \right) = O\left(\frac{\mathsf{w}_v}{\mathsf{W}} \sum_{u \in V} \mathsf{w}_u \right) = O(\mathsf{w}_v).$$

For the lower bound, $\Pr[u \sim v] = \Theta(\frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}})$ holds for all $\mathsf{w}_u \le \frac{\mathsf{W}}{\mathsf{w}_v}$. We set $w' := \frac{\mathsf{W}}{\mathsf{w}_v}$ and observe that $w' = \omega(1)$. Using Lemma 4.2, we obtain

$$\mathbb{E}[\deg(v)] \ge \sum_{u \neq v, u \in V_{\le w'}} \Pr[u \sim v] = \Omega\left(\frac{\mathsf{w}_v}{\mathsf{W}} \mathsf{W}_{\le w'} \right) = \Omega(\mathsf{w}_v).$$

□

As the expected degree of a vertex is roughly the same as its weight, it is no surprise that whp the degrees of all vertices with weight sufficiently large are concentrated around the expected value. The following lemma gives a precise statement.

Lemma 4.5. *The following properties hold whp for all $v \in [n]$.*

(i) $\deg(v) = O(\mathsf{w}_v + \log^2 n)$.

(ii) If $\mathsf{w}_v = \omega(\log^2 n)$, *then* $\deg(v) = (1 + o(1))\mathbb{E}[\deg(v)] = \Theta(\mathsf{w}_v)$.

(iii) $\sum_{v \in V_{\geq w}} \deg(v) = \Theta(\mathsf{W}_{\geq w})$ *for all* $w = \omega(\log^2 n)$.

Proof. Let $v \in V$ with fixed position $\mathsf{x}_v \in \mathcal{X}$ and let $\mu := \mathbb{E}[\deg(v) \mid \mathsf{x}_v] = \Theta(\mathsf{w}_v)$. By definition of the model, conditioned on the position x_v the degree of v is a sum of independent Bernoulli random variables. By Lemma 4.4 there exists a constant c such that $2e\mu < c \log^2 n$ holds for all vertices $v \in V_{\leq \log^2 n}$ and all positions $\mathsf{x}_v \in \mathcal{X}$. Thus, if $v \in V_{\leq \log^2 n}$, we apply a Chernoff bound (Theorem 3.1.(iii)), and obtain $\Pr[\deg(v) > c \log^2 n] \leq 2^{-c \log^2 n} = n^{-\omega(1)}$. If $v \in V_{\geq \log^2 n}$, we similarly obtain $\Pr[\deg(v) > 3\mu/2] \leq e^{-\Theta(\mu)} = n^{-\omega(1)}$ and $\mu = \Theta(\mathsf{w}_v)$ by Lemma 4.4. Then (i) follows by applying a union bound over all vertices.

For (ii), let $v \in V$ such that $\mathsf{w}_v = \omega(\log^2 n)$, let μ be as defined above and put $\varepsilon = \frac{\log n}{\sqrt{\mu}} = o(1)$. Thus by the Chernoff bound,

$$\Pr\left[|\deg(v) - \mu| > \varepsilon \cdot \mu\right] \leq e^{-\Theta(\varepsilon^2 \cdot \mu)} = n^{-\omega(1)},$$

and we obtain (ii) by applying Lemma 4.4 and a union bound over all such vertices. Finally, from (ii) we infer $\sum_{v \in V_{\geq w}} \deg(v) = \sum_{v \in V_{\geq w}} \Theta(\mathsf{w}_v) = \Theta(\mathsf{W}_{\geq w})$ for all $w = \omega(\log^2 n)$, which shows (iii). □

We conclude this section by proving that if we sample the weights randomly from an appropriate distribution, then almost surely the resulting weights satisfy our conditions on power-law weights.

Lemma 4.6. *Let* $\mathsf{w}_{\min} = \Theta(1)$ *and* $F = F_n : \mathbb{R} \to [0, 1]$ *be non-decreasing such that* $F(z) = 0$ *for all* $z \leq \mathsf{w}_{\min}$, *and* $F(z) = 1 - \Theta(z^{1-\beta})$ *for all* $z \in [\mathsf{w}_{\min}, n^{1/(\beta-1-\varepsilon)}]$, *where* $\varepsilon > 0$. *Suppose that for every vertex* $v \in [n]$, *we choose the weight* w_v *independently according to the cumulative probability distribution* F. *Then asymptotically almost surely the resulting weight vector* w *satisfies the power-law conditions (PL1) and (PL2) with* $\overline{w} = (n/\log^2 n)^{1/(\beta-1)}$. *Moreover, for any fixed function* $1 \geq \lambda(n) \geq n^{-o(1)}$ *the error probability is bounded by* $\lambda(n)$ *for sufficiently large* n.

Thus, any property that holds with probability $1 - q$ for weights satisfying (PL1) and (PL2) also holds for weights sampled according to $F(.)$ with probability at least $1 - q - \lambda(n) = 1 - q - n^{o(1)}$.

Proof. Condition (PL1) is fulfilled by definition of F. Now consider (PL2). Denote by Y_z the number of vertices with weight at least z. For all $z \in [\mathsf{w}_{\min}, n^{1/(\beta-1-\varepsilon)}]$ the expected number of vertices with weight at least z is

$$\mathbb{E}[Y_z] = n(1 - F(z)) = \Theta(nz^{1-\beta}), \tag{4}$$

i.e., we have $c_3 nz^{1-\beta} \leq \mathbb{E}[Y_z] \leq c_4 nz^{1-\beta}$ for some $0 < c_3 \leq c_4$.

For the lower bound of (PL2) we show that whp for all $\mathsf{w}_{\min} \leq z \leq \overline{w}$ we have $Y_z > 0.5c_3 nz^{1-\beta}$. Hence, the lower bound of (PL2) even holds for $\eta = 0$ und thus also for all $\eta > 0$. To this end, we note that for $z \leq \overline{w}$ we have $\mathbb{E}[Y_z] = \Omega(\log^2 n)$, so for any $\mathsf{w}_{\min} \leq z \leq \overline{w}$ the Chernoff bound (Theorem 3.1.(ii)) yields

$$\Pr[Y_z \leq 0.5c_3 nz^{1-\beta}] \leq \Pr[Y_z \leq 0.5\mathbb{E}[Y_z]] \leq \exp(-\Omega(\mathbb{E}[Y_z])) = n^{-\omega(1)}.$$

Note that Y_z is always an integer. Hence we can assume without loss of generality that either $z \in \{\mathsf{w}_{\min}, \overline{w}\}$ or $0.5c_3nz^{1-\beta}$ is an integer, because if $Y_z > 0.5c_3nz^{1-\beta}$ holds for these values of z, then it holds for all other values z as well. Thus, we can restrict z to a set of size $O(n)$, which allows to take a union bound, and the lower bound of (PL2) holds whp.

For the upper bound of (PL2), let $0 < \eta < \varepsilon/2$. We show that whp for all $z \geq \mathsf{w}_{\min}$ we have $Y_z < c_2nz^{1-\beta+\eta}$, where $c_2 \geq 6c_4 \max\{1, \mathsf{w}_{\min}^{-\varepsilon}\}$. We first consider $\mathsf{w}_{\min} \leq z \leq z_\eta :=$ $(n/\log^2 n)^{1/(\beta-1-\eta)}$. In this range, the intended bound is $c_2nz^{1-\beta+\eta} = \Omega(\log^2 n)$. The Chernoff bound (Theorem 3.1.(iii)) applies (since $6z^\eta \max\{1, \mathsf{w}_{\min}^{-\varepsilon}\} > 2e$) and yields

$$\Pr[Y_z \geq c_2nz^{1-\beta+\eta}] \leq 2^{-c_2nz^{1-\beta+\eta}} = n^{-\omega(1)}.$$

By the same argument as above, we can restrict z to $\{\mathsf{w}_{\min}, z_\eta\}$ and values where $c_2nz^{1-\beta+\eta}$ is integral, and thus we may use the union bound.

In the remaining case $z > z_\eta$ we use Markov's inequality to bound

$$\Pr[Y_z \geq c_2nz^{1-\beta+\eta}] = \Pr[Y_z \geq \Omega(z^\eta)\mathbb{E}[Y_z]] \leq O(z^{-\eta}) \leq O(z_\eta^{-\eta}) \leq n^{-\Omega(\eta)}. \tag{5}$$

By the same argument as above, we can restrict z to values where $c_2nz^{1-\beta+\eta}$ is integral, which happens for $O(\log^2 n)$ values above z_η. Note in particular that any such value satisfies $z \leq n^{1/(\beta-1-\eta)}$, so that in (5) we only use values for z in the valid range $[\mathsf{w}_{\min}, n^{1/(\beta-1-\varepsilon)}]$. Hence, we can use the union bound to obtain error probability $O(n^{-\Omega(\eta)} \log^2 n)$. For later reference we note that by the same argument for $\widehat{z} := n^{1/(\beta-1-\varepsilon)}$, whp $Y_{\widehat{z}} < 1$.

So far we have shown that the lower bound of (PL2) holds whp for all $\eta \geq 0$, while the upper bound holds for any $\eta > 0$ with probability $1 - O(n^{-\Omega(\eta)} \log^2 n)$. From this we can conclude that for any $\eta' = \eta'(n) \geq \omega(\log\log n/\log n)$, with probability $1 - O(n^{-\Omega(\eta')})$, (PL2) holds for all $\eta \geq \eta'$. Indeed, (1) if (PL2) holds for every $\eta \geq \eta'$ that is a power of 2 then it holds for all $\eta \geq \eta'$, and (2) by a union bound over all $O(\log(1/\eta'))$ powers of 2 between 1 and η', (PL2) holds for all such powers of 2 with probability $1 - O(n^{-\Omega(\eta')} \log^2(n) \cdot \log(1/\eta')) \geq 1 - n^{-\Omega(\eta')}$, since $\eta' \geq \omega(\log\log n/\log n)$.

In order to also cover all $\eta < \eta'$, we fix a continuous, strictly decreasing function $\eta'(n)$ with $o(1) \geq \eta'(n) \geq \omega(\log\log n/\log n)$ and inverse function $g(\eta)$. If $\eta < \eta'$, then η is a non-trivial function in n, so we may also choose $c_2 = c_2(\eta(n))$ as a function in n in (PL2). Formally, we may set $c_2 := (g(\eta))^2$, which implies $c_2 \geq n^2$ for $\eta < \eta'$. With this choice, the upper bound of (PL2) holds trivially for all $z < \widehat{z} = n^{1/(\beta-1-\varepsilon)}$, where we may assume $\varepsilon < \beta - 2$. On the other hand, we have seen before that whp $Y_{\widehat{z}} < 1$. Since Y_n only takes integral values, this implies the upper bound of (PL2) for all $z \geq \widehat{z}$. Altogether, for any fixed continuous, strictly decreasing function $o(1) \geq \eta'(n) \geq \omega(\log\log n/\log n)$ with probability $1 - n^{-\Omega(\eta')}$ condition (PL2) holds for all $\eta > 0$. In particular, (PL2) holds with probability $1 - \lambda(n)$. □

5 Giant Component, Diameter, and Average Distance

Throughout this section we assume $2 < \beta < 3$. Under this assumption we prove that whp our model has a giant component with diameter at most $(\log n)^{O(1)}$, and that all other components are only of polylogarithmic size. We will further show that the expected average distance of any two vertices in the giant is $(2 \pm o(1)) \log\log n/|\log(\beta - 2)|$. The same formula has been known to hold for various graph models, including Chung-Lu [13] and hyperbolic random graphs [1]. The lower bound follows from the first moment method on the number of paths of different types. Note that the probability that a fixed path $P = (v_1, \ldots, v_k)$ exists in our model is the same as in Chung-Lu random graphs, since the marginal probability of $v_i \sim v_{i+1}$ conditioned on the positions of $v_1, \dots v_i$ is $\Theta(\min\{1, \mathsf{w}_{v_i}\mathsf{w}_{v_{i+1}}/\mathsf{W}\})$, as in the Chung-Lu model. In particular, the expected number of paths coincides for both models (save the factors coming from the $\Theta(.)$-notation). Not surprisingly, the lower bound for the expected average distance follows from general statements on power-law graphs, bounding the expected number of too short paths by $o(1)$, cf. [16, Theorem 2]. The main contribution of this section is to prove a matching upper bound for the average distance.

In the whole section let G be a graph sampled from our model. We start by considering the subgraph induced by the *heavy vertices* $\bar{V} := V_{\geq \overline{w}}$, where $\overline{w}$ is given by the definition of power law weights, see condition (PL2). We call the induced subgraph $\bar{G} := G[\bar{V}]$ the *heavy core*.

Lemma 5.1 (Heavy core)**.** *Whp $\bar{G}$ is connected and has diameter $o(\log\log n)$.*

Proof. Let $\bar{n}$ be the number of vertices in the heavy core, and fix $0 < \eta < 3 - \beta$. Since $\overline{w} \leq n^{(1-\Omega(1))/(\beta-1)}$, we may bound $\bar{n} = \Omega(n\overline{w}^{1-\beta-\eta}) = n^{\Omega(1)}$. By (EP2), the connection probability for any heavy vertices u, v, regardless of their position, is at least

$$p_{uv}(\mathsf{x}_u, \mathsf{x}_v) \geq \left(\frac{n}{\overline{w}^{\beta-1-\eta}}\right)^{-1+\omega(1/\log\log n)} \geq \bar{n}^{-1+\omega(1/\log\log n)}.$$

Therefore, the diameter of the heavy core is at most the diameter of an Erdős-Rényi random graph $G_{\bar{n},p}$, with $p = \bar{n}^{-1+\omega(1/\log\log n)}$. With probability $1 - \bar{n}^{-\omega(1)}$, this diameter is $\Theta(\log\bar{n}/\log(p\bar{n})) = o(\log\log n)$ [18]. Since $\bar{n} = n^{\Omega(1)}$, this proves the lemma. □

Next we show that if we start at a vertex of weight w, going greedily to neighbors of largest weight yields a short path to the heavy core with a probability that approaches 1 as w increases.

Lemma 5.2 (Greedy path)**.** *Let $0 < \varepsilon < 1$, and let v be a vertex of weight $2 \leq w \leq \overline{w}$. Then with probability at least $1 - O\left(\exp\left\{-w^{\Omega(\varepsilon)}\right\}\right)$ there exists a path of length at most $(1+\varepsilon)\frac{\log\log n}{|\log(\beta-2)|}$ from v to the heavy core. In particular, for every $\varepsilon > 0$ there is a $C > 0$ such that whp for all $v \in V_{\geq(\log n)^C}$ there exists a path of length at most $(1+\varepsilon)\frac{\log\log n}{|\log(\beta-2)|}$ from v to the heavy core. Moreover, whp there are $\Omega(n)$ vertices in the same component as the heavy core.*

Proof. Recall from the proof of Lemma 5.1 that there are $\bar{n} = n^{\Omega(1)}$ heavy vertices. Let $\tau := (\beta-2)^{-1/(1+\varepsilon/2)}$. Note that $1 < \tau < 1/(\beta-2)$, and that $1/\log\tau = (1+\varepsilon/2)/|\log(\beta-2)|$. Set $v_0 := v$, and define recursively v_{i+1} to be the neighbor of v_i of highest weight. Moreover, let $w_i := \min\{w^{\tau^i}, \overline{w}\}$ for all $i \geq 0$. We will show that with sufficiently high probability $\mathsf{w}_{v_i} \geq w_i$ for all $0 \leq i \leq i_{\max}$, where $i_{\max} := \lceil\log_\tau(\log\overline{w}/\log w)\rceil$ is the smallest integer such that $w^{\tau^i} \geq \overline{w}$. Note that this implies that there is a path from v to the heavy core of length at most $i_{\max} \leq (1+\varepsilon/2)\log\log n/|\log(\beta-2)| + 1 \leq (1+\varepsilon)\log\log n/|\log(\beta-2)|$, for sufficiently large n.

Let $0 \leq i \leq i_{\max} - 1$, and assume that v_i has weight at least w_i. We need to show that v_i connects to a vertex of weight at least w_{i+1}. By condition (EP1), the edges from v_i to v, $v \in V_{\geq w_{i+1}}$, are independently present with probability $\Omega(\min\{\mathsf{w}_v w_i/\mathsf{W}, 1\})$, respectively. If $w_i w_{i+1} \geq \mathsf{W}$, this probability is $\Omega(1)$. Since there are at least $|V_{\geq w_{i+1}}| \geq \bar{n} = n^{\Omega(1)}$ vertices of weight at least w_{i+1}, the probability that v_i will connect to at least one of them is $1 - \exp\{-n^{\Omega(1)}\} = 1 - \exp\{-w_i^{\Omega(\varepsilon)}\}$. So assume $w_i w_{i+1} < \mathsf{W}$. Then we can bound the edge probability from below by $\Omega(w_i w_{i+1}/\mathsf{W})$. Thus, for any $\eta > 0$ the probability that v_i does *not* connect to a vertex of weight at least w_{i+1} is at most

$$p_i := \prod_{v\in V,\, \mathsf{w}_v \geq w_{i+1}} \left(1 - \Omega\left(\frac{w_i w_{i+1}}{\mathsf{W}}\right)\right) \leq \exp\left\{ - \Omega\left(\frac{w_i w_{i+1}}{\mathsf{W}} \cdot |V_{\geq w_{i+1}}|\right)\right\}$$
$$\overset{(PL2)}{\leq} \exp\left\{-\Omega\left(w_i w_{i+1}^{2-\beta-\eta}\right)\right\}.$$

Since $w_{i+1} \leq w_i^{\tau}$, we obtain

$$p_i \leq \exp\left\{-\Omega\left(w_i^{1-\tau(\beta-2+\eta)}\right)\right\}$$

Note that since $\tau < 1/(\beta-2)$, the exponent of w_i in this expression is positive for sufficiently small $\eta > 0$. More precisely, we have

$$1 - \tau(\beta-2) = 1 - (\beta-2)^{\varepsilon/(2+\varepsilon)} = \Omega(\varepsilon),$$

and thus for sufficiently small $\eta > 0$ we have

$$p_i \leq \exp\left\{-w_i^{\Omega(\varepsilon)}\right\}. \tag{6}$$

By the union bound, the probability that for *every* $0 \leq i \leq i_{\max} - 1$ the vertex v_i has a neighbor of weight at least w_{i+1} is at least $1 - \sum_i \exp\left\{-w_i^{\Omega(\varepsilon)}\right\} = 1 - \exp\left\{-w^{\Omega(\varepsilon)}\right\}$, which proves the first claim.

For the second claim, let $C = \Omega(1/\varepsilon)$ with sufficiently large hidden constant. If a vertex v has weight at least $(\log n)^C$ then the probability estimated above is at least $1 - e^{-\Omega((\log n)^{C'})} = 1 - n^{\omega(1)}$. The claim now follows from a union bound over all vertices of weight at least $(\log n)^C$.

For the size of the giant component, we apply the same arguments as before for $w = 2$. Fix $\eta > 0$ sufficiently small, and let $V_i := \{v \in V \mid w_i \leq \mathsf{w}_v \leq w_i^{1+\eta}\}$. For every $\eta > 0$, V_i contains at least $\Omega(nw_i^{1-\beta-\eta})$ and at most $O(nw_i^{1-\beta+\eta})$ vertices by condition (PL2). For every $v \in V_i$, let $\Gamma_i(v) := \{u \in V_{i+1} \mid v \sim u\}$, and let $E_i(v) := \mathbb{E}[|\Gamma_i(v)|]$. Moreover, let $\gamma := \tau(2-\beta-\eta)+1 > 0$. Then for every $\eta > 0$ and for every $v \in V_i$,

$$E_i(v) \geq \Omega\left(nw_{i+1}^{1-\beta-\eta} \cdot \frac{w_i w_{i+1}}{\mathsf{W}}\right) \geq \Omega(w_{i+1}^{2-\beta-\eta} w_i) = \Omega(w_i^{\tau(2-\beta-\eta)+1}) \geq \Omega(w_i^{\gamma}).$$

As this lower bound is independent of v, we also have $E_i := \min_{v\in V_i} E_i(v) = \Omega(w_i^{\gamma})$. Let $\mu := \min\{\gamma, \frac{1}{2C}\}$ and $B_i := \{v \in V_i \mid |\Gamma_i(v)| \leq E_i/2\}$. This set will play the role of "bad" vertices.

Claim 5.3. *There is a constant $c > 0$ such that $|B_i| \leq 2\exp\{-cw_i^{\mu}\} \cdot |V_i|$ holds whp for all $i \geq 0$ with $w_i \leq (\log n)^C$.*

We postpone the proof of Claim 5.3 (and Claim 5.4 below) until we have finished the main argument. We uncover the sets V_i one by one, starting with the largest weights. Let $\delta > 0$ be so small that $\tau(\mu - \delta) > \mu$. Note that we may replace the factor 2 in Claim 5.3 by any other factor $D_1 \geq 2$ without violating the claim. We will show by induction that if $D_1 = O(1)$ is sufficiently large, then whp the fraction of vertices in V_i with a weight-increasing path to the inner core is at least $1 - D_1 \exp\{-cw_i^{\mu-\delta}\}$. Note that for any $i_0 = i_0(c) = O(1)$ the statement is trivial for all $i \leq i_0$, if we choose $D_1 = D_1(i_0, c)$ sufficiently large. Also, if $w_i \geq (\log n)^C$ then we already know that whp all vertices in V_i are connected to the inner core with weight-increasing paths. So consider some $i_0 < i \leq i_{\max}$ such that $w_i \leq (\log n)^C$, and assume the claim is shown for $i+1$. Let V'_{i+1} be the set of vertices in V_{i+1} for which there is no weight-increasing path to the inner core, so by induction hypothesis $|V'_{i+1}| \leq D_1 \exp\{-cw_{i+1}^{\mu-\delta}\} \cdot |V_{i+1}|$. Now we uncover V_i.

Claim 5.4. *There exists $D_2 > 0$ such that whp $|E(V_i, V'_{i+1})| \leq D_1 \exp\{cw_{i+1}^{\mu-\delta}\} \cdot |V_i| \cdot E_i \cdot w_i^{D_2}$.*

Let $B'_i := \{v \in V_i \mid |E(\{v\}, V'_{i+1})| \geq E_i/2\}$. If the low-probability event of Claim 5.4 does not occur, it follows in particular with $w_{i+1}^{\mu-\delta} = w_i^{\mu+\Omega(1)}$ that

$$|B'_i| \leq \frac{2|E(V_i, V'_{i+1})|}{E_i} \leq 2D_1 \exp\{-cw_{i+1}^{\mu-\delta}\} \cdot |V_i| \cdot w_i^{D_2} \leq D_1 \exp\{-cw_i^{\mu}\} \cdot |V_i| \tag{7}$$

for all $i \geq i_0$, provided that $i_0 = i_0(c)$ (and thus, w_{i_0}) is a sufficiently large constant. It remains to observe that every vertex in $V_i \setminus (B_i \cup B'_i)$ has at least one edge into $V_{i+1} \setminus V'_{i+1}$. Since the latter vertices are all connected to the inner core, we have at least $|V_i| - |B_i| - |B'_i|$ vertices in V_i that are connected to the inner core. By Claim 5.3 and Equation (7), whp both B_i and B'_i have size at most $D_1 \exp\{-cw_i^{\mu}\}|V_i|$, so together they have size at most $D_1 \exp\{-cw_i^{\mu-\delta}\}|V_i|$, for all $i \geq i_0$ where i_0 is sufficiently large. This concludes the induction modulo Claims 5.3 and 5.4. The existence of the giant component now follows because whp a constant fraction of V_{i_0} is connected to the inner core, and V_{i_0} has linear size by (PL2). □

Proof of Claim 5.3. For fixed $v \in V_i$, the events "$v \sim u$" are independent for all $u \in V_{i+1}$. So by the Chernoff bound, there is a constant $c > 0$ such that $\Pr[v \in B_i] \leq \exp\{-cw_i^{\gamma}\}$ and $\mathbb{E}[|B_i|] \leq \exp\{-cw_i^{\gamma}\}|V_i|$. In order to prove concentration we will use Theorem 3.3. For this, we need to argue that the probability space of our random graph model is a product of *independent* random variables. Recall that we apply two different randomized processes to create the geometric graph. First, for every vertex v we choose $\mathsf{x}_v \in \mathcal{X}$ independently at random. Afterwards, every edge is present with some probability p_{uv}. So far, these random variables are not independent.

The n random variables $\mathsf{x}_1, \ldots, \mathsf{x}_n$ define the vertex set and the edge probabilities p_{uv}. We introduce a second set of $n-1$ independent random variables. For every $u \in \{2, \ldots, n\}$ we let $Y_u := (Y_u^1, \ldots, Y_u^{u-1})$, where every Y_u^v is independently and uniformly at random from $[0,1]$. Then for $v < u$, we include the edge $\{u, v\}$ in the graph if and only if

$$p_{uv} > Y_u^v.$$

We observe that indeed this implies $\Pr[u \sim v \mid \mathsf{x}_u, \mathsf{x}_v] = p_{uv}(\mathsf{x}_u, \mathsf{x}_v)$ as desired. Furthermore, the $2n-1$ random variables $\mathsf{x}_1, \ldots, \mathsf{x}_n, Y_2, \ldots Y_n$ are independent and define a product probability space Ω which is equivalent to our random graph model. Formally, every $\omega \in \Omega$ defines a graph $G(\omega)$. Now we consider the bad event

$$\mathcal{B} = \{\omega \in \Omega : \text{ in } G(\omega) \text{ there exists } v \in V_i \cup V_{i+1} \text{ such that } \deg(v) \geq (\log n)^{O(1)}\}.$$

By Lemma 4.5 and the assumption $w_i \leq (\log n)^C$, indeed we have $\Pr[\mathcal{B}] = n^{-\omega(1)}$. Moreover for all $\omega, \omega' \in \overline{\mathcal{B}}$ that differ in two coordinates we have $||B_i(\omega)| - |B_i(\omega')|| \leq (\log n)^{O(1)}$. Furthermore, choose $t = \exp\{-cw_i^{\mu}\} \cdot |V_i|$, and observe that $w_i^{\mu} \leq (\log n)^{1/2}$. Then Theorem 3.3 implies

$$\begin{aligned}\Pr[|B_i| - \mathbb{E}[|B_i|] \geq t] &\leq 2\exp\left(-\frac{t^2}{64|V_i|(\log n)^{O(1)}}\right) + n^{O(1)} \Pr[\mathcal{B}] \\ &\leq 2\exp\left(-\frac{e^{-2cw_i^{\mu}}|V_i|}{64(\log n)^{O(1)}}\right) + n^{O(1)} \Pr[\mathcal{B}] = n^{-\omega(1)}.\end{aligned}$$

Hence, whp we have $|B_i| \leq \mathbb{E}[|B_i|] + t \leq (\exp\{-cw_i^{\gamma}\} + \exp\{-cw_i^{\mu}\}) \cdot |V_i|$. The claim now follows since $\mu < \gamma$ and $w_i > 1$. □

Proof of Claim 5.4. Let $Z_i = |E(V_i, V'_{i+1})|$ be the random variable counting the edges between V_i and V'_{i+1}. By (EP1) the expectation of Z_i is at most

$$\begin{aligned}\mathbb{E}[Z_i] &\le |V'_{i+1}| \cdot |V_i| \cdot O\left(\frac{w_{i+1}^{1+\eta} w_i^{1+\eta}}{\mathsf{W}}\right) \le O\left(D_1 \exp\{-cw_{i+1}^{\mu-\delta}\} n w_{i+1}^{1-\beta+\eta} \cdot |V_i| \cdot \frac{w_{i+1}^{1+\eta} w_i^{1+\eta}}{\mathsf{W}}\right) \\ &\le D_1 \exp\{-cw_{i+1}^{\mu-\delta}\} \cdot |V_i| \cdot O\left(w_{i+1}^{2-\beta+2\eta} w_i^{1+\eta}\right) \\ &\le D_1 \exp\{-cw_{i+1}^{\mu-\delta}\} \cdot |V_i| \cdot O\left(w_i^{\tau(2-\beta)+1+\eta(2\tau+1)}\right) \\ &\le D_1 \exp\{-cw_{i+1}^{\mu-\delta}\} \cdot |V_i| \cdot E_i \cdot O\left(w_i^{\eta(3\tau+1)}\right).\end{aligned}$$

Since we assumed $w_i \ge 2$, we may upper bound the $O(\cdot)$-term by $0.5 w_i^{D_2}$ for a sufficiently large $D_2 > 0$.

We use the same bad event $\mathcal{B}$ as above in the proof of Claim 5.3 and observe that for all $\omega, \omega' \in \overline{\mathcal{B}}$ that differ in two coordinates we have again $|Z_i(\omega) - Z_i(\omega')| \le (\log n)^{O(1)}$. Then for $t = 0.5 D_1 \exp\{-cw_{i+1}^{\mu-\delta}\} \cdot |V_i| \cdot E_i \cdot w_i^{D_2}$ it follows similarly as in Claim 5.3 that

$$\Pr[Z_i - \mathbb{E}[Z_i] \ge t] = n^{-\omega(1)}.$$

$\square$

By Lemma 5.2, whp every vertex of weight at least $(\log n)^C$ has small distance from the heavy core. It remains to show that every vertex in the giant component has a large probability to connect to such a high-weight vertex in a small number of steps. The next lemma shows that the more vertices of small weight we have in the neighborhood of a vertex, the more likely it is that there is an edge from the neighborhood to a vertex of large weight.

Lemma 5.5 (Bulk lemma)**.** *Let $\varepsilon > 0$. Let $\mathsf{w}_{\min} \le w \le \overline{w}$ be a weight, and let $k \ge \max\{2, w^{\beta+\varepsilon}\}$ be an integer. For a vertex $v \in V_{<w}$, let N_v be the set of all vertices in distance at most k of v in the graph $G_{<w}$. Then for a random vertex $v \in V_{<w}$,*

$$\Pr[\mathrm{dist}(v, V_{\ge w}) > k \text{ and } |N_v| \ge k] \le O\left(e^{-w^{\Omega(1)}}\right).$$

Proof. We may assume $w \le n^{1/2}$, since otherwise $k > n$, and the statement is trivial. Let $c > 0$ be such that for all vertices u of weight at least w, all vertices $u' \in V$, and every fixed position $\mathsf{x}_{u'} \in \mathcal{X}$ we have $\Pr[u \sim u' \mid \mathsf{x}_{u'}] \ge cw/n$, i.e., c is the hidden constant of condition (EP1). Finally by the power-law assumption (PL2), for any sufficiently small $0 < \eta < 1$ we may choose $\tilde{w} = O(w^{1+\eta})$ such that there are at least $\Omega(n/w^{\beta-1+\eta})$ vertices with weights between w and $\tilde{w}$.

We first uncover the graph $G_{<w}$ induced by vertices of weight less than w. Let $N_v := N_v(k, w)$ be the k-neighborhood of v in $G_{<w}$. Once $G_{<w}$ is fixed, consider a random vertex u with weight in $[w, \tilde{w}]$. Let $R := R(v) := \Pr_u[u \sim N_v \mid G_{<w}]$.

Claim 5.6. $Q := \Pr_u\left[|N_v \cap \Gamma(u)| \ge cw|N_v|/(2nR) \;\middle|\; G_{<w}\right] \ge \frac{cw}{2n}$.

Proof. Let $x := cw|N_v|/(2nR)$. We first use $|N_v \cap \Gamma(u)| \le |N_v|$ to bound

$$\mathbb{E}\left[|N_v \cap \Gamma(u)| \;\middle|\; G_{<w}\right] \le Q|N_v| + (R - Q)x \le Q|N_v| + Rx.$$

On the other hand, the left hand side is at least $cw|N_v|/n$ by our choice of c. Together, $Q \ge cw/n - Rx/|N_v| = cw/(2n)$, proving the claim. $\square$

Now we distinguish three cases. *(1)* If $|N_v| < k$ then there is nothing to show. *(2)* If $R \geq w^\beta/n$, then

$$\mathbb{E}[\#u \text{ with } \mathsf{w}_u \in [w, \tilde{w}] \text{ and } u \sim N_v \mid G_{<w}, R \geq w^\beta/n] \geq \Omega\left(\frac{w^\beta}{n} \cdot \frac{n}{w^{\beta-1+\eta}}\right) = \Omega(w^{\Omega(1)}),$$

since the number of vertices of weight in $[w, \tilde{w}]$ is at least $\Omega(n/w^{\beta-1+\eta})$ by (PL2). Since every u draws its position and its neighbors independently from each other, we may apply the Chernoff bounds and obtain

$$\Pr[\exists u \text{ with } \mathsf{w}_u \in [w, \tilde{w}] \text{ and } u \sim N_v \mid G_{<w}, R \geq w^\beta/n] \geq 1 - O\left(e^{-w^{\Omega(1)}}\right), \tag{8}$$

as desired.

(3) For the last case, $|N_v| \geq k$ and $R < w^\beta/n$, we will show that it is very unlikely that this case occurs for a random v (over the random choices in $G_{<w}$). More precisely, let $V_R \subseteq V_{<w}$ be the set of vertices v of weight less than w for which $|N_v| \geq k$ and $R(v) < w^\beta/n$. Further, let $\mathcal{E}$ be the event that $|V_R| \geq ne^{-c'w}$, where c' is a constant to be fixed later. Then we will show that $\Pr[\mathcal{E}] = e^{-\Omega(w)}$. Note that with this statement, we can conclude the proof as follows. Let v be a random vertex of weight less than w. When we uncover $G_{<w}$, then $\mathcal{E}$ occurs only with probability $e^{-\Omega(w)}$. On the other hand, if $\mathcal{E}$ does not occur, then there at most $ne^{-c'w}$ vertices $v' \in V_{<w}$ for which $|N_v| \geq k$ and $R(v') < w^\beta/n$, and the probability that v is among them is at most $ne^{-c'w}/|V_{<w}| = O(e^{-c'w}w^{\beta-1+\eta}) = O(e^{-\Omega(w)})$ for any $\eta > 0$. Finally, if v is not among these vertices, then either $|N_v| < k$, and we are done, or $R(v) \geq w^\beta/n$, and then $N_v \not\sim V_{\geq w}$ with probability at most $O(e^{-w^{\Omega(1)}})$ by (8). Thus the theorem follows. So it remains to show the following claim.

Claim 5.7. *For $V_R := \{v \in V_{<w} \mid |N_v| \geq k$ and $R < w^\beta/n\}$, with $\mathcal{E}$ being the event that $|V_R| \geq ne^{-c'w}$,*

$$\Pr[\mathcal{E}] = O(e^{-\Omega(w)}). \tag{9}$$

Before we prove Claim 5.7, we need some preparation. Sort the vertices $v \in V_R$ decreasingly by $|N_v|$. We go through the list one by one, and pick greedily a set $V_{Gr} \subseteq V_R$ such that the N_v, $v \in V_{Gr}$ are pairwise disjoint. Then after this procedure, the following holds.

Claim 5.8. $\sum_{v \in V_{Gr}} 2|N_v|^5 \geq |V_R|$.

Proof of Claim 5.8. We prove Claim 5.8 by the following charging argument. Whenever we pick a vertex v to be included into V_{Gr}, we inductively define levels $L_s(v) \subseteq V_R$, $s \geq 0$ by $L_0(v) := \{u \in N_v \mid |N_u| \leq |N_v|^2\}$ and $L_{s+1}(v) := \bigcup_{v' \in L_s(v)} \{u \in N_{v'} \mid |N_u| \leq |N_v|^{2^{-s}}\}$. The vertex v pays one coin to each vertex in $\bigcup_{s \geq 0} L_s(v)$. We claim that *(i)* every vertex v that we pick pays at most $2|N_v|^5$ coins, and *(ii)* every vertex in V_R is paid at least one coin. Note that (i) and (ii) together will imply Claim 5.8.

To prove (i), we observe that $|L_0(v)| \leq |N_v|$ and $|L_1(v)|/|L_0(v)| \leq |N_v|^2$ by definition of $L_0(v)$, and $|L_{s+1}(v)|/|L_s(v)| \leq |N_v|^{2^{1-s}}$ for all $s \geq 1$ by definition of $L_s(v)$. Therefore, $|L_s(v)| \leq |N_v|^{1+2+\sum_{j=1}^{s-1} 2^{1-j}}$. Moreover, for all $s > s_0 := \lfloor \log_2 \log_k |N_v| \rfloor$ we have $|N_v|^{2^{-s}} < k$, so $L_{s+1} = \emptyset$ by definition of V_R. On the other hand, for all $s \leq s_0$ we have $|N_v|^{2^{-s}} \geq k \geq 2$, so the terms $|N_v|^{3+\sum_{j=1}^{s-1} 2^{1-j}}$ increase at least geometrically fast for $s \leq s_0$. Hence, $\sum_{s=0}^{s_0} |L_s(v)| \leq \sum_{s=0}^{s_0} |N_v|^{3+\sum_{j=1}^{s-1} 2^{1-j}} \leq 2|N_v|^{3+\sum_{j=1}^{s_0-1} 2^{1-j}} \leq 2|N_v|^5$, proving (i).

For (ii), we show the following statement inductively for all vertices v. After v has paid its coins, every vertex u which comes after v in the ordering, and for which $N_u \cap N_v \neq \emptyset$ holds, has received at least one coin. Note that it will follow that each vertex that we consider and that we do not pick has been paid by an earlier vertex. So assume that u comes after v in the ordering, and that $N_u \cap N_v \neq \emptyset$. Since we go through the vertices in descending order with respect to $|N_v|$, we have $|N_u| \leq |N_v|$. Let $v' \in N_u \cap N_v$. If $|N_{v'}| \leq |N_v|^2$, then $v' \in L_0$ and $u \in L_1$, so v pays to u. If $|N_{v'}| > |N_v|^2$, then we have considered v' before v. However, since we picked v, and since $v' \in N_v$ (and thus, $v \in N_{v'}$), v' was not picked. Therefore, by induction hypothesis v' had been paid by some earlier vertex v'', so $v' \in L_s(v'')$ for some $s \geq 0$. Since $|N_u| \leq |N_v| < |N_{v'}|^{1/2} \leq |N_{v''}|^{2^{-s}}$, we obtain $u \in L_{s+1}(v'')$, so u has been paid by v'' as well. This proves (ii), and thus concludes the proof of Claim 5.8. Note that $2\sum_{v \in V_{Gr}} |N_v|^5 \geq |V_R| \geq ne^{-c'w}$ if $\mathcal{E}$ holds. □

Proof of Claim 5.7. With Claim 5.8, we can finally prove Claim 5.7 as follows. Fix a vertex u such that $\mathsf{w}_u \leq \tilde{w}$. Then for each position x_u of u, the expected degree of u conditioned on x_u is in $O(\tilde{w})$, and it is the sum of independent random variables by Lemmas 4.3 and 4.4. Note that the hidden constant in the $O(.)$-notation is independent of w_u and of x_u. Therefore, by the Chernoff-Hoeffding bound, there are constants $c', C > 0$ independent of w_u and x_u such that $\Pr[\deg(u) \geq i] \leq e^{-2c'i}$ for all $i \geq C\tilde{w}$, and this also holds if u is a *random* vertex with weight in $[w, \tilde{w}]$. So let u be a random vertex with weight in $[w, \tilde{w}]$, and let $V_u := \{v \in V_{Gr} \mid |N_v \cap \Gamma(u)| \geq |N_v| cw^{1-\beta}/2\}$. Consider the random variables

$$S_1 := 2 \sum_{v \in V_u} |N_v|^5 \qquad \text{and} \qquad S_2 := \frac{cw^{1-\beta}}{2} \sum_{v \in V_u} |N_v|.$$

Note that $S_2 \leq \deg(u)$ by definition of V_u, and since all $v \in V_u \subseteq V_{Gr}$ have disjoint N_v. Hence, $\Pr[S_2 \geq i] \leq \Pr[\deg(u) \geq i] \leq e^{-2c'i}$ for all $i \geq C\tilde{w}$. Now consider the expectation of S_1 conditioned on $\mathcal{E}$. On the one hand, since we are in the case $R < w^\beta/n$, we have $|N_v|cw^{1-\beta}/2 < |N_v|cw/(2nR)$, and thus $\Pr[v \in V_u \mid v \in V_{Gr}] \geq cw/(2n)$ by Claim 5.6. Hence, $\mathbb{E}[S_1 \mid \mathcal{E}] \geq cw/n \cdot \sum_{v \in V_{Gr}} |N_v|^5 \geq cwe^{-c'w}/2$ by Claim 5.8. On the other hand, since $\sum_{v \in V_u} |N_v|^5 \leq (\sum_{v \in V_u} |N_v|)^5$, we may bound $S_1 \leq 2 \cdot \left(2w^{\beta-1}S_2/c\right)^5$. Both inequalities together yield

$$\begin{aligned} cwe^{-c'w}/2 \leq \mathbb{E}[S_1 \mid \mathcal{E}] &\leq 2 \cdot \left(\frac{2w^{\beta-1}}{c}\right)^5 \cdot \mathbb{E}[S_2^5 \mid \mathcal{E}] = 2 \cdot \left(\frac{2w^{\beta-1}}{c}\right)^5 \cdot \sum_{i \geq 1} i^5 \Pr[S_2 = i \mid \mathcal{E}] \\ &\leq 2 \cdot \left(\frac{2w^{\beta-1}}{c}\right)^5 \cdot \frac{\sum_{i \geq 1} i^5 \Pr[S_2 = i]}{\Pr[\mathcal{E}]}. \end{aligned}$$

Solving for $\Pr[\mathcal{E}]$ yields $\Pr[\mathcal{E}] \leq w^{O(1)} e^{c'w} \sum_{i \geq 1} i^5 \Pr[S_2 = i]$. Observe that $S_2 > 0$ already implies $S_2 > cw^{1-\beta}k/2 \gg \tilde{w}$, since $|N_v| \geq k$ for all $v \in V_{Gr}$. So if w is sufficiently large then the first $C\tilde{w}$ terms of $\sum_{i \geq 1} i^5 \Pr[S_2 = i]$ vanish. On the other hand, recall that $\Pr[S_2 \geq i] \leq e^{-2c'i}$ for all $i \geq C\tilde{w}$. Hence, if w is sufficiently large,

$$\Pr[\mathcal{E}] \leq w^{O(1)} e^{c'w} \sum_{i \geq C\tilde{w}} i^5 \Pr[S_2 \geq i] \leq w^{O(1)} e^{c'w} \sum_{i \geq C\tilde{w}} i^5 e^{-2c'i} = \tilde{w}^{O(1)} e^{-\Omega(\tilde{w})} = O(e^{-\Omega(w)}).$$

This concludes the proof of Claim 5.7, and thus of the lemma. □

□

The upper bounds on the diameter and the average distance now follow easily from the lemmas we proved so far. We collect the results in the following theorem, which reformulates Theorem 2.2 and Theorem 2.3.

Theorem 5.9 (Components and Distances)**.** *With high probability,*

(i) there is a giant component, i.e., a connected component which contains $\Omega(n)$ vertices;

(ii) all other components have at most polylogarithmic size;

(iii) the giant component has polylogarithmic diameter.

Moreover, the average distance (i.e., the expected distance of two uniformly random vertices in the largest component) is $(2 \pm o(1))\frac{\log\log n}{|\log(\beta-2)|}$ in expectation and with probability $1-o(1)$.

Proof. (i) has been proven in Lemma 5.2. For (ii) and (iii) we conclude from the same lemma that whp the giant contains all vertices of weight at least $w := (\log n)^C$, for a suitable constant $C > 0$, and that whp all such vertices have distance at most $(1+\varepsilon)\frac{\log\log n}{|\log(\beta-2)|}$ from the heavy core $\bar{V}$. Choose a sufficiently small constant $\varepsilon > 0$, and apply Lemma 5.5 with $\ell = w^{\beta+\varepsilon}$. Then a random vertex in $V_{<w}$ has probability at least $1 - e^{-w^{\Omega(1)}}$ to either be at distance at most ℓ of $V_{\geq w}$, or to be in a component of size less than ℓ. Note that for sufficiently large C this probability is at least $1 - n^{-\omega(1)}$. By the union bound, whp one of the two options happens for all vertices in $V_{<w}$. This already shows that whp all non-giant components are of size less than $\ell = (\log n)^{O(1)}$. For the diameter of the giant, recall that whp the heavy core has diameter $o(\log\log n)$ by Lemma 5.1. Therefore, whp the diameter of the giant component is $O(\ell + \log\log n) = (\log n)^{O(1)}$.

For the average distance, let $\varepsilon = \varepsilon(n) = o(1)$, and let $v \in [n]$. We set $\lambda_\varepsilon := (1+\varepsilon)\frac{\log\log n}{|\log(\beta-2)|}$. Fix $\ell \geq 3$, $\ell = n^{o(1)}$, and let $w := w(\ell) = \ell^{1/(\beta+1)}$. We uncover the graph in two steps: in a first step, we uncover $G_1 := G[V_{<w} \cup \{v\}]$, and in the second step, we uncover the rest. Consider the $\ell-1$-neighborhood Γ of v in G_1. If $\mathsf{w}_v \geq w$ then Γ trivially contains a vertex of weight at least w. Otherwise, by Lemma 5.5, with probability $1 - O(\exp\{-w^{\Omega(1)}\})$ either $\Gamma \sim V_{\geq w}$, or Γ is the whole connected component of v in G (which happens automatically in the case $\Gamma \not\sim V_{\geq w}$ and $|\Gamma| < \ell$). If Γ is the whole connected component, then v is not connected to the core, and there is nothing to show. Otherwise, there is a vertex $v' \in V_{\geq w}$ with $\mathrm{dist}(v,v') \leq \ell$, and by Lemma 5.2 with probability $1 - O(\exp\{-w^{\Omega(\varepsilon)}\})$ there is a path from v' to the heavy core of length at most λ_ε. Summarizing, we have shown that for every vertex v and every $\ell \geq 3$ with $\ell = n^{o(1)}$

$$\Pr\left[\infty > \mathrm{dist}(v, V_{\mathrm{core}}) \geq \ell + \lambda_\varepsilon\right] \leq e^{-\Omega(w(\ell)^{\Omega(\varepsilon)})} = O(e^{-\ell^{\Omega(\varepsilon)}}). \tag{10}$$

Let us first consider the expectation of the average distance, i.e., if u, u' denote random vertices in the largest component of a random graph G then we consider $\mathbb{E}_G[\mathbb{E}_{u,u'}[\mathrm{dist}(u,u')]]$. Since $\mathrm{dist}(u,u') \leq n$ we can condition on any event happening with probability $1-n^{-\omega(1)}$, in particular we can condition on the event $\mathcal{E}$ that G has a giant component containing V_{core}, all other components have size $(\log n)^{O(1)}$, G has diameter $(\log n)^{O(1)}$, and the core has diameter $d_{\mathrm{core}} = o(\log\log n)$. Moreover, by bounding $\mathrm{dist}(u,u') \leq \mathrm{dist}(u, V_{\mathrm{core}}) + \mathrm{dist}(u', V_{\mathrm{core}}) + d_{\mathrm{core}}$ it suffices to bound $2 \cdot \mathbb{E}_G[\mathbb{E}_u[\mathrm{dist}(u, V_{\mathrm{core}})] \mid \mathcal{E}] + d_{\mathrm{core}}$. Now we use $\mathbb{E}[X] = \sum_{\ell>0} \Pr[X \geq \ell]$ for a random variable X taking values in $\mathbb{N}$ to bound

$$\mathbb{E}_u[\mathrm{dist}(u, V_{\mathrm{core}})] \leq \lambda_\varepsilon + \sum_{\ell=1}^{(\log n)^{O(1)}} \Pr_u\left[\mathrm{dist}(u, V_{\mathrm{core}}) \geq \ell + \lambda_\varepsilon\right].$$

Note that conditioned on $\mathcal{E}$, since u is chosen uniformly at random from the giant component, $\mathrm{dist}(u, V_{\mathrm{core}}) < \infty$. Taking expectation over G, conditioned on $\mathcal{E}$, we may use (10) to bound the probability that $\mathrm{dist}(v, V_{\mathrm{core}})$ is too large for a vertex chosen uniformly at random from V. Since the giant has size $\Omega(n)$, this probability increases at most by a constant factor if we instead choose v uniformly at random from the giant. Hence, we obtain

$$\mathbb{E}_G[\mathbb{E}_u[\mathrm{dist}(u, V_{\mathrm{core}})] \mid \mathcal{E}] \leq \lambda_\varepsilon + \sum_{\ell=1}^{(\log n)^{O(1)}} O(e^{-\ell^{\Omega(\varepsilon)}}) + n^{-\omega(1)}.$$

For every *constant* $\varepsilon > 0$ the sum in the above expression is $O(1)$, so if $\varepsilon = o(1)$ tends to zero sufficiently slowly, then it is still $o(\log \log n)$. This yields the desired bound on the expected average distance of $2\lambda_{o(1)} + o(\log \log n) = (2 + o(1)) \frac{\log \log n}{|\log(\beta-2)|}$.

For concentration, we want to show $\Pr_G[\mathbb{E}_{u,u'}[\mathrm{dist}(u, u')] \geq 2\lambda_\varepsilon] = o(1)$ for some $\varepsilon = \varepsilon(n) = o(1)$. Similarly as before, we bound

$$\Pr_G[\mathbb{E}_{u,u'}[\mathrm{dist}(u, u')] \geq \lambda_\varepsilon] \leq n^{-\omega(1)} + \Pr_G[2 \cdot \mathbb{E}_u[\mathrm{dist}(u, V_{\mathrm{core}})] + d_{\mathrm{core}} \geq \lambda_\varepsilon \mid \mathcal{E}].$$

Let $\gamma > 0$ be sufficiently small and $\omega(1) \leq \rho \leq o(\log \log n)$. We claim that for sufficiently large n, $2 \cdot \mathbb{E}_u[\mathrm{dist}(u, V_{\mathrm{core}})] + d_{\mathrm{core}} \geq \lambda_\varepsilon$ can only happen if for some $\ell > \rho$ we have $\Pr_u[\mathrm{dist}(u, V_{\mathrm{core}}) \geq \ell + \lambda_{\varepsilon/2}] \geq e^{-\ell^{\gamma \cdot \varepsilon}}$. Indeed, otherwise we have (conditioned on $\mathcal{E}$), similarly as before

$$\mathbb{E}_u[\mathrm{dist}(u, V_{\mathrm{core}})] \leq \lambda_{\varepsilon/2} + \rho + \sum_{\ell=\rho}^{(\log n)^{O(1)}} \Pr_u[\mathrm{dist}(u, V_{\mathrm{core}}) \geq \ell + \lambda_{\varepsilon/2}] \leq \lambda_{\varepsilon/2} + \rho + O(1),$$

and $\mathbb{E}_{u,u'}[\mathrm{dist}(u, u')] \leq 2 \cdot \mathbb{E}_u[\mathrm{dist}(u, V_{\mathrm{core}})] + d_{\mathrm{core}} \leq 2\lambda_{\varepsilon/2} + o(\log \log n) < 2\lambda_\varepsilon$ (if $\varepsilon = o(1)$ decreases sufficiently slowly compared to $(\rho + d_{\mathrm{core}})/\log \log n$). Using the union bound over all $\rho \leq \ell \leq (\log n)^{O(1)}$, the desired probability is bounded from above by

$$\sum_{\ell=\rho}^{(\log n)^{O(1)}} \Pr_G \left[\Pr_u[\mathrm{dist}(u, V_{\mathrm{core}}) \geq \ell + \lambda_{\varepsilon/2}] \geq e^{-\ell^{\gamma \cdot \varepsilon}} \mid \mathcal{E} \right].$$

However, by (10) and Markov's inequality, for sufficiently small $\gamma > 0$ we have for v randomly chosen from V, $\Pr_G[\Pr_v[\infty > \mathrm{dist}(v, V_{\mathrm{core}}) \geq \ell + \lambda_{\varepsilon/2}] > e^{-\ell^{\gamma \cdot \varepsilon}}] \leq O(e^{-\ell^{\Omega(\varepsilon)}})$. Since the giant has linear size, this probability increases at most by a constant factor if we instead draw v from the giant component (conditioned on $\mathcal{E}$). Thus, the desired probability is bounded by

$$\sum_{\ell=\rho}^{(\log n)^{O(1)}} O(e^{-\ell^{\Omega(\varepsilon)}}),$$

which is $o(1)$, since $\rho = \omega(1)$. This finishes the proof. □

6 Degree Sequence

In this paper we assume that the weights follow a power law. We start with the maximum degree $\Delta(G)$ of a GIRG, which is a simple corollary of Lemma 4.5.

Corollary 6.1. *Whp, $\Delta(G) = \Theta(\mathsf{w}_{\max})$, where $\mathsf{w}_{\max} = \max\{\mathsf{w}_v \mid v \in V\}$. In particular, for all $\eta > 0$, whp, $\Delta(G) = \Omega(\overline{w})$ and $\Delta(G) = O(n^{1/(\beta-1-\eta)})$.*

Proof. We deduce from the model definition that $\omega(\log^2 n) \le \overline{w} \le \mathsf{w}_{\max} = O(n^{1/(\beta-1-\eta)})$. Then Lemma 4.5 directly implies the statement. $\square$

Next, we calculate the expected number of vertices with degree at least d.

Lemma 6.2. *For any sufficiently small $\eta > 0$ there exist constants $c_3, c_4 > 0$ such that for all integers $1 \le d \ll \overline{w}$, we have*

$$c_3 n d^{1-\beta-\eta} \le \mathbb{E}[\#\{v \in V \mid \deg(v) \ge d\}] \le c_4 n d^{1-\beta+\eta}.$$

Proof. Let η be sufficiently small. Recall that by Lemma 4.4 there exist constants $c_5, c_6 > 0$ such that for all vertices v, $c_5 w_v \le \mathbb{E}[\deg(v)] \le c_6 w_v$. Let $1 \le d \ll \overline{w}$ and let v be any vertex with $w_v \ge \frac{2}{c_5} d$. Thus $\mathbb{E}[\deg(v)] \ge 2d$, and by a Chernoff bound

$$\Pr[\deg(v) < d] \le \Pr[\deg(v) < 0.5\mathbb{E}[\deg(v)]] \le e^{-\mathbb{E}[\deg(v)]/8} \le e^{-d/4} \le e^{-1/4}.$$

As we have power-law weights, there are $\Omega(n d^{1-\beta-\eta})$ vertices with weight at least $\frac{2}{c_5} d$, and such a vertex has degree at least d with probability at least $1 - e^{-1/4}$. By linearity of expectation, $\mathbb{E}[\#\{v \in V \mid \deg(v) \ge d\}] = \sum_{v \in [n]} \Pr[\deg(v) \ge d] \ge c_3 n d^{1-\beta-\eta}$.

Next let v be a vertex with weight at most $w := \frac{d}{3ec_6}$, hence $2e\mathbb{E}[\deg(v)] \le \frac{2d}{3} < \frac{3d}{4}$. By a Chernoff bound (Theorem 3.1.(iii)) we obtain

$$\Pr[\deg(v) \ge d] \le \Pr[\deg(v) > 3d/4] \le 2^{-3d/4}.$$

Thus, for the upper bound it follows that

$$\begin{aligned} \mathbb{E}[\#\{v \in V \mid \deg(v) \ge d\}] &= \sum_{v \in [n]} \Pr[\deg(v) \ge d] \le |V_{\ge w}| + \sum_{v \in V_{\le w}} \Pr[\deg(v) \ge d] \\ &\le O(n w^{1-\beta+\eta}) + n \cdot 2^{-3d/4}. \end{aligned}$$

Note that $d^2 \le 3 \cdot 2^{3d/4}$ holds for all $d \ge 1$. This fact implies $n \cdot 2^{-3d/4} \le 3nd^{-2} < 3nd^{1-\beta+\eta}$. Hence indeed for $c_4 > 0$ large enough it holds $\mathbb{E}[\#\{v \in V \mid \deg(v) \ge d\}] \le c_4 n d^{1-\beta+\eta}$. $\square$

With these preparations, we come to the main theorem of this section, which is a more precise formulation of Theorem 2.1 and states that the degree sequence follows a power law with the same exponent as the weight sequence.

Theorem 6.3. *For all $\eta > 0$ there exist constants $c_7, c_8 > 0$ such that whp*

$$c_7 \frac{n}{d^{\beta-1+\eta}} \le \#\{v \in V \mid \deg(v) \ge d\} \le c_8 \frac{n}{d^{\beta-1-\eta}},$$

where the first inequality holds for all $1 \le d \le \overline{w}$ and the second inequality holds for all $d \ge 1$.

Before we prove Theorem 6.3, we note that together with our standard calculations from Section 4 we immediately obtain the average degree in the graph.

Corollary 6.4. *With high probability, $\frac{1}{n} \sum_{v \in V} \deg(v) = \Theta(1)$.*

Proof of Theorem 6.3. We first consider the case where d is larger than $\log^3 n = o(\overline{w})$. From Condition (PL2) on the vertex weights and Lemma 4.4 it follows that there exists a constant $c_9 > 0$ such that $c_9 \frac{n}{d^{\beta-1+\eta}} \leq \#\{v \in V \mid \mathbb{E}[\deg(v)] \geq 1.5d\}$ for all $\log^3 n \leq d \leq \overline{w}$. Then by Lemma 4.5, whp every vertex v with $\mathbb{E}[\deg(v)] \geq 1.5d$ has degree at least $(1-o(1))1.5d \geq d$ for n large enough. Hence whp there exist at least $c_9 \frac{n}{d^{\beta-1+\eta}}$ vertices with degree at least d.

Vice-versa, $\#\{v \in V \mid \mathbb{E}[\deg(v)] \geq 0.5d\} \leq c_{10} \frac{n}{d^{\beta-1-\eta}}$ for some constant $c_{10} > 0$. By the same arguments as above, whp every vertex v with $\mathbb{E}[\deg(v)] < 0.5d$ has degree at most $(1+o(1))0.5d < d$. Thus the total number of vertices with degree at least d can be at most $c_{10} \frac{n}{d^{\beta-1-\eta}}$. This proves the theorem for $d \geq \log^3 n$.

Let $1 \leq d \leq \log^3 n$, $\varepsilon > 0$ be sufficiently small, $V' := V_{\leq n^\varepsilon}$ be the set of small-weight vertices, and $G' := G[V']$. First, we introduce some notation and define the two random variables $g_d := \#\{v \in V \mid \deg(v) \geq d\}$ and $f_d := \#\{v \in V' \mid \deg_{G'}(v) \geq d\}$. Note that by Lemma 6.2, we already have $c_3 n d^{1-\beta-\eta} \leq \mathbb{E}[g_d] \leq c_4 n d^{1-\beta+\eta}$ and it remains to prove concentration. Clearly,

$$f_d \leq g_d \leq f_d + 2 \sum_{v \in V \setminus V'} \deg(v). \tag{11}$$

Next we apply Lemma 4.5 together with Lemma 4.2 and see that whp,

$$\sum_{v \in V \setminus V'} \deg(v) = \Theta\left(\mathsf{W}_{\geq n^\varepsilon}\right) = O\left(n^{1+(2-\beta+\eta)\varepsilon}\right) = n^{1-\Omega(1)}.$$

Recall that we assume $d \leq \log^3 n$, so in particular $\mathbb{E}[g_d] = \Omega(n/(\log n)^{3(\beta-1+\eta)})$. It follows that $\mathbb{E}\big[\sum_{v \in V \setminus V'} \deg(v)\big] = o(\mathbb{E}[g_d])$. Inequalities (11) thus imply $\mathbb{E}[f_d] = (1+o(1))\mathbb{E}[g_d]$. Hence, it is sufficient to prove that the random variable f_d is concentrated around its expectation, because this will transfer immediately to g_d.

We aim to show this concentration result again via 3.3. Analogouly to the proof of Claim 5.3, we can assume that the probability space Ω is a product space of independent random variables. For every $\omega \in \Omega$, we denote by $G(\omega)$ the resulting graph, and similarly we use $G' = G'(\omega)$ and $f_d = f_d(\omega)$. We introduce the bad event:

$$\mathcal{B} := \{\omega \in \Omega : \text{ the maximum degree in } G'(\omega) \text{ is at least } n^{2\varepsilon}\}. \tag{12}$$

We observe that $\Pr[\mathcal{B}] = n^{-\omega(1)}$, since whp every vertex $v \in V'$ has degree at most $O(\mathsf{w}_v + \log^2 n) = o(n^{2\varepsilon})$ by Lemma 4.5. Let $\omega, \omega' \in \overline{\mathcal{B}}$ such that they differ in at most two coordinates. We observe that changing one coordinate x_i or Y_i can influence only the degrees of i itself and of the vertices which are neighbors of i either before or after the coordinate change. It follows that $|f_d(\omega) - f_d(\omega')| \leq 4n^{2\varepsilon} =: c$. Therefore, f_d satisfies the Lipschitz condition of Theorem 3.3 with bad event $\mathcal{B}$. Let $t = n^{1-\varepsilon} = o(\mathbb{E}[f_d])$. Then since $n \Pr[\mathcal{B}] = n^{-\omega(1)}$, Theorem 3.3 implies

$$\Pr\left[|f_d - \mathbb{E}[f_d]| \geq t\right] \leq 2e^{-\frac{t^2}{64c^2 n}} + \left(\tfrac{4n^2}{c} + 1\right)\Pr[\mathcal{B}] = e^{-\Omega(n^{1-4\varepsilon})} + n^{-\omega(1)} = n^{-\omega(1)},$$

which proves the concentration and concludes the proof. $\square$

7 Example: GIRGs and generalizations

In this section, we further discuss the special cases of our model mentioned in Section 2.1. In particular, we show that the GIRG model introduced in [9] is a special case, and we discuss a non-metric example.

The distance model Consider the following situation, which will cover both of our examples. As our underlying geometry we consider the ground space $\mathcal{X} = [0,1]^d$, where $d \geq 1$ is a (constant) parameter of the model. We sample from this set according to the standard (Lebesgue) measure. This is in the spirit of the classical random geometric graphs [25].

To describe the distance of two points $x, y \in \mathcal{X}$, assume we have some measurable function $\|.\| : [-1/2, 1/2)^d \to \mathbb{R}_{\geq 0}$ such that $\|0\| = 0$ and $\|-x\| = \|x\|$ for all $x \in [-1/2, 1/2)^d$. Note that $\|.\|$ does not need to be a norm or seminorm. We extend $\|.\|$ to $\mathbb{R}^d$ via $\|z\| := \|z - u\|$, where $u \in \mathbb{Z}^d$ is the unique lattice point such that $z - u \in [-1/2, 1/2)^d$. For $r \geq 0$ and $x \in \mathcal{X}$, we define the r-ball around x to be $B_r(x) := \{x \in \mathcal{X} \mid \|x - y\| \leq r\}$, and we denote by $V(r)$ the volume of the r-ball around 0. Intuitively, $B_r(x)$ is the ball around x in $[0,1]^d$ with the torus geometry, i.e., with 0 and 1 identified in each coordinate. Assume that $V : \mathbb{R}_{\geq 0} \to [0,1]$ is surjective, i.e., for each $V_0 \in [0,1]$ there exists r such that $V(r) = V_0$.

Let $\alpha \in \mathbb{R}_{>0}$ be a parameter. Since the case $\alpha = 1$ deviates slightly from the general case, we assume $\alpha \neq 1$. Let p be any edge probability function that satisfies for all u, v and $\mathsf{x}_u, \mathsf{x}_v \in \mathcal{X} = [0,1]^d$,

$$p_{uv}(\mathsf{x}_u, \mathsf{x}_v) = \Theta\left(\min\left\{ 1, V(\|\mathsf{x}_u - \mathsf{x}_v\|)^{-\alpha} \cdot \left(\frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}} \right)^{\max\{\alpha, 1\}} \right\} \right). \tag{13}$$

Then, as we will prove later in Theorem 7.3, p satisfies conditions (EP1) and (EP2), so it is a special case of our model.

Example 7.1. *If we choose* $\|.\|$ *to be the Euclidean distance* $\|.\|_2$ *then we obtain the* GIRG *model introduced in [9], where the distance of two points in* $[0,1]^d$ *is given by their Euclidean distance on the torus. In [9] it was shown that a graph from such a GIRG model whp has clustering coefficient* $\Omega(1)$*, that it can be stored with* $O(n)$ *bits in expectation, and that it can be sampled in expected time* $O(n)$*. Moreover, it was shown that hyperbolic random graphs are contained in the GIRG model.*

The next distance measure is particularly useful to model social networks: assume that two individuals share one feature (e.g., they are in the same sports club), but are very different in many other features (work, music, ...). Then they are still likely to know each other, which is captured by the minimum component distance.

Example 7.2. *Let the* minimum component distance *be defined by*

$$\|x\|_{\min} := \min\{x_i \mid 1 \leq i \leq d\} \textit{ for } x = (x_1, \ldots, x_d) \in [-1/2, 1/2)^d.$$

Note that the minimum component distance is not a metric for $d \geq 2$*, since there are* $x, y, z \in \mathcal{X}$ *such that* x *and* y *are close in one component,* y *and* z *are close in one (different) component, but* x *and* z *are not close in any component. Thus the triangle inequality is not satisfied. However, it still satisfies the requirements specified above, so our results of this paper apply.*

Theorem 7.3. *In the geometric setting described above, let* p *be any function that satisfies Equation (13). Then conditions (EP1) and (EP2) are satisfied.*

Proof. Fix u, v, and x_u. Note that $V(r)$ is the cumulative probability distribution $\Pr_{\mathsf{x}_v}(\|\mathsf{x}_u - \mathsf{x}_v\| \leq r)$. The marginal edge probability is given by the Riemann-Stieltjes integral over r,

$$E := \mathbb{E}_{\mathsf{x}_v}[p_{uv}(\mathsf{x}_u, \mathsf{x}_v) \mid \mathsf{x}_u] = \Theta\left(\int_0^\infty \min\left\{ 1, V(r)^{-\alpha} \cdot \left(\frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}} \right)^{\max\{\alpha, 1\}} \right\} dV(r) \right).$$

In particular, for every sequence of partitions $r^{(t)} = \{0 = r_0^{(t)} < \ldots < r_{\ell(t)}^{(t)}\}$ with meshes tending to zero, the upper Darboux sum with respect to $r^{(t)}$ converges to the expectation,

$$E = \Theta\left(\lim_{t\to\infty}\sum_{s=1}^{\ell(t)}\left(\sup_{r_{s-1}^{(t)}\le r\le r_s^{(t)}}\min\left\{1, V(r)^{-\alpha}\cdot\left(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right)^{\max\{\alpha,1\}}\right\}\right)\left(V(r_{s+1}^{(t)}) - V(r_s^{(t)})\right)\right).$$

Since V is surjective, we may refine the meshes $r^{(t)}$ if necessary such that the meshes of the partitions $V^{(t)} = \{V(r_0^{(t)}), \ldots, V(r_{\ell(t)}^{(t)})\}$ also tend to zero. Hence,

$$\begin{aligned}
E &= \Theta\left(\lim_{t\to\infty}\sum_{s=1}^{\ell(t)}\min\left\{1, (V_s^{(t)})^{-\alpha}\cdot\left(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right)^{\max\{\alpha,1\}}\right\}\left(V_{s+1}^{(t)} - V_s^{(t)}\right)\right)\\
&= \Theta\Big(\int_0^1 \min\left\{1, V^{-\alpha}\cdot\left(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right)^{\max\{\alpha,1\}}\right\}dV\Big),
\end{aligned}$$

where the latter integral is an ordinary Riemann integral. If $\mathsf{w}_u\mathsf{w}_v/\mathsf{W} \ge 1$, the integrand is 1 and we obtain $E = \Theta(1) = \Theta\left(\min\left\{1, \frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right\}\right)$. On the other hand, if $\mathsf{w}_u\mathsf{w}_v/\mathsf{W} < 1$ then let $r_0 := (\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}})^{\max\{\alpha,1\}/\alpha} < 1$. Note that if $r_0 = \Theta(1)$, then also $r_0 = \Theta(\mathsf{w}_u\mathsf{w}_v/\mathsf{W})$. Therefore,

$$\begin{aligned}
E &= \Theta\Big(\int_0^{r_0} 1dV + \left(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right)^{\max\{\alpha,1\}}\int_{r_0}^1 V^{-\alpha}dV\Big)\\
&= \begin{cases}\Theta\Big(r_0 + \frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\left(1 - r_0^{1-\alpha}\right)\Big) = \Theta\Big(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\Big) & \text{, if } \alpha < 1, \text{ and}\\ \Theta\Big(r_0 + \left(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right)^{\alpha}\left(r_0^{1-\alpha} - 1\right)\Big) = \Theta\Big(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\Big) & \text{, if } \alpha > 1,\end{cases}
\end{aligned}$$

as required.

It remains to show that p satisfies (EP2), i.e., that $p_{uv} \ge \left(\frac{n}{\overline{w}^{\beta-1-\eta}}\right)^{-1+\omega(1/\log\log n)}$ for all vertices u, v with $\mathsf{w}_u, \mathsf{w}_v \ge \bar{w}$, all $\mathsf{x}_u, \mathsf{x}_v \in \mathcal{X}$, and all $\eta > 0$. Since $V(\|\mathsf{x}_u - \mathsf{x}_v\|_2) \le 1$, we may use Equation (13) to bound $p_{uv} \ge \Omega(\min\{1, (\mathsf{w}_u\mathsf{w}_v/\mathsf{W})^{\max\{\alpha,1\}}\})$. If $\mathsf{w}_u\mathsf{w}_v/\mathsf{W} \ge 1$ then there is nothing to show (since the right hand side of (EP2) is $o(1)$ by the upper bound on $\overline{w}$). Otherwise, if $\mathsf{w}_u\mathsf{w}_v/\mathsf{W} < 1$, then

$$p_{uv} \ge \Omega\left(\left(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\right)^{\max\{\alpha,1\}}\right) \ge \Omega\left(\frac{\overline{w}^2}{n}\right) \ge \left(\frac{n}{\overline{w}^{\beta-1-\eta}}\right)^{-1+\omega(1/\log\log n)},$$

where the last step follows from the lower bound on $\overline{w}$. This concludes the proof. □

Finally, we discuss a variation of Example 7.1 where we let $\alpha \to \infty$ and thus obtain a threshold function.

Example 7.4. *Let $\|.\|$ be the Euclidean distance $||.||_2$ and let p again satisfy (13), but this time we assume that $\alpha = \infty$. More precisely, we require*

$$p_{uv}(\mathsf{x}_u, \mathsf{x}_v) = \begin{cases}\Theta(1) & \text{if } \|\mathsf{x}_u - \mathsf{x}_v\| \le O\big(\big(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\big)^{1/d}\big)\\ 0 & \text{if } \|\mathsf{x}_u - \mathsf{x}_v\| \ge \Omega\big(\big(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}}\big)^{1/d}\big),\end{cases} \tag{14}$$

where the constants hidden by O and Ω do not have to match, i.e., there can be an interval $[c_1(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}})^{1/d}, c_2(\frac{\mathsf{w}_u\mathsf{w}_v}{\mathsf{W}})^{1/d}]$ for $\|\mathsf{x}_u - \mathsf{x}_v\|$ where the behaviour of $p_{uv}(\mathsf{x}_u, \mathsf{x}_v)$ is arbitrary. This *function p yields the case $\alpha = \infty$ of the* GIRG *model introduced in [9]. In [9] we proved that threshold hyperbolic random graphs are contained in this model, and furthermore that the model whp has clustering coefficient $\Omega(1)$, it can be stored with $O(n)$ bits in expectation, and that it can be sampled in expected time $O(n)$.*

Notice that the volume of a ball with radius $r_0 = \Theta((\frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}})^{\frac{1}{d}})$ around any fixed $x \in \mathcal{X}$ is $\Theta(\min\{1, \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}}\})$. Thus, by (14), for fixed x_u it follows directly that

$$\mathbb{E}_{\mathsf{x}_v}[p_{uv}(\mathsf{x}_u, \mathsf{x}_v) \mid \mathsf{x}_u] = \Theta\Big(\Pr_{\mathsf{x}_v}\big[\|\mathsf{x}_u - \mathsf{x}_v\|_2 \le r \mid \mathsf{x}_u\big]\Big) = \Theta\left(\min\left\{1, \frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}}\right\}\right).$$

In order to also satisfy (EP2), we additionally require that $2 < \beta < 3$ and $\overline{w} = \omega(n^{1/2})$. Then for all $\mathsf{w}_u, \mathsf{w}_v \ge \overline{w}$ we have $\frac{\mathsf{w}_u \mathsf{w}_v}{\mathsf{W}} = \omega(1)$. For all positions $\mathsf{x}_u, \mathsf{x}_v \in \mathcal{X}$ we thus obtain $p_{uv}(\mathsf{x}_u, \mathsf{x}_v) = \Theta(1)$ by (14). We emphasize that this additional assumption is only necessary for property (EP2). For the degree sequence this condition is not required.

8 Conclusion

We studied a class of random graphs that genericly augment Chung-Lu random graphs by an underlying ground space, i.e., every vertex has a random position in the ground space and edge probabilities may arbitrarily depend on the vertex positions, as long as marginal edge probabilities are preserved. Since our model is very general, it contains well-known special cases like hyperbolic random graphs [5, 24] and geometric inhomogeneous random graphs [9]. Beyond these well-studied models, our model also includes non-metric ground spaces, which are motivated by social networks, where two persons are likely to know each other if they share a hobby, regardless of their other hobbies.

Despite its generality, we show that all instantiations of our model have similar connectivity properties, assuming that vertex weights follow a power law with exponent $2 < \beta < 3$. In particular, there exists a unique giant component of linear size and the diameter is polylogarithmic. Surprisingly, for all instantiations of our model the average distance is the same as in Chung-Lu random graphs, namely $(2 \pm o(1))\frac{\log\log n}{|\log(\beta-2)|}$. In some sense, this shows universality of ultra-small worlds.

We leave it as an open problem to determine whether the diameter of our model is $O(\log n)$ for $2 < \beta < 3$.

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