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E_6 Grand Unified Theories

Doctoral Thesis

ADVISOR: prof. dr. Borut Bajc

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Teorije poenotenja z grupo E_6

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Izjava o avtorstvu in objavi elektronske oblike zaključnega dela

Podpisani Vasja Susič izjavljam:

- da sem doktorsko disertacijo z naslovom E_6 *Grand Unified Theories* (sl. *Teorije poenotenja z grupo* E_6) izdelal kot rezultat lastnega raziskovalnega dela pod mentorstvom prof. dr. Borut Bajca,
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Povzetek

V tej doktorski disertaciji preučujemo Teorije poenotenja z grupo E_6. Omejili smo se na renormalizabilne supersimetrične modele, kjer se fermioni Standardnega modela nahajajo v treh družinah fundamentalne upodobitve 27, medtem ko sestavljanje modelov poteka v Higgsovem sektorju. V vseh modelih predpostavimo $\mathbb{Z}_2$ simetrijo, pod katero so upodobitve fermionskega sektorja lihe, upodobitve Higgsovega sektorja pa sode.

Pred analizo samih modelov najprej pripravimo vso potrebno infrastrukturo za računanje z grupo E_6: definiramo njeno Lijevo algebro, zberemo potrebna orodja za opis njenih irreducibilnih upodobitev, poiščemo v teh upodobitvah stanja, ki nas zanimajo, in z njimi končno izračuamo invariante.

Z vsemi orodji za E_6 pripravljenimi, se nato posvetimo sistematičnemu študiju modelov od preprostejših h bolj kompliciranim, in skušamo med njimi najti realistične modele. Prvi kriterij je, ali lahko v modelu spontano zlomimo E_6 vse do grupe Standardnega modela v enem koraku. Izkaže se, da najpreprostejši modeli tega ne zmorejo. Model, ki ima v Higgsovem sektorju le upodobitve 27, $\overline{27}$ in 78, vsako v poljubno mnogo kopijah, lahko zlomi kvečjemu do SU(5). Prav tako so neustrzni modeli s Higgsovem sektorjem $351' \oplus \overline{351'}$, 650 term model s parom $351 \oplus \overline{351}$ s poljubno mnogo dodanimi 27 in $\overline{27}$.

Spondatni zlom simetrije pa je uspešen v model $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}$, ki mu pravimo prototipni model. V tem modelu naletimo na nepričakovano težavo, ko ne moremo opraviti razcepa dublet-triplet. Odpravimo jo lahko z dvema različnima nadgradnjama Higgsovega sektorja, ko dodamo še en par $27 \oplus \overline{27}$ (z dodatnimi omejitvami) ali pa upodobitev 78; dobljena modela po vrsti imenujemo model I in model II. V obeh modelih najdemo ustrezno rešitev za vakuum, uspešno opravimo razcep dublet-triplet, analiziramo Yukawin sektor in izračunamo masne matrike za lahke fermione. Čeprav je v Yukawinih sektorjih modela I in modela II mehanizem mešanja okusov različen, pa so ostale njune lastnosti podobne. V obeh modelih se izkaže, da spekter lahkih delcev ustreza stanjem v MSSM, medtem ko so družine vektorskih kvarkov in leptonov na skali poenotenja: naši modeli E_6 (generično) torej ne napovedujejo lahkih vektorskih stanj. Lahki nevtrini dobijo maso prek gugalničnega mehanizma tipa I in II, medtem ko sta desno-ročni nevtrino in singletni nevtrino približno na skali poenotenja. V obeh modelih smo tudi določili prispevke h protonskem razpadu dimenzije 5 in utemeljili, da so lahko dovolj majni, saj se nekateri tripleti mediatorji ne sklapljajo s fermioni SM, hkrati pa poenotenje konstant ne potrebuje več tripleta lažjega od skale poenotenja, saj imamo veliko stanj in s tem več možnih pragov.

Ugotovili torej smo, da oba modela izpolnjujeta minimalne kriterije, vsaj na analitičnem nivoju, da sta realistična, in zato možna kandidata za minimalni realističen E_6 modela v podmnožici renormalizabilnih supersimetričnih modelov. Modela imata po vrsti 3 in 2 Yukawine matrike, zato je model II verjetno bolj prediktiven.

Ključne besede: fenomenologija onkraj Standardnega modela, supersimetrične teorije poenotenja, E_6, spontani zlom simetrije, rešitev za vakuum, razcep dublet-triplet, Yukawin sektor, protonski razpad

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Abstract

In this PhD thesis we study Grand Unified Theories based on the E_6 group. We limit ourselves to renormalizable supersymmetric models, with the Standard Model fermions contained in three families of the fundamental representation 27, while model building takes place in the breaking sector. Also, similarly as in SU(5), we impose a $\mathbb{Z}_2$ matter parity under which the fermionic sector is odd, while the breaking sector is even.

First, we extensively prepare all the necessary infrastructure for computation in the E_6 group. We define the E_6 algebra, gather the necessary tools for the description of its irreducible representations, identify in these representation the states of interest, and then compute the invariants with these representations.

With all the E_6 tools prepared, we then systematically study various models, proceeding from the simple to more complicated, and assess whether they are realistic. The first criterion is whether the models permit a one stage breaking from E_6 to the Standard Model group. The simplest models fail in this regard. We show that the (renormalizable) models with breaking sectors consisting of representations 27, $\overline{27}$ and 78 cannot break beyond SU(5), regardless of the number of representations. Also, if we base the breaking sector on a pair $351 \oplus \overline{351}$, with an arbitrary number of copies of 27 and $\overline{27}$, or take the breaking sector to be $351' \oplus \overline{351'}$ or 650, we also cannot break into the Standard Model group.

A successful breaking does occur in the model $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}$, which we call the prototype model. An unexpected problem with this model, however, is its inability to perform doublet-triplet splitting by fine-tuning. We cure this problem in two extensions of this model, called model I and model II, where we introduce an extra $27 \oplus \overline{27}$ pair (with some restrictions) or an extra 78 into the breaking sector, respectively. In these models, we find solutions breaking to the Standard Model and use them to successfully perform doublet-triplet splitting and analyze the Yukawa sector and compute the mass matrices of the low-energy fermions. In the Yukawa sectors of model I and model II, the mechanisms of flavor mixing are different, but other features are similar. In particular, we find that the low-energy spectrum in both models is that of the MSSM, with families of vector-like quarks and leptons all at the GUT scale: the E_6 models under consideration thus do not predict light vector-like states. The light neutrinos get masses via type I and type II seesaw mechanism, while the right-handed neutrino and the singlet neutrino are roughly at the GUT scale. In both models, we also compute contributions to $D = 5$ proton decay; we argue that the decay rate can be sufficiently suppressed, since not all decay-mediating triplets of the model couple to the fermions, and furthermore coupling unification no longer requires one of the triplet states to be lighter than the GUT scale due to many possible heavy thresholds.

We find that both model I and model II satisfy the minimal criteria for being realistic (at least at the analytic level), and thus are possible candidates for the minimal E_6 model in the subset of renormalizable supersymmetric models. The two models have 3 and 2 Yukawa terms respectively, which means model II is likely more predictive.

Keywords: beyond the Standard Model Phenomenology, supersymmetric Grand Unified Theories, E_6, spontaneous symmetry breaking, vacuum solution, doublet-triplet splitting, Yukawa sector, proton decay

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Abbreviations and conventions

DT Doublet-triplet

EOM Equations of motion

EW Electroweak

GUT Grand Unified Theory

MSSM Minimal Supersymmetric Standard Model

QFT Quantum Field Theory

RG Renormalization Group

SM Standard Model

SUSY Supersymmetry

UV Ultraviolet

VEV Vacuum expectation value

We use the following conventions for the Pauli matrices σ^i and Gell-Mann matrices λ^a:

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

$$\begin{aligned}
\lambda^1 &:= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^2 &:= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^3 &:= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda^4 &:= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda^5 &:= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda^6 &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda^7 &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda^8 &:= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{2}$$

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1 Introduction

The culmination of particle physics in the 20th and 21st century is a theory called the Standard Model. This theory successfully describes the known elementary particles and all the known interactions between them except gravity; the Standard Model is the theory of the strong, the weak and the electromagnetic force. Although this theory boasts an unprecedented experimental success in describing processes at subatomic levels, there are various experimental and theoretical puzzles, which hint at physics beyond the Standard Model. Various extensions of the Standard Model have been proposed, and one intriguing possibility is that of unification of the Standard Model forces; these theories are called Grand Unified Theories, and they aim to unify the strong, weak and electromagnetic forces into a single type of force at high energies.

There are many proposed models of unification scenarios, with the most popular being those based on the SU(5) and SO(10) gauge groups. Another candidate is the group E_6, which is often cited as promising and was identified as such early (see [1]) into the GUT program, which began with [2]. Despite the potential promise of E_6, there are no complete and realistic models of E_6 GUT, at least to the author's knowledge, in the literature so far. With complete we mean a top-down analysis of a model, where one considers both symmetry breaking and the Yukawa sector simultaneously, as well as address issues such as doublet-triplet splitting; with realistic we mean that the models aim to describe the fermion Standard Model masses and mixing angles correctly, at least with no apparent deficiencies at the analytic level. There have been occasional studies of isolated topics of E_6 model building in the past; the Yukawa sector of E_6 models was for example studied in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], while symmetry breaking in some simple cases was also studied: a renormalizable supersymmetric case in [14], a non-renormalizable supersymmetric case in [15], and non-supersymmetric renormalizable cases in [16, 7].

There are probably two reasons for the lesser popularity of E_6. The first is the complexity of the E_6 group, which is bigger than both SU(5) and SO(10). It is not part of the orthogonal or unitary families, which lend themselves quite well to visualisation or at least intuition, and are easily constructed by generalization (e.g. such as for the SO(N) groups in [17]). The second reason is a lack of specific motivation: SU(5) is the smallest possible GUT group, while SO(10) has certain interesting features, such as the automatic inclusion of the right-handed neutrino and thus elegantly explaining neutrino masses. Although E_6 does contain some (phenomenologically) interesting features, such as the presence of vector-like quarks and leptons in its fundamental representation 27, it also rolls back some of the advantages gained from moving from SU(5) to SO(10). In SO(10), for example, there is automatic R-parity conservation (see [18, 19, 20]) with the 126 breaking rank ([21, 22, 23]), which is lost when SO(10) is embedded into E_6. Furthermore, the situation is also somewhat complicated by the fact that the simplest models do not break into the Standard Model group, such as the $27 \oplus \overline{27} \oplus 78$ renormalizable supersymmetric model of [14].

The goal of this doctoral dissertation is to fill the gap in top-down E_6 model building, with studying the simplest models one can write and finding realistic candidates, and also to gather some technical details on the E_6 group, which can be used as a good reference for any future work in E_6 GUT. Due to their greater simplicity, we shall focus on supersymmetric renormalizable models.

The dissertation is organized as follows: in section 2 we briefly introduce the various concepts from particle physics used later on without much hesitation. Section 3 is devoted to group theory and the E_6 group in particular, providing all the necessary infrastructure for model building. Here, we remind the reader of some basic group-theoretic facts and how E_6 fits into the picture of simple groups and its potential advantages. We then give the explicit form of its Lie algebra, analyze its lowest dimensional representations, review some computational tools and finally present some computations of invariants. We then study various models in section 4. First, we describe the general setup of the models and then present some unrealistic models with arguments why they cannot work. We then analyze a prototype model, $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}$, which is not realistic by itself due to what seems a minor glitch, but sprouts two realistic extensions: the models $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27} \oplus \widetilde{27} \oplus \overline{\widetilde{27}}$ and $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27} \oplus 78$, which we dub "model I" and "model II", respectively. We find that the low-energy spectrum in these models is that of MSSM, while the vector-like states are at the GUT scale.

The dissertation is partly based on the paper [24], and some additional work not yet published.

Note: sometimes we color code for greater clarity. The coloring of VEVs is according to their energy scale: red signifies the GUT scale, while blue signifies the EW scale.

2 Elements in Particle Theory

In this section, we make a brief presentation of the necessary concepts in particle theory (well-known from the literature), which will then be used in the model building section (section 4) as needed. This section partly functions also as a resource for the notation and conventions used later on.

The topics under consideration are QFT and Yang Mills theories in section 2.1, the Standard Model in section 2.2, Supersymmetry in section 2.3, GUTs with a few examples in section 2.4 and finally section 2.5 with various concepts used later on.

2.1 QFT and Yang-Mills Gauge theory

Since Quantum Field Theory and Yang-Mills gauge theory are textbook type topics, we make only a very brief introduction here, and refer the reader for more information to the standard literature, for example [25, 26].

Quantum Field Theory is a successful joining of both Quantum Theory and the Special Theory of Relativity. The theory is specified by giving a Lagrange density $\mathcal{L}$, which is integrated over all spacetime to obtain the action S:

$$S = \int d^4x \, \mathcal{L}. \tag{3}$$

The objects in the Lagrangian are *fields*, and they map from the spacetime manifold to a target space, which is a representation of the Lorentz group. The target space specifies the type of field: the simplest objects are scalar fields, spinor fields (with spin 1/2) or vector fields. Unlike in a classical theory, where the fields would obey the Euler-Lagrange equations, all degrees of freedom in a quantum field theory are second-quantized (via path integrals for example), with quanta representing particles associated with each type of field.

Analytical computation in QFT is undertaken perturbatively, where each term in the Lagrangian, which is not a kinetic term or a mass term, is treated as a small perturbation of the free theory. The perturbative expansion corresponds to an expansion in Feynman diagrams with the increasing number of loops. Contributions from Feynman diagrams, which have no loops, are called *tree-level*, and they represent the first approximation. In this PhD thesis, we will mostly be concerned with model building and not precision prediction; loop diagrams will not be computed at any point, since our interest is first and foremost in the qualitative aspects of models based on the group E_6.

Interactions in QFT are incorporated through gauge theory. Postulating that the theory is invariant under local transformations (or *gauge transformations*) of some Lie group G (different group elements are applied at different spacetime points). If ψ is the object being transformed, and it transforms as $U(x)\psi(x)$, then $\partial\psi$ will not transform with $U(x)$; for this reason, we define the *covariant* derivative D by

$$D_\mu := \partial_\mu - igA^a_\mu \hat{t}^a, \tag{4}$$

where ∂_μ is the usual gradient operator, t^a are the generators of the group G, g is a coupling constant, and A^a_μ are called *gauge fields*, which transform under a gauge transformation, such that the covariant derivative of the field transforms as $D_\mu\psi(x) \mapsto U(x)D_\mu\psi(x)$. The quanta of the field A^a_μ are called gauge bosons, and they are carriers of the interaction. In a gauge theory, the Lagrangian therefore needs to be invariant under local (gauge) transformations; this is achieved by adding terms formed from an invariant combination of fields, and a derivative of any field needs to be changed to a covariant derivative.

Defining the matrix A_μ as the linear combination of group generators,

$$A_\mu := A_\mu^a \, t^a, \tag{5}$$

we define the the *gauge field strength* $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{6}$$

In Yang-Mills gauge theory, the kinetic term of the gauge fields is written as

$$-\tfrac{1}{2c}\mathrm{Tr}(F^{\mu\nu}F^{\mu\nu}), \tag{7}$$

where c is such that the when expanding this expression, the $(\partial_\mu A_\nu)^2$ term has the normalizing factor $1/2$ in front of it. Expanding the field strength in the generators of the symmetry implicitly defines the component field strength $F^a_{\mu\nu}$:

$$F_{\mu\nu} = F^a_{\mu\nu} \, t^a. \tag{8}$$

While the gauge potentials A^a_μ are present (as degrees of freedom) in any gauge theory, the spinor field content (also called the matter content) and the scalar degrees of freedom depend on the model of the theory.

2.2 The Standard Model

The Standard Model is the theory of strong, weak and electromagnetic interactions. It is a non-abelian Yang-Mills theory based on the Standard Model gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}_Y$ (this is a textbook topic; see for example [27, 28]). The gauge bosons of the strong interaction $\mathrm{SU}(3)_C$ are called gluons, while the electroweak part $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ contains the weak interaction and the hypercharge $\mathrm{U}(1)_Y$, which is spontaneously broken into the electromagnetic $\mathrm{U}(1)_{\mathrm{EM}}$ (more on this a bit later). It is well-tested experimentally, and beside small hints like neutrino oscillations and dark matter, there are currently few clues of physics beyond the SM.

We label the representations of the SM group by (a, b, c), where a denotes the dimension of the $\mathrm{SU}(3)_C$ representation, b denotes the dimension of the $\mathrm{SU}(2)_L$ representation, and c denotes half-hypercharge $Y/2$. We thus have 8 gluons in $(8, 1, 0)$, and the 3 W and single Y gauge boson in $(1, 3, 0)$ and $(1, 1, 0)$, respectively. All in all, we have 12 gauge bosons in the Standard Model.

The matter content of the theory consists of three families, each of which has the following fermionic representations (all are written as left-handed Weyl fermions, so they consists of a particle and its antiparticle):

$$\begin{aligned} &Q \sim (3, 2, +1/6), \qquad && u^c \sim (\overline{3}, 1, -2/3), \qquad && d^c \sim (\overline{3}, 1, +1/3), \\ &L \sim (1, 2, -1/2), && e^c \sim (1, 1, +1). \end{aligned} \tag{9}$$

The upper row consists of quarks: the left-handed up and down quarks are in Q, while the right-handed up and down quarks are in u^c and d^c, respectively. The lower row consists of leptons: the left-handed electron and neutrino are in L, while the right-handed electron is in e^c. The Standard Model historically does not contain any right-handed neutrinos, although the phenomenon of neutrino oscillations point to non-vanishing neutrino masses, with the right-handed neutrino being one of the possible explanations. Note that we have used a common type of notation, where right-handed fields are written as conjugated left handed fields: $\psi^c := C\overline{\psi}^T$ (bispinor notation), where $C = i\gamma^2\gamma^0$ is the charge conjugation operator.

The Standard Model also contains a representation of scalars (the Higgs doublet):

$$H \sim (1, 2, +1/2). \tag{10}$$

This scalar field has the potential

$$V(H, H^\dagger) = -\mu^2\, H^\dagger H + \lambda\, (H^\dagger H)^2. \tag{11}$$

Writing

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}, \tag{12}$$

we take the minimum of the potential at

$$\langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \tag{13}$$

where $v = \mu/\sqrt{2\lambda}$ and it is called the vacuum expectation value. Due to a nonzero v, the theory is no longer $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetric (v is rotated into the second component) — we say the gauge theory was spontaneously broken by the *Higgs mechanism*. The remaining symmetry is only a U(1) symmetry (electromagnetism), under which h^0 is neutral; this symmetry has the generator $Q = t^3_L + Y/2$, where t^3_L is the third generator of $\mathrm{SU}(2)_L$, and Y is the generator of $\mathrm{U}(1)_Y$. We call this phenomenon symmetry breaking: although the theory (and the equations of motion) have the full symmetry, the vacuum solution (minimum of the potential) has a smaller symmetry. We look at two consequences of this symmetry breaking:

- The following terms in the Standard Model Lagrangian are called the Yukawa terms (written schematically without spinor or gauge indices):

  $$\mathcal{L}_{\text{Yukawa}} = Y^{ij}_U\,(Q)^i(u^c)^j\,H + Y^{ij}_D\,(Q)^i(d^c)^j\,H^* + Y^{ij}_E\,(L)^i(e^c)^j\,H^*, \tag{14}$$

  where summation of all gauge indices and spinor indices is implied. These terms represent an interaction of two fermions and the Higgs fields. Plugging in the VEV $\langle H \rangle$, these interactions give mass terms for the quarks and leptons.

  The Yukawa sector of the Standard Model has an important feature: *flavor mixing*. Once the EW VEV is present, we get the mass terms $Y_f\, v/\sqrt{2}$; the mass eigenstates (matrix m^f) are computed by biunitary transformations; one writes $m^f = V^{f\dagger}_L\,(Y_f v/\sqrt{2})\,V^f_R$, where V^f_L and V^f_R are the left and right unitary matrices ($f = U, D, E$). There are 3 Yukawa matrices, so $3+3$ unitary matrices V^f_L and V^f_R, but there are only 5 SM representations for which one can redefine the flavor basis by a rotation in the families. In the flavor basis, one of the Q-terms will thus always have off-diagonal values — hence flavor mixing interactions. The misalignment of the flavor basis and the mass basis can be described by the CKM matrix $V_{CKM} = V^{U\dagger}_L V^D_L$; this matrix can be parameterized (for 3 families of fermions) with 3 mixing angles and one CP-violating phase.

- The kinetic terms of the Higgs doublet

$$(D_\mu H)^\dagger (D^\mu H) \tag{15}$$

give gauge boson mass terms $\frac{m^2}{2} A_\mu A^\mu$, once $\langle H \rangle$ is plugged in. For every broken generator, which is not part of the symmetry of the solution, we get a massive gauge boson. This means that $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{EM}}$ breaking causes three gauge bosons to become massive (the W^+, W^- and Z^0), while the photon remains massless.

If a global symmetry is broken, the Goldstone Theorem states that there is a massless scalar state for each broken generator. In gauge theories with broken local symmetry, the massless states can be rotated away by a gauge transformation, thus disappearing from the theory, with the degree of freedom transferred to the longitudinal polarization of the massive gauge boson.

2.3 Supersymmetry

Here, we briefly present the idea of supersymmetry; more information on SUSY and MSSM can be found in [29, 30].

The Lorentz symmetry in Quantum Field Theory can be extended to include generators, which transform between particles of different spin. With this addition the symmetry generators form a structure called a *superalgebra*, which is a $\mathbb{Z}_2$ graded Lie algebra: the Lie algebra splits into $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ represents even (bosonic) elements and $\mathfrak{g}_1$ represents odd (fermionic) elements. The Lie bracket of two even or two odd elements is even, while the Lie bracket of an even and odd element is odd. The Lie super-bracket operation of two generators manifests as a commutator $[.,.]$ if at least one element is even, and as the anticommutator $\{.,.\}$ if both elements are odd.

In the Lorentz superalgebra, we supplement the angular momentum operators $M^{\mu\nu}$ and the linear momentum operators P^μ with pairs of odd operators Q_α and $\overline{Q}_{\dot\alpha}$, where α and $\dot\alpha$ are spinorial indices of Lorentz types $(1/2, 0)$ and $(0, 1/2)$, respectively. The $N = 1$ superalgebra, where we add only one pair of supercharges Q and $\overline{Q}$, is the one of phenomenological interest, since the representation theory of $N > 1$ Lorentz superalgebras shows that there are no appropriate chiral supermultiplets. The $N = 1$ SUSY algebra has the following (anti)commutation relations (writing only non-trivial ones):

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= -i(\eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma}), &(16)\\
[M^{\mu\nu}, P^\rho] &= -i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu), &(17)\\
[M^{\mu\nu}, Q_\alpha] &= \tfrac{i}{2}(\gamma^{\mu\nu} Q)_\alpha, &(18)\\
\{Q_\alpha, Q_\beta\} &= -2(\gamma^\mu C)_{\alpha\beta} P_\mu, &(19)
\end{aligned}$$

where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, with γ^μ the Dirac matrices satisfying the relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and C is the charge conjugation operator $((\gamma^\mu)^T = C\gamma^\mu C)$. Note that also $(Q_\alpha)^\dagger = \overline{Q}_{\dot\alpha}$.

In this extended symmetry, the multiplets are called supermultiplets. Since the supercharges Q and $\overline{Q}$ transform between states of different spin, the supermultiplets contain multiple particles of different spin. For $N = 1$ supersymmetry, there are two kinds of supermultiplets (below we consider on-shell degrees of freedom):

1. The *chiral multiplet* (ψ, ϕ): it contains one complex scalar ϕ and one left-handed spin 1/2 fermion ψ.

2. The *vector multiplet* (A_μ, λ): it contains the gauge boson A_μ, as well as a left-handed spin 1/2 fermion λ.

If we have a theory, which we want to make supersymmetric, we need to add so called superpartners: the fermionic superpartners of gauge bosons A_μ are called *gauginos* λ, the scalar superpartners of fermions ψ are called *sfermions* ϕ, and for any Higgs scalar fields, we need to add fermionic superpartners called *Higgsinos*.

Although there exist formalisms which are manifestly supersymmetric, we will not mention them here. We can simply view a supersymmetric theory as any other theory, where we include all the terms given the degrees of freedom we have available: supersymmetry then manifests itself in special relations among the coefficients of different operators. It turns out that a supersymmetric theory has the following Lagrangian (see for example [29]):

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{extra}} - V, \tag{20}$$

$$\begin{aligned}
\mathcal{L}_{\text{kinetic}} &= (D_\mu \phi)^\dagger_i \, (D^\mu \phi)^i + \tfrac{i}{2}\psi^i \sigma^\mu (D_\mu \overline{\psi})_i - \tfrac{i}{2}(D_\mu \psi)^i \sigma^\mu \overline{\psi}_i \\
&\quad - \tfrac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \tfrac{i}{2}\lambda^i \sigma^\mu (D_\mu \overline{\lambda})_i - \tfrac{i}{2}(D_\mu \lambda)^i \sigma^\mu \overline{\lambda}_i, && (21) \\
\mathcal{L}_{\text{Yukawa}} &= -\tfrac{1}{2} \frac{\partial^2 W(\phi)}{\partial \phi^i \, \partial \phi^j} \psi^i \psi^j - \tfrac{1}{2} \frac{\partial^2 \overline{W}(\overline{\phi})}{\partial \overline{\phi}_i \, \partial \overline{\phi}_j} \overline{\psi}_i \overline{\psi}_j, && (22) \\
\mathcal{L}_{\text{extra}} &= +ig\sqrt{2}(\overline{\psi}_i \overline{\lambda}^a)(t^a)^i{}_j \, \phi^j - ig\sqrt{2}\phi^\dagger_i \, (t^a)^i{}_j (\psi^j \lambda^a), && (23) \\
V(\phi^i, \overline{\phi}_j) &= \sum_i \left| \frac{\partial W}{\partial \phi^i} \right|^2 + \tfrac{1}{2} g^2 \sum_a \left(\phi^\dagger_i (t^a)^i{}_j \, \phi^j \right)^2 . && (24)
\end{aligned}$$

We have the kinetic terms, the Yukawa terms, the potential of the scalar fields consisting of F-terms and D-terms (respectively), and the extra terms with the fermion-sfermion-gaugino interaction. The generators t^a are in the same representation as the chiral multiplet scalars ϕ^i (we can view all the gauge group irreducible representations of chiral multiplets forming one big reducible representation under the gauge group). The function $W(\phi^i)$ is a holomorphic function of the scalar fields, has mass dimension 3, and is formed from the most general scalar terms compatible with gauge symmetry.

If we supersymmetrize SM in a minimal way, we get the "Minimal Supersymmetric Standard Model" or MSSM. Beside the vector supermultiplets $(8, 1, 0)$, $(1, 3, 0)$ and $(1, 1, 0)$ (we added 12 gauginos: 8 gluinos, 3 winos and 1 bino), we add the following matter content (per family) in chiral supermultiplets:

$$\begin{aligned}
Q &\sim (3, 2, +1/6), & d^c &\sim (\overline{3}, 1, +1/3), & u^c &\sim (\overline{3}, 1, -2/3), \\
L &\sim (1, 2, -1/2), & e^c &\sim (1, 1, 1), \\
H_u &\sim (1, 2, +1/2), & H_d &\sim (1, 2, -1/2). && (25)
\end{aligned}$$

We added new sfermion and Higgsino degrees of freedom. Furthermore, since the superpotential W is holomorphic and due to anomaly cancellation, we needed to introduce two Higgs fields H_u and H_d (unlike in SM, where we only have $H \sim (1, 2, +1/2)$). The superpotential of the MSSM is

$$W = \mu\, H_u H_d + Y_U^{ij}\,(Q)^i (u^c)^j\, H_u + Y_D^{ij}\,(Q)^i (d^c)^j\, H_d + Y_E^{ij}\,(L)^i (e^c)^j\, H_d, \qquad (26)$$

giving the Higgs mass via the μ terms, while the other terms are represent Yukawa interaction. Note that we have omitted the $u^c u^c d^c$, QLd^c and LLe^c terms, which could also be formed. These terms need to be avoided, because they lead to unacceptable phenomenological consequences (such as nucleon decay, since they violate lepton and baryon numbers). We can forbid them for example by a $\mathbb{Z}_2$ symmetry called R-parity, which can be defined by $(-1)^{3(B-L)+2s}$, where B and L are the baryon and lepton number, and s is the spin of the particle. Under R-parity, all SM fields are even, and the superpartners are odd.

Since the low-energy theory we observe is the Standard Model, and not MSSM, supersymmetry needs to be broken somewhere above 1 TeV. Although mechanisms for SUSY breaking exist, we will not go into them here; whatever the mechanism, we can imagine that below the SUSY breaking scale, we get extra terms of dimension less than 4, called the *soft terms*, which parametrize the breaking. The terms consist of gaugino masses, sfermion (scalar) masses, and trilinear scalar terms (A-terms). The scale of SUSY breaking is not uniquely predicted by theory (but it should be as low as possible, if we want to use SUSY for alleviating the hierarchy problem).

If a theory is supersymmetric, it becomes much simpler compared to a non-supersymmetric theory with the same degrees of freedom; in SUSY theories, for example, we have some nearly miraculous cancellations in the calculation. The main phenomenological motivation of supersymmetry is to alleviate the hierarchy problem (the scalar masses, for example, are now protected by the chiral symmetry of the fermions); low-energy SUSY also enables unification of gauge couplings in GUT theories without introducing any new degrees of freedom between the SUSY scale and the GUT scale. Also, the lightest supersymmetric partner (the one with odd R-parity) is a candidate for Dark Matter in such theories.

2.4 Grand Unified Theories

Grand Unified Theories are extensions of the Standard Model, where the strong, weak and electromagnetic interactions unify into one single type of interaction at high energies (first proposed in [2]; in general, see [31]). The main concept is to have a gauge theory with a (simple) symmetry group, which includes the SM group as its subgroup. The unified theory would then spontaneously break via the Higgs mechanism to the SM group at some high energy M_{GUT}, similar to how electroweak symmetry $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ breaks into electromagnetism $\mathrm{U}(1)_{\mathrm{EM}}$ at the EW scale $M_{\mathrm{EW}} \sim 100\,\mathrm{GeV}$.

Two important typical predictions of GUT models are the following:

- *Proton decay*: proton decay is mediated by color triplets $(3, 1, -1/3)$ and $(\overline{3}, 1, +1/3)$, which are hard to avoid in GUT models. Since the triplets will have typical mass $\sim M_{\mathrm{GUT}}$, and proton decay has not yet been experimentally observed with a half-life limit of $\gtrsim 10^{34}$ years, the GUT scale is predicted to be at very high energies above $\gtrsim 10^{16}\,\mathrm{GeV}$ in SUSY GUT theories.

- *Magnetic monopoles*: since in GUT theories, new degrees of freedom activate at very small scales, these theories typically predict magnetic monopoles or other monopole-type objects. Their non-observation can be explained by cosmic inflation.

In GUTs, particles in representations of the Standard Model join together (possibly with new particles) to form irreducible representations of the unification group. Also, since the unified theory has a simple group (cannot be decomposed into factors) as its symmetry, there is only one coupling constant g; this is in contrast with the Standard Model, where we have a gauge coupling g_i for each of the factors: the strong coupling g_3, the weak coupling g_2 and the hypercharge coupling $g_1 = g_Y$ (properly normalized).

In QFT, the couplings change their value depending on the (energy) scale μ that we investigate the theory at. We call this phenomenon RG running, and the gauge couplings change according to equations

$$\frac{dg_i}{d\,\ln(\mu)} = -\frac{\beta_i}{16\pi^2}\,g_i^3. \tag{27}$$

The coefficient β_i is called the beta function, and it depends on the matter content of the theory. It is computed to be (at leading order)

$$\beta_i = \tfrac{11}{3}T(G) - \tfrac{2}{3}T(F) - \tfrac{1}{6}T(S), \tag{28}$$

where G, F and S refer to representations of gauge bosons, (Weyl) fermions, and (real) scalars, respectively. The i refers to the factor in the Standard Model, but the formula holds generally for any model. The number $D_2(R)$ is called the Dynkin index of the representation R and it is defined by the normalization of the scalar product of generators in the representation R (denoted by t_R):

$$\mathrm{Tr}\big(t_R^a\, t_R^b\big) = D_2(R)\,\delta^{ab}. \tag{29}$$

It is computed for the representation R of the group of factor i, for which we are computing the coupling RG flow.

We compute Dynkin indices for all representations of gauge bosons, Weyl fermions and real scalars present in the theory below scale μ, and add them up according to equation (28). If there are new (non-singlet) particles at some scale μ, then the RG running is modified by adding their contribution to the beta function. In supersymmetric theories, gauge bosons are part of the vector supermultiplet, which also contains fermion superpartners, while the Weyl fermions and complex scalars are together in chiral supermultiplets. For this reason, the beta function in SUSY can be computed by

$$\beta = 3T(V) - T(C), \tag{30}$$

where V refers to the vector multiplet and C to the chiral multiplets.

We now give some important comments on RG flow in GUTs:

- For a GUT to be consistent (with single stage breaking), there needs to be one gauge coupling above M_{GUT}, while there are 3 different gauge couplings of the SM below M_{GUT}. The SM gauge couplings therefore need to unify (take the same value) at M_{GUT}. Although the running can be influenced at high energy by new degrees of freedom, the logarithm in RG equation (27) implies that the lighter the degree of freedom is, the more it will contribute to the RG running from EW to GUT scale. Assuming no extra light degrees of freedom, the coupling constants of the SM do not unify, but if we assume the existence of supersymmetric partners at scales somewhere above $\sim$ TeV, the MSSM RG flow will unify the couplings a few orders of magnitude below the Planck scale M_{Pl} ($M_{\mathrm{Pl}} = 1/\sqrt{G} \sim 10^{18}\,\mathrm{GeV}$, in units $c = \hbar = 1$ and G being Newton's gravitational constant), where we expect the GUT scale to be. For this reason, it seems that GUTs prefer supersymmetry; the E_6 models we will be considering will all be supersymmetric.

- We note that once the gauge group is chosen, we already chose the gauge bosons of the theory (they are in the adjoint representation), while "matter content" (fermions and scalar) further depend on the model. We see from equations (28) and (30) that matter content will always contribute negatively to the beta function. An overall positive beta function implies asymptotic freedom (at large energies), while a negative beta function can lead to a Landau pole. Typically, GUTs will have enough matter content for the beta function to be negative, so Landau poles can present a problem, especially if they occur before the Planck scale. In that case, the GUT can have a cut-off scale Λ just one or two orders of magnitude above M_{GUT}, which would also imply that non-renormalizable terms with M_{GUT}/Λ factors could also become important. Despite this worry, we shall analyze renormalizable models only (as these are the simplest), but due to the reasons described, we shall prefer models with as little matter content as possible.

The most popular unified theories are based on groups SU(5) and SO(10). Since both are subgroups of E_6, it is instructive to first look at these simpler examples.

2.4.1 SU(5) theory

In an SU(5) theory, we unify the matter content of each family in the following way:

- The right-handed down quarks and the lepton doublet fit into the $\overline{5}$ of SU(5) due to the following SU(5) $\to$ SM decomposition:

$$\overline{5} = (\overline{3}, 1, +1/3) \oplus (1, 2, -1/2). \tag{31}$$

  We label this representation by $\overline{5}_F$ and write

$$\overline{5}^i_F = (d^c)^i \oplus (L)^i. \tag{32}$$

- The left-handed quarks, and the right handed electron and up quarks are part of the 10 of SU(5):

$$10 = (3, 2, 1/6) \oplus (\overline{3}, 1, -2/3) \oplus (1, 1, +1). \tag{33}$$

  Writing this representation as 10_F, we have

$$10^i_F = (Q)^i \oplus (u^c)^i \oplus (e^c)^i. \tag{34}$$

If we consider a SUSY model of SU(5) GUT, we then add a pair $5_H \oplus \overline{5}_H$, which contains the $H_u \oplus H_d$ pair of Higgs fields, which acquire VEVs at the EW scale. With these fields, and assuming R-parity, we can write the Yukawa part of the superpotential schematically as

$$W\big|_{\mathrm{Yukawa}} = \sum_{i,j} Y_5^{ij}\, 10^i_F \cdot \overline{5}^j_F \cdot \overline{5}_H + Y_{10}^{ij}\, 10^i_F \cdot 10^j_F \cdot 5_H. \tag{35}$$

Since the Higgs fields $5_H \oplus \overline{5}_H$ do not have SM singlets, they cannot acquire GUT scale VEVs lest they break also the Standard Model group. We need to add another SU(5) representation, such as the adjoint 24_H, which can acquire a VEV: written as a 5×5 traceless matrix, $\langle 24 \rangle = v\,\mathrm{diag}(2,2,2,-3,-3)$, where v is determined by the equations of motion. The breaking part of the superpotential can then be written by the terms

$$m\,5_H \cdot \overline{5}_H + \lambda\,5_H \cdot 24_H \cdot \overline{5}_H + m'\,24_H^2 + \lambda'\,24_H^3, \tag{36}$$

which indeed gives $v \neq 0$.

Note that this is only one possible model based on SU(5). This model, for example, has the problematic prediction $M_D^T = M_E$ at the GUT scale (since both are controlled by Y_5), and a vanishing neutrino mass.

2.4.2 SO(10) theory

We consider here a renormalizable SUSY GUT model based on SO(10), which is considered the minimal realistic model of this type (this model is studied in more detail in [32, 33]).

In SO(10), we unify all the fermion content of one SM family into a single representation. We have the SO(10) → SU(5) decomposition

$$16 = 10 \oplus \overline{5} \oplus 1, \tag{37}$$

which means that beside the SM fermions from one family, we have an extra SU(5) singlet 1, which functions as a right-handed neutrino. In the SO(10) arena, neutrino masses thus arise naturally.

Beside having three copies 16_F^i, we also need to add some further SO(10) representations, which will contain the Higgs fields, as well as the GUT scale VEVs. The model in question also contains the Higgs fields $10_H \oplus 126_H \oplus \overline{126}_H \oplus 210_H$. With these fields, one can write the breaking sector by

$$\begin{aligned} W|_{\text{breaking}} &= m_{10}\,10_H^2 + m_{126}\,126_H \cdot \overline{126}_H + m_{210}\,210_H^2 \\ &\quad + \lambda 10_H \cdot 210_H \cdot 126_H + \bar{\lambda}\,10_H \cdot 210_H \cdot \overline{126}_H \\ &\quad + \lambda'\,126_H \cdot 210_H \cdot \overline{126}_H + \lambda''\,210_H^3. \end{aligned} \tag{38}$$

The superpotential of the Yukawa sector consists of two terms, which couple to the fermionic 16_F's:

$$W\big|_{\text{Yukawa}} = \sum_{i,j} Y_{10}^{ij}\,16_F^i \cdot 16_F^j \cdot 10_H + Y_{\overline{126}}^{ij}\,16_F^i \cdot 16_F^j \cdot \overline{126}_H. \tag{39}$$

Note that this model has the following features:

- R-parity in this model is automatic. It arises from the way SO(10) invariants are constructed: since the 16_F's are the only representations carrying spinor indices, they need to be present in pairs so that spinor indices can be contracted. This conclusion is true in models, where 126 and not 16 break the rank (see [18, 19, 20, 21, 22, 23]).

- The Yukawa sector has 2 terms; both terms are needed to describe flavor mixing. If we had only one term, the Yukawa matrix could be diagonalized and the flavor basis would align with the mass basis. The Higgs fields H_u and H_d therefore need to be present in both 10 and $\overline{126}$ simultaneously. For that to happen, a representation is needed which couples the 10 and the $\overline{126}$, and thus the 210 is added.

- The GUT scale breaking happens due to VEVs in the representations 210, 126 and $\overline{126}$. These are computed by solving the equations of motion.

2.4.3 Proton decay

In an effective theory (still with SM as the gauge group), operators which break baryon and lepton number can lead to proton decay (for proton decay in GUTs, see [34, 35, 36]). Schematically, these operators can be written as $qqql$, where q represents a quark (or antiquark), and l represents a lepton (or antilepton), so that the $B-L$ quantum number is preserved by these operators. Ultimately, both q and l are fermions, so the effective operator in the Lagrangian has dimension 6, and is suppressed by two masses m_1 and m_2 due to dimensional analysis:

$$\mathcal{L}_{\text{eff}} = \frac{qqql}{m^2}. \tag{40}$$

We have omitted here the family indices; for operators of this type in the superpotential, having multiple families is crucial, otherwise these terms in the superpotential vanish. Depending on the origin of this term, m_1 and m_2 can represent different mass scales of the theory; due to this fact, we talk of $D=6$, $D=5$ and $D=4$ proton decay. In the SUSY GUT context, they are defined below:

- $D=6$: here, $m_1 = m_2 = M_{\text{GUT}}$, so we effectively have

  $$\mathcal{L}_{\text{eff}} = \frac{qqql}{M_{\text{GUT}^2}}. \tag{41}$$

  Operators of this type arise due to exchange of a gauge boson (for example the $(3,2,-5/6)$ in SU(5) GUT), or through a scalar triplet-antitriplet pair $(3,1,-1/3)$–$(\overline{3},1,+1/3)$ or $(3,1,-4/3)$–$(\overline{3},1,+4/3)$ by mass insertion.

- $D=5$: this type of proton decay is present for example in SUSY theories and is represented by

  $$\mathcal{L}_{\text{eff}} = \frac{qqql}{M_{\text{SUSY}}\, M_{\text{GUT}}}. \tag{42}$$

  Here, the operator arises from integrating out a triplet-antitriplet pair from the superpotential, getting a $qqql$ term in the superpotential (4 scalars). Through the Yukawa terms of the SUSY theory, this then turns into an operator with two fermions and two scalars (sfermions); the diagram can then be closed into a 4-fermion diagram with the help of the SUSY fermion-sfermion-gaugino interaction vertices.

  Since in our models $M_{\text{SUSY}} \ll M_{\text{GUT}}$, this is the dangerous type of proton decay, which can lead to big decay rates compared to $D=6$. For this reason, in SUSY GUT theories we are primarily interested in $D=5$ proton decay operators [37, 38, 39, 40].

- $D = 4$: this type of decay would come from the operators of type

$$\mathcal{L}_{\text{eff}} = \frac{qqql}{M_{\text{SUSY}}^2}, \tag{43}$$

which leads to a too fast decay rate incompatible with observation. This type of decay would be mediated by a down-type squark and can be forbidden by imposing R-parity (or matter parity). Under this parity, one no longer has the interaction vertices of two SM fermions and a squark, since the SM fermions are even and the squark is odd under this parity.

In our models, $D = 4$ proton decay will be forbidden by imposing matter parity.

2.5 Various concepts and mechanisms

In this subsection, some important concepts used throughout later sections, are introduced in a brief manner.

- *Majorana mass*: mass terms couple left-handed particles of a certain type to right-handed particles of the same type. If the left-handed and right-handed degrees of freedom are separate, we call these particles Dirac particles. But there is another irreducible representation of the Lorentz group, where we take a field $\Psi \sim (1/2, 0) \oplus (0, 1/2)$ of the Lorentz group, which satisfies the constraint $\Psi = \Psi^c$ (where the definition of ^{c} has already been given in section 2.2). This constraint implies that these degrees of freedom are their own antiparticles. Writing the left-handed and right-handed components of Ψ in 2-component notation by ψ_L and ψ_R (a Weyl spinor column), respectively, the Dirac mass terms are $\psi_L^\dagger \psi_R$ and $\psi_R^\dagger \psi_L$, while the Majorana mass terms are $\psi_L^T \varepsilon_2 \psi_L$ and $\psi_R^T \varepsilon_2 \psi_R$ (where $\varepsilon_2 = i\sigma^2$). A Majorana particle can therefore form its own mass term.

- *Seesaw mechanism*: suppose one adds a right-handed neutrino ν^c to the Standard Mode; it is a singlet under the Standard Model and it can thus have a Majorana mass term, schematically $M\,\nu^c\nu^c$. Taking account also of the mass terms schematically written as $m\,\nu\nu^c$, which appear after EW symmetry breaking, one has the mass matrix for the $(\nu\,\nu^c)$ pair

$$\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}. \tag{44}$$

In the $m \ll M$ regime, the mass matrix has the approximate eigenvalues M and $-m\frac{m}{M}$. Taking m to be at the EW scale and M to be much higher, we thus get a light neutrino mass, where the EW scale is suppressed by the extra factor m/M, thus explaining the smallness of neutrino mass ($\lesssim 1\,\text{eV}$ for neutrino masses).

The important feature of the mass matrix in equation (44) that a bigger mass M means a smaller mass for the light neutrino, is called the seesaw mechanism. It need not involve a right-handed neutrino, and there are three possibilities at tree level how to obtain the effective $(LH)^2/M$ operator of neutrino mass, represented in Figure 1 and defined below:

  - Type I: $(1, 1, 0)$ fermions, aka right-handed neutrinos ν^c (denoting an unrelated degree of freedom to ν). Studied in [41, 42, 43, 44, 45].

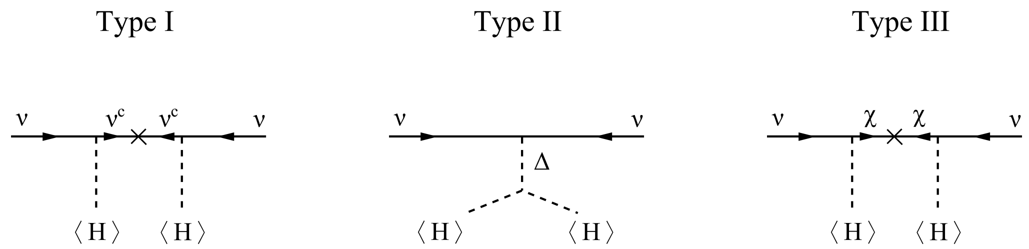

Figure 1: Different types of seesaw mechanism.

- Type II: $(1,3,\pm1)$ scalars, denoted Δ and $\overline{\Delta}$. Studied in [46, 47, 48, 49].
- Type III: $(1,3,0)$ fermions, denoted in this thesis by χ. Studied in [50].

- *Vector-like particles*: vector-like particles are particles in a (pseudo)real representation of the gauge group. This can ether be a (pseudo)real irreducible representation R (for example the 10 of SO(10)), or a direct sum $R \oplus \overline{R}$ where R is a complex irreducible representation (for example $5 \oplus \overline{5}$ of SU(5)). These representations can form their own mass term, i.e. the Lagrangian term $m_R\, R \cdot \overline{R}$, with the mass m_R independent of the EW Higgs mechanism, meaning they can be present at any scale. Within the context of GUT, including our E_6 models, the mass m_R is around the GUT scale.

- *Doublet-triplet splitting*: This is a problem present in many GUTs. We refer here to H_u and H_d type doublets $(1,2,\pm1/2)$ and triplets/antitriplets of the type $(3,1,-1/3)$ and $(\overline{3},1,+1/3)$. A doublet/antidoublet (at least one, we can also have a multi-Higgs low-energy theory) need to be light — of the order of the EW scale — corresponding to the SM Higgs, while triplets need to be heavy due to their contribution to the proton decay rate. This is sometimes hard to achieve, since most lowest dimensional representations of SU(5) (and other unification groups) contain either both the doublet and the triplet or none.

- *Missing partner mechanism*: This is a mechanism to achieve doublet-triplet splitting best understood in SU(5) language (see [51, 52, 53]). The representation 50 contains a triplet but not a doublet, which can be used to our advantage. The simplest case is to have the representations $5 \oplus \overline{5} \oplus 50 \oplus \overline{50} \oplus 75$; forbidding the $5 \cdot \overline{5}$ mass term, we have the mass term for the triplets coming from

$$\begin{pmatrix} 5 & 50 \end{pmatrix} \begin{pmatrix} 0 & \langle 75 \rangle \\ \langle 75 \rangle & m_{50} \end{pmatrix} \begin{pmatrix} \overline{5} \\ \overline{50} \end{pmatrix}. \tag{45}$$

  Notice that both triplets have mass due to m_{50} and $\langle 75 \rangle$ being nonzero, while the doublet mass matrix consists only of the upper-left entry (they are present only in $5 \oplus \overline{5}$), which has zero-mass. We thus have a zero-mass doublet–antidoublet pair and two massive triplets (the Higgs mass can be reintroduced as a small m_5 mass, but this still keeps both triplets heavy). Note that the presence of the representation 75 was crucial; without it, we cannot couple the 5 and the $\overline{50}$, laving a triplet massless.

  This mechanisms can be extended to bigger groups and to more representations, but the idea is the same and one can treat the arguments still in the SU(5) language. Suppose we have $2n$ triplets–antitriplet pairs (with one coming from

$50 \oplus \overline{50}$), and $2n-1$ doublet-antidoublet pairs, so that we form the triplet mass matrix (written in $n \times n$ blocks) by

$$\begin{pmatrix} 0 & C \\ B & A \end{pmatrix}, \tag{46}$$

where the last column and last line of this matrix are removed to obtain the doublet matrix. If this matric has full matrix rank (the number of independent columns, which is equal to the number of independent rows) $2n$, then blocks A, B and C need to have full rank n, and all triplets are massive (no zero eigenvalues of the matrix). The doublet matrix, however, has matrix rank $2n-2$: the $n-1$ remaining columns of A are independent, and the rank of the submatrix of the first n columns is also $n-1$ due to the big zero block and the B block having now only $n-1$ rows. For the doublets, only $2n-2$ of the $2n-1$ get a mass, and therefore the mechanism provides one massless double. To achieve full rank in the block B, a 75 of SU(5) needs to be present.

3 Group Theory and E_6

In this section, we make a brief introduction to group theory and look at where the E_6 group comes from. We then study the $\mathfrak{e}_6$ Lie algebra, how to perform computations with it, and present some explicitly computed results and definitions, which will later be relevant for model building. The subsections will thus gradually shift topics from more mathematically oriented to those motivated exclusively by physics. This section presents all the necessary details on the E_6 group; this will enable section 4 to focus on building specific models, while many issues and definitions common to all models are found in section 3.

The general material on group theory is based on [54, 55, 56]; for any computation with representations we will casually use Slansky [57]. Also, there exists a dedicated book on exceptional groups [58], while the computational tools of E_6 are based on [59]. Some efforts of computation within E_6 (such as the Clebsch-Gordan coefficients) were made in [60, 61, 62].

3.1 Preliminaries, the classification of simple Lie algebras

Within Gauge theory 2.1, interactions are modelled by internal local symmetries. At a given space-time point, the symmetries form a "continuous group of transformations", a so called *Lie group*. In this section, we briefly review the relevant results, tools and notation from the theory of Lie groups, by which we prepare the necessary background for a more focused E_6 study later on. This will include the well-known classification of simple Lie algebras, which will illuminate the role of E_6 in the bigger picture. Although the statements will be correct, we will adopt a more conversational style compared to what a mathematician would use.

A *Lie group* G is a group and a smooth manifold, in which the operations of multiplication and inversion are smooth. Implicitly assuming the use of charts, this means an element of G can be (locally) denoted by a set of real parameters, denoted here by α^a, whose number is equal to the dimension of the manifold: we will call this number simply the *dimension*.

Examples of Lie groups are the classical matrix groups, which we define below:

- The *general linear group* $\mathrm{GL}(n)$ is the group of all invertible $n \times n$ matrices. It can be further distinguished by adding a $\mathbb{R}$ or $\mathbb{C}$ to specify that we are considering real or complex matrices, respectively.

- The *special linear group* $\mathrm{SL}(n)$ is the group of all $n \times n$ matrices A with $\det(A) = 1$. We can consider either $\mathbb{R}$ or $\mathbb{C}$ matrices.

- The *orthogonal group* $\mathrm{O}(n)$ is the group of $n \times n$ matrices O, for which $OO^T = O^T O = I$. We can consider either $\mathbb{R}$ or $\mathbb{C}$ matrices.

- The *special orthogonal group* $\mathrm{SO}(n)$ is the group of all orthogonal $n \times n$ matrices O, for which $\det(O) = 1$.

- The *unitary group* $\mathrm{U(n)}$ is the group of $n \times n$ complex matrices U, for which $UU^\dagger = U^\dagger U = I$.

- The *special unitary group* $\mathrm{SU}(n)$ is the group of all unitary $n \times n$ matrices, for which $\det(U) = 1$.

- The *symplectic group* $\mathrm{Sp}(n)$ is the group of all $(2n) \times (2n)$ matrices (either $\mathbb{R}$ or $\mathbb{C}$), for which $AJA^\dagger = A^\dagger JA = J$, where J is the canonical skew-linear form:

$$J := \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}. \tag{47}$$

A *Lie algebra* $\mathfrak{g}$ is a vector space endowed with an extra operation $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket*, which needs to be bilinear, antisymmetric and for which the Jacobi identity holds:

$$[x,[y,z]] + [y,[z,x]] + [x,[x,y]] = 0, \tag{48}$$

for all $x, y, z \in \mathfrak{g}$. The Lie algebra can be real or complex, depending on whether the underlying vector space is real or complex.

The fundamental result of Lie theory states that the local structure of a Lie group is intimately connected to its *Lie algebra*. One of the results, which will concern us the most, is the existence of a locally bijective map called the exponential map, $\exp : \mathfrak{g} \to G$, which sends the zero vector $0 \in \mathfrak{g}$ to the group unit element $1 \in G$. Although one can construct this map in an abstract setting of smooth manifolds (via integral curves of left-invariant vector fields under the action of the group), the map becomes much more concrete if one imagines that we have realized the Lie algebra properties on a subset of matrices of a given size (we found a "representation"). If we choose a basis of vectors in the Lie algebra, and label them by $i\,t^a$ (with the index a taking values from 1 to the $\dim(G)$, of course), we can write an arbitrary element of the Lie algebra $\mathfrak{g}$ by a linear combination $\alpha_a t^a$, which we now view as a matrix. The matrices t^a can be complex, with the factor i in front for convenience (if the representation of the group is unitary, adding the i ensures that t^a are hermitian), while the parameters α_a take real values in a real Lie algebra, or complex values in a complex Lie algebra. One can then write the exponential map as

$$\exp : i\alpha_a t^a \quad \mapsto \quad e^{i\alpha_a\, t^a}, \tag{49}$$

with the exponential of the matrix defined by the Taylor series $e^A = \sum_{i=0}^{\infty} A^i/i!$ (which always converges). One can therefore label an element in the group by the parameters α_a:

$$G(\alpha_a) = e^{i\alpha_a\, t^a}. \tag{50}$$

In the concrete realization (representation) of the Lie group, we view $G(\alpha_a)$ as a matrix, which is invertible. Strictly speaking, every element of a connected finite-dimensional Lie group can be written as a finite product of such exponential terms. If we take the derivative with respect to one of the parameters a_0, we roughly get the basis vector t^a in the algebra:

$$\tfrac{d}{d\alpha_{a_0}} G(\alpha_a)\big|_{\alpha_a=0} = it^{a_0}. \tag{51}$$

This result is why we can view the basis t^a as infinitesimal transformations in the a-th direction and we call them *generators* of infinitesimal transformations.

Due to the bilinearity of the Lie bracket operation, it is sufficient to know the commutation relations of the generators:

$$[t^a, t^b] = i \sum_c f^{abc} t^c, \tag{52}$$

where the coefficients f^{abc} are called *structure constants*.

If G and H are two Lie groups, then $G \times H$ is the *product group*, where elements are formally written as ordered pairs (g, h), with $g \in G$ and $h \in H$. Multiplication and inversion are then defined by

$$(g_1, h_1)\,(g_2, h_2) := (g_1\, g_2, h_1\, h_2), \tag{53}$$
$$(g_1, h_1)^{-1} := (g_1^{-1}, h_1^{-1}). \tag{54}$$

At the algebra level, if the Lie algebras of G and H are $\mathfrak{g}$ and $\mathfrak{h}$, respectively, then the Lie algebra of $G \times H$ is $\mathfrak{g} \oplus \mathfrak{h}$, where the $\oplus$ symbol denotes the direct product of vector spaces, and the commutator of different terms in the direct sum being zero. This is equivalent to having $[t, t'] = 0$ for any generator t of $\mathfrak{g}$ and any generator t' of $\mathfrak{h}$.

The (Lie) subgroup of a Lie group G, is a subgroup which is also a submanifold (there are some topological subtleties here, which we will not go into). If H is a subgroup of G ($H \subset G$), and their corresponding Lie algebras are $\mathfrak{h}$ and $\mathfrak{g}$, respectively, then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ ($\mathfrak{g} \subset \mathfrak{g}$), i.e. $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ and $\mathfrak{h}$ is closed under the commutator.

By further studying Lie algebras of Lie groups, it is possible to get a full classification. We outline the summary of how this classification theorem arises, supplemented by definitions of important concepts which arise:

- The *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$, is the maximal subalgebra of $\mathfrak{g}$, in which all elements commute. Although the Cartan subalgebra is not unique, it has a unique dimension. We call it the *rank* of the group. We specify it by picking the maximal set of generators $\{t^a\}$, which commute among themselves; viewing the generators as matrices, the Cartan subalgebra generators can thus be simultaneously diagonalized. In the diagonal basis, the $i - th$ basis vector e_i is thus an eigenvector of all Cartan subalgebra generators: $(t^a)e_i = (w_i)^a e_i$ (no sum over i) for a Cartan generator t^a, where $(w_i)^a$ is called the a-th *weight* of the state e_i. We can collect the weights of a given state e_i into a vector $\vec{w}_i$, with its length equal to the rank of the group. The number of states depends on the dimensionality of the matrices t^a (depends on the "representation", see section 3.4). The set of all states $\{e_i\}$ can thus be represented as a set of vectors $\{\vec{w}_i\}$ in the space $\mathbb{R}^k$, where k is the rank of the group.

- Let the dimension of G be n and its rank k. The *adjoint representation* is a matrix form of the generators t^a, where they are of dimension $n \times n$ (the same as their number), and the action of the a-th generator t^a on the state e^b is given by

  $$\hat{t}^a\, e^b = [t^a, t^b]. \tag{55}$$

  By virtue of this formula, we can denote the basis states e^b simply by the generators t^b, and the action on such a state corresponds to the commutator.

We can realize the adjoint representation matrices by defining them in terms of the structure constants of the algebra:

$$(t^a)^{bc} := -i\, f^{abc}. \tag{56}$$

- The weights of the states in the adjoint representation are called *roots*. If G has dimension n and rank k, then we can represent the roots as n vectors in $\mathbb{R}^k$, which we call *the root system*. Since the Cartan subalgebra generators commute amongst themselves, their weights will be zero, so there are k zero-vectors in the root-system. It turns out that for a state with nonzero root $\vec{w}$, there is always a state with the opposite root $\vec{w}$. The $n-k$ nonzero roots can thus be divided into pairs. Choosing a $(k-1)$-dimensional hyperplane (which does not contain any of the nonzero roots) in $\mathbb{R}^k$ through the origin $\vec{0}$, we split the space $\mathbb{R}^k$ into two half-spaces; this also separates the nonzero roots into *positive roots* on one half-space and *negative roots* in the other half-space (it does not matter which half-space is defined as positive and which space is negative). The *simple roots* are defined as the smallest subset of positive roots, such that every positive root can be written as a linear combination of simple roots with non-negative integer coefficients. The number of simple roots is k, and they span the space $\mathbb{R}^k$. The set of simple roots is called *the simple root system*.

- An *invariant subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a subalgebra, for which $[h, g] \in \mathfrak{h}$ for any $h \in \mathfrak{h}$ and any $g \in \mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is *simple*, if it is not Abelian (the Lie bracket is non-trivial) and it contains no invariant subalgebras other than $\mathfrak{g}$ and 0. A Lie algebra $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras: $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$. Semisimple Lie algebras are essentially Lie algebras without any separate U(1) factors.

- It turns out that the simple root system of a semisimple Lie algebra has certain properties. The simple root systems of different factors $\mathfrak{g}_i$ (which are simple Lie algebras) in a semisimple Lie algebra $\mathfrak{g}$ are, for example, orthogonal. In a simple Lie algebra, pairs of simple roots can only have certain angles between them, and the angle (if it is not 90°) also determines the ratio of lengths of the two simple roots, as shown in Table 1. We can thus represent the simple root system by a *Dynkin diagram*: we draw a node for each simple root and then connect pairs of roots with a single, double, triple or no line, depending on the angle between them.

- We have thus deduced the following: if the Lie group contains no separate U(1) factors, it corresponds to a semisimple Lie algebra. The semisimple Lie algebra is a sum of simple Lie algebras, each of which has a root system and that in turn has a simple root system and a connected Dynkin diagram. The Dynkin diagrams of simple terms in a semisimple algebra are disconnected from each other. We will classify the possible semisimple Lie algebras by classifying all the possible connected Dynkin diagrams. Certain Dynkin diagrams are forbidden either to the properties of simple root systems, or they cannot be realized geometrically. We now state the result of the theorem: the possible Dynkin diagrams and their Dynkin labels are given in Figure 2.

The groups, which the Dynkin diagrams from Figure 2 represent, are identified in Table 2. In essence, the classification theorem states that there are 4 infinite families of

Dynkin diagrams, along with 5 so called "exceptional" diagrams. The 4 infinite families correspond to the special orthogonal groups $\mathrm{SO}(n)$ (even and odd, separately), special unitary groups $\mathrm{SU}(n)$ and symplectic groups $\mathrm{Sp}(2n)$, which correspond to rotations in the spaces $\mathbb{R}^n$ (real), $\mathbb{C}^n$ (complex) and $\mathbb{H}^n$ (quaternion), respectively. The 5 exceptional groups are called G_2, F_4, E_6, E_7 and E_8; in contrast with the four infinite families of classical matrix groups, they cannot be intuitively visualized, but they are all connected to the octonions $\mathbb{O}$ (for details, see [55]).

The Dynkin diagram of a simple Lie algebra can quickly reveal some of its properties. The number of nodes in the diagram equals the rank of the algebra, while the existence of a non-trivial symmetry axis which mirrors the Dynkin diagram into itself reveals the existence of complex representations for that Lie algebra (see section 3.4 for definition).

Table 1: The types of connections between nodes in a Dynkin diagram.

type of connection	angle	length ratio	directed?
none	$90°$	no constraint	n/a
single	$120°$	$1:1$	no
double	$135°$	$\sqrt{2}:1$	yes
triple	$150°$	$\sqrt{3}:1$	yes

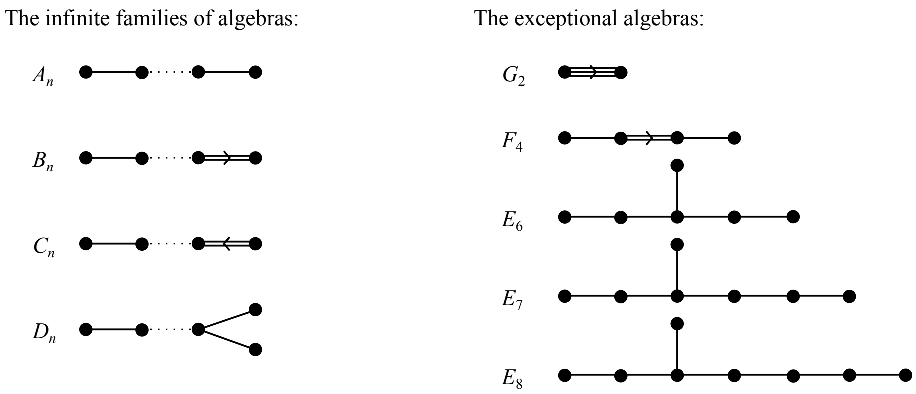

Figure 2: Classification of simple Lie algebras via Dynkin diagrams.

3.2 Motivation for E_6

Accepting the paradigm of GUT, we look for possible simple groups, which could describe the unification of forces from the Standard Model group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. The desired properties of the unification group G, largely due to phenomenological reasons, are the following:

- If we are looking for "true" unification of all SM forces, the group should be simple (or at least all SM forces unify in a single simple factor).
- Since G has to be spontaneously broken to the SM group, the $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ group need to be a subgroup of G.

Table 2: Identification of Dynkin labels with the names of the algebras in their complex and real form.

Dynkin label	complex form	real form	# generators	rank
A_n	$\mathfrak{sl}(n+1,\mathbb{C})$	$\mathfrak{su}(n+1)$	$(n+1)^2-1$	n
B_n	$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{so}(2n+1)$	$(2n+1)(2n)/2$	n
C_n	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(n)$	$n(2n+1)$	n
D_n	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{so}(2n)$	$(2n)(2n-1)/2$	n
E_6	complex $\mathfrak{e}_6$	real $\mathfrak{e}_6$	78	6
E_7	complex $\mathfrak{e}_7$	real $\mathfrak{e}_7$	133	7
E_8	complex $\mathfrak{e}_8$	real $\mathfrak{e}_8$	248	8
F_4	complex $\mathfrak{f}_4$	real $\mathfrak{f}_4$	52	4
G_2	complex $\mathfrak{g}_2$	real $\mathfrak{g}_2$	14	2

- Since all the SM fermions are in complex representations, with only one member of the conjugate pair present ("SM is a chiral theory"), we prefer a group G which has complex representations. If G does not have complex representations, we automatically have a doubling of states of the Standard Model, with the phenomenological constraint that one member of each conjugate pair of representations needs to be heavy.

- Smaller groups are preferred to larger ones due to simplicity. The big groups with big representations also have large Dynkin indices of their representations, so adding matter content can quickly lead to a Landau pole in the RG of the gauge coupling soon after the GUT scale.

Since there exists a classification of simple Lie algebras (see section 3.1), we can systematically go through the possible choices:

- The smallest unitary group, which contains the SM group, is SU(5). In this group, the SM fermion representations are gathered into two representations of SU(5): 10 and $\overline{5}$. Bigger unitary groups are also possible, but they contain usually unwanted extra states; the fundamental representation n of SU(n) (with $n > 5$) will contain the 5 of SU(5), as well as $n-5$ SU(5) singlets. The unitary groups have complex representations.

- Among the orthogonal groups, the odd ones SO($2n+1$) do not have complex representations. Among the even ones, it is the groups SO($4n+2$) which have a conjugate-pair of complex inequivalent representations. The smallest orthogonal group, which contains the SM group, is SO(10). This group has a 16 dimensional spinor representation, which contains all the SM fermions in one family, as well as a right-handed neutrino, which is phenomenologically very attractive due to nonzero neutrino masses. Higher orthogonal groups would be also possible, but their spinor representations (used for describing chiral fermions) will necessarily involve both 16's and $\overline{16}$'s of SO(10), which makes them less attractive.

- The symplectic groups do not have complex representations.

- Among the exceptional groups, G_2 and F_4 do not contain the SM groups, while E_7 and E_8 do not have complex representations. The only viable candidate here is E_6, which both contains the SM group and has complex representations. The fundamental representation 27 of E_6 is especially useful, since it contains all the SM fermions of one family. Model building is also simplified by the fact that E_6 is anomaly free.

We see that given the chosen criteria, the best candidates for a unification group are SU(5), SO(10) and E_6. While SU(5) and SO(10) theories are well studied in the literature, E_6 is less so, partly due to its complexity and partly due to it being an exceptional group and thus lending computation less intuitive.

Compared to SU(5) and SO(10), there are some features of E_6, which may be exploited in model building. Not all features may apply to any one model, but we list the possibilities below:

- Like in the 16 of SO(10), and unlike the SU(5), the 27 of E_6 provides matter unification of a family in a single representation. Furthermore, SO(10) and E_6 also naturally provide a right-handed neutrino.

- One could try to unify the fermions with the Higgs: the fermionic 16_F is joined with the H_u and H_d of MSSM, present in the 10 of SO(10), in a single representation 27.

- The 27 of E_6 automatically contains vector-like quarks and leptons, which is not automatic in SU(5) and SO(10).

- The extra states in the 27 could be used to try to fit part of the second family of fermions into the representation of the first family. This would provide a possible mechanism for small mixing between the $\bar{5}$'s of SU(5) of the first two generations, which is desirable due to small K–$\overline{K}$ mixing (the models would have contributions to box diagrams via fermion–sfermion–gaugino vertices).

- Unlike in SO(10), where the 16^4 exists, the 27^4 in E_6 is not an invariant. This would imply that any 27^4 effective operators of proton decay need to have extra factors of $\langle R\rangle/\Lambda$, where $\langle R\rangle$ is a GUT scale VEV of the representation R and Λ is the cut-off of the theory. These extra factor lead to suppressions in proton decay.

3.3 A practical approach to the E_6 Lie algebra

The $\mathfrak{e}_6$ Lie algebra is 78-dimensional and hence E_6 has 78 generators. This number is much larger than 12 of the Standard Model group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, or even the numbers 24 and 45 of the other popular GUT groups SU(5) and SO(10), respectively. Any kind of analysis and computation within E_6 and its representations will therefore be complicated and tedious; to make things as transparent and efficient as possible, it is clear we will have to organize the generators in a systematic way.

Since the classical groups, such as the orthogonal and unitary families, lend themselves much easier to an intuitive understanding and visualization, we will try to better understand $\mathfrak{e}_6$ through one of its maximal subalgebras, where these advantages can be used. Specifically, we will use the maximal "trinification" subalgebra $\mathfrak{su}(3)_C\times\mathfrak{su}(3)_L\times\mathfrak{su}(3)_R$ (some other maximal subgroups of E_6 are $\mathrm{SO}(10)\times\mathrm{U}(1)$ and $\mathrm{SU}(6)\times\mathrm{SU}(2)$). Similarly, we will denote the trinification group by $\mathrm{SU}(3)_C\times\mathrm{SU}(3)_L\times\mathrm{SU}(3)_R$. The indices C, L, R refer to *color*, *left* and *right*, which is already suggestive of the embedding of the SM group we will be using: the first factor contains the color $\mathrm{SU}(3)_C$ intact, the second factor contains the weak $\mathrm{SU}(2)_L$ interaction, while the $\mathrm{U}(1)_Y$ interaction of the SM is contained as a specific combination of generators partly in $\mathrm{SU}(3)_L$ and partly in $\mathrm{SU}(3)_R$, on which more detail will be given later. From now on, we shall sometimes refer to the trinification group simply as $\mathrm{SU}(3)^3$, with the order of the factors implicitly meant as was stated above.

The adjoint representation has the following decomposition into irreducible representations of the $\mathfrak{su}(3)^3$ subalgebra:

$$78 = (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \oplus (3,\bar{3},\bar{3}) \oplus (\bar{3},3,3). \tag{57}$$

We label the generators of the $\mathfrak{su}(3)$-factors by t^A_C, t^A_L and t^A_R, where the index A is the adjoint index of the $\mathfrak{su}(3)$ algebra ($A = 1,\ldots,8$). We label the remaining $27+27$ generators as $t^{\alpha}{}_{aa'}$ and $\bar{t}_{\alpha}{}^{aa'}$, where we shall denote the $\mathfrak{su}(3)_C$ indices by $\alpha,\beta,\gamma,\ldots$, the $\mathfrak{su}(3)_L$ indices by $a,b,c,\ldots$, and the $\mathfrak{su}(3)_R$ indices by $a',b',c',\ldots$. The mentioned indices are fundamental (upper) or anti-fundamental (lower) indices of the $\mathfrak{su}(3)$ algebras, so they go from 1 to 3. The t and $\bar{t}$ generators are "complex generators" in the sense of lowering and raising operators $t^1 \pm i\,t^2$ in the $\mathfrak{su}(2)$ algebra. As a consequence, not all the structure constants in this basis of generators are real. It is possible to transform to a real basis of generators by taking $\frac{1}{2}(t^{\alpha}{}_{aa'} + \bar{t}_{\alpha}{}^{aa'})$ and $\frac{1}{2i}(t^{\alpha}{}_{aa'} - \bar{t}_{\alpha}{}^{aa'})$, analogous to the $\mathfrak{su}(2)$ case. The transformation properties of the real generators under $\mathfrak{su}(3)^3$, however, are obscured.

An arbitrary element x in the $\mathfrak{e}_6$ algebra can be written as

$$\mathfrak{e}_6 \ni\ x = X^A_C\, t^A_C + X^A_L\, t^A_L + X^A_R\, t^A_R + Z_{\alpha}{}^{aa'}\, t^{\alpha}{}_{aa'} + Z^{*\alpha}{}_{aa'}\, \bar{t}_{\alpha}{}^{aa'}, \tag{58}$$

where X^A_C, X^A_L, X^A_R are real coefficients and $Z_{\alpha}{}^{aa'}$ are complex coefficients, with * denoting complex conjugation. The total number of real degrees of freedom of these coefficients is 78: 24 come from X_C, X_L and X_R, while $27 = 3^3$ complex degrees of freedom come from Z's.

Note that the maximal subalgebra $\mathfrak{su}(3)^3$ is embedded in such a way, that it conforms to Slansky [57] and has the same conventions as left-right models. To that effect, the commutation relations and the action on the fundamental representation from [59] must be appropriately modified. Since the modification consists of applying conjugation on the embedding of the factors $\mathfrak{su}(3)_L$ and $\mathfrak{su}(3)_R$, we have to exchange Gell-Mann matrices λ_A originating from these two factors (identified by the use of indices a,b or a',b') with $-(\lambda_A)^*$ and changing the height position of all indices of the type a,b,c and a',b',c'.

The commutation relations of the $\mathfrak{e}_6$ generators, which provide the definition of the $\mathfrak{e}_6$ Lie algebra, are the following:

$$\left[t^A_C, t^B_R\right] = \left[t^A_R, t^B_L\right] = \left[t^A_L, t^B_C\right] = 0, \tag{59}$$

$$\left[t^A_C, t^B_C\right] = i f^{ABC}\, t^C_C, \tag{60}$$

$$\left[t^A_L, t^B_L\right] = i f^{ABC}\, t^C_L, \tag{61}$$

$$\left[t^A_R, t^B_R\right] = i f^{ABC}\, t^C_R, \tag{62}$$

$$\begin{aligned}
\left[t^A_C, t^\alpha{}_{aa'}\right] &= -\tfrac{1}{2}(\lambda_A)^\alpha{}_\beta \, t^\beta{}_{aa'}, && (63)\\
\left[t^A_L, t^\alpha{}_{aa'}\right] &= \tfrac{1}{2}(\lambda_A)^b{}_a \, t^\alpha{}_{ba'}, && (64)\\
\left[t^A_R, t^\alpha{}_{aa'}\right] &= \tfrac{1}{2}(\lambda_A)^{b'}{}_{a'} \, t^\alpha{}_{ab'}, && (65)\\
\left[t^A_C, \bar{t}_\alpha{}^{aa'}\right] &= \tfrac{1}{2}(\lambda_A)^\beta{}_\alpha \, \bar{t}_\beta{}^{aa'}, && (66)\\
\left[t^A_L, \bar{t}_\alpha{}^{aa'}\right] &= -\tfrac{1}{2}(\lambda_A)^a{}_b \, \bar{t}_\alpha{}^{ba'}, && (67)\\
\left[t^A_R, \bar{t}_\alpha{}^{aa'}\right] &= -\tfrac{1}{2}(\lambda_A)^{a'}{}_{b'} \, \bar{t}_\alpha{}^{ab'}, && (68)
\end{aligned}$$

$$\begin{aligned}
\left[t^\alpha{}_{aa'}, t^\beta{}_{bb'}\right] &= -\varepsilon^{\alpha\beta\gamma} \, \varepsilon_{abc} \, \varepsilon_{a'b'c'} \, \bar{t}_\gamma{}^{cc'}, && (69)\\
\left[\bar{t}_\alpha{}^{aa'}, \bar{t}_\beta{}^{bb'}\right] &= \varepsilon_{\alpha\beta\gamma} \, \varepsilon^{abc} \, \varepsilon^{a'b'c'} \, t^\gamma{}_{cc'}, && (70)
\end{aligned}$$

$$\left[\bar{t}_\alpha{}^{aa'}, t^\beta{}_{bb'}\right] = (\lambda_A)^\beta{}_\alpha \, \delta^a{}_b \, \delta^{a'}{}_{b'} \, t^A_C - \delta^\beta{}_\alpha \, (\lambda_A)^a{}_b \, \delta^{a'}{}_{b'} \, t^A_L - \delta^\beta{}_\alpha \, \delta^a{}_b \, (\lambda_A)^{a'}{}_{b'} \, t^A_R. \qquad (71)$$

The numbers f^{ABC} are the $\mathfrak{su}(3)$ structure constants, λ_A are the 8 Gell-Mann matrices, while the δ and ε are the Kronecker and Levi-Civita symbols, respectively. We follow the convention $\varepsilon_{123} = \varepsilon^{123} = 1$.

Schematically, we can separate the generators into 5 sectors: the t_C, the t_L, the t_R, the t and the $\bar{t}$. The result of the commutators among the five sectors is represented in figure 3.

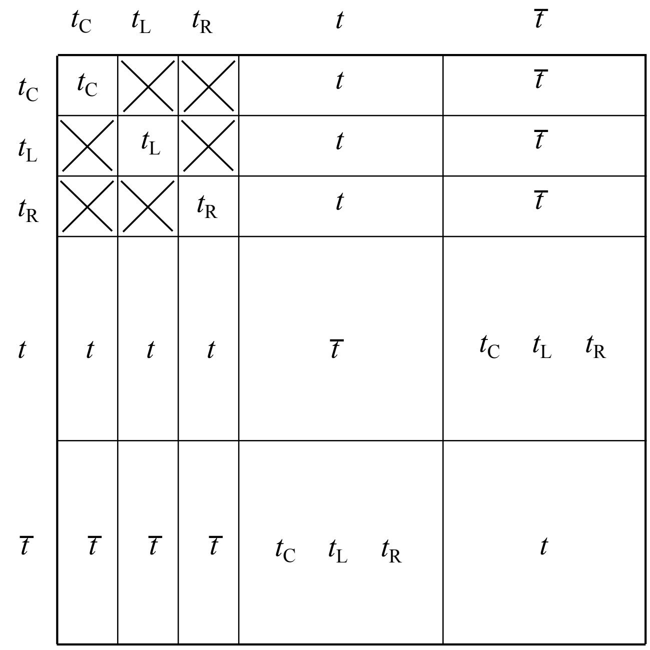

Figure 3: Schematic presentation of the commutation relation between the different sectors of generators. The lengths are in proportion to the number of generators of each type.

The first computational objective is to determine these generators explicitly, which will provide the necessary infrastructure for further computation. Although the commutation relations can be used to extract the $\mathfrak{e}_6$ structure constants and thus get the explicit form of the generators in the adjoint representation via

$$(t^a)^b{}_c = -i\, f_{abc}, \tag{72}$$

where a, b, c are adjoint indices, having the generators in the fundamental 27 dimensional representation is more useful. To that end, one has to study the action of the 78 generators on the representation 27. The decomposition of the fundamental representation 27 in terms of the irreducible representations of the maximal subalgebra $\mathfrak{su}(3)^3$ is

$$27 = (3, 3, 1) \oplus (1, \bar{3}, 3) \oplus (\bar{3}, 1, \bar{3}). \tag{73}$$

The above irreducible representations of $\mathfrak{su}(3)^3$ can be written as 3×3 matrices, which we respectively label by $L^{\alpha a}$, $M_a{}^{a'}$ and $N_{a'\alpha}$ (conforming to the notation in [59], modified to respect the convention in [57]). The position and type of the indices reveal the transformation properties of these matrices. The action of the generators on a state in the fundamental representation (L, M, N) is the following:

$$(t^A_C\, L)^{\alpha a} = \tfrac{1}{2}(\lambda_A)^\alpha{}_\beta\, L^{\beta a}, \tag{74}$$
$$(t^A_C\, M)_a{}^{a'} = 0, \tag{75}$$
$$(t^A_C\, N)_{a'\alpha} = -\tfrac{1}{2}(\lambda^*_A)_\alpha{}^\beta\, N_{a'\beta}, \tag{76}$$

$$(t^A_L\, L)^{\alpha a} = \tfrac{1}{2}(\lambda_A)^a{}_b\, L^{\alpha b}, \tag{77}$$
$$(t^A_L\, M)_a{}^{a'} = -\tfrac{1}{2}(\lambda^*_A)_a{}^b\, M_b{}^{a'}, \tag{78}$$
$$(t^A_L\, N)_{a'\alpha} = 0, \tag{79}$$

$$(t^A_R\, L)^{\alpha a} = 0, \tag{80}$$
$$(t^A_R\, M)_a{}^{a'} = \tfrac{1}{2}(\lambda_A)^{a'}{}_{b'}\, M_a{}^{b'}, \tag{81}$$
$$(t^A_R\, N)_{a'\alpha} = -\tfrac{1}{2}(\lambda^*_A)_{a'}{}^{b'} N_{b'\alpha}, \tag{82}$$

$$(t^\alpha{}_{aa'}\, L)^{\beta b} = \varepsilon^{\alpha\beta\gamma}\, \delta^b{}_a\, N_{a'\gamma}, \tag{83}$$
$$(t^\alpha{}_{aa'}\, M)_b{}^{b'} = -\varepsilon_{abc}\, \delta^{b'}{}_{a'}\, L^{\alpha c}, \tag{84}$$
$$(t^\alpha{}_{aa'}\, N)_{b'\beta} = -\varepsilon_{a'b'c'}\, \delta^\alpha{}_\beta\, M_a{}^{c'}, \tag{85}$$
$$(\bar{t}_\alpha{}^{aa'}\, L)^{\beta b} = \varepsilon^{abc}\, \delta^\beta{}_\alpha\, M_c{}^{a'}, \tag{86}$$
$$(\bar{t}_\alpha{}^{aa'}\, M)_b{}^{b'} = \varepsilon^{a'b'c'}\, \delta^a{}_b\, N_{c'\alpha}, \tag{87}$$
$$(\bar{t}_\alpha{}^{aa'}\, N)_{b'\beta} = -\varepsilon_{\alpha\beta\gamma}\, \delta^{a'}{}_{b'}\, L^{\gamma a}. \tag{88}$$

Expressions such as $(t^\alpha{}_{aa'}\, L)^{\beta b}$ should be interpreted as "the βb-th component in the L-matrix part of the new state, obtained by a transformation via $t^\alpha{}_{aa'}$". The action of the generators on the fundamental representation also suggests a simple interpretation of all the generators: while t_C^A, t_L^A and t_R^A perform only transformations within each of the matrices (each of the types t_C, t_L and t_R performs an $\mathfrak{su}(3)$ rotation on the columns of one of the matrices, and a rotation of the rows on another). The $t^\alpha{}_{aa'}$ and $\bar{t}_\alpha{}^{aa'}$ type generators exchange numbers between the matrices L, M and N; more specifically, the generators $t^\alpha{}_{aa'}$ push the numbers from L to M, and similarly from M to N and from N to N to L; these pushes between the L, M, N matrices can be compactly written as $L \to M \to N \to L$. The $\bar{t}_\alpha{}^{aa'}$ generators push in the opposite way: $L \leftarrow M \leftarrow N \leftarrow L$ in compact notation. We can therefore view the $\mathfrak{e}_6$ subalgebra as part of the $\mathfrak{su}(27)$ algebra (albeit a small part); while the SU(27) group can rotate between all the 27 states independently, the group E_6 can also perform these rotations, but not independently (example: $SU(3)_C$ rotates both the rows of L as well as the columns of N).

Note that the action of generators on the fundamental representation in equations (74)-(88) contains all the necessary information to reconstruct the generators themselves as 27×27 matrices. To see this, consider an analogous equation for the components ϕ^i of a state $\phi = \phi^i e_i$ of the type

$$(t^a \phi)^i = A^i{}_j \phi^j, \tag{89}$$

$$(t^a \phi)^i e_i = A^i{}_j \phi^j e_i, \tag{90}$$

where e_i denotes the basis. The equation (90) can be rewritten in the Dirac notation with basis states $\mid j\rangle$ as

$$t^a \mid j\rangle = A^i{}_j \mid j\rangle. \tag{91}$$

Multiplying by $\langle i \mid$ on the left, we get

$$\langle i \mid t^a \mid j\rangle = A^i{}_j, \tag{92}$$

which is exactly the matrix representation of the operator t^a in the basis e_i.

Similarly, in equations (74)-(88), we can identify the coefficient in front of the matrices L, M, N on the right hand sides as the matrix elements of the generator. The analysis is complicated by the fact that the labels of the states are rather complicated, since the states do not lie in a column of numbers, but in the 3×3 matrices L, M, N. They are not denoted by just one number as ϕ^i in equation (92), but rather by two numbers (position in the matrix) as well as a letter (L, M, N). This does not prevent us from constructing the matrices for the generators in the computer, provided we fix the numbering of the 27 states. We chose the following labeling:

- The states in $L^{\alpha a}$ are numbered 1 through 9 in lexicographical order, i.e. the state $L^{\alpha a}$ has the number $3(\alpha - 1) + a$.
- The states in $M_a{}^{a'}$ are numbered 10 through 18 in lexicographical order, i.e. the state $M_a{}^{a'}$ has the number $9 + 3(a - 1) + a'$.
- The states in $N_{a'\alpha}$ are numbered 19 through 24 in lexicographical order, i.e. the state $N_{a'\alpha}$ has the number $18 + 3(a' - 1) + \alpha$.

A similar labeling scheme is also used for the 78 generators:

- The generators t_C^A: A, numbers 1-8.

- The generators t_L^A: $8+A$, numbers 9-18.
- The generators t_R^A: $16+A$, numbers 19-24.
- The generators $T^\alpha{}_{aa'}$: $24+9(\alpha-1)+3(a-1)+a'$, numbers 25-51.
- The generators $\bar{t}_\alpha{}^{aa'}$: $51+9(\alpha-1)+3(a-1)+a'$, numbers 52-78.

Once the computer code for the generators in the form of 27×27 matrices generators was successfully implemented, we also checked the commutation relations (59)-(71).

3.4 Representation theory and E_6

Rigourously, a representation of the group G on a vector space V is a mapping $R: G \to \mathrm{Aut}(V)$, which respects

$$R(g_1)R(g_2) = R(g_1 g_2). \tag{93}$$

Here, $\mathrm{Aut}(V)$ denotes the set of all automorphisms (bijective linear maps) $V \to V$, while g_1 and g_2 are arbitrary elements of the group G. In equation (93), the multiplication operation is the composition of automorphisms, while the operation on the right-hand side is multiplication of elements within the group G. Choosing a basis of V, the "states" in V can be denoted by columns of coefficients in the linear expansion using this basis. Automorphisms can then represented as invertible matrices.

Similarly, a representation of a Lie algebra $\mathfrak{g}$ is a mapping $r: \mathfrak{g} \to \mathrm{End}(V)$ satisfying

$$r([x,y]) = r(x)r(y) - r(y)r(x), \tag{94}$$

with $\mathrm{End}(V)$ denoting the set of all endomorphisms and $x, y \in \mathfrak{g}$. In a given basis, the endomorphisms can be represented as matrices. Indeed, one has all the necessary information for computation, once the matrix forms of the generators are specified in a given basis. One can then check the commutation relations, and construct group elements in the form of invertible matrices via the exponential map. Because of this close relation, we will often use sloppy language: when talking about representations, we will sometimes talk about R or r, sometimes about their respective images, but most of the time "representations" will actually refer to the underlying vector spaces V. What is being referred to as "representation" in a particular situation should be apparent from the context.

The representation R is *reducible*, if there exists a similarity transformation, which brings all the group elements into block diagonal form, i.e. there exists a matrix S, such that $SR(g)S^{-1}$ has a given block form for all elements g in the Lie group G. At the level of the Lie algebra, the representation r is reducible if there exists a similarity transformation S, such that all generators t^a can be brought into block diagonal form by $S\,t^a\,S^{-1}$. The representation is *irreducible*, if it is not reducible. Intuitively, reducible representations can be split into distinct parts, which transform independently of one another, while irreducible transformations cannot be simplified in this way. We say that the underlying vector space V of a reducible representation R splits via $V = V_1 \oplus \ldots \oplus V_m$, which suggests the notation $R = R_1 \oplus \ldots R_m$; we say the representation R is decomposed into its irreducible representations R_1 to R_m.

Two representations R_1 and R_2 can be combined to form a new representation called the *tensor product* representation and denoted by $R_1 \otimes R_2$. If V_1 and V_2 are the underlying vector spaces of representations R_1 and R_2, then the underlying vector space of $R_1 \otimes R_2$ is the tensor product of vector spaces $V_1 \otimes V_2$. As a consequence, if the dimensions of V_1 and V_2 are n_1 and n_2, the dimension of the representation $R_1 \otimes R_2$ is $n_1 n_2$. Labeling the representations by their dimensionality, we thus write the tensor product representation as $n_1 \otimes n_2$. The transformation matrices can be written (in double index notation) as

$$\big[(R_1 \otimes R_2)(g)\big]_{ik,jl} = \big[R_1(g)\big]_{ij} \big[R_2(g)\big]_{kl}, \tag{95}$$

which translates to a single generator t^a having the representation

$$\big[t^a_{R_1 \otimes R_2}\big]_{ik,jl} = \big[t^a_{R_1}\big]_{ij} \delta_{kl} + \delta_{ij} \big[t^a_{R_2}\big]_{kl}. \tag{96}$$

The tensor product representation need not be irreducible, so it may be possible to decompose it into irreducible representations. We shall write these decompositions many times in the course of this work.

The *conjugate representation* of a representation R is a representation, where the generators t^a are changed into $-t^{a*}$, where $*$ denotes complex conjugation. The element $e^{i\alpha_a t^a}$ thus becomes the element $e^{-i\alpha_a t^{a*}}$ in the conjugate representation. We denote the conjugate representation by $\overline{R}$; it has the same dimensionality as the representation R. We say that the representation $\overline{R}$ is equivalent to R, if there exists a similarity transformation S, such that it brings one into the other, i.e. $S\, t^a\, S^{-1} = -t^{a*}$ for all generators t^a. From the point of view of the equivalence property, there are three different types of representations:

1. The representation R is *complex* if R and $\overline{R}$ are not equivalent.
2. The representation R is *real* if R and $\overline{R}$ are equivalent and the generators t^a can be written as matrices with real entries only.
3. The representation R is *pseudo-real* if R and $\overline{R}$ are equivalent and the generators t^a cannot be written as matrices with only real entries (at least one t^a will have complex entries, for example σ^2 in the Pauli matrices).

The adjoint representation is for example always real. We will not specifically distinguish between real or pseudo-real representations, but will be mostly interested whether R and $\overline{R}$ are equivalent or not, which we will simply call real and complex.

Let the column ϕ^i denote a vector in V in a given basis, and $X^i{}_j$ and $t^i{}_j$ denote the matrices in the same basis of a group element and a generator in the Lie algebra, respectively.[1] The action of the group element and the generator on a state ϕ^i in V are then respectively written as $X^i{}_j\ \phi^j$ and $t^i{}_j\ \phi^j$. The indices i and j always go from 1 to the dimension of the representation at hand. One obvious drawback of treating all the representations in this way is that for each representation the generators (seen here as $n \times n$ matrices) have to be computed separately. The treatment of representations is therefore separate, with no way to connect different representations within the formalism.

For group-theoretic computations, it is therefore much more convenient to use the formalism of tensor methods (see [56] for general methods, [59] for E$_6$). Using tensor

[1] We denote the conjugate representation by $\overline{\phi}_i$ with a lower index.

products of representations, one can construct higher representations from lower ones. A (reducible) representation written as a multi-index object $\Phi^{i_1\cdots i_n}{}_{j_1\cdots j_m}$ with n upper and m lower indices lives in the tensor space $V^{\otimes n}\otimes(V^*)^{\otimes m}$. To know the action of the generator $\hat{t}$ or the group element $\hat{X}$, we only need to know their matrix representations $t^i{}_j$ and $X^i{}_j$ on the vector space V: the multi-index object transforms as

$$(\hat{t}\,\Phi)^{i_1\cdots i_n}{}_{j_1\cdots j_m} = \sum_{r=1}^{n} t^{i_r}{}_{k_r}\,\Phi^{i_1\cdots k_r\cdots i_n}{}_{j_1\cdots j_m} - \sum_{r=1}^{m} (t^*)_{j_r}{}^{l_r}\,\Phi^{i_1\cdots i_n}{}_{j_1\cdots l_r\cdots j_m}, \tag{97}$$

$$(\hat{X}\,\Phi)^{i_1\cdots i_n}{}_{j_1\cdots j_m} = X^{i_1}{}_{k_1}\cdots X^{i_n}{}_{k_n}\,(X^\dagger)^{l_1}{}_{j_1}\cdots(X^\dagger)^{l_m}{}_{j_m}\,\Phi^{k_1\cdots k_n}{}_{l_1\cdots l_m}. \tag{98}$$

Since the symmetric and antisymmetric parts of products transform separately already under the general linear group GL(n), they will also transform separately under G, which is represented only with some of the matrices from GL(n). Sometimes, as in the case of the representation $351'$ in E$_6$, further relations have to be imposed to project out the irreducible representations. Ultimately, irreducible representations can therefore be described by multi-index objects, which satisfy certain symmetric/antisymmetric properties under the exchange of indices and possibly some additional relations.

A well-known way how to label the irreducible representations is the *Dynkin notation* (see for example [57]), which is based on the Dynkin diagram of the Lie algebra. Each node in the Dynkin diagram represents a simple root vector in the root system. Each irreducible representation is uniquely determined by its highest weight, which can be written as a linear combination of the simple roots with non-negative integer coefficients. The root system of E$_6$ is 6-dimensional, so we can use 6-number sequences to label the irreducible representations, once we decide upon the order of the nodes (we use the convention shown in Figure 4). In fact, a theorem of representation theory states that there is a one-to-one correspondence between non-negative integer sequences of a given length and irreducible representations for any finite-dimensional (semi)simple Lie algebra.

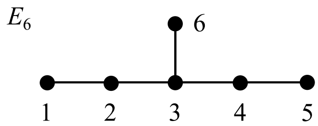

Figure 4: Numbering convention for the nodes in the Dynkin diagram of E$_6$.

We will write the 6-number sequences for E$_6$ in parentheses, with a small gap left for the last number, which corresponds to the node of the E$_6$ Dynkin diagram which is not in the same line with the others. For example, (00000 0) labels the trivial representation, which we will also call "the singlet". The lowest-dimensional nontrivial representations are written in Table 4. Coincidentally, in E$_6$, there are two inequivalent complex representations of dimensionality 351, which we differentiate by giving a prime to one of them. Note that our labels conform for the most part to Slansky [57], but we do exchange the barred and non-barred labels of 351, as well as for $351'$, so that the unbarred labels correspond to upper-index representations and the barred ones to lower-index representations.

The Dynkin labels allow for a quick determination of some of the properties of the representations. Although these are in principle well-known from theorems in representation theory, it will be instructive to go through them in the specific case of E_6. The E_6 Dynkin diagram has a non-trivial mirror symmetry over the central vertical axis. This in turn reverses the order of the first 5 integers in the Dynkin label. Representations with symmetric labels are real (or pseudoreal), while representations with asymmetric labels are complex. Furthermore, the labels suggest the tensorial construction of these representations. Imagine the following naive scheme of using indices: suppose we use 6 different types of indices, where the i-th type of index corresponds to the *basic representation* with the only non-zero value in the Dynkin label being a 1 in the i-th place; the i-th type of index therefore runs from 1 to the dimension of this i-th basic representation. The Dynkin label of a representation then describes how many indices of each type we need in the tensor formalism for this representation; for example, if the first integer in the Dynkin label for a representation is m, we will need m indices of the first type. One can further deduce that these m indices will have to be symmetric: the Dynkin label corresponds to the state with the highest weight in the representation (in the Dynkin basis — the basis of simple root vectors), and this state can be tensorially constructed only by taking the maximum weight of the first basic representation in each of the tensor factors, which is present in the completely symmetric (as far as the first type of index is concerned) part of the tensor.

The tensor methods become even simpler by realizing only one type of index, and not six, is necessary for the description — this will be the fundamental index, which is the 1-st type of index in our labeling. The 4-th and 5-th type of indices are redundant due to the mirror-symmetry of the Dynkin diagram: they can be merely written as lower indices of type 2 and 1, respectively. The first five indices, which correspond to the horizontal chain of nodes (which correspond to the $\mathfrak{su}(6)$ subalgebra of $\mathfrak{e}_6$), have the same properties as the indices in the unitary groups: the 2-nd type of index can be substituted by an antisymmetric pair of 1-st type indices, while the 3-rd type of index can be substituted by three antisymmetric indices of the 1-st type. Similarly, following the same logic from the right side of the horizontal chain of nodes, the 4-th type index is equivalent to two antisymmetric indices of type 5, which is equivalent to two antisymmetric lower indices of type 1. The only remaining puzzle is the 6-th type index, which is the adjoint index (running from 1 to 78). This can be eliminated with the standard trick: the adjoint representation can be encoded as a linear combination of the generators, for example as a 27×27 matrix ${\phi^i}_j$ with the help of the formula

$$ {\phi^i}_j = \phi^a {(t^a)^i}_j, \tag{99} $$

where a is the adjoint index, ϕ^a are the components in the adjoint representation, and i and j are upper and lower fundamental indices; Einstein summation convention applies.

In summary, a representation given by a Dynkin label can be found in a tensor, which has as many symmetric indices of a given type as the integer in the corresponding place in the Dynkin label indicates, and then we can further replace these indices with the fundamental indices as instructed in Table 3. Notice that these properties can also be read from the tensor products of the representations (tables of these are in [57]). Also, note that a so constructed tensor does not necessarily contain only the irreducible representation given by the above Dynkin label; to use only the correct degrees of freedom, further relations may have to be imposed (one can guess these relation by noting that their form has to be basis independent).

Table 3: Translating the six types of index into using only fundamental/antifundamental indices.

index type	translates into
1	one upper
2	two antisymmetric upper
3	three antisymmetric upper
4	two antisymmetric lower
5	one lower
6	one upper and one lower via the generators

Table 4: Some lowest-dimensional nontrivial irreducible representations of E_6: their Dynkin labels and corresponding dimensionality. None are omitted below dimension 1000.

Dynkin label	dimensionality	complex?	comment
(10000 0)	27	yes	fundamental
(00001 0)	$\overline{27}$	yes	anti-fundamental
(20000 0)	$351'$	yes	
(00002 0)	$\overline{351'}$	yes	
(01000 0)	351	yes	
(00010 0)	$\overline{351}$	yes	
(00100 0)	2925	no	
(00000 1)	78	no	adjoint
(10001 0)	650	no	

The lowest dimensional representations, which will also be the most important for GUT model building, are the 27, $\overline{27}$, 78, $351'$, $\overline{351'}$, and also 351, $\overline{351}$ and 650 to a lesser degree. Model building will require us to know the SM content of these representations; in particular, the following will be very important: the number of SM singlets (denoted S, they can acquire VEVs), and the number of weak doublets/antidoublets $(1, 2, +1/2)$ and $(1, 2, -1/2)$ (denoted by D and $\overline{D}$, respectively), as well as proton decay mediating triplets/antitriplets $(3, 1, -1/3)$ and $(\overline{3}, 1, +1/3)$ (denoted by T and $\overline{T}$, respectively). For this reason, Table 5 contains this information for the representations we will be using, as well as our convention for the labels, their tensor construction and the Dynkin index D_2 for these representations (defined by equation (29); values from [57], where we took into account the normalization of the 27 to be 3 in our case). Further details of these representations will now follow separately, with a detailed analysis of the states of significance in subsection 3.4.6.

3.4.1 The (anti)fundamental representation: 27 and $\overline{27}$

The fundamental representation 27 in E_6 has the following decomposition into irreducible representations of SO(10):

$$27 = 16 \oplus 10 \oplus 1. \tag{100}$$

We shall now discuss the content of these representations further in terms of SU(5)

Table 5: Construction and labels of the irreducible reps of dimension below 1000 and their SM content: numbers of singlets, doublets, antidoublets, triplets and antitriplets.

rep dim.	tensor	restrictions	S	D	$\overline{D}$	T	$\overline{T}$	D_2
27	ψ^i		2	1	2	1	2	3
$\overline{27}$	$\overline{\psi}_i$		2	2	1	2	1	3
78	${\phi^i}_j$	${\phi^i}_j = \phi^a\,{(t^a)^i}_j$	5	1	1	1	1	12
351	Ξ^{ij}	$\Xi^{ij} = -\Xi^{ji}$	5	6	5	6	5	75
$\overline{351}$	$\overline{\Xi}_{ij}$	$\overline{\Xi}_{ij} = -\overline{\Xi}_{ji}$	5	5	6	5	6	75
$351'$	Θ^{ij}	$\Theta^{ij} = \Theta^{ji}$, $d_{ijk}\,\Theta^{jk} = 0$	5	4	4	4	5	84
$\overline{351'}$	$\overline{\Theta}_{ij}$	$\overline{\Theta}_{ij} = \overline{\Theta}_{ji}$, $d^{ijk}\,\overline{\Theta}_{jk} = 0$	5	4	4	5	4	84
650	${X^i}_j$	$\mathrm{Tr}(X) = \mathrm{Tr}(t^a\,X) = 0$	11	7	7	7	7	150

and the SM group, with the knowledge of further decompositions under these groups from subsections 2.4.2 and 2.4.1.

Since the representation 16 contains exactly all the fermion particles from one generation of the Standard Model, with the addition of a right-handed neutrino, it seems that the fundamental representation will be useful for describing fermionic particles. Alongside the 16, we also have exotic states in a 10 and 1 of SO(10); an E_6 GUT model will therefore also contain these extra states, which will need to be made heavy. Due to the SO(10) $\to$ SU(5) decomposition $10 = 5 \oplus \overline{5}$, the exotics will consist of vector-like quarks and vector-like leptons, with the singlet 1 having the role of another right-handed neutrino. We expect these exotics to acquire masses of the order of M_{GUT}, which indeed they typically will, with some further details depending on the model.

The fundamental representation 27, when thought of as a fermionic representation, will therefore contain the following representations in terms of the SM group:

- The standard right- and left-handed quarks: $Q \sim (3, 2, +1/6)$, $d^c \sim (\overline{3}, 1, +1/3)$, $u^c \sim (\overline{3}, 1, -2/3)$.
- The exotic vector-like quarks $d' \oplus d'^c \sim (3, 1, -1/3) \oplus (\overline{3}, 1, +1/3)$.
- The standard left-handed lepton doublet and the right-handed electron: $L \sim (1, 2, -1/2)$, $e^c \sim (1, 1, 1)$.
- The vector-like lepton doublets $L' \oplus L'^c \sim (1, 2, -1/2) \oplus (1, 1, +1/2)$.
- The SO(10) right-handed neutrino ν^c and the exotic E_6 singlet neutrino s.

This content will form one generation of fermions in our models, so we will need 3 copies of the representation 27. For some purposes, it will be easier to think of the states in the 27 under the decomposition $E_6 \to SU(3)^3$:

$$27 = (3, 3, 1) \oplus (1, \overline{3}, 3) \oplus (3, 1, 3) = L \oplus M \oplus N. \tag{101}$$

The states are then collected into the 3×3 matrices L, M and N, for which the explicit transformations under E_6 were already given in subsection 3.3. We collect more information on all of the fundamental states in Figure 5, with some further elaboration in the following:

- The group E_6 has rank 6, so it has a 6 dimensional Cartan subalgebra of diagonal generators. Noting that $SU(3)^3$ is one of its maximal subgroups, we can pick two diagonal generators from each factor. As usual in $SU(3)$, we pick the 3-rd and 8-th generator, so the diagonal generators will be t^3_C, t^8_C, t^3_L, t^8_L, t^3_R and t^8_R. A state in the fundamental representation can be uniquely specified by specifying its eigenvalues under these diagonal generators. These are the numbers in the first two columns of Figure 5. To get half-integer numbers for the eigenvalues, we choose to specify $\sqrt{3}t^8_C$, $\sqrt{3}t^8_L$, $\sqrt{3}t^8_R$ instead of respectively t^8_C, t^8_L, t^8_R.

- Certain linear combinations of the diagonal generators have special significance from the point of view of the embedding chain

$$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \subseteq \mathrm{SU}(5), \tag{102}$$
$$\mathrm{SU}(5) \times \mathrm{U}(1)' \subseteq \mathrm{SO}(10), \tag{103}$$
$$\mathrm{SO}(10) \times \mathrm{U}(1)'' \subseteq \mathrm{E}_6. \tag{104}$$

We see that t^3_C, t^8_C and t^3_L are part of the SM quantum numbers, but the remaining 3 diagonal generators can be rewritten to form independent (and pairwise orthogonal) combinations:

$$Y/2 = \tfrac{1}{\sqrt{3}}\, t^8_L + t^3_R + \tfrac{1}{\sqrt{3}}\, t^8_R, \tag{105}$$
$$\mathrm{U}(1)' = -2\sqrt{3}\, t^8_L + 4\, t^3_R - 2\sqrt{3}\, t^8_R, \tag{106}$$
$$\mathrm{U}(1)'' = 2\sqrt{3}\, t^8_L - 2\sqrt{3}\, t^8_R. \tag{107}$$

The half-hypercharge $Y/2$ is of course also part of the Standard Model. Notice that both $Y/2$ and $\mathrm{U}(1)'$ are present in SO(10); since the left-right symmetry group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$ is a subgroup of SO(10), the diagonal generators of SO(10) will need to span only a left-right symmetric part of the combinations. The t^3_L is present in the SM group, so t^3_R can stand on its own in $Y/2$, but the t^8_L and t^8_R are present only in a left-right symmetric combination in $Y/2$ and $\mathrm{U}(1)'$. In contrast, The last diagonal generator $\mathrm{U}(1)''$, which is outside SO(10), will form the remaining antisymmetric combination of t^8_L and t^8_R. The complete set of diagonal generators in E_6 of course also has to span a left-right symmetric space (E_6 is an even bigger symmetry than SO(10), which is already left-right symmetric), which is most easily see in the original basis of generators adapted to $SU(3)^3$. The normalizations of the factors $\mathrm{U}(1)'$ and $\mathrm{U}(1)''$ conform to the conventions in representation decompositions $E_6 \to \mathrm{SO}(10) \times \mathrm{U}(1)''$ and $\mathrm{SO}(10) \to \mathrm{SU}(5) \times \mathrm{U}(1)'$ in Slansky [57].

There are two further linear combinations of diagonal generators of physical interest: one is the electric charge Q, with the second being the $B-L$ charge of the left-right models, which corresponds to the difference of the baryon number B and lepton number L. Since we know

$$Q = t^3_L + Y/2 = t^3_L + t^3_R + \tfrac{1}{2}(B-L), \tag{108}$$

we obtain

$$Q = t^3_L + \tfrac{1}{\sqrt{3}}\, t^8_L + t^3_R + \tfrac{1}{\sqrt{3}}\, t^8_R, \tag{109}$$
$$B - L = \tfrac{2}{\sqrt{3}}\, t^8_L + \tfrac{2}{\sqrt{3}}\, t^8_R. \tag{110}$$

Note: in the literature, the $\mathrm{U}(1)'$ factor is often denoted $\mathrm{U}(1)_\chi$, while $\mathrm{U}(1)''$ is denoted by $\mathrm{U}(1)_\psi$ [63] or sometimes by $\mathrm{U}(1)_X$ [15].

- The particle labels in Figure 5 for the fermionic 27 have for the most part already been specified previously: the quarks in Q are labeled by u and d (each in three colors), while the content of the lepton doublet L is the neutrino ν and electron e (similarly L' contains ν' and e', L^c contains ν'^c and e'^c). A quick intuitive check that the particles are identified correctly can be performed by considering the decomposition of the matrices L, M and N into SM representations. The color triplets have to be in L, the color anti-triplets in N, while the leptons are in the matrix M. The various SU(3) factors have the following effect:
  - The $\mathrm{SU}(3)_C$ factor rotates between the rows of the matrix L and between the columns of the matrix N.
  - The $\mathrm{SU}(3)_L$ factor rotates between the rows of the matrix M and between the columns of the matrix L.
  - The $\mathrm{SU}(3)_R$ factor rotates between the rows of the matrix N and between the columns of the matrix M.

  The color triplets are therefore positioned vertically in the matrix L, while the color anti-triplets are positioned horizontally in the matrix N. As far as the second factor is concerned, only the $\mathrm{SU}(2)_L$ part of $\mathrm{SU}(3)_L$ (with the standard embedding) is part of the Standard Model, so the SM group only rotates between the first two rows in M and between the first two columns in L, while all other states in 27 are weak singlets.

- Notice that some of the particles are defined with a minus in front. In principle, the minus is irrelevant, since all the fields (complex scalar or spinor) are defined only up to a complex phase. The minuses are there for the same reason as the minuses in the representation $\overline{5}$ in SU(5) GUT: to be compatible with the usual labels in the Standard Model. More specifically, they come from the details of the embedding of $\mathrm{SU}(2)_L$. Note that in SU(5), the standard embedding of $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L$ is such that the fundamental representation 5 decomposes as $(3,1) \oplus (1,2)$, with the first three entries rotated by $\mathrm{SU}(3)_C$, while the last two entries rotate with $\mathrm{SU}(2)_L$. This means the $\overline{5}$ of SU(5) actually contains $\overline{2}$ of $\mathrm{SU}(2)_L$; the $\overline{2}$ is equivalent to a 2 via $\overline{2}_i = \varepsilon_{ij} 2^j$, where ε_{ij} is the two index antisymmetric tensor, hence one minus sign in the states e and e'. We use an analogous reason for the states d^c due to the flipped embedding of $\mathrm{SU}(2)_R$ in the matrix N in Figure 5.

- The # symbol represents the number of the state in a 27-entry long column ψ^i: the order of these entries will specify the ordered basis, in which we write our generators.

The anti-fundamental representation $\overline{27}$ will have analogous states, but with all quantum numbers opposite. It contains the conjugate representation of the ones in 27, so we use the following self-explanatory labels for its contents: $\bar{Q}$, $\bar{L}$, $\bar{d}^c$, $\bar{u}^c$, $\bar{e}^c$, $\bar{\nu}^c$, $\bar{L}'$, $\bar{L}'^c$, $\bar{d}'^c$, $\bar{d}'$ and $\bar{s}$, which can be arranged into a 27-entry column $\overline{\psi}_i$ with a lower index.

3.4.2 The adjoint: 78

The representation 78 is the adjoint representation. It is most simply written in a 27×27 matrix $\phi^i{}_j$, by taking a linear combination of the generators in the fundamental representation:

$$\phi^i{}_j = \phi^a\ (t^a)^i{}_j. \tag{111}$$

The index a goes from 1 to 78, so the states in the adjoint are actually the coefficients ϕ^a. The meaning of these coefficients depends on the basis chosen for the generators t^a. A classification of the generators was already elaborated on in subsection 3.3: due to the $\mathrm{E}_6 \to \mathrm{SU}(3)_C \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ decomposition

$$78 = (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \oplus (3,\overline{3},\overline{3}) \oplus (\overline{3},3,3), \tag{112}$$

we label the generators respectively by t_C^A, t_L^A, t_R^A, $t^\alpha{}_{aa'}$ and $\bar{t}_\alpha{}^{aa'}$, with A being the SU(3) adjoint index and α, a and a' being the SU(3) fundamental indices of the factors C, L and R, respectively. As already discussed in subsection 3.3, this basis is actually complex due to the presence of the generators $t^\alpha{}_{aa'}$ and $\bar{t}_\alpha{}^{aa'}$. A real basis would instead consist of the linear combinations

$$t_1^{\alpha aa'} := \tfrac{1}{2}(t^\alpha{}_{aa'} + \bar{t}_\alpha{}^{aa'}), \tag{113}$$

$$t_2^{\alpha aa'} := \tfrac{1}{2i}(t^\alpha{}_{aa'} - \bar{t}_\alpha{}^{aa'}). \tag{114}$$

Conversely,

$$t^\alpha{}_{aa'} = t_1^{\alpha aa'} + i\, t_2^{\alpha aa'}, \tag{115}$$

$$\bar{t}_\alpha{}^{aa'} = t_1^{\alpha aa'} - i\, t_2^{\alpha aa'}. \tag{116}$$

Note that the upper position of the indices in t_1 and t_2 is just a convention and does not hold any significance, since these generators do not transform as neither triplets nor antitriplets under the factors of the trinification group.

The real and the complex basis each have their advantages and drawbacks. The complex basis far better represents the transformation properties of the generators (and the states associated to them) under the subgroups, most notably under the trinification group $\mathrm{SU}(3)^3$, for which this formalism was designed in the first place. The complex generators t and $\bar{t}$ transform as either triplets or antitriplets under each of the factors, while the transformation rules are obfuscated when using the real basis. The use of complex generators does require some care, though. When the Lie algebra is over the field of real numbers, one must not forget that the complex generators will always come in pairs and with complex coefficients, which are related to each other via complex conjugation: we have the relation

$$Z_\alpha{}^{aa'}\, t^\alpha{}_{aa'} + Z^{*\alpha}{}_{aa'}\, \bar{t}_\alpha{}^{aa'} = X_{\alpha aa'}\, t_1^{\alpha aa'} + Y_{\alpha aa'}\, t_2^{\alpha aa'}, \tag{117}$$

with

$$Z_\alpha{}^{aa'} = \tfrac{1}{2}(X_{\alpha aa'} - i\, Y_{\alpha aa'}). \tag{118}$$

Here, the Z coefficients are complex, while X and Y coefficients are real. We can use either one and still have the same number of degrees of freedom. In the complex case, the Z coefficients determine the Z^* coefficients by complex conjugation, and so the coefficients in front of t and $\bar{t}$ type generators are not independent in a real Lie algebra (this is somewhat analogous to how left-handed spinors behave compared to right-handed spinors). Another real/complex issue is the normalization of the generators. In the fundamental representation, where the generators are constructed as 27×27 matrices, the real generators form an orthogonal basis:

$$\mathrm{Tr}(t^A\, t^B) = D_2\, \delta^{AB}, \tag{119}$$

where δ^{AB} is the Kronecker delta, and the indices A and B go from 1 to 78, and t^A are the set of all real generators in the $\mathfrak{e}_6$ algebra in matrix form. The factor D_2 is called the Dynkin index, and it depends on the representation; for the fundamental representation, we have $D_2 = 3$. In the complex basis, however, we have

$$\mathrm{Tr}(t^\alpha{}_{aa'}\, t^\beta{}_{bb'}) = 0, \tag{120}$$
$$\mathrm{Tr}(\bar{t}_\alpha{}^{aa'}\, \bar{t}_\beta{}^{bb'}) = 0, \tag{121}$$
$$\mathrm{Tr}(t^\alpha{}_{aa'}\, \bar{t}_\beta{}^{bb'}) = 2D_2\, \delta^\alpha{}_\beta\, \delta_a{}^b\, \delta_{a'}{}^{b'}. \tag{122}$$

Since our GUT models will break E_6 all the way down to the SM group, it is important to consider the decomposition of 78 into SM irreducible representations. In the context of GUT, it is most instructive to use the breaking chain $E_6 \to SO(10) \to SU(5) \to SM$. Under $E_6 \to SO(10)$ we have

$$78 = 45 \oplus 16 \oplus \overline{16} \oplus 1, \tag{123}$$

under $SO(10) \to SU(5)$ we have

$$45 = 24 \oplus 10 \oplus \overline{10} \oplus 1, \tag{124}$$
$$16 = 10 \oplus \overline{5} \oplus 1, \tag{125}$$
$$\tag{126}$$

and under $SU(5) \to SM$ we have

$$24 = (8,1,0) \,\oplus\, (1,3,0) \,\oplus\, (1,1,0) \,\oplus\, (3,2,-5/6) \,\oplus\, (\overline{3},2,+5/6), \tag{127}$$
$$10 = (3,2,+1/6) \,\oplus\, (\overline{3},1,-4/6) \,\oplus\, (1,1,+6/6), \tag{128}$$
$$5 = (3,1,-2/6) \,\oplus\, (1,2,+3/6). \tag{129}$$

Which of the generators is which can be computed on the basis of the quantum numbers of the generators, determined by the adjoint action (the commutator) of the diagonal generators. After some tedious computation, one arrives at the result presented in Figure 6, which shows the decomposition of the states in 78. Some further clarification of this figure is given below:

- The generators are grouped according to their type: t_C, t_L, t_R, t and $\bar{t}$. The use of color is intended to give visual clarity.
- It is the complex basis which gives states with well-defined transformation properties under the SM group. Not only are types t and $\bar{t}$ needed, but one

also needs to consider particular combinations in the L and R sectors: we have defined

$$t_R^{12+} := t_R^1 + i\, t_R^2, \tag{130}$$
$$t_R^{12-} := t_R^1 - i\, t_R^2, \tag{131}$$

and analogously for $t_L^{12\pm}$ and $t_L^{45\pm}$.

- The figure describes how **states** in the representation 78 transform. The easiest way to denote the state, however, is by their corresponding generators: the states are actually the coefficients in front of the generators, but we nevertheless use the generator labeling of states. There is a related subtlety in the case of complex generators: we would expect the t-type generators (the t^α) to be color triplets due to the upper α index, and indeed they are. But their accompanying states are actually the coefficients $Z_\alpha{}^{aa'}$, which are color anti-triplets. The coefficients always have their quantum numbers opposite to those of the generators, completely analogous and for the same reasons as in vector analysis, where the basis vectors transform covariantly, and their coefficients (with the upper indices) transform contravariantly, so that the full linear combination gives the same "physical" vector in every basis. One should also remember that the commutation relations of the Lie algebra tell us how the adjoint basis states transform, and not really the generators themselves: this explains the placement of minuses in equations (63)–(68), which opposite to what one would naively expect.

- The last quantum number written for the SM representations is actually $6(Y/2) = 3Y$. This normalization gives all the hypercharges as integers.

In E_6 GUT models, spontaneous symmetry breaking will play an important role. In order to identify the broken symmetry, it is important to study how the different subgroups are embedded (in the standard way) into E_6. The generators of the Standard Model group, of the Pati-Salam group $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, of SU(5) and of SO(10) are identified in Figure 7. Some further comments on the figure are given below:

- The generators, which are present in the specified subgroup only in a particular linear combination, are colored less intensely. Remember that the Standard Model group has 12 generators and the SU(5) has 24 generators, both include the linear combination $\frac{1}{\sqrt{3}}\, t_L^8 + t_R^3 + \frac{1}{\sqrt{3}}\, t_R^8$. Pati-Salam has 21 generators and SO(10) has 45, both also include the linear combination $t_L^8 + t_R^8$.

- The identification of the subgroup generators is consistent with their hierarchy: the SM group is present in both Pati-Salam and SU(5), and both of these are in turn subgroups of SO(10).

- The subgroup generators can be identified by considering, which representations of the Standard Model from Figure 6 are present in the adjoints of various subgroups. We know, for example, that SU(5) consists of the SM generators, amended by the lepto-quark generators $(3, 2, -5/6)$ and $(\overline{3}, 2, +5/6)$. For Pati-Salam and SO(10), one can further consider the fact that they contain the left-right group factor $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ as part of their symmetry. When we consider the generators of type t and $\bar{t}$ in the scheme of Figure 7, they are grouped into 3×3 arrays; in each array, generators of $SU(3)_L$ will rotate between the rows (they act on the second index a), while $SU(3)_R$ will rotate between the rows of each array (it acts on the third index a'). The $SU(2)_L$ part will therefore rotate between the first two rows, and $SU(2)_R$ between the first two columns. If the left-right symmetry factor is part of a group, having the 11 entry in the 3-by-3 array automatically includes also the entries 12, 21 and 22 entries, which are accessible through rotations. In the same way, the entries 13 and 23 are connected, as are 31 and 32. The 33 entry is a singlet under $SU(2)_L \times SU(2)_R$, so it can be present by itself. These patterns can be seen in the figure.

3.4.3 The 351 and $\overline{351}$

The construction of the representation 351 is most easily performed through the tensor product of the fundamental representations:

$$27 \otimes 27 = \underbrace{351}_{a} \oplus \underbrace{351' \oplus \overline{27}}_{s}. \tag{132}$$

The anti-symmetric part of the product will therefore coincide exactly with the representation 351. The states of this representation can therefore be written in a 27×27 matrix, which we label Ξ^{ij}. We have $\Xi^{ij} = -\Xi^{ji}$. Similarly, the conjugate representation is a two-index antisymmetric matrix $\overline{\Xi}_{ij}$.

The representation 351 has the following $E_6 \to SO(10)$ decomposition:

$$351 = \overline{144} \oplus 120 \oplus 45 \oplus 16 \oplus \overline{16} \oplus 10. \tag{133}$$

It can therefore be roughly viewed as analogous to the 120 of SO(10). Note that this is the SO(10) representation, which can form an antisymmetric Yukawa term with two spinor representations 16. There is some further discussion on this in section 4.

The states in the 351 can be labeled similarly to the states in the fundamental representation 27. Using the notation from Figure 5, one can simply write the state in 351 by taking two successive labeled states in 27. For example, one can write a state with the dual-label $u^3 e$, which can be translated into the matrix $\Xi_{ij} = A^{i,j}_{7,12}$, where we used the antisymmetric symbol $A^{i,j}_{a,b} := \delta^i{}_a\, \delta^j{}_b - \delta^i{}_b\, \delta^j{}_a$. The dual-label states correspond to different basis antisymmetric entries in the matrix. The dual-labels have the advantage, that one can immediately compute their SM quantum numbers by adding the quantum numbers of both labels. Similarly, we can use dual-labels with the states from $\overline{27}$ (denoted with a bar and with opposite quantum numbers) to describe the representation $\overline{351}$.

3.4.4 The $351'$ and $\overline{351}'$

For constructing the representation $351'$, we again make use of the tensor product $27 \otimes 27$ in equation (132). We see that $351'$ is found in the symmetric part, alongside a $\overline{27}$. The symmetric two-index matrix $\Theta^i j$ will therefore represent a reducible representation, with both the $351'$ and the unwanted degrees of freedom from the $\overline{27}$. One can remove the $\overline{27}$ by imposing the following relation [59]:

$$d_{ijk}\,\Theta^{jk} = 0. \tag{134}$$

As always, i, j, k go from 1 to 27, and d_{ijk} is the *invariant tensor* of E$_6$. More information on this tensor can be found in subsection 3.5.1; it will be sufficient for now to know that this tensor does not change its numerical values under any E$_6$ transformation. As a consequence, equation (134) is in an E$_6$ invariant form: if it holds before an arbitrary E$_6$ transformation on the matrix Θ, it also holds after, splitting the reducible representation into parts which transform separately. Since the equation (134) has one free index, it represents 27 constraints on the matrix Θ, reducing the number of degrees of freedom from $27{\cdot}28/2$ (a symmetric matrix) by 27 to exactly what is needed: 351. Similarly, the conjugate representation $\overline{351'}$ is represented by a symmetric matrix $\overline{\Theta}_{ij}$, which satisfies

$$d^{ijk}\,\overline{\Theta}_{jk} = 0, \tag{135}$$

completely analogous to equation (134). The d-tensor with all indices being upper has the same numerical values as the d-tensor with all indices being lower.

The representation $351'$ has the following $\mathrm{E}_6 \to \mathrm{SO}(10)$ decomposition:

$$351' = \overline{144} \oplus 126 \oplus 54 \oplus 16 \oplus 10 \oplus 1. \tag{136}$$

For model building purposes, the $351'$ of E$_6$ is to a large extent analogous to the 126 of SO(10). Note that 126 is the SO(10) representation, which can form a symmetric Yukawa term with two spinor representations 16.

One can label the states in $351'$ with double-labels in the same way one does in the representation 351. For example, u^3e would now label a symmetric combination of these two states, which corresponds to the matrix $\Theta^{ij} = S^{i,j}_{7,12}$, where we used the symmetric symbol $S^{i,j}_{a,b} := \delta^i{}_a\,\delta^j{}_b + \delta^i{}_b\,\delta^j{}_a$. But here, we have the extra complication that not all states written in such a way are part of $351'$, but only those, for which equation (134) is satisfied. Since the matrix Θ forms the reducible representation $351' \oplus \overline{27}$, and due to the $\mathrm{E}_6 \to \mathrm{SO}(10)$ decomposition

$$\overline{27} = \overline{16} \oplus 10 \oplus 1, \tag{137}$$

ambiguities can arise only for SO(10) representations 10 and 1. It is only for SO(10) multiplets 10 or 1, for which (134) is not already automatically satisfied. Note, however, that states with well-defined transformation properties under SO(10) do not always correspond to the basis states in the dual-label notation; in practice, one needs to evaluate the condition (134) explicitly.

3.4.5 The 650

Consider the following tensor product of E$_6$ representations:

$$27 \otimes \overline{27} = 650 \oplus 78 \oplus 1. \tag{138}$$

This demonstrates that a matrix $X^i{}_j$, with one upper and one lower index, will contain the 650 representation, along with the unwanted 78 and 1. Similar to the representation $351'$, we will project the unwanted degrees of freedom out from this matrix.

The singlet 1 is formed with the linear combination $\psi^i \overline{\psi}_i$, which can also be written as $\sum_i \delta^i{}_i$. This is the identity matrix $X^i{}_j = \delta^i{}_j$, which can be projected out by using the trace. For the adjoint 78, we know we can write the states with the generators in the fundamental representations 27, i.e. $\phi^i{}_j = \phi^a (t^a)^i{}_j$. These will be projected out by the standard scalar product in the vector space of 27×27 matrices: $(A, B) = \mathrm{Tr}(AB)$. With this, we can get rid of components proportional to the generators. The projection conditions can therefore be written as

$$X^i{}_i = 0, \qquad\qquad (t^a)^i{}_j \, X^j_i = 0, \tag{139}$$

or alternatively in matrix notation as

$$\mathrm{Tr}(X) = 0, \qquad\qquad \mathrm{Tr}(t^a \, X) = 0. \tag{140}$$

The index a goes from 1 to 78, so both conditions together project out $79 = 78 + 1$ degrees of freedom, which is what we want: $650 = 27^2 - 79$. The projection conditions are independent from each other, since the generators t^a are traceless (one can for example use the argument that $E_6 \subset SU(27)$, and the generators of SU(27) are all traceless).

Note that 650 is not a complex representation, so there is no independent $\overline{650}$. The $E_6 \to SO(10)$ decomposition of 650 is written as

$$650 = 210 \oplus 144 \oplus \overline{144} \oplus 54 \oplus 45 \oplus 16 \oplus \overline{16} \oplus 10 \oplus 10 \oplus 1. \tag{141}$$

For model building purposes, the 650 of E_6 is analogous to the representation 210 of SO(10). Furthermore, both of these representations are (pseudo)real.

One can denote the states in the 650 with dual-label notation, with the first label from 27 and the second label from $\overline{27}$. These states are subject to the projection conditions in equation (139) though.

3.4.6 Identification of singlets, doublets and triplets

In the E_6 models we will be considering later on, certain representations (of the Standard Model group) will have special significance, because they will be involved in one way or another in measurable (low-energy) phenomena. All such states will need to be located in the representations we have considered thus far; exactly this is the purpose of this subsection. The important Standard Model representations will be the following:

- The singlets $(1, 1, 0)$: these states will be important for considerations of spontaneous symmetry breaking. We aim for the low energy theory of our GUT models to be reduced to the Standard Model, so it is only the Standard Model singlet states which can acquire a VEV (otherwise we break the SM group), and at the same time, it will be exactly the VEVs of these singlets which will break the bigger E_6 symmetry.

- The doublets $(1, 2, +1/2)$ and antidoublets $(1, 2, -1/2)$: since our models will be supersymmetric, we have two MSSM Higgs doublets, the H_u and the H_d, which are a doublet and an antidoublet, respectively. These states will be a linear combination of the doublet/antidoublet states. For our model to be phenomenologically acceptable, the H_u and H_d will need to have low masses $\mathcal{O}(M_W)$ compared to the rest of the doublets and antidoublets, with typical masses $\mathcal{O}(M_{\mathrm{GUT}})$.

- The triplets $(3, 1, -1/3)$ and antitriplets $(\overline{3}, 1, +1/3)$: these states mediate proton decay, which has experimentally not been observed thus far. This means we need to keep all the triplets heavy, of the scale $\mathcal{O}(M_{\mathrm{GUT}})$.

- There will also be important weak triplet scalar fields $(1, 3, \pm 1)$ for type II seesaw. We do not deal with them in this section, but instead as they arise (section 4.4.3).

These states can be identified with the help of Figure 8, where a $E_6 \to SO(10) \to SU(5)$ decomposition is performed. We have not included the conjugate representations $\overline{27}$, $\overline{351}$ and $\overline{351'}$, since their decomposition can be deduced by conjugating the decompositions already written. Note that the states of interest can be found in the following SU(5) representations, which are present in the decompositions:

- The SM singlets are contained in: 1, 24, 75.

- The weak doublets are contained in: 5, 45.

- The weak antidoublets are contained in: $\overline{5}$, $\overline{45}$.

- The triplets are contained in: 5, 45, 50.

- The antitriplets are contained in: $\overline{5}$, $\overline{45}$, $\overline{50}$.

Counting of the states of interest produces Table 5, which was already written down in the previous subsections. Note that with the exception of representation $351'$ (and $\overline{351'}$), which contains a triplet-only 50 ($\overline{50}$)of SU(5), all the representations contain the same number of doublets and triplets, as well as the same number of antidoublets and antitriplets.

All the singlets in the irreducible representations of E_6 below dimension 1000 are identified in Table 6 and 8. Some comments and observations on this tables are due:

- Notice the labels for the singlets we have chosen. The singlets in 27 are denoted with the letter c and the singlets in $\overline{27}$ with the letter d. There is historical precedent for these singlets, so we conform to the labeling in [14]. In some of the other representations, we decided to proceed further down the alphabet: the singlets in $351'$ and $\overline{351'}$ are denoted by the letters e and f, respectively, while the singlets in 351 and $\overline{351}$ are denoted by the letters g and h, respectively. The singlets in 650 are denoted by x. The choice of letters for the adjoint 78, while present in the work [14], was changed. In this work, the 5 SM singlets in that representation are denoted by

$$a_1 \sim t_R^6 + it_R^7, \tag{142}$$
$$a_2 \sim t_R^6 - it_R^7, \tag{143}$$
$$a_3 \sim \sqrt{2}\, t_R^8, \tag{144}$$
$$a_4 \sim \sqrt{2}\, t_R^3, \tag{145}$$
$$b_3 \sim \sqrt{2}\, t_L^8, \tag{146}$$

where $\sim$ means "corresponds to the following algebra element". Therefore, the original labels were using the letters a and b, but the labels do not directly correspond to states which transform in a well-defined way under the SU(5) subgroup of E_6. For this reason, we defined new linear combinations of these, found in Table 6, and denoted by u_1, u_2, v, w and y (letters from the end of the alphabet).

- All the SM singlets can be found in either 1, 24 or 75 of SU(5). Note that the labeling is such that the singlets in the 24's have an index 4 or 5 in the representations 351, $351'$ and their conjugates, while they have indices from 6 to 10 in the 650. The lone SM singlet in the 75 of SU(5) has the index 11 in the 650. This makes it easier to remember, which singlet transforms under which SU(5) representation.

- To identify the singlets in the tensor formalism, "particle notation" was used. Except for the adjoint representation, the labels of the fundamental representation 27 from Table 5 are used. The $\overline{27}$ uses the same labels, but with a bar, while higher representations such as $351'$ and 351 and 650 use the double label notation (with bars on the labels where appropriate), which was already commented on in each specific case in subsections 3.4.3, 3.4.4 and 3.4.5. For the representations $351'$ and $\overline{351'}$ we implicitly symmetrize with respect to the two labels, in 351 and $\overline{351}$ we antisymmetrize, while in the 650 we use one unbarred and one barred label. For brevity, the labels of SM representations are used, with standard summation applied. For example, by writing $d'\, d^c$ we mean $(d')^1(d^c)_1 + (d')^2(d^c)_2 + (d')^3(d^c)_3$ and by writing LL'^c we mean $\varepsilon_{ij}(L)^i(L'^c)^j = \nu\,\nu'^c - e\,e'^c$. The double label scheme, after possible symmetrization or antisymmetrization, then needs to be normalized with a positive constant, such that we have a Kähler normalization, as in equations (166)–(173). A few full examples of how the compactly written states in double label notation are translated into the tensor formalism, are given below: in the symmetric $351'$, we have

$$s\, s \to \qquad \Theta^{ij} = \delta^i{}_{18}\, \delta^j{}_{18}, \tag{147}$$
$$\nu^c\, s \to \qquad \Theta^{ij} = \tfrac{1}{\sqrt{2}}\, S^{i,j}_{17,18}, \tag{148}$$
$$L'\, L'^c + d'^c\, d' \to \qquad -\tfrac{1}{\sqrt{10}} \left(\left(S^{i,j}_{14,10} + S^{i,j}_{11,13} \right) + \left(S^{i,j}_{25,3} + S^{i,j}_{26,6} + S^{i,j}_{27,9} \right) \right), \tag{149}$$

while in the antisymmetric $\overline{351}$, we have

$$\bar\nu^c\, \bar s \to \qquad \Xi_{ij} = \tfrac{1}{\sqrt{2}}\, A^{17,18}_{i,j}, \tag{150}$$
$$\bar L\, \bar L'^c - \tfrac{2}{3}\bar d^c\, \bar d' \to \qquad \Xi_{ij} = -\sqrt{\tfrac{3}{20}} \left(\left(A^{15,10}_{i,j} - A^{12,13}_{i,j} \right) - \tfrac{2}{3} \left(A^{22,3}_{i,j} + A^{23,6}_{i,j} + A^{24,9}_{i,j} \right) \right). \tag{151}$$

Notice that equations (149) and (151) have a minus in front of the normalizing factor, which comes from a minus in each of the terms due to the definition of some states with minuses in Figure 5.

- The identification of states in the adjoint can be written via the generators. We therefore specify the singlet in "particle notation" by writing the specific linear combination of generators, which this states corresponds to. The linear combinations already have the proper normalization. Notice from equations (105)–(107) that y is proportional to the hypercharge $Y/2$, while v and w are proportional to $U(1)'$ and $U(1)''$, respectively.

- We specified in which representations of E_6, SO(10), SU(5) and Pati-Salam (PS) the singlets can be found. The order of the factors in Pati-Salam is $SU(4)_C \times SU(2)_L \times SU(2)_R$. Some singlets do not have a well defined transformation property under the Pati-Salam group: these are the singlets in the 45 of SO(10), which contains the SM singlets in the PS representations $(1, 1, 3)$ and $(15, 1, 1)$. Taking into account which generators belong to $SU(4)_C$ in Figure 7, we can compute that the $SU(4)_C$ singlet combination corresponds to the combination $\sqrt{6}v - 2y$ in the 78, $\sqrt{3/2}g_3 - g_5$ in 351 and $\sqrt{3/2}h_3 - h_5$ in $\overline{351}$.

With similar rules of "particle notation" as for the singlet states, we also list the doublets $(1, 2, +1/2)$, antidoublets $(1, 2, -1/2)$, triplets $(3, 1, -1/3)$ and antitriplets $(\bar{3}, 1, +1/3)$ in Table 9. Some comments:

- We only defined the doublet and triplet states in the representations 78, 27, $\overline{27}$, $351'$ and $\overline{351'}$, since they will be relevant later on.

- Excluding the representation 78, this table provides particle notation only for states in the first column (for the representations 27 and $351'$). The states in the third column (in the conjugate representations) have equivalent particle notation, we only need to change all the unbarred labels from Figure 5 to barred labels.

- Particle notation again assumes implicit summation of indices where applicable. But since these states are no longer SM singlets, they may also carry a color or a weak index. If we label (just in this paragraph) the $SU(3)_C$ color indices by a, b, c and the $SU(2)_L$ indices by i, j, QL is translated into $\varepsilon_{ij}\, Q^{ai}\, L^j$ and has a running index a, since the state is a triplet. Other examples would be the doublet $Qd^c \to Q^{ai}(d^c)_a$, or the antitriplet $QQ \to \varepsilon_{abc}\, \varepsilon_{ij}\, Q^{bi}\, Q^{cj}$.

- For the representation 78, particle notation implies the use of generators as labels. For the doublets D_0 and $\bar{D}_0$, the two labels in fact represent the neutral "Higgs" component in each, and not the components of a doublet.

3.5 Invariants in E_6

Invariants are singlets in a tensor product of representations. In other words, they are a special linear combination of products of states, so that they do not change under the action of the group (they stay *invariant*).

In the tensor formalism, the indices tell us how an object transforms. A product of representations will be an invariant, when all the indices are contracted and there are no remaining free indices. In the construction of E_6 invariants, one can also use the invariant d-tensor, which has either three symmetric upper indices or three symmetric lower indices (see subsection 3.5.1 for more details). Building an invariant of E_6 will therefore proceed very similarly to building invariant in any other group:

1. Choose the E_6 representations and take the outer product, with all the indices different.
2. Multiply (again outer product) with an arbitrary number of d-tensor with all-upper and all-lower indices.
3. Contract indices in pairs, each consisting of an upper and a lower index. Continue contracting as long as possible; if no free indices remain, the object is an invariant.

In order to perform step 3 and obtain an invariant, the number of lower indices has to be equal to the number of upper indices, since each contraction reduces the number of upper and lower indices by the same amount (by one). The d-tensors at step 2 always involve three indices, which implies that the prerequisite for an invariant at step 1 is that the difference in the number of lower and upper indices is a multiple of 3 (either positive or negative), which can be used as a quick criterion whether a certain combination of representations is a candidate for an invariant.

Although all objects obtained in the above manner are invariant, not all of them are non-trivial (they can give zero), subject to the symmetry and antisymmetry properties of the indices involved. On can for example take the representations $351'$ and $\overline{351}$, and form the expression $\Theta^{ij}\,\overline{\Xi}_{ij}$, but this gives zero (the first factor has symmetric indices, while the second factor has antisymmetric indices), which is confirmed by looking at Table 5 of tensor products of representations and observing that the product $351' \times \overline{351}$ does not contain any singlets.

Moreover, we are interested only in invariants, which are (linearly) independent. Given a combination of representations in step 1, one can then arbitrarily increase the complexity of the contraction in the invariant by adding pairs of d-tensors (one with upper, one with lower indices) in step 2. One should note however, that contractions among the d-tensors are subject to certain relations (investigated in subsection 3.5.1) and thus often yield simple results; adding d-tensor pairs will therefore not necessarily enable finding new independent invariants. Indeed, most of the tensor products of 2 or 3 representations contain only one singlet, so one need consider only the simplest non-trivial case of adding d-tensors. In practice, we therefore always compute the number of singlets in a given tensor product, and then consider combinations of d-tensors in step 2 only as long as we have not found that number of independent invariants.

3.5.1 The invariant d-tensor

In this subsection, we look into the concept of an invariant tensor and more specifically, we gather all the details on the d-tensor in E_6.

An invariant tensor is a powerful computational tool. Suppose we have a Lie group G, which leaves a tensor T invariant. Then one can build an invariant with the help of this tensor; if the tensor T is a two index invariant tensor, i.e. $T_{ij} = T'_{ij}$, then $T_{ij}\phi^i\psi^j$ is an invariant polynomial in the entries of the representations ϕ^i and ψ^j, since

$$T_{ij}\phi^i\psi^j = T'_{ij}\phi'^i\psi'^j = T_{ij}\phi'^i\psi'^j. \tag{152}$$

An invariant tensor is called *primitive*, if it cannot be expressed as a sum of terms, constructed with products and possibly contraction of indices of lower rank primitive tensors. We also include the Kronecker Delta as primitive. More on primitive tensors and this approach to Lie algebras can be found in [64].

The group SU(n) contains the following primitive invariant tensors: $\delta^i{}_j$, $\varepsilon_{i_1 \ldots i_n}$ and $\varepsilon^{i_1 \ldots i_n}$, where the upper (lower) indices are fundamental (antifundamental). Other invariant tensors can be constructed from these. The primitive invariant tensors in the group E_6 are $\delta^i{}_j$, d_{ijk} and d^{ijk}. Since the d-tensors have three indices, the building of invariants in E6 somewhat resembles the invariant theory in SU(3), where ε_{ijk} also has three indices (which run from 1 to 3). The major differences between the constructing invariants in SU(3) and E_6 are the following:

1. The d-tensor is completely symmetric in the indices, while the Levi-Civita symbol ε in completely antisymmetric.
2. While $\varepsilon^{ijm}\varepsilon_{mkl} = \delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k$, the tensor $D^{ik}_{kl} := d^{ijm}\, d_{mkl}$ is an invariant tensor independent of Kronecker delta's. The tensor D^{ij}_{kl} is symmetric in both upper and lower indices.

There is a relation, which the d-tensor has to satisfy and allows us to reconstruct the tensor completely. The relation connects the actions of the d-tensor between two different descriptions of the fundamental representation: as a column ψ^i and as matrices L, M, N. The relation reads [59]

$$\tfrac{1}{6} d_{ijk}\psi^i\psi^j\psi^k = -\det L + \det M - \det N - \mathrm{Tr}(LMN), \tag{153}$$

where the minuses in front of $\det L$ and $\det N$ are modifications due to the changed embedding (matrices L and N have an odd number of indices of the type a and a', which change position in the embedding), and the factor $\frac{1}{6}$ turns out to fix the normalization

$$d^{ikl}d_{jkl} = 10\delta^i{}_j. \tag{154}$$

Equation (153) yields the reconstruction of the d-tensor, as well as some of its properties:

- First notice that the right hand side of equation (153) is zero, if ψ^i contains two nonzero entries or less. Namely, two nonzero entries in the 3×3 matrices L, M, N give the determinants $\det L = \det M = \det N = 0$. Also, at least one of the matrices L, M, N has to be null, which implies $LMN = 0$ and finally $\mathrm{Tr}(LMN) = 0$.
- Taking one non-zero value $\psi^i = \delta^i{}_{i_0}$, i.e. ψ^i is the i_0-th basis state, we get $\frac{1}{6} d_{i_0 i_0 i_0} = 0$ (for an arbitrary $i_0 = 1, \ldots, 27$). The d-tensor therefore has a null diagonal.
- Taking two nonzero values, i.e. $\psi^i = \delta^i{}_{i_1} \pm \delta^i{}_{i_2}$, we get

$$\tfrac{1}{6}(d_{i_1 i_1 i_1} \pm 3d_{i_1 i_1 i_2} + 3d_{i_1 i_2 i_2} \pm d_{i_2 i_2 i_2}) = 0, \tag{155}$$

$$d_{i_1 i_2 i_2} \pm d_{i_1 i_1 i_2} = 0. \tag{156}$$

We have used the complete symmetry of the d-tensor under index exchange, as well as the null diagonal property. The last equation implies that the d-tensor values, where two of the indices are equal, are zero:

$$d_{i_1 i_2 i_2} = d_{i_2 i_1 i_2} = d_{i_2 i_2 i_1} = 0. \tag{157}$$

- We now take three nonzero entries in the column: $\psi^i = \delta^i{}_{i_1} + \delta^i{}_{i_2} + \delta^i{}_{i_3}$. The complete symmetry of the d-tensor, together with same index entries being zero, give

$$\tfrac{3!}{6} d_{i_1 i_2 i_3} = -\det L + \det M - \det N - \mathrm{Tr}(LMN), \tag{158}$$
$$d_{i_1 i_2 i_3} = -\det L + \det M - \det N - \mathrm{Tr}(LMN). \tag{159}$$

The equation (159) is an explicit formula for the computation of the d-tensor entries.

Qualitatively, nonzero entries in equation (153) can potentially result only in two cases: either all three indices give values in only one of the L, M, N matrices or each of the L, M, N matrices contain exactly one of the states, referred to by the indices. The former case can lead to one of the determinants being non-zero, while the latter can lead to a nonzero trace.

Furthermore, with the normalization in equation (154), the entries of the d-tensor can be found to have only three possible numerical values: 0, 1 or -1.

Once the explicit form of the d-tensor is computed, one can also compute some further identities (listed below). The full list of the d-tensor properties is thus the following:

- The tensor d^{ijk} and d_{ijk} have the same numerical values, so they have the same properties.

- The tensor d^{ijk} is completely symmetric under the exchange of indices (unlike the Levi-Civita tensor).

- The tensor d^{ijk} gives zero, as soon as two indices have the same value (like the Levi-Civita tensor).

- The only non-zero values of the tensor d^{ijk} are 1 and -1 (like in the Levi-Civita tensor).

- It has the normalization

$$d^{ikl}\, d_{jkl} = 10\, \delta^i{}_j. \tag{160}$$

- Unlike the Levi-Civita tensor, contracting just one index gives an independent tensor, which is symmetric in the upper indices and symmetric in the lower indices:

$$d^{ijm}\, d_{klm} =: \; D^{ij}_{kl}. \tag{161}$$

- The following identities are computed to hold:

$$d_{ijk}\, d_{lmn}\, d^{ail}\, d^{bjm}\, d^{ckn} = -30\, d^{abc}, \tag{162}$$

$$d^{aij}\, d^{bkl}\, d_{cik}\, d_{djl} = -4\, D^{ab}_{cd} + 5\, \delta^a{}_c\, \delta^b{}_d + 5\, \delta^a{}_d\, \delta^b{}_c. \tag{163}$$

- Under any E_6 transformation $U^i{}_j$, the d-tensor is invariant: $d^{ijk} = U^i{}_a\, U^j{}_b\, U^k{}_c\, d^{abc}$. This means an action of any generator t^a on the d-tensor is vanishing:

$$(t^a)^i{}_l\, d^{ljk} + (t^a)^j{}_l\, d^{ilk} + (t^a)^k{}_l\, d^{ijl} = 0. \tag{164}$$

- The sum of quadratic terms of the generators t^a in the fundamental representation 27 has the following relation:

$$(t^a)^i{}_j\, (t^a)^k{}_l = \tfrac{1}{6}\, \delta^i{}_j\, \delta^k{}_l + \tfrac{1}{2}\, \delta^i{}_l\, \delta^k{}_j - \tfrac{1}{2}\, D^{ij}_{kl}. \tag{165}$$

  This equation represents the "completeness relation" of the E_6 generators in the fundamental representation, similar to the relation $(\sigma^a)^i{}_j(\sigma^a)^k{}_l = 2\delta^i{}_l\delta^k{}_j - \delta^i{}_j\delta^k{}_l$ for the Pauli matrices, which are the generators of the fundamental representation 2 of SU(2). Generalizations of the completeness relation exists also for SU(N) [65]. The E_6 is different in this regard, since the completeness relation also contains the tensor D^{ij}_{kl}.

3.5.2 Explicit computation of invariants

In this section, we list the lowest order invariants formed from the representations 27, 78, 351, 351′, 650 and their conjugates (when non-equivalent). We will therefore have a catalogue, where we can look up which invariants can be formed with a given set of representations, as well as see how to explicitly compute each of the invariants in tensor notation. The computations can then be made with a computer; the full explicit forms are of course too complicated to write down, so we will write-down only those terms, which involve Standard Model singlets only. The singlet terms will be important for computations of spontaneous symmetry breaking in our E_6 models. The labels of the singlets are the same as in subsection 3.4.6. The normalization of the singlets is such that we have the standard Kähler normalization

$$\langle \psi^i\, (\psi^*)_i \rangle = |c_1|^2 + |c_2|^2, \tag{166}$$

$$\langle \overline{\psi}_i\, (\overline{\psi}^*)^i \rangle = |d_1|^2 + |d_2|^2, \tag{167}$$

$$\langle \phi^i{}_j\, (\phi^*)_i{}^j \rangle = |u_1|^2 + |u_2|^2 + |v|^2 + |w|^2 + |y|^2, \tag{168}$$

$$\langle \Theta^{ij}\, (\Theta^*)_{ij} \rangle = |e_1|^2 + |e_2|^2 + |e_3|^2 + |e_4|^2 + |e_5|^2, \tag{169}$$

$$\langle \overline{\Theta}_{ij}\, (\overline{\Theta}^*)^{ij} \rangle = |f_1|^2 + |f_2|^2 + |f_3|^2 + |f_4|^2 + |f_5|^2, \tag{170}$$

$$\langle \Xi^{ij}\, (\Xi^*)_{ij} \rangle = |g_1|^2 + |g_2|^2 + |g_3|^2 + |g_4|^2 + |g_5|^2, \tag{171}$$

$$\langle \overline{\Xi}_{ij}\, (\overline{\Xi}^*)^{ij} \rangle = |h_1|^2 + |h_2|^2 + |h_3|^2 + |h_4|^2 + |h_5|^2, \tag{172}$$

$$\begin{aligned}\langle X^i{}_j\, (X^*)_i{}^j \rangle = {} & |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 + |x_6|^2 \\ & + |x_7|^2 + |x_8|^2 + |x_9|^2 + |x_{10}|^2 + |x_{11}|^2. \end{aligned} \tag{173}$$

We start with dimension 2 invariants, which are basically just the mass terms. These will therefore always be of the form $R \otimes \overline{R}$ for a complex representation R, or $R \otimes R$ for a real representation R, while the products of representations unrelated by complex conjugation will not yield invariants. We list all the possibilities from our given set of representations in table 11. Notice, that the singlet definitions and labels in subsection 3.4.6 were chosen, so that the quadratic invariants have as simple a form as possible: in complex representations, a singlet from the representation is always paired with the corresponding singlet in the conjugate representation, while for (pseudo)real representations, the singlets are either in a quadratic terms or are a product of a conjugate pair of singlets. In hope of of greater clarity, we use the red color to write the singlets.

The dimension 3 invariants are more complicated, since we now also use the d-tensor. We consider all the invariants with the representations of dimension below 500, so the 650 is excluded. We split these invariants into two groups:

1. The first group consists of those, where either all three representations are of different types or a single copy of each representation present multiple times suffices to produce a non-zero invariant. The invariants are listed in Table 12, with the simplest tensor expression to compute them. The explicit singlet-only terms can be found in equations (174)–(190). There are 17 of these invariants.

2. The second group consists of those invariants, where the same type of representation is present in two or three of the factors, but in an antisymmetric combination. This means we need different copies of the same type of representation to produce a non-zero invariant. These invariants, and their tensor expression, are present in Table 14. The SM singlet terms are written in equations (191)–(195), where we chose the following labels: the first copy of the representation has the usual labels, the second copy has one prime on the Standard Model singlets, and the third copy has two primes on the singlets. We see that the invariants with three different copies of the same representation are antisymmetric in any pair of factors. There are 2 invariants with a double factor and 3 invariants with a triple factor.

$$I_{27^3} = 0, \tag{174}$$

$$I_{\overline{27}^3} = 0, \tag{175}$$

$$I_{27\otimes 78\otimes\overline{27}} = \tfrac{1}{\sqrt{6}}\left(c_1 d_2 u_1 + c_2 d_1 u_2\right) - \tfrac{\sqrt{5}}{2\sqrt{6}} c_2 d_2 v - \tfrac{\sqrt{2}}{3} c_1 d_1 w - \tfrac{1}{6\sqrt{2}} c_2 d_2 w, \tag{176}$$

$$I_{351'^3} = 3\left(e_3 {e_4}^2 + e_5\left(-\sqrt{2} e_2 e_4 + e_1 e_5\right)\right), \tag{177}$$

$$I_{\overline{351'}^3} = 3\left(f_3 {f_4}^2 + f_5\left(-\sqrt{2} f_2 f_4 + f_1 f_5\right)\right), \tag{178}$$

$$I_{351'\otimes\overline{27}^2} = {d_2}^2 e_1 + \sqrt{2} d_1 d_2 e_2 + {d_1}^2 e_3, \tag{179}$$

$$I_{\overline{351'}\otimes 27^2} = {c_2}^2 f_1 + \sqrt{2} c_1 c_2 f_2 + {c_1}^2 f_3, \tag{180}$$

$$\begin{aligned} I_{351'\otimes 78\otimes\overline{351'}} = \tfrac{1}{24}\Big(&-2\sqrt{30} e_1 f_1 v - 2\sqrt{2} e_1 f_1 w + 4\sqrt{3} e_1 f_2 u_2 \\ &+ e_2\left(4\sqrt{3} f_1 u_1 - \sqrt{30} f_2 v - 5\sqrt{2} f_2 w + 4\sqrt{3} f_3 u_2\right) \\ &+ \sqrt{2}\left(4 e_4 f_4 w + 2\sqrt{3} e_5 f_4 u_1 + \sqrt{15} e_5 f_5 v + e_5 f_5 w\right) \\ &+ 4 e_3\left(\sqrt{3} f_2 u_1 - 2\sqrt{2} f_3 w\right) + 2\sqrt{6} e_4 f_5 u_2\Big), \end{aligned} \tag{181}$$

$$I_{351^2\otimes\overline{27}} = 2\sqrt{5}g_1(d_1g_2+d_2g_3), \tag{182}$$

$$I_{\overline{351}^2\otimes 27} = 2\sqrt{5}h_1(c_1h_2+c_2h_3), \tag{183}$$

$$\begin{aligned} I_{351\otimes 27\otimes 78} = \tfrac{5}{12}\Big(&2\sqrt{3}c_1\left(-\sqrt{5}g_2u_1+2g_3v+2g_5y\right)\\ &+c_2\left(\sqrt{3}g_2v-3\sqrt{5}g_2w+2\sqrt{15}g_3u_2-4\sqrt{3}g_4y\right)\Big), \end{aligned} \tag{184}$$

$$\begin{aligned} I_{\overline{351}\otimes\overline{27}\otimes 78} = \tfrac{5}{12}\Big(&2\sqrt{3}d_1\left(-\sqrt{5}h_2u_2+2h_3v+2h_5y\right)\\ &+d_2\left(\sqrt{3}h_2v-3\sqrt{5}h_2w+2\sqrt{15}h_3u_1-4\sqrt{3}h_4y\right)\Big), \end{aligned} \tag{185}$$

$$I_{351^2\otimes 351'} = \tfrac{1}{2\sqrt{2}}\left(g_1(e_4g_4-e_5g_5)\right), \tag{186}$$

$$I_{\overline{351}\otimes\overline{351'}} = \tfrac{1}{2\sqrt{2}}\left(h_1(f_4h_4-f_5h_5)\right), \tag{187}$$

$$\begin{aligned} I_{351\otimes 78\otimes\overline{351'}} = \tfrac{1}{120}\Big(&20\sqrt{3}f_1g_1u_1\\ &+5\left(\sqrt{30}f_2g_1v-3\sqrt{2}f_2g_1w-4\sqrt{3}f_3g_1u_2+2\sqrt{6}f_5g_5u_2\right)\\ &+2f_4\left(5\sqrt{6}g_4u_1-2\sqrt{5}\left(\sqrt{6}g_3y+\sqrt{6}g_5v+g_5y\right)\right)\\ &+f_5\left(g_4\left(\sqrt{30}v-15\sqrt{2}w-4\sqrt{5}y\right)-4\sqrt{30}g_2y\right)\Big), \end{aligned} \tag{188}$$

$$\begin{aligned} I_{351'\otimes 78\otimes\overline{351}} = \tfrac{1}{120}\Big(&-20\sqrt{3}e_3h_1u_1-4\sqrt{5}e_4\left(\sqrt{6}h_3y+h_5\left(\sqrt{6}v+y\right)\right)\\ &+5\left(\sqrt{30}e_2h_1v-3\sqrt{2}e_2h_1w+4\sqrt{3}e_1h_1u_2+2\sqrt{6}e_4h_4u_2\right)\\ &+e_5\left(-4\sqrt{30}h_2y+\sqrt{30}h_4v-15\sqrt{2}h_4w-4\sqrt{5}h_4y+10\sqrt{6}h_5u_1\right)\Big), \end{aligned} \tag{189}$$

$$\begin{aligned} I_{351\otimes 78\otimes\overline{351}} = \tfrac{1}{12\sqrt{2}}\Big(&+w(g_2h_2+4g_3h_3+g_4h_4+4g_5h_5)-g_1h_1\left(\sqrt{15}v+5w\right)\\ &+\sqrt{3}\left(\sqrt{5}v(g_2h_2+g_4h_4)+2u_1(g_2h_3+g_4h_5)\right)+2\sqrt{3}u_2(g_3h_2+g_5h_4)\Big). \end{aligned} \tag{190}$$

$$I_{351\otimes\overline{27}_1\otimes\overline{27}_2} = \tfrac{1}{\sqrt{2}}\left(d_1'd_2-d_1d_2'\right)g_1, \tag{191}$$

$$I_{\overline{351}\otimes 27_1\otimes 27_2} = \tfrac{1}{\sqrt{2}}\left(c_1'c_2-c_1c_2'\right)h_1, \tag{192}$$

$$\begin{aligned} I_{351_1\otimes 351_2\otimes 351_3} = \tfrac{1}{\sqrt{2}}\big(&g_1\left(-3g_2'g_3''+3g_2''g_3'+2g_4'g_5''-2g_4''g_5'\right)\\ &+g_1'\left(3g_2g_3''-3g_2''g_3-2g_4g_5''+2g_4''g_5\right)\\ &+g_1''\left(-3g_2g_3'+3g_2'g_3+2g_4g_5'-2g_4'g_5\right)\big), \end{aligned} \tag{193}$$

$$\begin{aligned} I_{\overline{351}_1\otimes\overline{351}_2\otimes\overline{351}_3} = \tfrac{1}{\sqrt{2}}\big(&h_1\left(-3h_2'h_3''+3h_2''h_3'+2h_4'h_5''-2h_4''h_5'\right)\\ &+h_1'\left(3h_2h_3''-3h_2''h_3-2h_4h_5''+2h_4''h_5\right)\\ &+h_1''\left(-3h_2h_3'+3h_2'h_3+2h_4h_5'-2h_4'h_5\right)\big), \end{aligned} \tag{194}$$

$$\begin{aligned} I_{78_1\otimes 78_2\otimes 78_3} = \tfrac{1}{12\sqrt{2}}\Big(&u_1\left(-\sqrt{15}u_2'v''+3u_2'w''+\sqrt{15}u_2''v'-3u_2''w'\right)\\ &+u_1'\left(\sqrt{15}u_2v''-3u_2w''-\sqrt{15}u_2''v+3u_2''w\right)\\ &+u_1''\left(-\sqrt{15}u_2v'+3u_2w'+\sqrt{15}u_2'v-3u_2'w\right)\Big). \end{aligned} \tag{195}$$

The invariants with representation 650 are rather complicated, so we will not give here their explicit form. As a curiosity, we only note in passing that the representation

650 is the first representation of E_6 (counting with increasing dimensions), which can give at the renormalizable level two independent invariants with the same representation factors. For example, we have two independent cubic invariants 650^3, computed by

$$I_{650^3} = \mathrm{Tr}(X\,X\,X) = X^i{}_j\, X^j{}_k\, X^k{}_i, \tag{196}$$

$$I'_{650^3} = d_{ijk}\, d^{lmn}\, X^i{}_l\, X^j{}_m\, X^k{}_n. \tag{197}$$

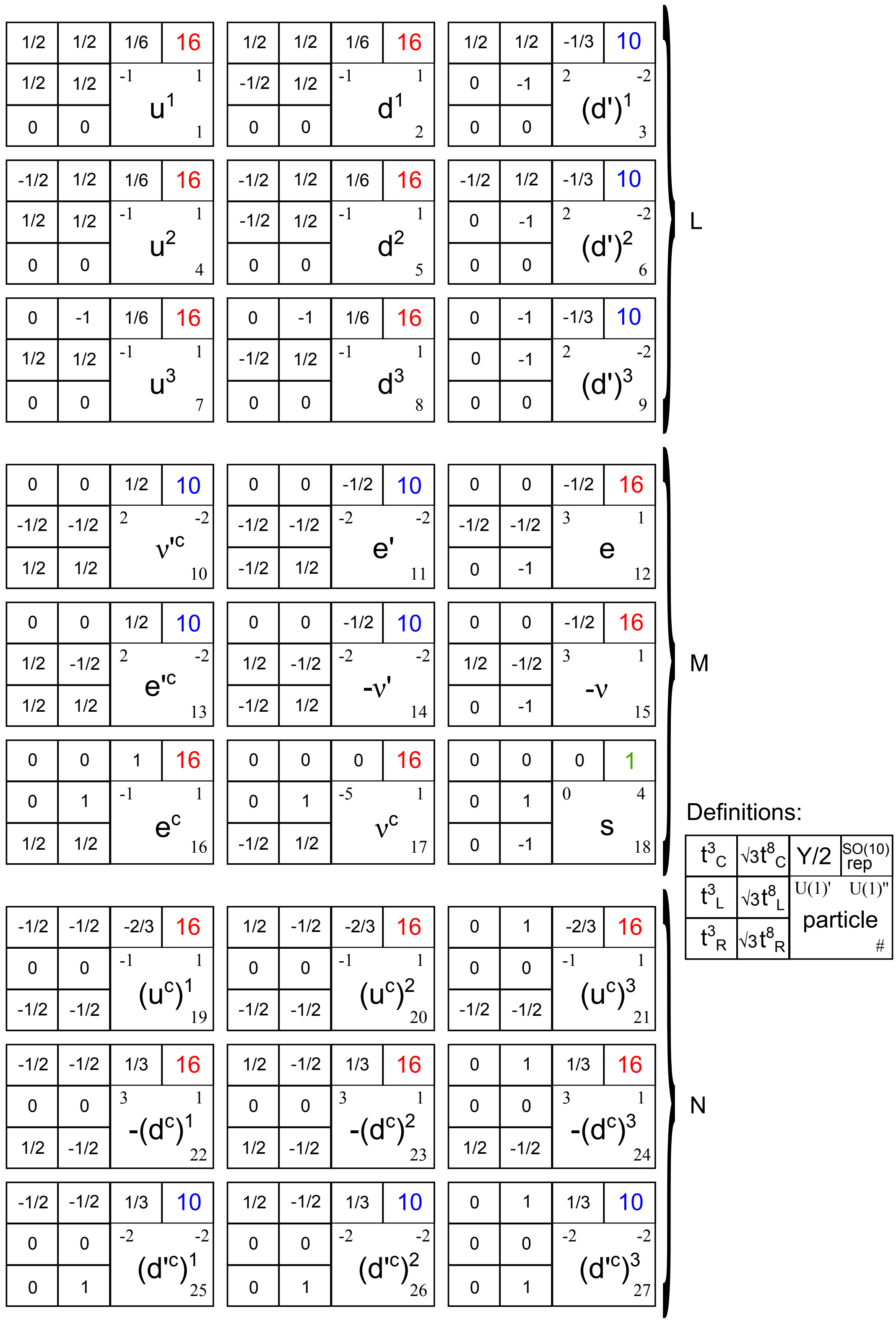

Figure 5: The quantum numbers of the states in the fundamental 27.

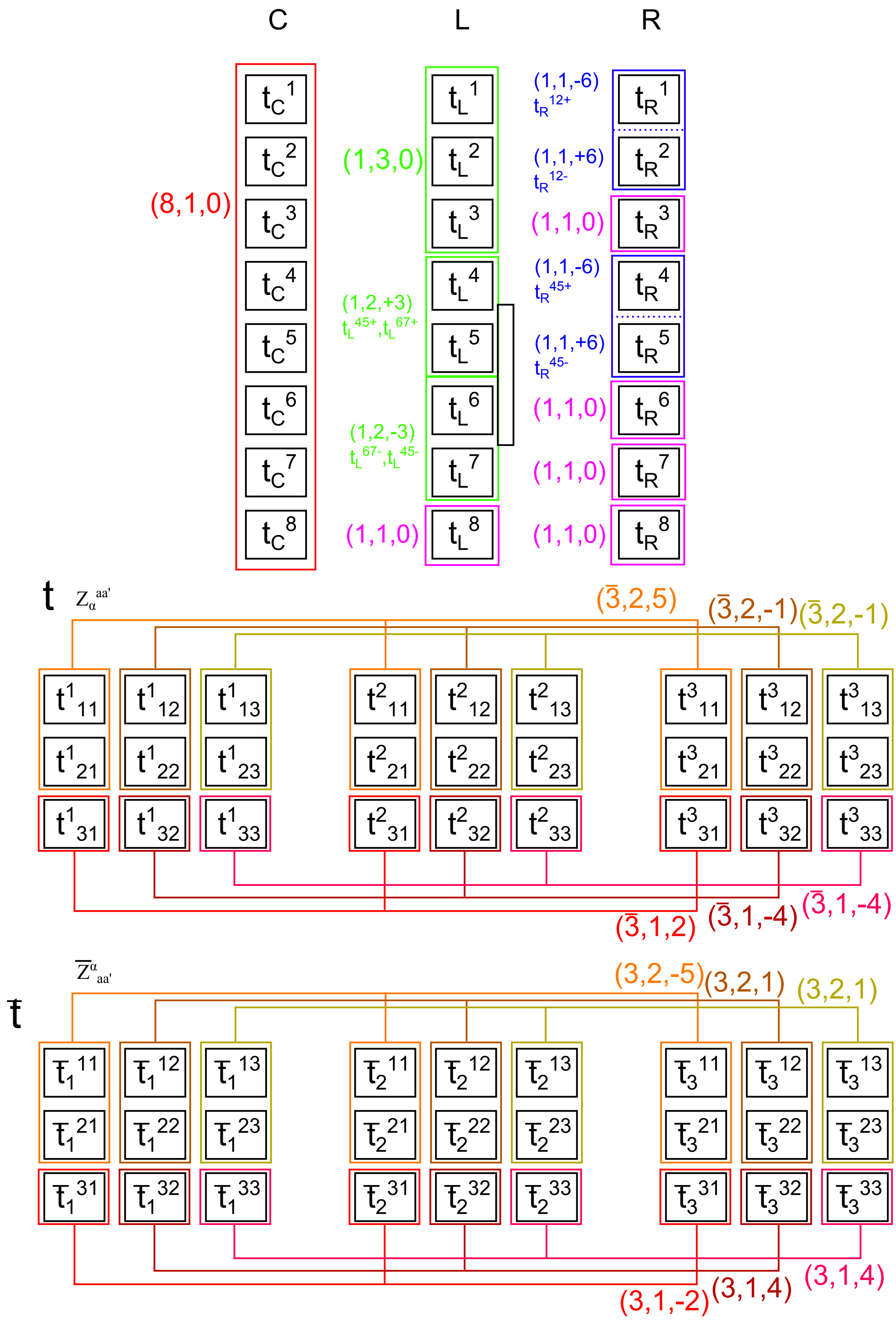

Figure 6: The quantum numbers of the **states** in the adjoint representation 78.

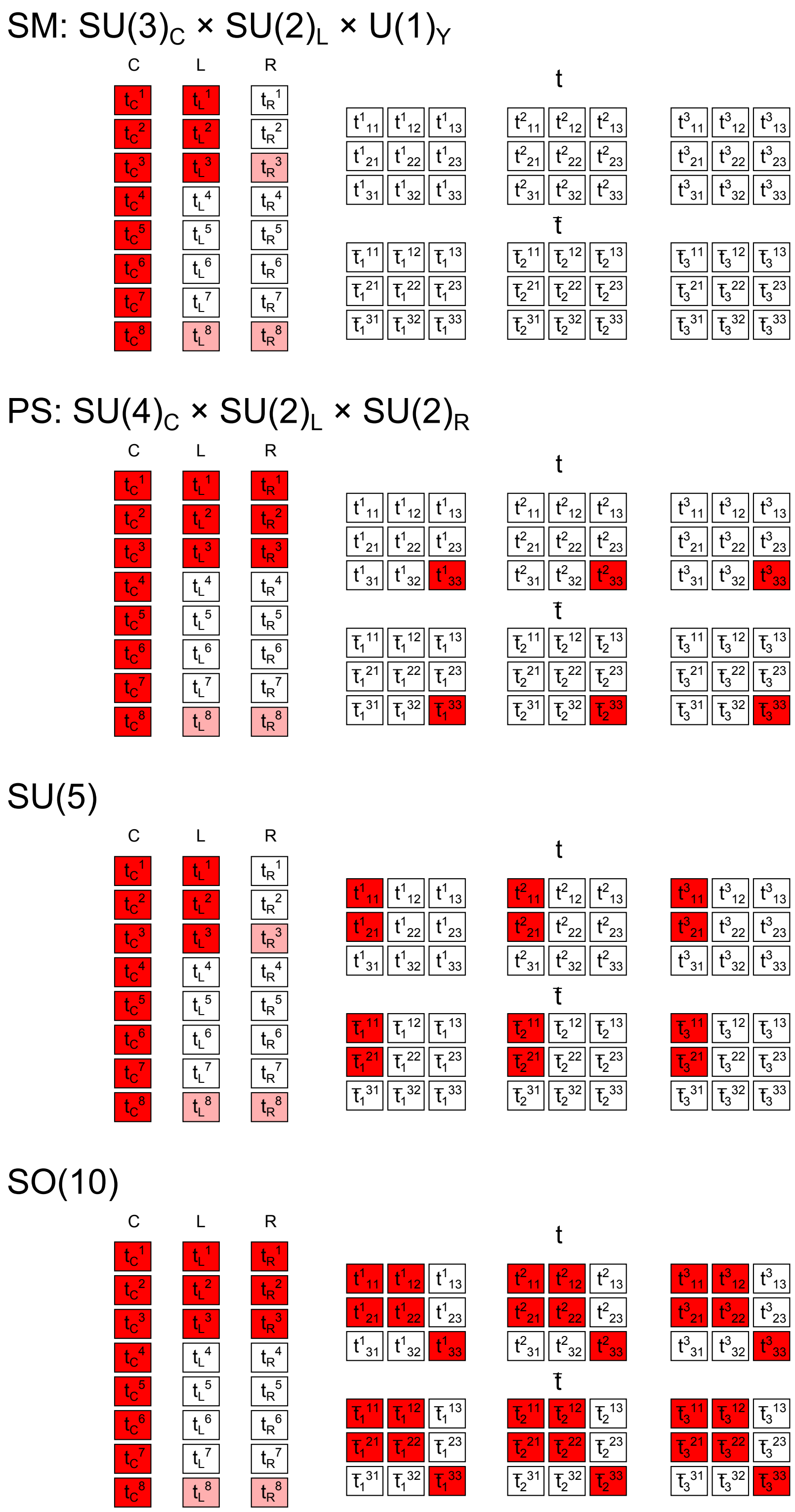

Figure 7: Standard embeddings of the Standard Model group, Pati-Salam group, SU(5) and SO(10) into E_6 at the algebra level.

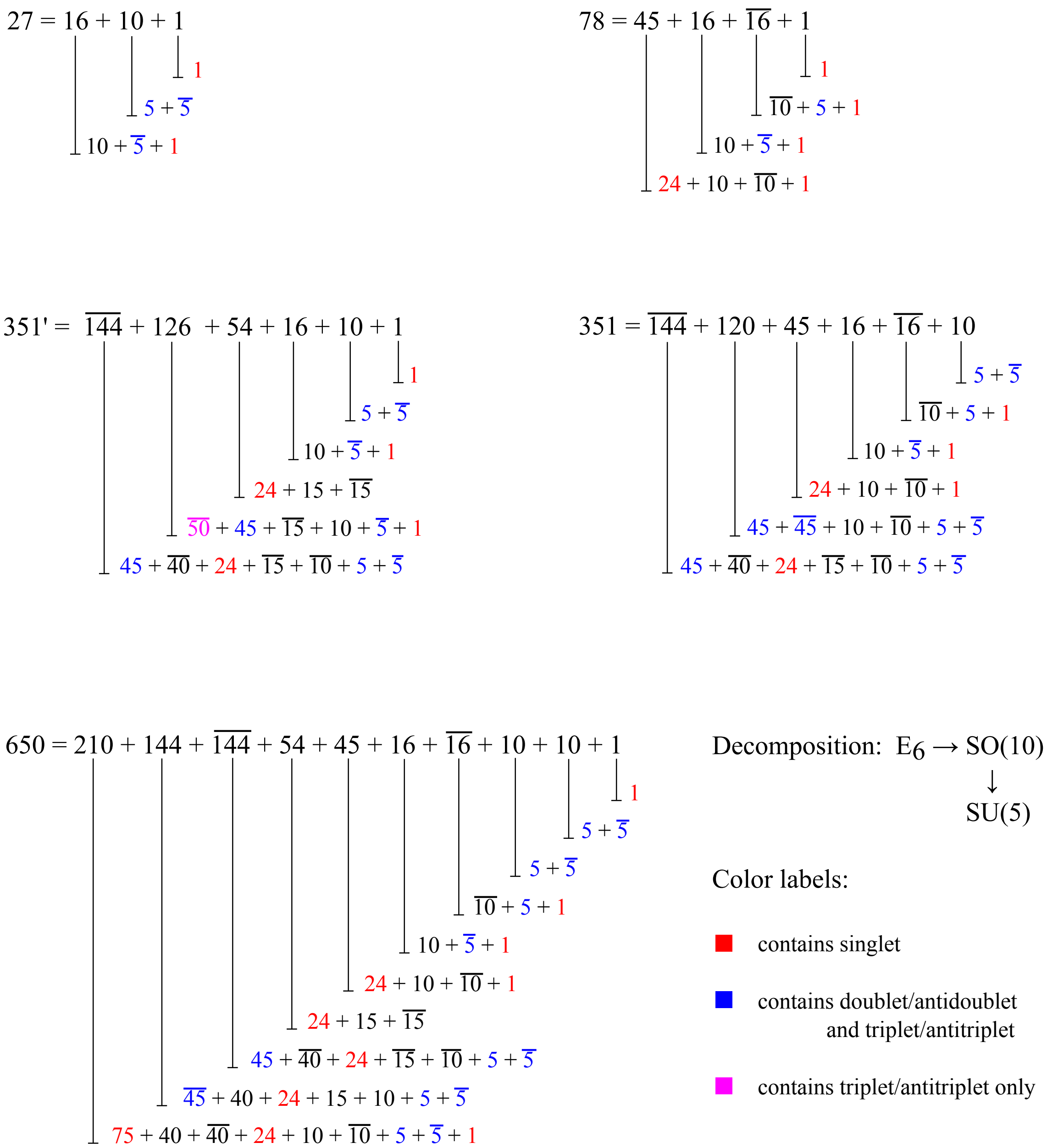

Figure 8: Counting of singlets, doublets and triplets in various representations.

Table 6: Identification in particle notation and labeling of singlets in E_6 irreducible representations of dim. < 500.

label	$E_6 \supseteq$ SO(10) $\supseteq$ SU(5)	$\supseteq$ PS	particle notation	U(1)$'$	U(1)$''$
c_1	$27 \supseteq 1 \supseteq 1$	$(1,1,1)$	s	0	+4
c_2	$27 \supseteq 16 \supseteq 1$	$(\bar{4},1,2)$	ν^c	-5	1
d_1	$\overline{27} \supseteq 1 \supseteq 1$	$(1,1,1)$	$\bar{s}$	0	-4
d_2	$\overline{27} \supseteq \overline{16} \supseteq 1$	$(4,1,2)$	$\bar{\nu}^c$	+5	-1
u_1	$78 \supseteq 16 \supseteq 1$	$(\bar{4},1,2)$	$\frac{1}{\sqrt{6}}\,(t_R^6 + i t_R^7)$	-5	-3
u_2	$78 \supseteq \overline{16} \supseteq 1$	$(4,1,2)$	$\frac{1}{\sqrt{6}}\,(t_R^6 - i t_R^7)$	+5	+3
v	$78 \supseteq 45 \supseteq 1$	/	$\frac{2}{\sqrt{30}}\,t_R^3 - \frac{1}{\sqrt{10}}\,t_L^8 - \frac{1}{\sqrt{10}}\,t_R^8$	0	0
w	$78 \supseteq 1 \supseteq 1$	$(1,1,1)$	$\frac{1}{\sqrt{6}}\,(-t_L^8 + t_R^8)$	0	0
y	$78 \supseteq 45 \supseteq 24$	/	$\frac{1}{\sqrt{15}}(\sqrt{3}\,t_R^3 + t_L^8 + t_R^8)$	0	0
e_1	$351' \supseteq 126 \supseteq 1$	$(10,1,3)$	$\nu^c\nu^c$	-10	+2
e_2	$351' \supseteq 16 \supseteq 1$	$(\bar{4},1,2)$	$\nu^c s$	-5	+5
e_3	$351' \supseteq 1 \supseteq 1$	$(1,1,1)$	$s\,s$	0	+8
e_4	$351' \supseteq 54 \supseteq 24$	$(1,1,1)$	$L'L'^c - \frac{2}{3}d'^c d'$	0	-4
e_5	$351' \supseteq \overline{144} \supseteq 24$	$(4,1,2)$	$L\,L'^c - \frac{2}{3}d^c\,d'$	+5	-1
f_1	$\overline{351'} \supseteq \overline{126} \supseteq 1$	$(\overline{10},1,3)$	$\bar{\nu}^c\bar{\nu}^c$	+10	-2
f_2	$\overline{351'} \supseteq \overline{16} \supseteq 1$	$(4,1,2)$	$\bar{\nu}^c\bar{s}$	+5	-5
f_3	$\overline{351'} \supseteq 1 \supseteq 1$	$(1,1,1)$	$\bar{s}\,\bar{s}$	0	-8
f_4	$\overline{351'} \supseteq 54 \supseteq 24$	$(1,1,1)$	$\bar{L}'\bar{L}'^c - \frac{2}{3}\bar{d}'^c\bar{d}'$	0	+4
f_5	$\overline{351'} \supseteq 144 \supseteq 24$	$(\bar{4},1,2)$	$\bar{L}\,\bar{L}'^c - \frac{2}{3}\bar{d}^c\,\bar{d}'$	-5	+1
g_1	$351 \supseteq 16 \supseteq 1$	$(\bar{4},1,2)$	$\nu^c\,s$	-5	+5
g_2	$351 \supseteq \overline{16} \supseteq 1$	$(4,1,2)$	$L\;L'^c + d^c\;d'$	+5	-1
g_3	$351 \supseteq 45 \supseteq 1$	/	$L'\,L'^c + d'^c\,d'$	0	-4
g_4	$351 \supseteq \overline{144} \supseteq 24$	$(4,1,2)$	$L\;L'^c - \frac{2}{3}d^c\;d'$	+5	-1
g_5	$351 \supseteq 45 \supseteq 24$	/	$L'\,L'^c - \frac{2}{3}d'^c\,d'$	0	-4
h_1	$\overline{351} \supseteq \overline{16} \supseteq 1$	$(4,1,2)$	$\bar{\nu}^c\,\bar{s}$	+5	-5
h_2	$\overline{351} \supseteq 16 \supseteq 1$	$(\bar{4},1,2)$	$\bar{L}\;\bar{L}'^c + \bar{d}^c\;\bar{d}'$	-5	+1
h_3	$\overline{351} \supseteq 45 \supseteq 1$	/	$\bar{L}'\,\bar{L}'^c + \bar{d}'^c\,\bar{d}'$	0	+4
h_4	$\overline{351} \supseteq 144 \supseteq 24$	$(\bar{4},1,2)$	$\bar{L}\;\bar{L}'^c - \frac{2}{3}\bar{d}^c\;\bar{d}'$	-5	+1
h_5	$\overline{351} \supseteq 45 \supseteq 24$	/	$\bar{L}'\,\bar{L}'^c - \frac{2}{3}\bar{d}'^c\,\bar{d}'$	0	+4

Table 8: Identification in particle notation and labeling of singlets in the 650 of E_6.

label	$E_6 \supseteq \mathrm{SO}(10) \supseteq \mathrm{SU}(5)$	particle notation	U(1)$'$	U(1)$''$
x_1	$650 \supseteq 1 \supseteq 1$	$5\left(Q\bar{Q}+u^c\bar{u}^c+e^c\bar{e}^c+d^c\bar{d}^c+L\bar{L}+\nu^c\bar{\nu}^c\right)$ $-4\left(d'\bar{d}'+L'^c\bar{L}'^c+d'^c\bar{d}'^c+L'\bar{L}'\right)-40s\bar{s}$	0	0
x_2	$650 \supseteq 16 \supseteq 1$	$-\left(d'^c\bar{d}^c+L'\bar{L}\right)+5\nu^c\bar{s}$	-5	-3
x_3	$650 \supseteq \overline{16} \supseteq 1$	$-\left(d^c\bar{d}'^c+L\bar{L}'\right)+5s\bar{\nu}^c$	$+5$	$+3$
x_4	$650 \supseteq 45 \supseteq 1$	$-\left(Q\bar{Q}+u^c\bar{u}^c+e^c\bar{e}^c\right)+3\left(d^c\bar{d}^c+L\bar{L}\right)$ $+4\left(d'^c\bar{d}'^c+L'\bar{L}'\right)-4\left(d'\bar{d}'+L'^c\bar{L}'^c\right)-5\nu^c\bar{\nu}^c$	$+0$	$+0$
x_5	$650 \supseteq 210 \supseteq 1$	$\left(Q\bar{Q}+u^c\bar{u}^c+e^c\bar{e}^c\right)-\left(d^c\bar{d}^c+L\bar{L}\right)-5\nu^c\bar{\nu}^c$	$+0$	$+0$
x_6	$650 \supseteq 45 \supseteq 24$	$-Q\bar{Q}+4\left(u^c\bar{u}^c-d'\bar{d}'+d'^c\bar{d}'^c\right)+3L\bar{L}$ $+6\left(L'^c\bar{L}'^c-L'\bar{L}'-e^c\bar{e}^c\right)-2d^c\bar{d}^c$	$+0$	$+0$
x_7	$650 \supseteq 54 \supseteq 24$	$-2\left(d'\bar{d}'+d'^c\bar{d}'^c\right)+3\left(L'^c\bar{L}'^c+L'\bar{L}'\right)$	$+0$	$+0$
x_8	$650 \supseteq 144 \supseteq 24$	$-2d'^c\bar{d}^c+3L'\bar{L}$	-5	-3
x_9	$650 \supseteq \overline{144} \supseteq 24$	$-2d^c\bar{d}'^c+3L\bar{L}'$	$+5$	$+3$
x_{10}	$650 \supseteq 210 \supseteq 24$	$-Q\bar{Q}+6\left(d^c\bar{d}^c-e^c\bar{e}^c\right)-9L\bar{L}+4u^c\bar{u}^c$	$+0$	$+0$
x_{11}	$650 \supseteq 210 \supseteq 75$	$-Q\bar{Q}+u^c\bar{u}^c+3e^c\bar{e}^c$	$+0$	$+0$

Table 9: Identification in particle notation and labeling of doublets and triplets in select representations of E_6.

label	$E_6 \supseteq$ SO(10) $\supseteq$ SU(5)	label	$E_6 \supseteq$ SO(10) $\supseteq$ SU(5)	doublet in p.n. triplet in p.n.
$\overline{D}_0, \overline{T}_0$	$78 \supseteq 16 \supseteq \overline{5}$	D_0, T_0	$78 \supseteq \overline{16} \supseteq 5$	$\frac{1}{\sqrt{12}}(t^6_L \pm i\, t^7_L)$ $\frac{1}{\sqrt{12}}\bar{t}_1{}^{31}, \frac{1}{\sqrt{12}}t^1{}_{31}$
D_1, T_1	$27 \supseteq 10 \supseteq 5$	$\overline{D}_1, \overline{T}_1$	$\overline{27} \supseteq 10 \supseteq \overline{5}$	L'^c d'
$\overline{D}_2, \overline{T}_2$	$27 \supseteq 10 \supseteq \overline{5}$	D_2, T_2	$\overline{27} \supseteq 10 \supseteq 5$	L' d'^c
$\overline{D}_3, \overline{T}_3$	$27 \supseteq 16 \supseteq \overline{5}$	D_3, T_3	$\overline{27} \supseteq \overline{16} \supseteq 5$	L d^c
D_4, T_4	$351' \supseteq 10 \supseteq 5$	$\overline{D}_4, \overline{T}_4$	$\overline{351'} \supseteq 10 \supseteq \overline{5}$	$Qd^c - Le^c - 4L'^c\nu^c$ $QL - u^c d^c - 4d's$
$\overline{D}_5, \overline{T}_5$	$351' \supseteq 10 \supseteq \overline{5}$	D_5, T_5	$\overline{351'} \supseteq 10 \supseteq 5$	$Qu^c - L\nu^c - 4L's$ $u^c e^c - d^c\nu^c + QQ - 4d'^c s$
$\overline{D}_6, \overline{T}_6$	$351' \supseteq 16 \supseteq \overline{5}$	D_6, T_6	$\overline{351'} \supseteq \overline{16} \supseteq 5$	$-Ls$ $-d^c s$
$\overline{D}_7, \overline{T}_7$	$351' \supseteq 126 \supseteq \overline{5}$	D_7, T_7	$\overline{351'} \supseteq \overline{126} \supseteq 5$	$-Qu^c - 3L\nu^c$ $-u^c e^c - 3d^c\nu^c - QQ$
D_8, T_8	$351' \supseteq 126 \supseteq 45$	$\overline{D}_8, \overline{T}_8$	$\overline{351'} \supseteq \overline{126} \supseteq \overline{45}$	$Qd^c + 3Le^c$ $QL + u^c d^c$
D_9, T_9	$351' \supseteq \overline{144} \supseteq 5$	$\overline{D}_9, \overline{T}_9$	$\overline{351'} \supseteq 144 \supseteq \overline{5}$	$-Qd'^c + 4L'^c\nu^c + L'e^c$ $-QL' + u^c d'^c + 4d'\nu^c$
$\overline{D}_{10}, \overline{T}_{10}$	$351' \supseteq \overline{144} \supseteq \overline{5}$	D_{10}, T_{10}	$\overline{351'} \supseteq 144 \supseteq 5$	$-L'\nu^c$ $-d'^c\nu^c$
D_{11}, T_{11}	$351' \supseteq \overline{144} \supseteq 45$	$\overline{D}_{11}, \overline{T}_{11}$	$\overline{351'} \supseteq 144 \supseteq \overline{45}$	$-dd'^c - 3e'e^c$ $-QL' - u^c d'^c$
$\overline{T}_{12}$	$351' \supseteq 126 \supseteq \overline{50}$	T_{12}	$\overline{351'} \supseteq \overline{126} \supseteq 50$	/ $2u^c e^c - QQ$

Table 11: All quadratic invariants in E$_6$ with dimensions of representations < 1000.

product	tensor notation	singlet terms
$27 \otimes \overline{27}$	$\psi^i\, \overline{\psi}_i$	$c_1 d_1 + c_2 d_2$
$78 \otimes 78$	$\phi^i{}_j\, \phi^j{}_i$	$2u_1 u_2 + v^2 + w^2 + y^2$
$351' \otimes \overline{351'}$	$\Theta^{ij}\, \overline{\Theta}_{ij}$	$e_1 f_1 + e_2 f_2 + e_3 f_3 + e_4 f_4 + e_5 f_5$
$351 \otimes \overline{351}$	$\Xi^{ij}\, \overline{\Xi}_{ij}$	$g_1 h_1 + g_2 h_2 + g_3 h_3 + g_4 h_4 + g_5 h_5$
$650 \otimes 650$	$X^i{}_j\, X^j{}_i$	${x_1}^2 + 2x_2 x_3 + {x_4}^2 + {x_5}^2 + {x_6}^2 + {x_7}^2 + 2x_8 x_9 + {x_{10}}^2 + {x_{11}}^2$

Table 12: All cubic invariants in E$_6$ with single copies of representations (dim. < 500).

product	tensor notation
$27 \otimes 27 \otimes 27$	$d_{ijk}\, \psi^i\, \psi^j\, \psi^k$
$\overline{27} \otimes \overline{27} \otimes \overline{27}$	$d^{ijk}\, \overline{\psi}_i\, \overline{\psi}_j\, \overline{\psi}_k$
$27 \otimes 78 \otimes \overline{27}$	$\overline{\psi}_i\, \phi^i{}_j\, \psi^j$
$351' \otimes 351' \otimes 351'$	$d_{ijk}\, d_{lmn}\, \Theta^{il}\, \Theta^{jm}\, \Theta^{kn}$
$\overline{351'} \otimes \overline{351'} \otimes \overline{351'}$	$d^{ijk}\, d^{lmn}\, \overline{\Theta}_{il}\, \overline{\Theta}_{jm}\, \overline{\Theta}_{kn}$
$351' \otimes \overline{27} \otimes \overline{27}$	$\Theta^{ij}\, \overline{\psi}_i\, \overline{\psi}_j$
$\overline{351'} \otimes 27 \otimes 27$	$\overline{\Theta}_{ij}\, \psi^i\, \psi^j$
$351' \otimes 78 \otimes \overline{351'}$	$\overline{\Theta}_{ij}\, \phi^j{}_k\, \Theta^{ki}$
$351 \otimes 351 \otimes \overline{27}$	$d^{iab}\, d_{akl}\, d_{bmn}\, \overline{\psi}_i\, \Xi^{km}\, \Xi^{ln}$
$\overline{351} \otimes \overline{351} \otimes 27$	$d_{iab}\, d^{akl}\, d^{bmn}\, \psi^i\, \overline{\Xi}_{km}\, \overline{\Xi}_{ln}$
$351 \otimes 27 \otimes 78$	$d^{abc}\, d_{aij}\, d_{bkl}\, \Xi^{ik}\, \psi^j\, \phi^l{}_c$
$\overline{351} \otimes \overline{27} \otimes 78$	$d_{abc}\, d^{aij}\, d^{bkl}\, \overline{\Xi}_{ik}\, \overline{\psi}_j\, \phi^c{}_l$
$351 \otimes 351 \otimes 351'$	$d_{ijk}\, d_{lmn}\, \Xi^{il}\, \Xi^{jm}\, \Theta^{kn}$
$\overline{351} \otimes \overline{351} \otimes \overline{351'}$	$d^{ijk}\, d^{lmn}\, \overline{\Xi}_{il}\, \overline{\Xi}_{jm}\, \overline{\Theta}_{kn}$
$351 \otimes 78 \otimes \overline{351'}$	$\overline{\Theta}_{ij}\, \phi^j{}_k\, \Xi^{ki}$
$351' \otimes 78 \otimes \overline{351}$	$\overline{\Xi}_{ij}\, \phi^j{}_k\, \Theta^{ki}$
$351 \otimes 78 \otimes \overline{351}$	$\overline{\Xi}_{ij}\, \phi^j{}_k\, \Xi^{ki}$

Table 14: All cubic invariants in E_6 which require multiple copies of representations (dim. < 500).

product	tensor notation
$351 \otimes \overline{27}_1 \otimes \overline{27}_2$	$\Xi^{ij}\ (\overline{\psi_1})_i\ (\overline{\psi_2})_j$
$\overline{351} \otimes 27_1 \otimes 27_2$	$\overline{\Xi}_{ij}\ (\psi_1)^i\ (\psi_2)_j$
$351_1 \otimes 351_2 \otimes 351_3$	$d^{abc}\ d_{ila}\ d_{knb}\ d_{mjc}\ \Xi^{ij}\ \Xi^{kl}\ \Xi^{mn}$
$\overline{351}_1 \otimes \overline{351}_2 \otimes \overline{351}_3$	$d_{abc}\ d^{ila}\ d^{knb}\ d^{mjc}\ \overline{\Xi}_{ij}\ \overline{\Xi}_{kl}\ \overline{\Xi}_{mn}$
$78_1 \otimes 78_2 \otimes 78_3$	$(\phi_1)^i{}_j\ (\phi_2)^j{}_k\ (\phi_3)^k{}_l$

4 Renormalizable E_6 SUSY GUT models

In this section, we gather all the E_6 model building done in this PhD thesis. We organize the material as follows: in subsection 4.1, we first look at the philosophy of our model building and determine the goals of models, which aspire to be realistic. Then, in subsection 4.2, we list a number of models which are simple, but not realistic, but they need to be at least considered when exploring the landscape of minimal E_6 models. In subsection 4.3, we then give a prototype model, which seems the simplest renormalizable SUSY model based on E_6, which gives a realistic vacuum solution; the prototype model, however, fails in an unexpected way, but suggests extension models, which are realistic. These extensions are model I and model II in subsections 4.4 and 4.5, respectively.

4.1 Preliminary considerations

4.1.1 The general setup

In this section, we shall investigate various renormalizable supersymmetric E_6 GUT models. It will turn out that the simplest models are not viable, and phenomenological acceptability puts restrictions on how simple these models can be. What we will be playing with is the types and number of representations we will include in the model, and then try to analyze the viability of the model in as general terms as possible. The analysis will always be top-down in the sense that we will be writing down the models, perform all the necessary computation, and check whether the low energy phenomenology is viable. This is the opposite of bottom-up approaches, where we write extra low energy terms, motivated by a specific phenomenon (such as the enhancement of a certain cross-section), and are sometimes not really interested in the specifics of the UV theory.

Although some literature on E_6 GUTs does exist([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]), E_6 is not very well explored compared to SU(5) and SO(10) GUTs, and, at least to the author's knowledge, no model was really investigated top-down with simultaneous consideration of both symmetry breaking and the Yukawa sectors. Our goal is therefore not just to find viable E_6 models and analyze them in more detail, but also to systematically study the landscape of models and find which are the necessary ingredients in E_6 model building.

All the models we will be considering, will have some common features, which we will now list:

- The breaking scenario, which we have in mind, is a **single stage breaking** from E_6 to the Standard Model group. Multiple stage breaking $E_6 \to G \to \mathrm{SM}$ could well take place, but in these cases one could describe the intermediate theory as an effective theory with the intermediate group G as its gauge group. Suppose the intermediate scale, where the intermediate breaking happens, is denoted by M_{i}. This scale would need to be between the EW scale M_{EW} and the GUT scale M_{GUT}. If the intermediate group is a simple group, such that we have in the intermediate stage an SU(5) or an SO(10) GUT, the scale M_{i} would need to be $\gtrsim 10^{17}\,\mathrm{GeV}$ due to proton-decay constraints, and since M_{GUT} is below the Planck scale M_{Pl}, the energy-scale window of where the intermediate effective theory is applicable is in fact very narrow, and one could roughly describe such a scenario with a single stage breaking. If, however, the intermediate scale M_{i} is closer to M_{EW}, we could have intermediate groups such as the left-right group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_{B-L}$ or the Pati-Salam group $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. Such scenarios, especially from the experimental point of view, would be better investigated with effective theories of these intermediate groups, consequently yielding these scenario of less interest to us.

- They will be **supersymmetric**, which means we will be considering the superpotential W instead of the potential V. The superpotential W contains only the scalars (from the chiral supermultiplets), has mass dimension 3, and has a local symmetry under the gauge group E_6. Due to the non-renormalization theorems, which perturbatively hold in supersymmetric theories for the superpotential (see for example [29]), we are in principle allowed to put the value of a coupling constant in front of an operator in the superpotential to zero, since the RG flow will not bring it to a non-zero value and will thus not render the symmetry breaking inconsistent. In general, we will try to avoid such simplifications and consider the most general cases of parameters.

  We will not be considering SUSY breaking, since this is an orthogonal problem to the symmetry breaking of the gauge group, and usually involves model building with an additional sector just for this purpose (there exist exceptions, see [66, 67]). It will suffice to imagine SUSY to be softly broken roughly somewhere above TeV. The E_6 models under considerations will therefore reduce to MSSM in the low energy limit.

  The reason to consider SUSY models and not non-SUSY models, beside any other appealing features of supersymmetry, is the well known fact that the MSSM coupling constants automatically unify somewhere around where M_{GUT} should be, which is not the case for the RG flow in the usual SM (without SUSY), where the couplings do not unify without additional degrees of freedom. The fact that unification occurs in the simple MSSM case also strengthens the argument for models with a single stage breaking. Also, a practical reason for considering SUSY models is their relative simplicity; this work represents, after all, the first steps into top-down E_6 model building.

- We will be considering only **renormalizable** models, which means the invariants included in the superpotential will be of order at most 3. Thus, we shall have only the (quadratic) mass terms, whose mass parameters we will label with the letter m, and the cubic invariants, whose coupling constants we will label with the letter λ.

- Since the 27 of E_6 contains the 16 of SO(10), we will include three copies of the fundamental representation 27 in our models, where the Standard Model fermions will be found (one copy per generation). We label the copies by 27^i_F, with $i = 1, 2, 3$. We shall refer to these 27's as *fermionic*. The remaining MSSM degrees of freedom are the Higgs supermultiplets H_u and H_d. If they are also part of the 27_F (perhaps they can be partly found in the 10 of SO(10), the operator $27_F\, 27_F\, 27_F$ leads to R-parity violating terms, which we would like to avoid. The problem lies in the fact that unlike in the SO(10) group, R-parity is not automatic in E_6, and we have to impose a discrete $\mathbb{Z}_2$ symmetry by hand. We shall refer to the imposed symmetry as **matter parity**, and it will take $27_F \mapsto -27_F$, and

$R \mapsto R$ for all non-fermionic representations R. This matter parity will in fact require, that the fermionic 27_F's are always in pairs, while everything else has no restrictions.

Is it possible that the fermionic 27_F contain nonzero VEVs at the GUT scale or parts of the Higgs doublets H_u or H_d and EW VEVs? Any VEV in the fermionic 27_F would imply, after insertion in a cubic term $27_F\, 27_F\, R$, a mass term of the form $27_F\, R$. That would mean that Standard Model particles, which are mass eigenstates, would also be partly found in the non-fermionic representations. The simplest solution is therefore to always have a division into the **fermionic** and **breaking sector**: the particle contents of the model will thus look like

$$\underbrace{27_F^1 \oplus 27_F^2 \oplus 27_F^3}_{\text{fermionic sector}} \oplus \underbrace{R_1 \oplus \ldots \oplus R_n}_{\text{breaking sector}}, \tag{198}$$

where the types and number of representations R_i are what we change from model to model. All the VEVs, either from GUT or EW breaking, are therefore found in the breaking sector. This means we can forget about the fermionic sector, when we are considering symmetry breakings. Indeed, taking the VEVs in the fermionic sector to be zero after them being considered in the equations of motion is equivalent to not considering them at all, since the F-terms have one derivative, but the fermionic 27_F's are always present in pairs, so plugging in zeros for their VEVs will lead to the disappearance of the fermionic terms from EOM.

The properties of the two sectors, in short:

fermionic sector Contains all the low energy MSSM fermions of our model. Has matter parity -1. Has no VEVs.

breaking sector No MSSM fermions, but contains all the VEVs: responsible for GUT and EW breaking. Has matter parity $+1$.

- The group is E_6 **anomaly free**, a well known result (see for example [68]) which we also independently checked by explicitly computing the following relation among the generators in the fundamental representation as well as the adjoint representation:

$$\mathrm{Tr}(\{t^a, t^b\}\, t^c) = 0, \tag{199}$$

where a, b, c run from 1 to 78 and $\{.,.\}$ denotes the anticommutator. Since the other representations are built from the 27 and 78 (and by complex conjugation), this relation will be inherited by the other representations, rendering E_6 anomaly free. These means the anomalies are automatically cancelled by each generation we add, so there are no restrictions for model building, and we can construct the breaking sector from any combinations of representations we wish.

Above, we have described what kind of models we will be considering. Now we list the phenomenological constraints we want to satisfy:

- The breaking sector actually needs to be able to perform the **spontaneous symmetry breaking** from E_6 to the Standard Model group. We will see this by analyzing the equations of motion, finding nontrivial solutions, and afterward compute the gauge boson masses to identify the resulting symmetry group. If this group cannot be the Standard Model, then the model is not viable.

- We will consider the Yukawa sector and see whether we can get the low-energy pattern of **fermion masses** from the Standard Model. More specifically, some crucial features are flavor mixing, the exotic states being heavy (order M_{GUT}), and the neutrinos being light ($\lesssim 1\,\mathrm{eV}$).
- The H_u and H_d of MSSM need to have masses of the order of the EW scale, while the remaining doublets and triplets are heavy. Proton decay needs to be sufficiently suppressed, however, so all the color triplets are heavy. This feature of having a light doublet and antidoublet, but the (anti)triplets and the remaining (anti)doublets heavy, is called **doublet-triplet splitting**. Although there are natural mechanisms to achieve this in special setups, we will resort to a fine-tuning of parameters in the superpotential.

We look into each of these in a bit more detail.

4.1.2 Symmetry breaking

We perform spontaneous symmetry breaking by solving the equations of motion. The EOM have to be solved for every degree of freedom, but by taking a SM ansatz, where only SM singlets can acquire a non-vanishing VEV, we need to write down only the EOM (and the terms in them) involving the VEVs. We generically label the VEVs by S.

In a non-supersymmetric theory, we would only need to minimize the potential V. The system of equations would thus be the condition for the stationary point of V:

$$\frac{\partial V}{\partial S} = 0, \tag{200}$$

where S goes over all VEVs. But the potential V has a special form in SUSY theories (see also section 2.3), namely

$$V = \sum_S |F_s|^2 + \tfrac{1}{2}\sum_a (D^a)^2. \tag{201}$$

The F-terms are defined by

$$F_S := \frac{\partial W}{\partial S}, \tag{202}$$

where S goes over all the VEVs and W is the superpotential, which is a holomorphic function of the VEVs and is formed by taking all the invariants from the matter superfields in the theory (up to renormalizable order).

The D-terms defined by

$$D^a := -g \sum_i \phi_i^\dagger\ (\hat{t}^a \phi_i), \tag{203}$$

where ϕ is a representation of the breaking sector, with the index i running over all the representations in the breaking sector, while $\hat{t}^a$ is the action of the a-th generator. We plug the VEVs into the representations ϕ_i.

Looking at equation (201), we see that due to the quadratic nature of the F and D-terms, the minimum of the potential V is at $V = 0$. We reach the minimum exactly when all F and D are zero: we get the EOM (studied in detail in [69, 70, 71])

$$F_S = 0, \tag{204}$$
$$D^a = 0. \tag{205}$$

We see that the number of F-terms equals the number of VEVs in the breaking sector, while the number of F-terms is equal to the number of generators of the E_6 group.

When a solution for the EOM is obtained (and it need not be unique), one needs a way to determine, what is the remaining symmetry group when plugging in the VEVs. This can easily be figured out by computing the masses of the gauge bosons ${A_\mu}^a$. These come from the kinetic terms of the scalar fields:

$$\sum_i (D_\mu\,\phi)^\dagger (D^\mu\,\phi), \tag{206}$$

where ϕ_i are all the scalar representations of the theory and $D_\mu := \partial_\mu - ig{A_\mu}^a \hat{t}^a$ is the covariant derivative. For the scalar representations, the VEVs obtained by solving the EOM need to be plugged-in. The gauge boson mass terms can then be written as

$$\mathcal{L}_{mass} = g^2 A_\mu^{\,a} M^{ab} A^{\mu\,b}, \tag{207}$$

where g is the E_6 gauge coupling constant, and the the mass-square matrix M^{ab} is defined as

$$M^{ab} := \sum_i (\hat{t}^a \phi_i)^\dagger (\hat{t}^b \phi_i). \tag{208}$$

The symbol $\hat{t}^a$ denotes the action of the a-th generator on the representation ϕ_i, and we sum up the contributions of all scalar representations ϕ_i with the VEVs plugged-in. The broken generators get a nonzero mass, while the unbroken generators remain massless. By figuring out which generators remain massless, one can determine the symmetry of the VEV solution.

A note on complex generators: in the decomposition of 78 under the Standard Model group, sometimes it is the complex generators which have well defined transformation properties. The usual mass term of a gauge boson corresponding to a (real) broken generator is simply

$$\frac{m^2}{2} A_\mu\, A^\mu. \tag{209}$$

Suppose now we have two gauge bosons $(A_1)_\mu$ and $(A_2)_\mu$ corresponding to real generators, where the complex combinations $A_- := \frac{1}{\sqrt{2}}(A_1 + iA_2)$ and $A_+ := \frac{1}{\sqrt{2}}(A_1 - iA_2)$ have well defined transformation properties, with each now having their own mass term (this is analogous to the $W^\pm$ in the Standard Model). Since ${A_+}^\dagger = A_-$ and ${A_-}^\dagger = A_+$, we have

$$\begin{aligned} m_+^2\ (A_+)_\mu (A_+)^{\dagger\mu} + m_-^2\ (A_-)_\mu (A_-)^{\dagger\mu} &= (m_+^2 + m_-^2)\ (A_+)_\mu (A_-)^\mu \\ &= \tfrac{m_+^2 + m_-^2}{2}\ \left((A_1)_\mu (A_1)^\mu + (A_2)_\mu (A_2)^\mu\right). \end{aligned} \tag{210}$$

The gauge bosons A_+ and A_-, or equivalently A_1 and A_2, both have the same square of the mass, namely $m_+^2 + m_-^2$. This fact was taken into consideration when computing Tables 16 and 19.

4.1.3 DT splitting

The triplets $(3, 1, -1/3)$ and antitriplets $(\overline{3}, 1, +1/3)$ mediate $D = 5$ proton decay (see section 2.4.3), so they need to stay heavy. The doublet $(1, 2, +1/2)$ need a light H_u

from the MSSM among them, and similarly the antidoublets $(1, 2, -1/2)$ will need a light MSSM H_d among them. This means we need one light doublet-antidoublet pair, while the triplets need to remain heavy.

We will be performing doublet-triplet splitting by fine-tuning of parameters. Considering the mass matrices $\mathcal{M}_{\text{doublets}}$ and $\mathcal{M}_{\text{triplets}}$, we will put a constraint on the parameters, such that one doublet will become massless, which will be a constraint on $\mathcal{M}_{\text{doublets}}$ (if there are no massless doublets already, we take the constraint to be that its determinant is zero). Let us look at a simple SU(5) example: suppose we have we have a superpotential term $m\overline{5} \cdot 5 + \lambda\,\overline{5} \cdot \langle 24 \rangle \cdot 5$, with $\langle 24 \rangle = v\,\mathrm{diag}(2, 2, 2, -3, -3)$. Written in blocks of triplets and doublets, the terms are

$$\begin{pmatrix} \overline{T} & \overline{D} \end{pmatrix} \begin{pmatrix} m + 2\lambda v\, I_{3\times 3} & 0 \\ 0 & m - 3\lambda v\, I_{2\times 2} \end{pmatrix} \begin{pmatrix} T \\ D \end{pmatrix} = (m + 2\lambda v)\,\overline{T}T + (m - 3\lambda v)\,\overline{D}D. \tag{211}$$

Taking

$$m \approx 3\lambda v, \tag{212}$$

we can make the doublet as light as we want, while the mass of the triplet becomes $\approx 5m/3$ and thus stays heavy. Although v is a solution to the EOM and itself depends on parameters of the Lagrangian, we can plug the EOM solution into equation (212) and get a pure constraint among the parameters. This specific case demonstrates the general principle, but it will be computationally more difficult in the cases we consider.

Although there exist mechanisms for DT splitting, which do not involve fine-tuning, they operate only in very specific models set up for them; since these mechanisms will not be possible in generic models, we will not depend on them, but instead rely on the less attractive fine-tuning.

4.1.4 Yukawa sector

Our models will not involve any flavor model building. We will do the most common thing in GUTs. Since the SM fermions of the i-th family are in 27^i_F, we need to look at which E_6 representations couple to two fermionic 27's. Since we have the decomposition

$$27 \otimes 27 = \underbrace{351' \oplus \overline{27}}_{\text{symmetric}} \oplus \underbrace{351}_{\text{antisymmetric}}, \tag{213}$$

we see that the SM Yukawa terms in renormalizable models can come only from the terms

$$W_{\text{Yukawa}} = \sum_{i,j} 27^i_F\, 27^j_F\, (Y^{ij}_{\overline{351'}}\, \overline{351'} + Y^{ij}_{27}\, 27 + Y^{ij}_{\overline{351}}\, \overline{351}). \tag{214}$$

The relevant representations in the breaking sector, with the help of which we can build a Yukawa sector, are therefore $\overline{351'}$, 27 and $\overline{351}$. Note that due to the what constitutes the symmetric and antisymmetric parts in the decomposition of equation (213), the Yukawa matrices $Y_{\overline{351'}}$ and Y_{27} are symmetric, while the $Y_{\overline{351}}$ is antisymmetric. In GUT, we usually build a Yukawa sector from two symmetric matrices (in SO(10) for example [32, 33]), since an antisymmetric matrix usually does not have enough parameters to fit the SM fermion masses (some attempts in different models were made in [72, 73]). Also, we need at least two Yukawa terms; if we have a single Yukawa term, we can rotate in family space with a U(3) rotation and thus diagonalize the single Yukawa matrix Y, thus eliminating any flavor mixing. Since we also need to reconstruct the CKM matrix of the SM, one Yukawa term will therefore not suffice (and the Higgses need to be present in at least two Yukawa terms). One also needs to keep in mind that multiple terms, where the fermionic 27_F's couple to the same type of representation, do not solve the lack of flavor mixing. If, for example, we have two 27's in the breaking sector, the two terms can be added together, and $Y_1\langle 27_1\rangle + Y_2\langle 27_2\rangle$ can be rewritten as a single term $Y\langle 27\rangle$, since the VEVs of the two 27's couple to the fermionic sector in exactly the same way.

It therefore seems that the best way to construct a realistic Yukawa sector is to have two symmetric Yukawa matrices for the 27 and $\overline{351'}$ (at least). Indeed, both Model I in section 4.4 and Model II in section 4.5 will follow this general pattern, but they will differ in the details substantially.

The goal in the Yukawa sections is to see whether the exotics can be heavy, and to compute masses for low-energy fermions of the SM. Writing the MSSM terms schematically as

$$W_{\text{Yukawa}} = M_U\; Qu^c\, H_u + M_D\; Qd^c\, H_d + M_E\; Le^c\, H_d + M_N\; (L\, H_u)^2/\Lambda, \tag{215}$$

with the up-type and down-type Higgses H_u and H_d getting VEVs, we have labeled the mass matrix in the up-sector by M_U, in the down-sector by M_D, in the charged lepton sector by M_E and in the neutrino sector by M_N. The Lagrangian mass terms of the fermions are then obtained by

$$-\frac{1}{2}\,\frac{\partial^2 W(\phi)}{\partial\phi_i\,\partial\phi_j}\psi_i\psi_j - \frac{1}{2}\,\frac{\partial^2 \overline{W}(\phi^\dagger)}{\partial\phi_i^\dagger\,\partial\phi_j^\dagger}\overline{\psi}_i\overline{\psi}_j, \tag{216}$$

where the sum over i and j is implied, with ϕ_i being scalar fields in the chiral supermultiplets and ψ_i the corresponding Weyl fermions in the chiral supermultiplets. We omitted the writing of spinor indices of ψ and $\overline{\psi}$. By this computation, we predict the tree-level masses of the fermions in the theory. We assume that contributions from SUSY threshold corrections are negligible. The scalar masses, however, also get contributions from SUSY-breaking soft terms (which involve new parameters, and we will not be dealing with them) — typically bringing their masses to somewhere around the SUSY breaking scale.

Although a realistic Yukawa sector is necessary for a viable model, we will still analyze symmetry breaking in models with insufficiently rich breaking sectors, since we are also interested in the needed ingredients for a successful symmetry breaking and we study it systematically. Furthermore, it is in principle possible that certain representations get only EW scale VEVs and do not need to be considered in GUT breaking.

4.2 Unsuccessful models

4.2.1 Breaking sector $n_{27}\,27\ \oplus\ n_{\overline{27}}\,\overline{27}\ \oplus n_{78}\,78$

We first consider models, where the breaking sector is constructed from an arbitrary number of copies of the fundamental 27, antifundamental $\overline{27}$ and the adjoint 78 representations of E_6. These are the simplest models one can construct. We write the contents of the breaking sector as

$$n_{27}\,27 \;\oplus\; n_{\overline{27}}\,\overline{27} \;\oplus n_{78}\,78, \tag{217}$$

where n_{27}, $n_{\overline{27}}$, n_{78} are non-negative integers and tell us the number of copies of each representation. For $n_{27} = n_{\overline{27}} = n_{78} = 1$, the result can already be found in the literature [14]: one can spontaneously break the gauge group at most to SO(10) at the renormalizable level. Here, we will generalize this result and show that we cannot spontaneously break E_6 into the Standard Model, even in the case of multiple copies. The arguments will be group-theoretical in nature.

If we use the SM ansatz, it is only the SM singlets which can acquire non-zero VEVs. Looking at Table 6, we see that representations 27 and $\overline{27}$ contain two SM singlets each, which are incidentally also SU(5) singlets. With no 78 present, all possible VEVs would be SU(5) singlets and we could thus break E_6 at most to SU(5). The presence of the 78 is therefore crucial.

Again reading from Table 6, we see the adjoint 78 has five singlets, with all but one being SU(5) singlets also. To break the group beyond SU(5) all the way to the SM group, one has to make use of the other singlet, which is a 24 under SU(5), and is denoted by y. When we have multiple copies of 78's, at least one of the y singlets will need to have a non-zero VEV, otherwise the unbroken group is SU(5). Is it possible for a y VEV to be non-zero? To answer this, we will need to consider the renormalizable invariants in the models, and then the F terms.

The only invariants at the renormalizable level, which we can construct from the representations 27, $\overline{27}$ and 78, are schematically the following (see subsection 3.5.2):

$$27\cdot\overline{27},\qquad 27\cdot 27\cdot 27,\qquad \overline{27}\cdot\overline{27}\cdot\overline{27}, \tag{218}$$

$$78\cdot 78,\qquad 27\cdot 78\cdot\overline{27},\qquad 78\cdot 78\cdot 78. \tag{219}$$

The F-terms corresponding to y-type singlets are computed by the partial derivative $\partial W/\partial y_i$, where y_i is the y-type singlet in the representation 78_i. For these terms, only invariants in equation (219) are relevant — those with at least one 78 factor. The quadratic mass terms $78\cdot 78$ contain the y-type singlet, while the other two do not:

- The invariant $27\cdot 78\cdot\overline{27}$ has no term with the singlet VEV y, as we can see from equation (176). Group-theoretically, one can obtain the same conclusion by analyzing this invariant in SU(5) language, where irreducible representations of E_6 are reducible in SU(5), with the invariant now looked at as a SU(5) invariant. The all-singlet term would need to be found as part of a product of three SU(5) irreducible representations, which contain these SM singlets. The 27 and $\overline{27}$ contain SM singlets only in a 1 of SU(5), while y is found in 24 of SU(5), so the terms we are looking for are of the form

$$1\;\otimes\;1\;\otimes\;24. \tag{220}$$

  But such terms do not contain any SU(5) singlets (we cannot form an SU(5) invariant with these factor), so the desired term with y cannot be present in the invariant.

- The invariant $78 \cdot 78 \cdot 78$ is antisymmetric in the factors; such invariants can be present only in models where $n_{78} \geq 3$. We see there are no y-type singlets in this invariant from equation (195) (no y, y' or y'' present). This can also be deduced from group-theoretic arguments. The 78 contains SM singlets in 1's and a 24 of SU(5). Possible SM singlet-only terms with at least one y-type singlet can be written in SU(5) language as

$$24 \otimes 1 \otimes 1, \tag{221}$$
$$24 \otimes 24 \otimes 1, \tag{222}$$
$$24 \otimes 24 \otimes 24. \tag{223}$$

  The first combination of representations in equation (221) cannot form an SU(5) invariant, because the tensor product is irreducible and it transforms as a 24. The double and triple product of 24's in equations 222 and (223) do contain an SU(5) invariant, but only in their symmetric part; but it is the antisymmetric part which is relevant, since the products in equations (221)–(223) come from the antisymmetric product of the 78's. The terms with y-type singlets are therefore not present in this invariant.

We conclude that the only terms, which have a presence of y-type singlets can be the mass terms for the representations 78. This means there can be no expectation values for these singlets: $\langle 24 \rangle = 0$ in SU(5) language. This conclusion can also be shown explicitly. If we label these representations by 78_i, or ϕ_i, the mass terms are then written as

$$m_{ij}\, 78_i\, 78_j = m_{ij}\, \mathrm{Tr}(\phi_i \phi_j) = m_{ij}\, y_i\, y_j + \ldots, \tag{224}$$

where we have written all the terms that contain y-type VEVs, the indices i, j run from 1 to n_{78} and summation over i and j is assumed. The fundamental and antifundamental indices of E_6 in the matrix ϕ are suppressed in this notation. The coefficients m_{ij} form a $n_{78} \times n_{78}$ matrix, which can be assumed to be symmetric, since the trace is symmetric under the exchange of ϕ_i and ϕ_j. Real symmetric matrices can be diagonalized, and in this new diagonal basis (with the new VEVs denoted by y'_i and the mass eigenvalues by m'_i), the mass terms can be written as

$$W = \sum_{i=1}^{n_{78}} m'_i y'^2_i + \ldots, \tag{225}$$

which gives the F-terms F_{y_i} to be

$$\frac{\partial}{\partial y'_i} = 2m'_i y'_i = 0. \tag{226}$$

All the masses must be nonzero, otherwise we have a massless SM singlet state, which is phenomenologically unacceptable, which means all $y'_i = 0$, and thus all $y_i = 0$ in the old basis. All the y-type VEVs are zero and the smallest possible unbroken group is therefore SU(5). We were able to prove that none of these models are viable, if we include only renormalizable terms, no matter the number of copies of 27, $\overline{27}$ and 78 we use. To find a viable model, one must therefore necessarily look include at least on higher dimensional representations, 351 or $351'$ for example.

As a small aside, we compare the obtained results in E_6 with some well known results from SU(5) GUT. Suppose we have a 5, $\overline{5}$ and 24, which are the fundamental, antifundamental and adjoint representations of SU(5), respectively, which is equivalent to the considered E_6 case. In SU(5), the 5 and $\overline{5}$ do not contain any SM singlets, so the only two invariants which we need consider for symmetry breaking are the quadratic and cubic invariant of 24. If we denote $24 = \Sigma$ the 5×5 matrix of the adjoint representation in SU(5), and the VEV by $\langle 24 \rangle = s\ \mathrm{diag}(2, 2, 2, -3, -3)$, the relevant superpotential terms would be

$$W = m_{24}\, 24 \cdot 24 + \lambda\, 24 \cdot 24 \cdot 24 + \ldots = m_{24}\, \mathrm{Tr}(\Sigma^2) + \lambda\, \mathrm{Tr}(\Sigma^3) + \ldots, \tag{227}$$

$$W = 30 m_{27}\, s^2 - 30\lambda\, s^3 + \ldots, \tag{228}$$

giving the F-term

$$F_s = \partial W/\partial s = 30 s(2m_{24} - 3s^2) = 0, \tag{229}$$

which has a non-trivial solution $s = \frac{2m_{24}}{3\lambda}$, breaking the SU(5) group into the SM group. In the case of SU(5), it is therefore possible to break to the Standard Model at the renormalizable level using only the adjoint 24. The different result from the E_6 case comes from the fact that the cubic invariant 24^3 contributes non-trivially, since it is symmetric in the factors, while the antisymmetric 78^3 in E_6 does not.

4.2.2 Breaking sector $351 \oplus \overline{351} \oplus n_{27}\, 27 \oplus n_{\overline{27}}\, \overline{27}$

In this model, we make use of the two-index antisymmetric representations 351 and $\overline{351}$. Beside these two, we can also add an arbitrary number of copies n_{27} of the representation 27, and $n_{\overline{27}}$ of the representation $\overline{27}$, so that the contents of the breaking sector can be written as

$$351 \oplus \overline{351} \oplus n_{27}\, 27 \oplus n_{\overline{27}}\, \overline{27}. \tag{230}$$

Looking at Table 6, we see that each of the representations in the pair $351 \oplus \overline{351}$ has 5 SM singlets, with three being SU(5) singlets, and two being in the 24 of SU(5). Each copy of either 27 or $\overline{27}$ contains two SM singlets, each being also SU(5) singlets. To break beyond SU(5) all the way into the Standard Model group, at least one of the four SM singlets in the 24's of SU(5) will need to acquire a nonzero VEV: at least one of the VEVs g_4, g_5, h_4 or h_5 need to be non-zero.

The different types of invariants one can form (see subsection 3.5.2) in this model are schematically written as

$$27 \cdot \overline{27}, \tag{231}$$

$$351 \cdot \overline{351}, \qquad 351 \cdot \overline{27}_i \cdot \overline{27}_j, \qquad \overline{351} \cdot 27_i \cdot 27_j. \tag{232}$$

The invariants which could potentially hold terms containing the $\langle 24 \rangle$'s are written in equation 232. The mass term $351 \cdot \overline{351}$ will have these terms, while the other two cubic terms will not. As already discussed in the previous unsuccessful model of subsection 4.2.1, one can see that most simply in the SU(5) language. Since the 27's contain VEVs which are also SU(5) singlets, and the only 24's are in the 351's, the only possible all-singlet term could come from the product $24 \otimes 1 \otimes 1$, but this does not contain the invariant.

Since the only term containing the singlet of type $\langle 24 \rangle$ are the mass terms, the superpotential has the form

$$W = m_{351}\, (g_4 h_4 + g_5 h_5) + \ldots, \tag{233}$$

which gives the F-terms

$$F_{g_4} = \tfrac{d}{dg_4} W = m_{351}\, h_4 = 0, \qquad F_{h_4} = \tfrac{d}{dh_4} W = m_{351}\, g_4 = 0, \tag{234}$$

$$F_{g_5} = \tfrac{d}{dg_5} W = m_{351}\, h_5 = 0, \qquad F_{h_5} = \tfrac{d}{dh_5} W = m_{351}\, g_5 = 0. \tag{235}$$

These imply $g_4 = h_4 = g_5 = h_5 = 0$, which means these models can break E_6 at most to SU(5). These models are therefore not viable, no matter how many copies of the 27 and $\overline{27}$ we take.

Some additional comments:

- Note that due to 351 ($\overline{351}$) being antisymmetric in the exchange of indices, the second and third invariants in equation (232) are antisymmetric in the two $\overline{27}$'s (27's). For $n_{\overline{27}} \leq 1$ or $n_{27} \leq 1$, these invariants can be trivially zero. For $n_{27}, n_{\overline{27}} > 2$, both types of invariants are present and are nonzero, but they do not contain any terms with the $\langle 24 \rangle$'s.

- Notice that we cannot construct the cubic invariants 351^3 and $\overline{351}$ with only single copies of these representations, since these two cubic invariants are antisymmetric in their factors. For these invariants to become non-trivial, we would need at least 3 different copies of the representations 351 and its conjugate. We will not consider these cases, since this greatly increases the number of degrees of freedom beyond what we actually need for the simplest viable models. These cubic invariants would contain terms with the $\langle 24 \rangle$'s though.

4.2.3 Breaking sector $351' \oplus \overline{351'}$

This model has the breaking sector $351' \oplus \overline{351'}$. Each of the representations contains five SM singlets, three of those being a 1 under $SU(5)$, and two of those a 24. At the renormalizable level, one can build the mass term and the two cubic invariants:

$$351' \cdot \overline{351'}, \qquad 351' \cdot 351' \cdot 351' \qquad \overline{351'} \cdot \overline{351'} \cdot \overline{351'}. \tag{236}$$

It turns out that this model also cannot break to the Standard Model group, but leaves invariant the Pati-Salam group $\mathrm{SU}(4)_C \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, and is thus not viable. Deducing this fact involves an analysis of the equations of motion; we were not able to find simple group-theoretic arguments to explain this, if they indeed exist. A thorough analysis of a more complicated model will be performed in subsection 4.3, in which the analysis of this model will come for free as a special case. For this reason, we postpone the computation, which is done in section 4.3.1.5.

4.2.4 Varia

We now list some further observations, which do not necessarily depend on a single model:

- The representation 650 is a real representation. Consequently, it cannot by itself break E_6 to the SM, since it cannot break the rank of the original E_6 group (similarly to how the adjoint representation alone cannot break the rank of any group). The rank would of course need to be broken from the 6 of E_6 to 4 of the Standard Model.

One can easily confirm that the model with the breaking sector 650 cannot break into the SM by observing that the generator t^8_L acts trivially on the VEVs of this representation: we compute that

$$(\hat{t}^8_L \langle X \rangle)^i{}_j = (t^8_L)^i{}_k \langle X \rangle^k{}_j - (t^{8*}_L)_j{}^k \langle X \rangle^i{}_k = 0. \tag{237}$$

The gauge boson mass matrix M^{ab} (see section 4.1.2 for details) is in the 650 model computed by $(\hat{t}^a 650)^\dagger (\hat{t}^b 650)$. Equation (237) thus implies that the gauge boson corresponding to the E_6 generator t^8_L is massless, and is thus part of the remaining symmetry into which we break. We note that t^8_L is a SM singlet, and unlike the linear combination $Y/2 = \frac{1}{\sqrt{3}}\, t^8_L + t^3_R + \frac{1}{\sqrt{3}}\, t^8_R$, it is not part of the Standard Model. This implies that we have (at least) one additional U(1) factor unbroken, and the 650 model is not viable due to symmetry breaking alone.

To have a realistic renormalizable model in its Yukawa sector, a 650 model would need to also be extended by a two representations (and their conjugates), which couple to the SM fermions in 27_F. This means we would probably need to add two of the three pairs $27 \oplus \overline{27}$, $351' \oplus \overline{351'}$ and $351 \oplus \overline{351}$. The model would be very complicated, having many invariants and the resulting EOM would likely not be solvable analytically. We will find much simpler realistic models, called model I and model II in sections 4.4 and 4.5, respectively. Beside these models being simple enough to be at least somewhat handled non-numerically, the smaller (by absolute value) β function of the running couplings also makes them less problematic. For these reasons, we shall not study models with the 650 in detail.

- We have not considered in detail "asymmetric" models, when some representation R is present in the breaking sector, while the conjugate representation $\bar{R}$ is not. In these cases, we have found that within supersymmetric models, it is sometimes hard for the unpaired representation to get a non-vanishing VEV. A more easily apparent problem, however, is the lack of the mass term $R\bar{R}$. For the model to have realistic masses, there must not be any massless fermions; in the absence of a mass term $R\bar{R}$, it is for example very hard to imagine how we could guarantee all the states in R to have non-vanishing mass. Even with terms like $R \cdot R_1 \cdot \langle R_2 \rangle$, we can at most give masses to those SM representations in R, which have their conjugate SM representations present in R_1. And considering cubic invariants R^3, even if we can manage a vacuum with $\langle R \rangle \neq 0$, only those SM representations present in R as conjugate pairs can get a mass, and we know there are parts of R, which are not-conjugate symmetric, since we are considering complex representations R. For this reason, we disfavor models with asymmetric breaking sectors.

  We can consider the asymmetric cases as special cases of symmetric models, by taking the ansatz $\langle \bar{R} \rangle = 0$ and also make all the parameters, which are in front of invariants containing $\bar{R}$, vanish.

- We have seen in subsection 4.2.1 that the $\langle 24 \rangle$ of SU(5) in the adjoint 78 is vanishing in renormalizable models, where we also have pairs of $27 \oplus \overline{27}$. This fact can further be generalized to models, where we also add pairs $351' \oplus \overline{351'}$. We can again prove that by invoking only group-theoretic arguments.

  We are considering models with $78 \;\oplus\; 27 \oplus \overline{27} \;\oplus\; 351' \oplus \overline{351'}$, where one can have an arbitrary number of copies of any representation. We claim that all

$y = \langle 24 \rangle = 0$ in the 78's. It is sufficient to show that these y's are not present in any other invariant than in the 78^2 mass term. In the absence of the $351'$, we already proved this. Adding now the $351'$ representations and looking at Tables 12 and 14, we see there is only one new type of invariant containing both the 78 and the $351'$:

$$351' \cdot 78 \cdot \overline{351'}. \tag{238}$$

Naively, one would expect y to be present in the 24^3 part in SU(5) language. But this invariant is computed by

$$\overline{\Theta}_{ij}\, \phi^j{}_k\, \Theta^{ki}, \tag{239}$$

or alternatively in matrix notation as

$$\mathrm{Tr}(\overline{\Theta}\, \phi\, \Theta). \tag{240}$$

Since the y singlet state resides in the adjoint of E_6, it can be written in matrix form as $y\,(t^Y)^i{}_j$, where t^Y is the corresponding generator of hypercharge. Next, we consider the action of the t^Y generator only on the VEVs: $\langle\Theta\rangle$. Since t^Y is a SM generator, its action on $\langle\Theta\rangle$ must therefore give zero, since the VEVs are SM singlets and do not transform under SM generators:

$$(\hat{t}^Y \Theta)^{ij} = 0 = (t^Y)^i{}_k\, \Theta^{kj} + (t^Y)^j{}_k\, \Theta^{ik}. \tag{241}$$

We omitted the VEV brackets $\langle\rangle$ in the notation. Note that $\hat{t}^Y$ (with a hat) denotes the action of the generator, while t^Y simply denotes the matrix representation of the generator in a basis. We choose a basis, in which t^Y is diagonal and therefore symmetric: $(t^Y)^T = t^Y$ componentwise. Written in matrix notation, equation (241) becomes more transparent:

$$0 = t^Y\, \Theta + \Theta\, (t^Y)^T. \tag{242}$$

Furthermore, using the symmetry of Θ, we get

$$0 = t^Y\, \Theta + \Theta^T\, (t^Y)^T = t^Y\, \Theta + (t^Y\, \Theta)^T. \tag{243}$$

The symmetric part of $t^Y\,\Theta$ (with only the SM singlet VEVs present) thus vanishes, thus making $t^Y\,\Theta$ antisymmetric, which is multiplied by the symmetric $\overline{\Theta}$ in equation (239), leaving the y term to vanish, as stated. One can check this also by computation: there is no y-term in the explicit form of this invariant in equation (181).

The analogous argument can be made also for the two-index antisymmetric representation $\Xi = 351$ and the invariant $\mathrm{Tr}(\overline{\Xi}\phi\Xi) = 351 \cdot 78 \cdot \overline{351}$: there is no y-term present in VEV only terms, because the null-action $\hat{t}^Y\Xi = 0$ yields $t^Y\Xi$ to be symmetric, while Ξ is antisymmetric. In the 351 case, we also have the invariant $27 \cdot 78 \cdot 351$, which does have an y-term present, so we cannot conclude $y = 0$ in 351 models. Also note that the y-term is present in the mixed invariants $351 \cdot 78 \cdot \overline{351'}$ and $351' \cdot 78 \cdot \overline{351}$, since for example $t^Y\Theta$ gives an antisymmetric matrix, while $\overline{\Xi}$ is also antisymmetric, exactly what is needed for the $\mathrm{Tr}(\overline{\Xi}\phi\Theta)$ to give a non-vanishing y term. All these statements can be cross-checked with the explicit computation in equations (174)–(190).

As a final point, we comment on another apparent possibility of correctly contracting the indices to form invariants of type $\overline{351}\times78\times351$ and $\overline{351'}\times78\times351'$. The use of the invariant tensor $d_{\mu\nu\lambda}$ allows us to write

$$\overline{\Xi}_{ab}\,\Phi^{i}{}_{c}\,\Xi^{jk}\,d_{ijk}\,d^{abc}, \tag{244}$$

$$\overline{\Theta}_{ab}\,\Phi^{i}{}_{c}\,\Theta^{jk}\,d_{ijk}\,d^{abc}. \tag{245}$$

These two invariants are trivially zero. Since d is symmetric under the exchange of indices, and Ξ is antisymmetric, the sums $d_{\lambda\mu\nu}\,\Xi^{\mu\nu}$ vanish trivially. The $351'$ representation Θ on the other hand has two symmetric indices, but $d_{kij}\,\Theta^{ij}=0$ is exactly the projection condition for this representation.

4.3 Almost successful prototype: $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}$

This model has a breaking sector, which consists of

$$351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}. \tag{246}$$

It turns out that this model is almost viable, and it fails solely for its inability to successfully perform doublet-triplet splitting, even by a fine tuning of parameters. In this model, multiple things will be computed, so we shall divide this subsection into the following parts:

1. Analyzing the symmetry breaking possibilities in this model.

2. Trying to perform doublet triplet splitting.

3. A discussion and summary of results.

Taking account of matter parity, we write the most general renormalizable superpotential of this model as

$$\begin{aligned} W = \quad & m_{351'}\ I_{351'\otimes\overline{351'}} + m_{27}\ I_{27\otimes\overline{27}} \\ & + \lambda_1\ I_{351'^3} + \lambda_2\ I_{\overline{351'}^3} \\ & + \lambda_3\ I_{27^2\otimes\overline{351'}} + \lambda_4\ I_{\overline{27}^2\otimes 351'} \\ & + \lambda_5\ I_{27^3} + \lambda_6\ I_{\overline{27}^3} \\ & + \sum_{i,j=1}^{3} \tfrac{1}{2}\left(Y_{27}^{ij}\ I_{27^i_F\otimes 27^j_F\otimes 27} + Y_{\overline{351'}}^{ij}\ I_{27^i_F\otimes 27^j_F\otimes\overline{351'}}\right). \end{aligned} \tag{247}$$

The last line represents the Yukawa terms, with the factor 1/2 in front for convenience.

4.3.1 Symmetry breaking

4.3.1.1 Equations of motion In supersymmetric models, the equations of motion consist of both F-terms and D-terms. We can take the Standard Model ansatz, where only SM singlets can acquire VEVs, and the equations of motion are written only with these VEVs. Furthermore, we can assume $\langle 27^i_F\rangle = 0$ due to matter parity, and thus the Yukawa sector is not involved in the breaking. The explicit form of invariants (with singlet-only terms) of the superpotential in equation (247) can be found in Table 11 and equations (174)–(195).

As we can see from Table 6, the breaking sector of the model contains $14 = 5+5+2+2$ singlets: the representation $351'$ contains five e's, the $\overline{351'}$ contains five f's, the 27 contains two c's and the $\overline{27}$ contains two d's. The F-terms take the explicit form

$$0 = \frac{\partial W}{\partial c_1} = m_{27} d_1 + \sqrt{2}\lambda_3 c_2 f_2 + 2\lambda_3 c_1 f_3, \tag{248}$$
$$0 = \frac{\partial W}{\partial d_1} = m_{27} c_1 + \sqrt{2}\lambda_4 d_2 e_2 + 2\lambda_4 d_1 e_3, \tag{249}$$
$$0 = \frac{\partial W}{\partial c_2} = m_{27} d_2 + 2\lambda_3 c_2 f_1 + \sqrt{2}\lambda_3 c_1 f_2, \tag{250}$$
$$0 = \frac{\partial W}{\partial d_2} = m_{27} c_2 + 2\lambda_4 d_2 e_1 + \sqrt{2}\lambda_4 d_1 e_2, \tag{251}$$
$$0 = \frac{\partial W}{\partial e_1} = m_{351'} f_1 + 3\lambda_1 {e_5}^2 + \lambda_4 {d_2}^2, \tag{252}$$
$$0 = \frac{\partial W}{\partial f_1} = m_{351'} e_1 + 3\lambda_2 {f_5}^2 + \lambda_3 {c_2}^2, \tag{253}$$
$$0 = \frac{\partial W}{\partial e_2} = m_{351'} f_2 - 3\sqrt{2}\lambda_1 e_4 e_5 + \sqrt{2}\lambda_4 d_1 d_2, \tag{254}$$
$$0 = \frac{\partial W}{\partial f_2} = m_{351'} e_2 - 3\sqrt{2}\lambda_2 f_4 f_5 + \sqrt{2}\lambda_3 c_1 c_2, \tag{255}$$
$$0 = \frac{\partial W}{\partial e_3} = m_{351'} f_3 + 3\lambda_1 {e_4}^2 + \lambda_4 {d_1}^2, \tag{256}$$
$$0 = \frac{\partial W}{\partial f_3} = m_{351'} e_3 + 3\lambda_2 {f_4}^2 + \lambda_3 {c_1}^2, \tag{257}$$
$$0 = \frac{\partial W}{\partial e_4} = m_{351'} f_4 + 6\lambda_1 e_3 e_4 - 3\sqrt{2}\lambda_1 e_2 e_5, \tag{258}$$
$$0 = \frac{\partial W}{\partial f_4} = m_{351'} e_4 + 6\lambda_2 f_3 f_4 - 3\sqrt{2}\lambda_2 f_2 f_5, \tag{259}$$
$$0 = \frac{\partial W}{\partial e_5} = m_{351'} f_5 + 6\lambda_1 e_1 e_5 - 3\sqrt{2}\lambda_1 e_2 e_4, \tag{260}$$
$$0 = \frac{\partial W}{\partial f_5} = m_{351'} e_5 + 6\lambda_2 f_1 f_5 - 3\sqrt{2}\lambda_2 f_2 f_4. \tag{261}$$

The D-terms have the schematic form

$$\begin{aligned} D^a = \quad & (27^\dagger)_i\, (\hat{t}^a\, 27)^i + (\overline{27}^\dagger)^i\, (\hat{t}^a\, \overline{27})_i \\ & + (351'^\dagger)_{ij}\, (\hat{t}^a\, 351')^{ij} + (\overline{351'}^\dagger)^{ij}\, (\hat{t}^a\, \overline{351'})_{ij}, \end{aligned} \tag{262}$$

where $\hat{t}^a$ is the action of the a-th generator. Since E_6 has 78 generators, there are in principle 78 D-terms, but only 5 are nontrivial. They correspond to the generators t^8_L, t^8_R, t^6_R, t^7_R and t^8_R. We label the D-terms accordingly as D^8_L, D^3_R, D^6_R, D^7_R and D^8_R.

$$0 = D^8_L = \frac{1}{\sqrt{3}}\Big(|c_1|^2 + |c_2|^2 + 2|e_1|^2 + 2|e_2|^2 + 2|e_3|^2 - |e_4|^2 - |e_5|^2$$
$$- |d_1|^2 - |d_2|^2 - 2|f_1|^2 - 2|f_2|^2 - 2|f_3|^2 + |f_4|^2 + |f_5|^2\Big), \tag{263}$$
$$0 = D^3_R = \frac{1}{2}\Big(- |c_2|^2 + |d_2|^2 - 2|e_1|^2 - |e_2|^2 + |e_5|^2 + 2|f_1|^2 + |f_2|^2 - |f_5|^2\Big), \tag{264}$$
$$0 = D^6_R = \frac{1}{2}\Big(c_2c_1^* + c_1c_2^* + \sqrt{2}e_2e_1^* + \sqrt{2}e_1e_2^* + \sqrt{2}e_3e_2^* + \sqrt{2}e_2e_3^* + e_5e_4^* + e_4e_5^*$$
$$- d_2d_1^* - d_1d_2^* - \sqrt{2}f_2f_1^* - \sqrt{2}f_1f_2^* - \sqrt{2}f_3f_2^* - \sqrt{2}f_2f_3^* - f_5f_4^* - f_4f_5^*\Big), \tag{265}$$
$$0 = D^7_R = \frac{i}{2}\Big(c_2c_1^* - c_1c_2^* - \sqrt{2}e_2e_1^* + \sqrt{2}e_1e_2^* - \sqrt{2}e_3e_2^* + \sqrt{2}e_2e_3^* - e_5e_4^* + e_4e_5^*$$
$$+ d_2d_1^* - d_1d_2^* - \sqrt{2}f_2f_1^* + \sqrt{2}f_1f_2^* - \sqrt{2}f_3f_2^* + \sqrt{2}f_2f_3^* - f_5f_4^* + f_4f_5^*\Big), \tag{266}$$
$$0 = D^8_R = \frac{1}{2\sqrt{3}}\Big(- 2|c_1|^2 + |c_2|^2 + 2|e_1|^2 - |e_2|^2 - 4|e_3|^2 + 2|e_4|^2 - |e_5|^2$$
$$+ 2|d_1|^2 - |d_2|^2 - 2|f_1|^2 + |f_2|^2 + 4|f_3|^2 - 2|f_4|^2 + |f_5|^2\Big). \tag{267}$$

We know that $D^8_L + \sqrt{3}D^3_R + D^8_R. = 0$, since this D-term corresponds to the hypercharge generator $Y/2$ of the Standard Model, which remains unbroken. Therefore, we need not consider D^{24} since it is not independent from the other D-term equations. Also, superficially D^6_R and D^7_R look like two complex equations. But the equations $D^6_R + i\, D^7_R$ and $D^6_R - i\, D^7_R$ are complex conjugates of each other, so D^6_R and D^7_R form just one independent complex equation. This is in accordance with the fact that D-terms are real equations, as opposed to the holomorphic F-terms. Some further simplification is also possible, since we can combine D^{16} and D^{19} into two other linear combinations, which are by themselves much simpler. All the 4 independent real D-term constraints can be compactly written in two real conditions D^I and D^{II}, as well as one complex condition D^{III}:

$$D^I \equiv \sqrt{3}D^8_L + 2D^3_R = |c_1|^2 - |d_1|^2 + |e_2|^2 - |f_2|^2 + 2|e_3|^2 - 2|f_3|^2 - |e_4|^2 + |f_4|^2, \tag{268}$$
$$D^{II} \equiv \qquad - 2D^3_R = |c_2|^2 - |d_2|^2 + |e_2|^2 - |f_2|^2 + 2|e_1|^2 - 2|f_1|^2 - |e_5|^2 + |f_5|^2, \tag{269}$$
$$D^{III} \equiv \quad D^6_R + iD^7_R = \quad c_1{c_2}^* - {d_1}^*d_2 + \sqrt{2}{e_1}^*e_2 - \sqrt{2}f_1{f_2}^*$$
$$+ \sqrt{2}{e_2}^*e_3 - \sqrt{2}f_2{f_3}^* + {e_4}^*e_5 - f_4{f_5}^*. \tag{270}$$

4.3.1.2 Symmetries of EOM Before plunging full-force into solving the system of equations, it will be very useful to make a few observations. Looking at the F-terms in equations (248)–(261) and the D-terms in equations (268)–(270), we can recognize two symmetries:

1. *Conjugation symmetry*: the breaking sector contains representations in complex conjugate pairs. Suppose we perform a sort of complex conjugation, where we exchanges between the representation and its conjugate, e.g. $27 \leftrightarrow \overline{27}$ and

$351' \leftrightarrow \overline{351'}$. The superpotential, however, contains invariants which do not respect this symmetry, such as the cubic invariants $351'^3$, so we need to exchange the parameters in front of the invariants. Explicitly, conjugation symmetry can be written as

$$c_i \leftrightarrow d_i, \tag{271}$$
$$e_i \leftrightarrow f_i, \tag{272}$$
$$\lambda_1 \leftrightarrow \lambda_2, \tag{273}$$
$$\lambda_3 \leftrightarrow \lambda_4, \tag{274}$$
$$\lambda_5 \leftrightarrow \lambda_6, \tag{275}$$

where $\leftrightarrow$ denotes the exchange of the quantities we have on the left- and right-hand side. Under the conjugation symmetry operation, the F-terms remain the same set of equations; in fact, it is easiest to observe that the superpotential W is invariant under the conjugation symmetry, which implies no change to the F-terms. The D-terms change according to the rules $D^I \mapsto -D^I$, $D^{II} \mapsto -D^{II}$, $D^{III} \mapsto -D^{III*}$, which is again an equivalent set of D-terms.

The conjugate symmetry observation will have an impact on how we approach the solving of the EOM. Its especially important feature is the exchange of the parameters λ in front of invariants. Without it, we could start with an ansatz $\langle 351' \rangle = \langle \overline{351'} \rangle$ and $\langle 27 \rangle = \langle \overline{27} \rangle$ (we mean $c_i = d_i$ and $e_i = f_i$ at the level of specific VEVs). Notice that it automatically solves the D-terms. But due to the exchange in λ's, this ansatz leads to a consistent set of F-terms only if the VEVs vanish or we make an exact fine-tuning $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. Taking the conjugate VEV pairs to have the same value thus forces us to take the same values for the conjugate pairs of parameters. But we are trying to avoid any *a priori* relations among parameters, especially exact fine-tunings; to avoid these relations among parameters, we thus abandon this route of a symmetric ansatz for the D-terms, but try to solve the F-terms first. The D-terms will then be solved in a non-trivial way, which is not *a priori* obvious.

2. *Alignment symmetry*: there is another symmetry in the EOM, defined by the exchanges

$$c_1 \leftrightarrow c_2, \qquad d_1 \leftrightarrow d_2, \tag{276}$$
$$e_1 \leftrightarrow e_3, \qquad f_1 \leftrightarrow f_3, \tag{277}$$
$$e_4 \leftrightarrow e_5, \qquad f_4 \leftrightarrow f_5. \tag{278}$$

In the breaking part of the superpotential W of equation (247), every invariant by itself remains unchanged under alignment symmetry, so the whole superpotential W and the EOM are unchanged as well. Furthermore, the D-terms exchange under alignment symmetry as $D^I \leftrightarrow D^{II}$ and $D^{III} \leftrightarrow D^{III*}$. The EOM system thus indeed remains unchanged.

While conjugation symmetry was similar to performing a complex conjugation operation, alignment symmetry is more tricky to understand intuitively. Somewhat superficially, what we are in fact doing is exchanging the two $\bar{5}$'s of SU(5) in the representation 27, as well as the two 1's in the 27; this naturally also has an impact on the $351'$, since it can be constructed with the help of the symmetric tensor product $27 \otimes 27$, and by extension also to the conjugate representations $\overline{27}$ and $\overline{351'}$.

This exchange of 1's changes the embedding of some subgroups of E_6, however. From the point of view of SO(10), exchange of $\bar{5}$'s represents exchange of parts of representations 16 and 10, and more importantly, also the SU(5) singlets in the 1 and 16 of SO(10); since this operation exchanges which quantity is SO(10) invariant, the alignment symmetry necessarily changes its embedding. Similarly, the standard embedding of the Pati-Salam group is also changed, since for example the Pati-Salam singlet e_3 is exchanged with the Pati-Salam nonsinglet e_1 in the $351'$. The SU(5) embedding does not change, and so the SM embedding also does not change.

We can elucidate the alignment symmetry much further by considering, how the change of embedding actually looks like at the Lie algebra level. We will see that the symmetry operation is actually a specific 90° rotation, which is also part of the E_6 group. To find this rotation, consider the fact that the SM embedding remains unchanged, which means that the rotation commutes with all the Standard Model generators. Since the commutator $[t, .]$ can be seed as the action of the generator t on the adjoint representation, the generators corresponding to the mystery rotation will be SM singlets. We know there are 5 SM singlets in the adnoint 78: t^8_L, t^3_R, t^6_R, t^7_R and t^8_R. We guess that the mystery rotation is part of a SU(2) group, denoted as $\mathrm{SU}(2)'_R$ and defined by the generators t^6_R, t^7_R and $t^3_R - \sqrt{3}t^8_R$. This $\mathrm{SU}(2)'_R$ is a subgroup of $\mathrm{SU}(3)_R$ in E_6, which rotates the second and third component in the 3 of $\mathrm{SU}(3)_R$. We confirm our guess by looking at the properties of these $\mathrm{SU}(2)'_R$ rotations. By construction, the $\mathrm{SU}(2)'_R$ rotations commute with the Standard Model generators, and in fact with the SU(5) generators as well, so the SM and SU(5) embeddings into E_6 are not changed, exactly as we wanted. But $\mathrm{SU}(2)'_R$ rotations do not commute with the standard $\mathrm{SU}(2)_R$ embedding into $\mathrm{SU}(3)_R$, so the embedding of $\mathrm{SU}(2)_R$ into EE is changed. Therefore, the $\mathrm{SU}(2)'_R$ has an impact on the left-right group, the Pati-Salam group and SO(10), since they all contain $\mathrm{SU}(2)_R$.

An $\mathrm{SU}(2)'_R$ real rotation between the second and third place of 90° essentially exchanges the second and the third place in the $\mathrm{SU}(3)_R$ triplet (up to a minus sign). The new $\mathrm{SU}(2)_R$ will therefore rotate between the first and third component of the triplet instead of the usual rotation between the first and second. The new $\mathrm{SU}(2)_R$ will thus consist of ladder operators $t^{45\pm}_R$ instead of ladder operators $t^{12\pm}_R$. To understand how the new embeddings of the Pati-Salam and SO(10) look graphically in Figures 6 and 7, we exchange generators t^1_R and t^2_R by t^4_R and t^5_R, respectively, while in each 3-by-3 block of the complex generators, the second column is exchanged with the third column.

4.3.1.3 The main branch of solutions To find a symmetry breaking solution, we follow the strategy outlined in the discussion on conjugation symmetry: we start by solving the F-terms in equations (248)-(261). That is a holomorphic system of 14 equations with 14 variables, with perhaps not all equations independent of each other, especially if we choose an ansatz with some VEVs zero; remember that we also have the D-terms, and the whole system of EOM is not overconstrained. Since the superpotential is renormalizable, the the highest order of invariants is the cubic order, which leads to the F-terms being quadratic equations. Since there is no universal way of symbolically computing solutions of a system of quadratic equations, we have to proceed according to the specifics of our system. A general strategy, however, involves finding a variable in an equation, which is present only with linear terms, and then solving the equation for that variable and eliminating it in the rest of the system. The advantage of finding variables with only linear terms lies in the fact that the linear equation can be solved uniquely and without resorting to square roots, which after insertion make the remaining equations much more complicated.

One solution, which always exists, is the trivial solution, where all the VEVs are zero and E_6 remains unbroken. Since we want to break all the way to the standard model, we would like to have as many SM singlets to have non-zero VEVs as possible. Therefore, assumptions that certain VEVs are considered to be "general" (almost always true), while assumptions that certain VEVs are vanishing are to be considered "specific".

It turns out there are two main branches of solutions: the first branch makes the general assumptions $c_1, d_1, e_5, f_5 \neq 0$, while the second branch assumes $c_2, d_2, e_4, f_4 \neq 0$. The two main branches partly overlap, since the two sets of assumptions are not mutually exclusive, so there may exist a solution, which conforms to both. We call them the *the main branches*, since their assumptions are general, and all non-special cases (which we will check later) fall in at least one of the two branches. Note that under alignment symmetry, the two branches are exchanged, since the two sets of assumptions are exchanged. We can therefore, without loss of generality, consider only the first branch; any conclusions we will draw can then be translated into analogous conclusions of the second branch via alignment symmetry.

We can now start with systematically solving the system of F-terms. We assume we are in the first branch:

$$c_1 d_1 e_5 f_5 \neq 0. \tag{279}$$

- First, we express e_1, f_1, e_3, f_3 from the terms F_{f_1}, F_{e_1}, F_{f_3} and F_{e_3}, respectively, with all the expressions being linear.

$$e_1 = \frac{-\lambda_3 {c_2}^2 - 3\lambda_2 {f_5}^2}{m_{351'}}, \qquad f_1 = \frac{-\lambda_4 {d_2}^2 - 3\lambda_1 {e_5}^2}{m_{351'}}, \tag{280}$$

$$e_3 = \frac{-\lambda_3 {c_1}^2 - 3\lambda_2 {f_4}^2}{m_{351'}}, \qquad f_3 = \frac{-\lambda_4 {d_1}^2 - 3\lambda_1 {e_4}^2}{m_{351'}}. \tag{281}$$

- We use the assumptions $c_1 \neq 0$ and $d_1 \neq 0$ to express e_2 and f_2 from F_{d_2} and F_{c_2}, respectively. The expressions are again linear, with the assumptions used so that we can place c_1 and d_1 into the denominator.

$$e_2 = -\frac{m_{351'} m_{27} c_2 - 2\lambda_3\lambda_4 d_2 {c_2}^2 - 6\lambda_2\lambda_4 d_2 {f_5}^2}{\sqrt{2} m_{351'} \lambda_4\, d_1}, \tag{282}$$

$$f_2 = -\frac{m_{351'} m_{27} d_2 - 2\lambda_3\lambda_4 c_2 d2^2 - 6\lambda_1\lambda_3 c_2 {e_5}^2}{\sqrt{2} m_{351'} \lambda_3\, c_1}. \tag{283}$$

- We express e_4 and f_4 from equations F_{e_2}, F_{f_2}, respectively. We make use of the assumptions $e_5 \neq 0$ and $f_5 \neq 0$ in the denominator.

$$e_4 = \frac{2\lambda_3\lambda_4 c_2 {d_2}^2 - m_{351'} m_{27} d_2 + 2\lambda_3\lambda_4 c_1 d_1 d_2 + 6\lambda_1\lambda_3 c_2 {e_5}^2}{6\lambda_1\lambda_3 c_1 e_5}, \tag{284}$$

$$f_4 = \frac{2\lambda_3\lambda_4 d_2 {c_2}^2 - m_{351'} m_{27} c_2 + 2\lambda_3\lambda_4 c_1 d_1 c_2 + 6\lambda_2\lambda_4 d_2 {f_5}^2}{6\lambda_2\lambda_4 d_1 f_5}. \tag{285}$$

- We express d_1 from F_{c_1}. This F-term contains two factors, out of which we can express d_1 linearly, so there are two possible solutions. We choose
$$d_1 = \frac{m_{351'} m_{27} - 2\lambda_3\lambda_4 c_2 d_2}{2\lambda_3\lambda_4 c_1}. \tag{286}$$
The alternative leads to an inconsistency, which is shown below. The alternative solution of d_1 from F_{c_1} would be
$$d_1 = \frac{m_{351'} m_{27} {d_2}^2 - 6\lambda_1\lambda_3 c_2 d_2 {e_5}^2 - 2\lambda_3\lambda_4 c_2 {d_2}^3}{2\lambda_3 c_1 \left(3\lambda_1 {e_5}^2 + \lambda_4 {d_2}^2\right)}. \tag{287}$$
Note that if the denominator in equation 287 is zero, that leads us back to the first solution of d_1. Assuming the second d_1 solution and proceeding further, we could then solve the F_{e_5} term by expressing f_5:
$$f_5 = \frac{3m_{27}\lambda_1 c_2 e_5}{m_{351'}\lambda_4 d_2}. \tag{288}$$
The F_{d_1} term then becomes
$$\frac{m_{351'} c_1 d_2 {e_5}^2 \left(m_{351'}^2\lambda_3\lambda_4 - 9m_{27}^2\lambda_1\lambda_2\right)}{3\lambda_2\left(3\lambda_1 {e_5}^2 + \lambda_4 {d_2}^2\right)\left(2\lambda_3 c_2\left(3\lambda_1 {e_5}^2 + \lambda_4 {d_2}^2\right) - m_{351'} m_{27} d_2\right)} = 0. \tag{289}$$
Since we assume no relations among the parameters, equation (289) cannot be solved: $c_1 \neq 0$ and $e_5 \neq 0$ due to the first main branch assumptions, while $d_2 = 0$ leads to an inconsistency due to an infinity in equation (288). This forces us to take the first solution of d_1.

- We express for example f_5 from any of the remaining four F-terms, such as F_{f_5}. This automatically solves also the remaining three F-terms.

Following the instructions above, we arrive at a solution, which solves all the F-terms:
$$d_1 = \frac{m_{351'} m_{27} - 2\lambda_3\lambda_4 c_2 d_2}{2\lambda_3\lambda_4 c_1}, \tag{290}$$
$$e_1 = -\frac{\lambda_3 {c_2}^2 + \frac{m_{351'}^2 (m_{351'} m_{27} - 2\lambda_3\lambda_4 c_2 d_2)^2}{108 m_{27}^2 \lambda_1^2 \lambda_2 {e_5}^2}}{m_{351'}}, \tag{291}$$
$$f_1 = -\frac{\lambda_4 {d_2}^2 + 3\lambda_1 {e_5}^2}{m_{351'}}, \tag{292}$$
$$e_2 = \frac{\lambda_3 c_1 \left(m_{27}\lambda_4 d_2 m_{351'}^3 - 2\lambda_3\lambda_4^2 c_2 {d_2}^2 m_{351'}^2 - 54 m_{27}^2\lambda_1^2\lambda_2 c_2 {e_5}^2\right)}{27\sqrt{2} m_{351'} m_{27}^2 \lambda_1^2 \lambda_2 {e_5}^2}, \tag{293}$$
$$f_2 = \frac{2\lambda_3 c_2 \left(\lambda_4 {d_2}^2 + 3\lambda_1 {e_5}^2\right) - m_{351'} m_{27} d_2}{\sqrt{2} m_{351'} \lambda_3 c_1}, \tag{294}$$
$$e_3 = \frac{\lambda_3 {c_1}^2 \left(-\frac{m_{351'}^2 \lambda_3 \lambda_4^2 {d_2}^2}{m_{27}^2 \lambda_1^2 \lambda_2 {e_5}^2} - 27\right)}{27 m_{351'}}, \tag{295}$$
$$f_3 = -\frac{m_{351'}^2 m_{27}^2 - 4m_{351'}\lambda_3\lambda_4 c_2 d_2 m_{27} + 4\lambda_3^2\lambda_4 {c_2}^2\left(\lambda_4 {d_2}^2 + 3\lambda_1 {e_5}^2\right)}{4 m_{351'}\lambda_3^2\lambda_4 {c_1}^2}, \tag{296}$$
$$e_4 = \frac{c_2 e_5}{c_1}, \tag{297}$$
$$f_4 = \frac{m_{351'}\lambda_3\lambda_4 c_1 d_2}{9 m_{27}\lambda_1\lambda_2 e_5}, \tag{298}$$
$$f_5 = \frac{m_{351'}(m_{351'} m_{27} - 2\lambda_3\lambda_4 c_2 d_2)}{18 m_{27}\lambda_1\lambda_2 e_5}. \tag{299}$$

Note that the above equations (290)–(299) are only a partial solution, since the VEVs c_1, c_2, d_2 and e_5 remain undetermined. They are determined by considering the D-terms in equations (268)–(270) with the partial solution plugged-in. Obtaining all the solutions in the first main branch would require finding all the solutions to the D-terms, which is a very complicated system of non-holomorphic polynomials. For now, we will be satisfied with finding one simple solution: assuming $c_2 = d_2 = 0$, equation D^{III} is then solved trivially, while equation D^{II} determines e_5. We get a specific solution

$$c_2 = 0, \qquad d_2 = 0, \tag{300}$$
$$e_2 = 0, \qquad f_2 = 0, \tag{301}$$
$$e_4 = 0, \qquad f_4 = 0, \tag{302}$$
$$d_1 = \frac{m_{351'}m_{27}}{2\lambda_3\lambda_4 c_1}, \tag{303}$$
$$e_1 = -\frac{m_{351'}}{6\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_1 = -\frac{m_{351'}}{6\lambda_1^{1/3}\lambda_2^{2/3}}, \tag{304}$$
$$e_3 = -\lambda_3 {c_1}^2/m_{351'}, \qquad f_3 = -\frac{m_{351'}m_{27}^2}{4\lambda_3^2\lambda_4 {c_1}^2}, \tag{305}$$
$$e_5 = \frac{m_{351'}}{3\sqrt{2}\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_5 = \frac{m_{351'}}{3\sqrt{2}\lambda_1^{1/3}\lambda_2^{2/3}}. \tag{306}$$

The only remaining term D^I becomes a polynomial condition for $|c_1|^2$:

$$\begin{aligned} 0 = {}& |m_{351'}|^4|m_{27}|^4 + 2|m_{351'}|^4|m_{27}|^2|\lambda_3|^2|c_1|^2 \\ &- 8|m_{351'}|^2|\lambda_3|^4|\lambda_4|^2|c_1|^6 - 16|\lambda_3|^6|\lambda_4|^2|c_1|^8. \end{aligned} \tag{307}$$

Note that the $|c_1|^0$ coefficient is positive, while the coefficient of the highest power $|c_1|^8$ is negative. This will ensure that the polynomial always has a positive solution for $|c_1|$, since the value of the polynomial is positive at $|c_0| = 0$, and becomes negative for large enough $|c_1|$, so it has to be zero for some intermediate value of $|c_1|$ (since polynomials are continuous functions). Knowing that a solution for $|c_1|$ will suffice, its explicit will fortunately not be needed.

We get the other main branch of solutions, if we perform the alignment symmetry operation on the ansatz for the first main branch in equations (290)–(299). We can again use alignment symmetry, to get a specific solution to the D-terms of the second branch by applying it onto the specific solution of the first branch in equations (300)–(306). Remember that this would correspond to a 90° real rotation by $\mathrm{SU}(2)'_R$, which brings the second entry of the 3 of $\mathrm{SU}(3)_R$ to the third entry. A 45° $\mathrm{SU}(2)'_R$ rotation of the original specific solution would also give a solution to the D-terms; it would correspond to the symmetric ansatz $c_1 = d_1$, $c_2 = d_2$, $e_1 = e_3$, $f_1 = f_3$, $e_4 = e_5$, $f_4 = f_5$. Notice that this last solution, which is alignment symmetric, has all VEVs nonzero and can be found in the overlap of the two main branches. All three specific solutions, which correspond to angles 0°, 45° and 90° are equivalent, since the choice of the solution merely chooses the embedding of $\mathrm{SU}(2)_R$ into $\mathrm{SU}(3)_R$. In any further calculations, we will be using the 0° solution.

4.3.1.4 Details of the specific solution We obtained a specific solution in our model in equations (300)–(306). This solution has certain vanishing VEVs:

$c_2 = d_2 = e_2 = f_2 = e_4 = f_4 = 0$, but all the other VEV's are nonvanishing for generic values of masses m and parameters λ.

The claim is that the found solution corresponds to the breaking $\mathrm{E}_6 \to \mathrm{SM}$. To see this, we compute the masses of the gauge bosons, with only the vanishing VEVs already plugged, and the non-vanishing VEVs not inserted for simplicity. We list the masses, classified by the SM representations the gauge bosons transform under, in Table 16. All the gauge bosons, except for those in the Standard Model, acquire non-zero masses.

Another important thing to consider is whether the specific solution is an isolated point. If it were not, we could have a combination of VEVs undetermined by the breaking, and therefore a flat directions in the F-terms ($F_{\tilde{s}} = \partial W/\partial \tilde{s} = 0$ trivially for some singlet mode $\tilde{s}$). This would imply a physical massless mode in the singlets (since the mass matrix is given by double derivatives of the superpotential), which is phenomenologically unacceptable. We can check the presence of the massless modes, and thus whether the solution is an isolated point, by computing the mass matrix of the SM VEV-acquiring singlets found in the breaking sector. We use the labels s_x for the singlet states, where x labels the corresponding VEV. The singlet mass matrix is a 14×14 matrix related to the states s_{c_i}, s_{d_i}, s_{e_j} and s_{f_j}, where $i = 1, 2$ and $j = 1, \ldots, 5$. The mass term in the Lagrangian can be written as

$$\frac{1}{2} \begin{pmatrix} s_{d_i} & s_{c_i} & s_{f_j} & s_{e_j} \end{pmatrix} \mathcal{M}_{\text{singlets}} \begin{pmatrix} s_{c_i} \\ s_{d_i} \\ s_{e_j} \\ s_{f_j} \end{pmatrix}, \tag{308}$$

where $\mathcal{M}_{\text{singlets}}$ is the matrix

$$\begin{pmatrix}
m_{27} & 0 & 2\lambda_4 e_3 & \sqrt{2}\lambda_4 e_2 & 0 & \sqrt{2}\lambda_4 d_2 & 2\lambda_4 d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & m_{27} & \sqrt{2}\lambda_4 e_2 & 2\lambda_4 e_1 & 2\lambda_4 d_2 & \sqrt{2}\lambda_4 d_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\lambda_3 f_3 & \sqrt{2}\lambda_3 f_2 & m_{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\lambda_3 c_2 & 2\lambda_3 c_1 & 0 & 0 \\
\sqrt{2}\lambda_3 f_2 & 2\lambda_3 f_1 & 0 & m_{27} & 0 & 0 & 0 & 0 & 0 & 2\lambda_3 c_2 & \sqrt{2}\lambda_3 c_1 & 0 & 0 & 0 \\
0 & 2\lambda_3 c_2 & 0 & 0 & m_{351'} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6\lambda_2 f_5 \\
\sqrt{2}\lambda_3 c_2 & \sqrt{2}\lambda_3 c_1 & 0 & 0 & 0 & m_{351'} & 0 & 0 & 0 & 0 & 0 & 0 & -3\sqrt{2}\lambda_2 f_5 & -3\sqrt{2}\lambda_2 f_4 \\
2\lambda_3 c_1 & 0 & 0 & 0 & 0 & 0 & m_{351'} & 0 & 0 & 0 & 0 & 0 & 6\lambda_2 f_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{351'} & 0 & 0 & -3\sqrt{2}\lambda_2 f_5 & 6\lambda_2 f_4 & 6\lambda_2 f_3 & -3\sqrt{2}\lambda_2 f_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{351'} & 6\lambda_2 f_5 & -3\sqrt{2}\lambda_2 f_4 & 0 & -3\sqrt{2}\lambda_2 f_2 & 6\lambda_2 f_1 \\
0 & 0 & 0 & 2\lambda_4 d_2 & 0 & 0 & 0 & 0 & 6\lambda_1 e_5 & m_{351'} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{2}\lambda_4 d_2 & \sqrt{2}\lambda_4 d_1 & 0 & 0 & 0 & -3\sqrt{2}\lambda_1 e_5 & -3\sqrt{2}\lambda_1 e_4 & 0 & m_{351'} & 0 & 0 & 0 \\
0 & 0 & 2\lambda_4 d_1 & 0 & 0 & 0 & 0 & 6\lambda_1 e_4 & 0 & 0 & 0 & m_{351'} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3\sqrt{2}\lambda_1 e_5 & 6\lambda_1 e_4 & 6\lambda_1 e_3 & -3\sqrt{2}\lambda_1 e_2 & 0 & 0 & 0 & m_{351'} & 0 \\
0 & 0 & 0 & 0 & 6\lambda_1 e_5 & -3\sqrt{2}\lambda_1 e_4 & 0 & -3\sqrt{2}\lambda_1 e_2 & 6\lambda_1 e_1 & 0 & 0 & 0 & 0 & m_{351'}
\end{pmatrix}. \tag{309}$$

If we plug-in the specific solution from equations (300)–(306), we can compute that the eigenspace of null vectors has dimension 4. This implies 4 massless singlet modes. Remember, however, that the adjoint 78 of E_6 contains five SM singlets, with only one of those (the hypercharge) in the Standard Model, and the remaining four corresponding to broken generators. The Higgs mechanism will then ensure that 4 massless scalar degrees of freedom will be eaten-up by the 4 singlet gauge bosons, which will acquire masses. The 4 massless modes in the matrix are thus only would-be Goldstone bosons, and there are no massless singlet states, which are physical. We conclude the obtained solution is isolated and therefore valid.

4.3.1.5 Alternative solutions We have shown that there are two main branches of solutions, which are exchanged if an alignment symmetry transformation is applied. There are of course other possible solutions, present in neither of the two branches. In this subsection, we will analyze all other solutions of this model and show that the main branch solutions are the only ones which break into the Standard Model group.

Table 16: Masses-squared for the specific solution in the prototype model of gauge bosons in SM representations using $c_2 = d_2 = e_2 = f_2 = e_4 = f_4 = 0$.

SO(10) $\supset$	SU(5) $\supset$	SM $\supset$	$(\text{mass})^2/g^2$
45	24	$(8, 1, 0)$	0
45	24	$(1, 3, 0)$	0
45	24	$(1, 1, 0)$	0
45	24	$(3, 2, +\frac{5}{6})$ $(\overline{3}, 2, -\frac{5}{6})$	$\frac{5}{6}\lvert e_5\rvert^2 + \frac{5}{6}\lvert f_5\rvert^2$
45	10 $\overline{10}$	$(3, 2, +\frac{1}{6})$ $(\overline{3}, 2, -\frac{1}{6})$	$\lvert e_1\rvert^2 + \lvert f_1\rvert^2 + \frac{1}{2}\lvert e_5\rvert^2 + \frac{1}{2}\lvert f_5\rvert^2$
45	10 $\overline{10}$	$(\overline{3}, 1, -\frac{2}{3})$ $(3, 1, +\frac{2}{3})$	$\lvert e_1\rvert^2 + \lvert f_1\rvert^2 + \frac{1}{2}\lvert e_5\rvert^2 + \frac{1}{2}\lvert f_5\rvert^2$
45	10 $\overline{10}$	$(1, 1, +1)$ $(1, 1, -1)$	$\lvert e_1\rvert^2 + \lvert f_1\rvert^2 + \frac{1}{2}\lvert e_5\rvert^2 + \frac{1}{2}\lvert f_5\rvert^2$
16 $\overline{16}$	10 $\overline{10}$	$(3, 2, +\frac{1}{6})$ $(\overline{3}, 2, -\frac{1}{6})$	$\frac{1}{2}\lvert c_1\rvert^2 + \frac{1}{2}\lvert d_1\rvert^2 + \lvert e_3\rvert^2 + \lvert f_3\rvert^2 + \frac{5}{6}\lvert e_5\rvert^2 + \frac{5}{6}\lvert f_5\rvert^2$
16 $\overline{16}$	10 $\overline{10}$	$(\overline{3}, 1, -\frac{2}{3})$ $(3, 1, +\frac{2}{3})$	$\frac{1}{2}\lvert c_1\rvert^2 + \frac{1}{2}\lvert d_1\rvert^2 + \lvert e_3\rvert^2 + \lvert f_3\rvert^2$
16 $\overline{16}$	10 $\overline{10}$	$(1, 1, +1)$ $(1, 1, -1)$	$\frac{1}{2}\lvert c_1\rvert^2 + \frac{1}{2}\lvert c_1\rvert^2 + \lvert e_3\rvert^2 + \lvert f_3\rvert^2$
16 $\overline{16}$	$\overline{5}$ 5	$(\overline{3}, 1, +\frac{1}{3})$ $(3, 1, -\frac{1}{3})$	$\frac{1}{2}\lvert c_1\rvert^2 + \frac{1}{2}\lvert d_1\rvert^2 + \lvert e_1\rvert^2 + \lvert f_1\rvert^2 +$ $+\lvert e_3\rvert^2 + \lvert f_3\rvert^2 + \frac{1}{2}\lvert e_5\rvert^2 + \frac{1}{2}\lvert f_5\rvert^2$
16 $\overline{16}$	$\overline{5}$ 5	$(1, 2, -\frac{1}{2})$ $(1, 2, +\frac{1}{2})$	$\frac{1}{2}\lvert c_1\rvert^2 + \frac{1}{2}\lvert d_1\rvert^2 + \lvert e_1\rvert^2 + \lvert f_1\rvert^2 +$ $+\lvert e_3\rvert^2 + \lvert f_3\rvert^2 + \frac{1}{2}\lvert e_5\rvert^2 + \frac{1}{2}\lvert f_5\rvert^2$
45 1	1 1	$(1, 1, 0)$ $(1, 1, 0)$	They mix: $\frac{2}{3}\Big((A+B) \pm \sqrt{(A+B)^2 - \frac{15}{4}AB}\,\Big),$ $A \equiv 4\lvert e_1\rvert^2 + 4\lvert f_1\rvert^2 + \lvert e_5\rvert^2 + \lvert f_5\rvert^2$ $B \equiv 4\lvert e_3\rvert^2 + 4\lvert f_3\rvert^2 + \lvert c_1\rvert^2 + \lvert d_1\rvert^2$
16 $\overline{16}$	1 1	$(1, 1, 0)$ $(1, 1, 0)$	They mix: $\frac{1}{2}\Big((C+D) \pm \sqrt{(C-D)^2 + 16\lvert E\rvert^2}\,\Big),$ $C \equiv \lvert c_1\rvert^2 + 2\lvert f_1\rvert^2 + 2\lvert e_3\rvert^2 + \lvert e_5\rvert^2$ $D \equiv \lvert d_1\rvert^2 + 2\lvert e_1\rvert^2 + 2\lvert f_3\rvert^2 + \lvert f_5\rvert^2$ $E \equiv e_1 {e_3}^* + {f_1}^* f_3$

We will get the alternative solutions by carefully proceeding through the assumptions of the main branches and selectively violating them, which is a bit tedious.

Remember that under the assumption $m_{351'} \neq 0$, we can always determine the VEVs e_1, f_1, e_3, f_3 via the F-terms F_{f_1}, F_{e_1}, F_{f_3} and F_{e_3}, respectively:

$$e_1 := -\frac{\lambda_3 {c_2}^2 + 3\lambda_2 {f_5}^2}{m_{351'}}, \tag{310}$$

$$f_1 := -\frac{\lambda_4 {d_2}^2 + 3\lambda_1 {e_5}^2}{m_{351'}}, \tag{311}$$

$$e_3 := -\frac{\lambda_3 {c_1}^2 + 3\lambda_2 {f_4}^2}{m_{351'}}, \tag{312}$$

$$f_3 := -\frac{\lambda_4 {d_1}^2 + 3\lambda_1 {e_4}^2}{m_{351'}}. \tag{313}$$

Case 1 The first step in the main branch was to assume $c_1, d_1 \neq 0$ so that we could determine e_2 and f_2 from the F-terms F_{d_2} and F_{c_2} respectively. The other branch has the initial assumption $c_2, d_2 \neq 0$ (determining the e_2 and f_2 VEVs from F_{d_1} and F_{c_1}, respectively). To avoid these assumptions altogether, we need to properly negate the statement that $c_1, d_1 \neq 0$ or $c_2, d_2 \neq 0$:

$$\neg\Big((c_1 \neq 0 \wedge d_1 \neq 0) \vee (c_2 \neq 0 \wedge d_2 \neq 0)\Big) \Leftrightarrow (c_1 = 0 \vee d_1 = 0) \wedge (c_2 = 0 \vee d_2 = 0). \tag{314}$$

Case 1.1 Assuming symmetrically $c_1 = c_2 = 0$, we get the following F terms:

$$F_{c_1} = m_{351'} d_1 = 0, \tag{315}$$

$$F_{c_2} = m_{351'} d_2 = 0. \tag{316}$$

We conclude $d_1 = d_2 = 0$. That means that the 27 and $\overline{27}$ have no VEVs: $\langle 27\rangle, \langle\overline{27}\rangle = 0$. Note the model then reduces to the $351' \oplus \overline{351}'$ model from section 4.2.3. The F-terms are solved by expressing the VEVs e_1, f_1, e_2, f_2, e_3, f_3 from the terms F_{f_1}, F_{e_1}, F_{f_2}, F_{e_2}, F_{f_3} and F_{e_3}, respectively. We get

$$c_1 = 0, \qquad d_1 = 0, \tag{317}$$

$$c_2 = 0, \qquad d_2 = 0, \tag{318}$$

$$e_1 = -\frac{3\lambda_2 {f_5}^2}{m_{351'}}, \qquad f_1 = -\frac{3\lambda_1 {e_5}^2}{m_{351'}}, \tag{319}$$

$$e_2 = \frac{3\sqrt{2}\lambda_2 f_4 f_5}{m_{351'}}, \qquad f_2 = \frac{3\sqrt{2}\lambda_1 e_4 e_5}{m_{351'}}, \tag{320}$$

$$e_3 = -\frac{3\lambda_2 {f_4}^2}{m_{351'}}, \qquad f_3 = -\frac{3\lambda_1 {e_4}^2}{m_{351'}}, \tag{321}$$

$$f_4 = \frac{m_{351'}^2 - 18\lambda_1\lambda_2 e_5 f_5}{18\lambda_1\lambda_2 e_4}. \tag{322}$$

The above solution solves all the F-terms in this case. Computing the masses of the gauge bosons gives already 21 massless gauge bosons, corresponding to the Pati-Salam group. Using rotations of $\mathrm{SU}(2)'_R$, we can choose a particular solution by the ansatz $e_5 = f_5 = 0$:

$$c_1 = 0, \qquad d_1 = 0, \tag{323}$$
$$c_2 = 0, \qquad d_2 = 0, \tag{324}$$
$$e_1 = 0, \qquad f_1 = 0, \tag{325}$$
$$e_2 = 0, \qquad f_2 = 0, \tag{326}$$
$$e_3 = -\frac{m_{351'}}{6\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_3 = -\frac{m_{351'}}{6\lambda_1^{1/3}\lambda_2^{2/3}}, \tag{327}$$
$$e_4 = \frac{m_{351'}}{3\sqrt{2}\,\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_4 = \frac{m_{351'}}{3\sqrt{2}\,\lambda_1^{1/3}\lambda_2^{2/3}}, \tag{328}$$
$$e_5 = 0, \qquad f_5 = 0. \tag{329}$$

From this solution, the unbroken Pati-Salam symmetry is obvious, since the only non-vanishing VEVs e_3, f_3, e_4 and f_4 are all Pati-Salam singlets under the standard Pati-Salam embedding into E_6 (see Table-6).

The ansatz (317)-(322) assumed that $e_4 \neq 0$. Alignment symmetry and conjugate symmetry can transform this condition to either $f_4 \neq$, $e_5 \neq 0$ or $f_5 \neq 0$. If none of these is true, then $e_4 = f_4 = e_5 = f_5 = 0$ and consequently all VEVs are zero.

The alternative assumptions $e_5 \neq 0$ or $f_5 \neq 0$ lead to a specific solution for the D-terms with the ansatz $e_4 = f_4 = 0$. In this alignment, the assumptions are compatible with those in Table 16, so we can easily see that with only $e_5, f_5, e_1, f_1 \neq 0$, we get 9 more massless gauge bosons over the SM: they consists of exactly the correct SM representations so that the unbroken group is Pati-Salam.

Case 1.2 Assuming asymmetrically $c_1 = d_2 = 0$, we get the following relevant F-terms:

$$F_{d_1} = 2\lambda_4 d_1 e_3 = 0, \tag{330}$$
$$F_{c_2} = 2\lambda_3 c_2 f_1 = 0. \tag{331}$$

If $d_1 = 0$ or $c_2 = 0$, it reduces to Case 1.1, so we conclude $e_3 = f_1 = 0$. Plugging this into the F-terms and solving further, we uniquely obtain all VEVs to be zero —

$$0 = \langle 27 \rangle = \langle \overline{27} \rangle = \langle 351' \rangle = \langle \overline{351'} \rangle, \tag{332}$$

so no breaking occurs.

Case 2 In case 1 the initial assumptions of nonzero c_1, d_1 or c_2, d_2 in the two main branches were bypassed. In case 2, we accept the first assumption of the branches and bypass the second assumption on e_5, f_5 or e_4, f_4.

Let us assume that $c_1 \neq 0$ and $d_1 \neq 0$, so that we proceed in accordance with the first main branch. There is no loss of generality, since the assumptions $c_2 \neq 0$ and $d_2 \neq 0$ would instead lead us to the second branch, which is equivalent due to alignment symmetry. We can determine e_2 and f_2 from the terms F_{d_2} and F_{c_2}, respectively:

$$e_2 := -\frac{-2\lambda_3\lambda_4 d_2 {c_2}^2 + m_{351'}m_{27}c_2 - 6\lambda_2\lambda_4 d_2 {f_5}^2}{\sqrt{2}m_{351'}\lambda_4 d_1}, \tag{333}$$

$$f_2 := -\frac{-2\lambda_3\lambda_4 c_2 {d_2}^2 + m_{351'}m_{27}d_2 - 6\lambda_1\lambda_3 c_2 {e_5}^2}{\sqrt{2}m_{351'}\lambda_3 c_1}. \tag{334}$$

The only remaining assumptions of the first main branch are $e_5 \neq 0$ and $f_5 \neq 0$. We violate them by putting

$$e_5 = 0. \tag{335}$$

Note that the other possible assumption $f_5 = 0$ would lead to an analogous analysis due to the conjugation symmetry of the equations of motion.

With the assumption $e_5 = 0$, we get the following equation for F_{e_2}:

$$F_{e_2} = 0 = \frac{d_2}{\sqrt{2}\lambda_3 c_1}\big(2\lambda_3\lambda_4(c_1 d_1 + c_2 d_2) - m_{351'}m_{27}\big). \tag{336}$$

We now proceed to systematically find all possible solutions that satisfy equation (336): either the first or the second factor have to be zero.

Case 2.1 Suppose we solve equation (336) in the simplest possible manner by assuming

$$d_2 = 0. \tag{337}$$

We can then determine f_5 from equation F_{e_5}

$$f_5 = -\frac{3m_{27}\lambda_1 c_2 e_4}{m_{351'}\lambda_4 d_1}. \tag{338}$$

This leads to the following equation for F_{f_2}:

$$F_{f_2} = 0 = \frac{c_2}{\sqrt{2}m_{351'}\lambda_4 d_1}\left(-m_{27}m_{351'}^2 + 2\lambda_3\lambda_4 c_1 d_1 m_{351'} + 18 m_{27}\lambda_1\lambda_2 e_4 f_4\right). \tag{339}$$

We are again forced to check multiple possibilities.

Case 2.1.1 Assume that the first factor in equation (339) is zero:

$$c_2 = 0. \tag{340}$$

It is then possible to determine e_4^2 and f_4^2 from F_{d_1} and F_{d_2} respectively. The VEVs e_4 and e_5 are then

$$e_4 = \pm\sqrt{\frac{d_1}{6\lambda_1\lambda_3 c_1}(m_{351'}m_{27} - 2\lambda_3\lambda_4 c_1 d_1)}, \tag{341}$$

$$f_4 = \pm\sqrt{\frac{c_1}{6\lambda_2\lambda_4 d_1}(m_{351'}m_{27} - 2\lambda_3\lambda_4 c_1 d_1)}. \tag{342}$$

The remaining two F-terms F_{e_4} and F_{f_4} become

$$F_{e_4} = 0 = \frac{m_{351'}\sqrt{\lambda_3\lambda_4} - 3m_{27}\sqrt{\lambda_1\lambda_2}}{\lambda_4\sqrt{6\lambda_2\lambda_3}}\sqrt{\frac{c_1}{d_1}}\sqrt{m_{351'}m_{27} - 2\lambda_3\lambda_4 c_1 d_1}, \tag{343}$$

$$F_{f_4} = 0 = \frac{m_{351'}\sqrt{\lambda_3\lambda_4} - 3m_{27}\sqrt{\lambda_1\lambda_2}}{\lambda_3\sqrt{6\lambda_1\lambda_4}}\sqrt{\frac{d_1}{c_1}}\sqrt{m_{351'}m_{27} - 2\lambda_3\lambda_4 c_1 d_1}. \tag{344}$$

Assuming no special relations among superpotential parameters, they can only be solved by the condition

$$\sqrt{m_{351'}m_{27} - 2\lambda_3\lambda_4 c_1 d_1} = 0. \tag{345}$$

This condition implies $e_4 = f_4 = 0$, which we see from equations (341) and (342). Also, $e_5 = 0$ due to the assumption of case 2. Furthermore, the current case 2.1.1 assumes $c_2 = 0$. The VEV f_5 was determined in equation (338) of case 2.1 to be proportional to c_2, so $f_5 = 0$. Since all VEVs $\langle 24\rangle$ are now zero, the unbroken group will be contain SU(5), so potential solutions of this case are not of interest.

Case 2.1.2 Assume that the second factor in equation (339) is zero. We can then determine d_1 to be

$$d_1 = \frac{m_{27}\left(m_{351'}^2 - 18\lambda_1\lambda_2 e_4 f_4\right)}{2m_{351'}\lambda_3\lambda_4 c_1}. \tag{346}$$

The F_{c_1} term then takes the form

$$F_{c_1} = 0 = -\frac{3\lambda_1 e_4\left(\left(2\lambda_3^2\lambda_4 {c_1}^2 e_4 - 3m_{27}^2\lambda_2 f_4\right)m_{351'}^2 + 54m_{27}^2\lambda_1\lambda_2^2 e_4 {f_4}^2\right)}{m_{351'}^3\lambda_3\lambda_4 c_1}. \tag{347}$$

This term can again be solved in multiple ways.

Case 2.1.2.1 Assume that the first factor in equation (347) is zero. Then

$$e_4 = 0, \tag{348}$$

and due to equation (338) also

$$f_5 = 0. \tag{349}$$

We already know that $e_5 = 0$, since this is the assumption of all cases under case 2. Also, $f_4 = 0$, which can be computed from the F_{e_4} term:

$$F_{e_4} = 0 = m_{351'} f_4. \tag{350}$$

We again have $\langle 24\rangle = 0$ in the SU(5) language, so this branch of (potential) solutions leaves at least the group SU(5) unbroken.

Case 2.1.2.2 Assume that the second factor in equation (347) is zero. We can then determine e_4 to be

$$e_4 = \frac{3m_{351'}^2 m_{27}^2\lambda_2 f_4}{2m_{351'}^2\lambda_3^2\lambda_4 {c_1}^2 + 54m_{27}^2\lambda_1\lambda_2^2 {f_4}^2}. \tag{351}$$

The F_{d_1} term becomes

$$F_{d_1} = 0 = \frac{3m_{27}\lambda_2\left(9m_{27}^2\lambda_1\lambda_2 - m_{351'}^2\lambda_3\lambda_4\right)}{m_{351'}^2\lambda_3^2\lambda_4 c_1{}^2 + 27m_{27}^2\lambda_1\lambda_2^2 f_4{}^2}\, c_1 f_4{}^2. \tag{352}$$

Since $c_1 \neq 0$ by assumption, we must have $f_4 = 0$. Then $e_4 = 0$ from equation (351) and consequently $f_5 = 0$ from equation (338). We also have e_5 as an assumption in all of case 2. Since all $\langle 24 \rangle = 0$, the group SU(5) remains unbroken.

One possible loophole is the above argument is the possibility, where the denominator in equation (351) is zero. That implies

$$f_4 := \pm\frac{m_{351'}\lambda_3 c_1}{3m_{27}\lambda_2}\sqrt{-\frac{\lambda_4}{3\lambda_1}}. \tag{353}$$

For F_{c_1} we get

$$F_{c_1} = 0 = \pm m_{27} e_4 \sqrt{\frac{-3\lambda_1}{\lambda_4}}. \tag{354}$$

Therefore $e_4 = 0$. With that equation F_{d_1} becomes

$$F_{d_1} = 0 = \frac{m_{351'}^2\lambda_3\lambda_4}{9m_{27}\lambda_1\lambda_2}\, c_1. \tag{355}$$

This equation cannot be solved for $c_1 \neq 0$, which we assumed in case 2. The loophole is therefore closed.

Case 2.2 We return all the way back to equation (336), the last equation of case 2. Instead of taking the first factor to be zero, as in case 2.1, we take the second factor to be zero and determine the VEV d_1:

$$d_1 := \frac{m_{351'}m_{27} - 2\lambda_3\lambda_4 c_2 d_2}{2\lambda_3\lambda_4 c_1}. \tag{356}$$

The term F_{c_1} then becomes

$$F_{c_1} = 0 = -\frac{6\lambda_1\lambda_3 c_1 e_4{}^2}{m_{351'}}. \tag{357}$$

Due to the assumption $c_1 \neq 0$ of case 2, we conclude $e_4 = 0$. This implies

$$F_{e_4} = 0 = m_{351'} f_4, \tag{358}$$

$$F_{e_5} = 0 = m_{351'} f_5. \tag{359}$$

We again get $\langle 24 \rangle = 0$ and the SU(5) group remains unbroken.

This exhausts all the possible avenues of finding a solution, proving that all the solutions, which break to the Standard Model, are found in the two main branches. In fact, all but one of the alternative solutions leave the group SU(5) unbroken. The exception is Case 1.1, where we are using the ansatz $\langle 27 \rangle = \langle \overline{27} \rangle = 0$. This case corresponds to solving the model with the breaking sector $351' \oplus \overline{351'}$ from section 4.2.3 and the solution breaks E_6 to the Pati-Salam group.

4.3.2 Doublet-triplet splitting

If we have the breaking sector $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}$, the fermionic 27_F's will couple to the 27 and the $\overline{351'}$. The MSSM Higgses H_u and H_d need to be present in both the 27 and the $\overline{351'}$, as already discussed in section 4.1.4. The mass terms, which connect weak doublets $(1, 2, +\frac{1}{2})$ to weak antidoublets $(1, 2, -\frac{1}{2})$ and the color triplets $(3, 1, -\frac{1}{3})$ to color antitriplets $(\overline{3}, 1, +\frac{1}{3})$, come from the breaking part of the superpotential in equation (247). Note that even though the fermionic 27^i_F also have (anti)doublets and (anti)triplets with the correct quantum numbers, the mass matrices of the fermionic sector and breaking sector do not mix due to $\mathbb{Z}_2$ matter parity.

All doublets and triplets in the representations of the breaking sector are defined in Table 9. The doublets (antidoublets) are denoted by D $(\overline{D})$, and the triplets (antitriplets) by T $(\overline{T})$. There are 11 doublet-antidoublet pairs and 12 triplet-antitriplet pairs, so the mass matrices have dimensions 11×11 and 12×12, respectively. If we write the mass terms as

$$\begin{pmatrix} D_1 & \cdots & D_{11} \end{pmatrix} \mathcal{M}_{\text{doublets}} \begin{pmatrix} \overline{D}_1 \\ \vdots \\ \overline{D}_{11} \end{pmatrix} + \begin{pmatrix} T_1 & \cdots & T_{12} \end{pmatrix} \mathcal{M}_{\text{triplets}} \begin{pmatrix} \overline{T}_1 \\ \vdots \\ \overline{T}_{12} \end{pmatrix}, \tag{360}$$

the two mass matrices $\mathcal{M}_{\text{doublets}}$ and $\mathcal{M}_{\text{triplets}}$ can be computed and are compactly written in equation (361). The parameters α and β control which of the mass matrices we want to write: for the triplet matrix take $\alpha = \beta = 2$, and for the doublet matrix remove the last row and column and take $\alpha = -3$ and $\beta = -\sqrt{3}$. The last row and column represent the triplet-antitriplet pair in the representations 50 and $\overline{50}$ of SU(5) with no doublet-antidoublet counterpart pair. The α and β are related to the Clebsch-Gordan coefficients of the doublet or triplets in the SU(5) representations 5 and 45. The values for α come directly from the VEV $\langle 24\rangle \propto \text{diag}(2, 2, 2, -3, -3)$, which couples a 5 and a $\overline{5}$. The parameter β comes from the terms coupling a 5 to a $\overline{45}$ of SU(5) (or the conjugate of that). The strange $\sqrt{3}$ is present in the β for the triplets due to the normalization of these states in the representations 45 and $\overline{45}$.

Notice that the parameters α and β are only in front of entries with VEVs e_4, e_5, f_4, f_5, since these are the SU(5)-breaking VEVs. We see that e_1, e_2, e_3 and f_1, f_2, f_3 from $351'$ and $\overline{351'}$ are not present in the matrix at all, while the c_1, c_2 and d_1, d_2 from 27 and $\overline{27}$ are SU(5) singlets, so the coefficients of the doublets and triplets in front of these VEVs are the same (no parameters α and β).

$$\left(\begin{array}{cccccccccccc}
m_{27} & \alpha\lambda_3\frac{f_4}{\sqrt{15}}-6\lambda_5 c_1 & \alpha\lambda_3\frac{f_5}{\sqrt{15}}+6\lambda_5 c_2 & -\sqrt{\frac{8}{5}}\lambda_3 c_1 & 0 & 0 & 0 & 0 & \sqrt{\frac{8}{5}}\lambda_3 c_2 & 0 & 0 & 0 \\
\alpha\lambda_4\frac{e_4}{\sqrt{15}}-6\lambda_6 d_1 & m_{27} & 0 & 0 & -\sqrt{\frac{8}{5}}\lambda_4 d_1 & 0 & 0 & 0 & 0 & -\sqrt{2}\lambda_4 d_2 & 0 & 0 \\
\alpha\lambda_4\frac{e_5}{\sqrt{15}}+6\lambda_6 d_2 & 0 & m_{27} & 0 & -\lambda_4\frac{d_2}{\sqrt{10}} & -\sqrt{2}\lambda_4 d_1 & -\sqrt{\frac{3}{2}}\lambda_4 d_2 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{\frac{8}{5}}\lambda_4 d_1 & 0 & 0 & m_{351'} & \alpha\sqrt{\frac{3}{5}}\lambda_1 e_4 & 0 & 0 & 0 & 0 & -\alpha\frac{\sqrt{3}}{2}\lambda_1 e_5 & 0 & 0 \\
0 & -\sqrt{\frac{8}{5}}\lambda_3 c_1 & -\lambda_3\frac{c_2}{\sqrt{10}} & \alpha\sqrt{\frac{3}{5}}\lambda_2 f_4 & m_{351'} & 0 & 0 & 0 & -\alpha\frac{1}{4}\sqrt{\frac{3}{5}}\lambda_2 f_5 & 0 & -\beta\frac{5\sqrt{3}}{4}\lambda_2 f_5 & 0 \\
0 & 0 & -\sqrt{2}\lambda_3 c_1 & 0 & 0 & m_{351'} & 0 & 0 & \alpha\frac{\sqrt{3}}{2}\lambda_2 f_4 & 0 & \beta\frac{\sqrt{15}}{2}\lambda_2 f_4 & 0 \\
0 & 0 & -\sqrt{\frac{3}{2}}\lambda_3 c_2 & 0 & 0 & 0 & m_{351'} & \beta\sqrt{5}\lambda_2 f_4 & -\alpha\frac{3}{4}\lambda_2 f_5 & 0 & \beta\frac{\sqrt{5}}{4}\lambda_2 f_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta\sqrt{5}\lambda_1 e_4 & m_{351'} & 0 & -\beta\frac{\sqrt{15}}{2}\lambda_1 e_5 & 0 & \alpha\sqrt{10}\lambda_1 e_4 \\
\sqrt{\frac{8}{5}}\lambda_4 d_2 & 0 & 0 & 0 & -\alpha\frac{1}{4}\sqrt{\frac{3}{5}}\lambda_1 e_5 & \alpha\frac{\sqrt{3}}{2}\lambda_1 e_4 & -\alpha\frac{3}{4}\lambda_1 e_5 & 0 & m_{351'} & 0 & 0 & 0 \\
0 & -\sqrt{2}\lambda_3 c_2 & 0 & -\alpha\frac{\sqrt{3}}{2}\lambda_2 f_5 & 0 & 0 & 0 & -\beta\frac{\sqrt{15}}{2}\lambda_2 f_5 & 0 & m_{351'} & 0 & 0 \\
0 & 0 & 0 & 0 & -\beta\frac{5\sqrt{3}}{4}\lambda_1 e_5 & \beta\frac{\sqrt{15}}{2}\lambda_1 e_4 & \beta\frac{\sqrt{5}}{4}\lambda_1 e_5 & 0 & 0 & 0 & m_{351'} & \alpha\sqrt{10}\lambda_1 e_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha\sqrt{10}\lambda_2 f_4 & 0 & 0 & \alpha\sqrt{10}\lambda_2 f_5 & m_{351'}
\end{array}\right). \tag{361}$$

Observe the particular form of the $\mathcal{M}_{ij}$ matrix in equation (361). For index values $i, j = 1, 2, 3$, the doublets and triplets come from the pair $27 \oplus \overline{27}$, while indices

$i, j = 4, \ldots, n$ correspond to fields coming from the pair $351' \oplus \overline{351'}$, where $n = 11$ for doublets and $n = 12$ for triplets. The mass matrix therefore has a block structure:

$$\begin{pmatrix} M_{3\times 3} & M_{3\times(n-3)} \\ M_{(n-3)\times 3} & M_{(n-3)\times(n-3)} \end{pmatrix}. \tag{362}$$

The blocks are populated by the following invariants:

- Block $M_{3\times 3}$ is populated by the terms from $27^2 \times \langle 27, \overline{351'}\rangle$ and $\overline{27}^2 \times \langle \overline{27}, 351'\rangle$.
- Blocks $M_{3\times(n-3)}$ and $M_{(n-3)\times 3}$ are populated by the terms from $27 \times \overline{351'} \times \langle 27\rangle$ and $\overline{27} \times 351' \times \langle \overline{27}\rangle$.
- Block $M_{(n-3)\times(n-3)}$ is populated by the terms from $351'^2 \times \langle 351'\rangle$ and $\overline{351'}^2 \times \langle \overline{351'}\rangle$.
- The 3×3 block and $(n-3) \times (n-3)$ block also contain the mass terms.

We now have to perform doublet-triplet splitting in the matrices encoded in equation (361). The simplest possible way to do this is to perform a fine-tuning of the parameters in the model: $m_{351'}$, m_{27} and λ_i. The fine-tuning procedure is usually (in model building of this type in general) specified by the requirement that all triplets remain heavy, while one doublet mode becomes massless: this massless doublet-antidoublet pair then corresponds to H_u and H_d. By relaxing the fine-tuning, so that a condition is satisfied only approximately (up to order $M_{\mathrm{EW}}/M_{\mathrm{GUT}}$), one can obtain a small EW-scale mass μ with the term $\mu\, H_u H_d$.

In our specific case, the procedure is a little more complicated due to the Higgs mechanism. Among the broken generators going from E_6 to the Standard Model, a doublet-antidoublet pair and a triplet-antitriplet pair of generators in the $16 \oplus \overline{16}$ of SO(10) is broken; consequently, in a solution breaking to the SM, there will already be a doublet and a triplets massless mode present in the two mass matrices, corresponding to would-be Goldstone bosons. One can check this explicitly by plugging-in the general VEV solution from equations (290)–(299) into the mass matrices in equation (361) and compute the dimensions of the left and right null-eigenspaces to be 1.

Doublet-triplet splitting in our case thus involves making a second doublet-antidoublet pair to be massless (the MSSM Higgses), while keeping the remaining triplets heavy. The masses of the scalars doublet and triplets can be in principle computed from the squared-mass matrices $\mathcal{M}^\dagger_{\mathrm{doublets}}\mathcal{M}_{\mathrm{doublets}}$ and $\mathcal{M}^\dagger_{\mathrm{triplets}}\mathcal{M}_{\mathrm{triplets}}$. In our case, we are interested only in the zero-modes, so we shall use methods applicable to the matrices $\mathcal{M}_{\mathrm{doublets}}$ and $\mathcal{M}_{\mathrm{triplets}}$ directly, and shall avoid squaring the mass matrices and unnecessarily complicating the calculation. We write this method on the generic matrix $\mathcal{M}$ below.

Having a massless mode already present in the square $\mathcal{M}^\dagger\mathcal{M}$ implies

$$\det \mathcal{M} = 0. \tag{363}$$

The condition for an additional massless mode in $\mathcal{M}^\dagger\mathcal{M}$ can be written as (see [74])

$$\mathrm{Cond}(\mathcal{M}) := \frac{\lim_{\epsilon\to 0} \det(\mathcal{M} + \epsilon I)/\epsilon}{\langle f|e\rangle} = 0, \tag{364}$$

where I is the identity matrix, with $|e\rangle$ and $|f\rangle$ the already present right and left zero-mass eigenmodes of $\mathcal{M}$:

$$\mathcal{M}|e\rangle = \mathcal{M}^\dagger|f\rangle = 0. \tag{365}$$

We now apply this method on our specific case. Using the general vacuum solution in equations (290)–(299) and confirming the presence of would-be Goldstone bosons by

$$\det \mathcal{M}_{\text{doublets}} = \det \mathcal{M}_{\text{triplets}} = 0, \tag{366}$$

the DT splitting conditions read

$$\text{Cond}(\mathcal{M}_{\text{doublets}}) = \frac{1}{72}\, m_{351'}^{9} m_{27}\, \frac{\lambda_3\lambda_4}{\lambda_1\lambda_2} = 0, \tag{367}$$

$$\text{Cond}(\mathcal{M}_{\text{triplets}}) = \frac{4}{243}\, m_{351'}^{10} m_{27}\, \frac{\lambda_3\lambda_4}{\lambda_1\lambda_2} \neq 0. \tag{368}$$

We see the obtained fine-tuning conditions are very simple, which is catastrophic. The expressions are merely a product of the Lagrangian parameters, so we cannot perform a fine-tuning on the doublets independently from the triplets: making a doublet mode massless would involve taking one of the masses or λ's to be zero, but that would also imply a massless triplet. The usual procedure of DT splitting via fine-tuning is therefore **not possible** in this case, which is a very surprising result. We further discuss and summarize the results in the next subsection.

As a last remark, we take special notice of the fact that the model contains one more triplet-antitriplet pair than a doublet-antidoublet pair. One could thus naively hope to incorporate the missing partner mechanism [51, 52, 53] for doublet-triplet splitting. The mechanism requires though, apart from a very specific setup of the mass matrix, an E_6 representation with a 75 of SU(5) (see section 2.5), with the smallest such representation being the 650 of E_6 (see Figure 8). An implementation of the missing partner mechanism in E_6 thus leads to a prohibitively complicated model.

4.3.3 Discussion and summary of the prototype model

The model with the breaking sector $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}$ does not seem to be viable.

A successful breaking $E_6 \to$ SM in this model is possible, as shown in subsection 4.3.1.3, but only through the main branches of solutions, as shown in subsection 4.3.1.5. The only solutions to break to the SM model do not allow, however, for a DT splitting by fine-tuning, as shown in subsection 4.3.2. Is it possible to save the model? We study minimal extensions of the prototype model in sections 4.4 and 4.5 and name them model I and model II, respectively. In the extensions, the DT splitting problem is cured and the models are viable.

4.4 Model I: $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27} \oplus \widetilde{27} \oplus \overline{\widetilde{27}}$

In this model, we extend the prototype model from section 4.3 with an additional fundamental-antifundamental pair of representations, which we denote by an overhead tilde symbol: $\widetilde{27} \oplus \overline{\widetilde{27}}$. The breaking sector thus consists of

$$351' \oplus \overline{351'} \;\oplus\; 27 \oplus \overline{27} \;\oplus\; \widetilde{27} \oplus \overline{\widetilde{27}}. \tag{369}$$

Alongside the usual $\mathbb{Z}_2$ matter parity, another restriction will be imposed in this model, otherwise the EOM become severely complicated to the point that an analytic solution is hard to obtain. In this simplification, we shall effectively separate the tilde and the non-tilde fields, so that the GUT-scale VEVs are in the non-tilde fields of the breaking sector, while the EW-scale VEVs are in the tilde fields. The light Higgs doublets will therefore be present solely in the tilde sector.

The separation of the tilde and non-tilde fields will be achieved similar to how matter parity separates the fermionic and breaking sector. We assume that in the breaking sector, the tilde fields need to be present in pairs, as if they have parity -1 under a $\mathbb{Z}_2$ symmetry. But since H_u and H_d are in the tilde fields, and they MSSM Higsses need to couple to fermions, we also need a $27_F \cdot 27_F \cdot \widetilde{27}$ term. Since the exotic fermions of the model need to be heavy, the fermionic representations also need to couple to the regular 27 and $\overline{351'}$. Therefore we cannot describe the restrictions on the tilde fields in terms of a symmetry group (such as a $\mathbb{Z}_2$ parity), since the tilde field $\widetilde{27}$ would need to correspond to the same group element as the 27 and $\overline{351'}$, thus negating the restrictions in the breaking sector. The restrictions thus have to be viewed not as a symmetry, but as setting certain parameters in the superpotential to zero. Remember that in SUSY theories, the superpotential is subject to the non-renormalization theorem, so a non-presence of an operator at one scale also implies non-presence at another. Although setting certain parameters to zero is a simplification, it is therefore not inconsistent to build a symmetry breaking solution with this ansatz.

The $\mathbb{Z}_2$ matter parity and the extra restrictions on the tilde fields yield the following superpotential of this model:

$$W = W_{\text{SSB}} + W_{DT} + W_{\text{Yukawa}}, \tag{370}$$

with

$$\begin{aligned} W_{\text{SSB}} = \;& m_{351'}\, I_{351' \otimes \overline{351'}} + m_{27}\, I_{27 \otimes \overline{27}} \\ &+ \lambda_1\, I_{351'^3} + \lambda_2\, I_{\overline{351'}^3} + \lambda_3\, I_{27^2 \otimes \overline{351'}} + \lambda_4\, I_{\overline{27}^2 \otimes 351'} \\ &+ \lambda_5\, I_{27^3} + \lambda_6\, I_{\overline{27}^3}, \end{aligned} \tag{371}$$

$$W_{DT} = m_{\widetilde{27}}\, I_{\widetilde{27} \otimes \overline{\widetilde{27}}} + \kappa_1\, I_{\widetilde{27}^2 \otimes \overline{351'}} + \kappa_2\, I_{\overline{\widetilde{27}}^2 \otimes 351'} + \kappa_3\, I_{\widetilde{27}^2 \otimes 27} + \kappa_4\, I_{\overline{\widetilde{27}}^2 \otimes \overline{27}}, \tag{372}$$

$$W_{\text{Yukawa}} = \sum_{i,j=1}^{3} \tfrac{1}{2} \left(Y_{27}^{ij}\, I_{27_F^i \otimes 27_F^j \otimes 27} + Y_{\overline{351'}}^{ij}\, I_{27_F^i \otimes 27_F^j \otimes \overline{351'}} + Y_{\widetilde{27}}^{ij}\, I_{27_F^i \otimes 27_F^j \otimes \widetilde{27}} \right). \tag{373}$$

The W_{SSB} is responsible for the spontaneous symmetry breaking, W_{DT} for DT splitting, while W_{Yukawa} is the Yukawa sector in this model. Comparing this with the superpotential of the prototype model in equation (247), we see that the W_{DT} terms were added, along with an extra term in the Yukawa sector.

4.4.1 Symmetry breaking

Due to the extra restrictions on the tilde fields, one can ignore these additions in the spontaneous symmetry breaking. Indeed, taking the ansatz

$$0 = \langle 27^i_F \rangle = \langle \widetilde{27} \rangle = \langle \overline{\widetilde{27}} \rangle, \tag{374}$$

the terms in equations (373) and (372) for W_{Yukawa} and Y_{DT} do not contribute to the F-term equations, and the EOM are reduced to the familiar case of the prototype model in equations (268)–(270) for the D-terms and equations (248)–(261) for the F-terms. This system of equations was already solved and analyzed in subsection 4.3.1, so we will merely copy the specific solution from equations (300)–(306):

$$c_2 = 0, \qquad d_2 = 0, \tag{375}$$

$$e_2 = 0, \qquad f_2 = 0, \tag{376}$$

$$e_4 = 0, \qquad f_4 = 0, \tag{377}$$

$$d_1 = \frac{m_{351'} m_{27}}{2\lambda_3 \lambda_4 c_1}, \tag{378}$$

$$e_1 = -\frac{m_{351'}}{6\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_1 = -\frac{m_{351'}}{6\lambda_1^{1/3}\lambda_2^{2/3}}, \tag{379}$$

$$e_3 = -\lambda_3 {c_1}^2 / m_{351'}, \qquad f_3 = -\frac{m_{351'} m_{27}^2}{4\lambda_3^2 \lambda_4 {c_1}^2}, \tag{380}$$

$$e_5 = \frac{m_{351'}}{3\sqrt{2}\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_5 = \frac{m_{351'}}{3\sqrt{2}\lambda_1^{1/3}\lambda_2^{2/3}}, \tag{381}$$

with $|c_1|^2$ determined by the polynomial condition

$$\begin{aligned} 0 = {} & |m_{351'}|^4 |m_{27}|^4 + 2|m_{351'}|^4 |m_{27}|^2 |\lambda_3|^2 |c_1|^2 \\ & - 8|m_{351'}|^2 |\lambda_3|^4 |\lambda_4|^2 |c_1|^6 - 16|\lambda_3|^6 |\lambda_4|^2 |c_1|^8. \end{aligned} \tag{382}$$

We already know this solution does indeed break to the Standard Model.

4.4.2 Doublet-triplet splitting

Note that the breaking sector of this model contains the previous 11 doublet-antidoublet pairs and 12 triplet-antitriplet pairs from the prototype model (see subsection 4.3.2), along with 3 new doublet-antidoublet pairs and 3 triplet-antitriplet pairs from the tilde fields $\widetilde{27}$ and $\overline{\widetilde{27}}$. We denote the new doublets D and triplets T with the same indices as we did in the 27 and $\overline{27}$, but with a tilde on top, as shown in Table 17.

Since the fermionic 27^i_F and tilde fields $\widetilde{27}$ and $\overline{\widetilde{27}}$ do not have GUT-scale VEVs, and due to the form of the superpotential in equation (370), there is no mixing between the tilde and the non-tilde mass matrices. In fact, there are three separate blocks of mass matrices for doublets and triplets: from the fermionic sector, from the tilde part of the breaking sector, and the non-tilde part of the breaking sector. These three block are populated exactly by the W_{Yukawa}, W_{DT} and W_{SSB} parts of the superpotential, respectively.

We already computed the mass matrix block of the non-tilde fields of the breaking sector in equation (361). Since the model I (the tilde model) uses the same symmetry braking of E_6 as the prototype model, we can apply the conclusions of section 4.3.2:

Table 17: Labels of doublets and triplets in the tilde fields. The corresponding EW-VEVs are also labeled.

doublet,triplet	$\subset$ SU(5)	$\subset$ SO(10)	$\subset E_6$	doublet VEV
$\widetilde{D}_1, \widetilde{T}_1$	5	10	$\widetilde{27}$	v_1
$\widetilde{D}_2, \widetilde{T}_2$	5	10	$\overline{\widetilde{27}}$	v_2
$\widetilde{D}_3, \widetilde{T}_3$	5	$\overline{16}$	$\overline{\widetilde{27}}$	v_3
$\overline{\widetilde{D}}_1, \overline{\widetilde{T}}_1$	$\overline{5}$	10	$\overline{\widetilde{27}}$	$\bar{v}_1$
$\overline{\widetilde{D}}_2, \overline{\widetilde{T}}_2$	$\overline{5}$	10	$\widetilde{27}$	$\bar{v}_2$
$\overline{\widetilde{D}}_3, \overline{\widetilde{T}}_3$	$\overline{5}$	16	$\widetilde{27}$	$\bar{v}_3$

the non-tilde block contains the would be Goldstone bosons, but fine tuning cannot be performed there. We now compute the new 3×3 mass matrices for the tilde doublets and triplets. Fine-tuning will be successfully accomplished in the tilde sector with the new κ parameters. The MSSM Higgses H_u and H_d will therefore have components in the tilde doublets and antidoublets, respectively, so these states will have electroweak VEVs. We denote the EW VEVs by v's and $\bar{v}$'s, as shown in Table 17.

The mass matrix terms are written as

$$\begin{pmatrix} \widetilde{D}_1 & \widetilde{D}_2 & \widetilde{D}_3 \end{pmatrix} \widetilde{\mathcal{M}}_{\text{doublets}} \begin{pmatrix} \overline{\widetilde{D}}_1 \\ \overline{\widetilde{D}}_2 \\ \overline{\widetilde{D}}_3 \end{pmatrix} + \begin{pmatrix} \widetilde{T}_1 & \widetilde{T}_2 & \widetilde{T}_3 \end{pmatrix} \widetilde{\mathcal{M}}_{\text{triplets}} \begin{pmatrix} \overline{\widetilde{T}}_1 \\ \overline{\widetilde{T}}_2 \\ \overline{\widetilde{T}}_3 \end{pmatrix}, \tag{383}$$

with the mass matrices explicitly written in compact form as

$$\widetilde{\mathcal{M}} = \begin{pmatrix} m_{\widetilde{27}} & -2\kappa_3 c_1 + \alpha\kappa_1 \frac{f_4}{\sqrt{15}} & 2\kappa_3 c_2 + \alpha\kappa_1 \frac{f_5}{\sqrt{15}} \\ -2\kappa_4 d_1 + \alpha\kappa_2 \frac{e_4}{\sqrt{15}} & m_{\widetilde{27}} & 0 \\ 2\kappa_4 d_2 + \alpha\kappa_2 \frac{e_5}{\sqrt{15}} & 0 & m_{\widetilde{27}} \end{pmatrix}. \tag{384}$$

with $\alpha = -3$ for $\widetilde{\mathcal{M}}_{\text{doublets}}$ and $\alpha = 2$ for $\widetilde{\mathcal{M}}_{\text{triplets}}$. The α values are the values from the VEV $\langle 24 \rangle$ of SU(5).

Since the would-be Goldstone bosons of the Higgs mechanism are in the other mass matrix block of the non-tilde fields, there are generically no massless modes in $\widetilde{\mathcal{M}}_{\text{doublets}}$ $\widetilde{\mathcal{M}}_{\text{triplets}}$, so the zero-mode condition in the doublets and the violation of that condition in the triplets is written as

$$\det\left(\widetilde{\mathcal{M}}_{\text{doublets}}\right) = 0, \tag{385}$$

$$\det\left(\widetilde{\mathcal{M}}_{\text{triplets}}\right) \neq 0. \tag{386}$$

Plugging in the vacuum solution from equations (375)–(381), we get the fine-tuning conditions

$$0 = m^3_{\widetilde{27}} - \frac{1}{30}\, m_{\widetilde{27}} m^2_{351'} \frac{\kappa_1\kappa_2}{\lambda_1\lambda_2} - 2 m_{\widetilde{27}} m_{351'} m_{27} \frac{\kappa_3\kappa_4}{\lambda_3\lambda_4}, \tag{387}$$

$$0 \neq m^3_{\widetilde{27}} - \frac{2}{135}\, m_{\widetilde{27}} m^2_{351'} \frac{\kappa_1\kappa_2}{\lambda_1\lambda_2} - 2 m_{\widetilde{27}} m_{351'} m_{27} \frac{\kappa_3\kappa_4}{\lambda_3\lambda_4}. \tag{388}$$

Unlike the non-tilde block, both conditions can be simultaneously satisfied, for example by fixing κ_1:

$$\kappa_1 \approx 30\,(m_{\widetilde{27}}^2\lambda_3\lambda_4 - 2m_{351'}m_{27}\kappa_3\kappa_4)\frac{\lambda_1\lambda_2}{m_{351'}^2\lambda_3\lambda_4\kappa_2}. \tag{389}$$

This fine-tuning of κ_1 ensures a zero-mode doublet as the left eigenvector and a zero-mode antidoublet mode as the right eigenvector of $\widetilde{\mathcal{M}}_{\text{doublets}}$. These zero-modes correspond to H_u and H_d of MSSM, respectively. We compute them to be

$$\begin{aligned} H_u \propto\ & \frac{\sqrt{1/30}\; m_{\widetilde{27}}m_{351'}\lambda_1^{-2/3}\lambda_2^{-1/3}\lambda_3\lambda_4\kappa_2}{m_{\widetilde{27}}^2\lambda_3\lambda_4 - 2m_{351'}m_{27}\kappa_3\kappa_4}\,\widetilde{D}_1 \\ &+ \frac{\sqrt{2/15}\; m_{351'}c_1\lambda_1^{-2/3}\lambda_2^{-1/3}\lambda_3\lambda_4\kappa_2\kappa_3}{m_{\widetilde{27}}^2\lambda_3\lambda_4 - 2m_{351'}m_{27}\kappa_3\kappa_4}\,\widetilde{D}_2 + \widetilde{D}_3, \end{aligned} \tag{390}$$

$$H_d \propto \frac{\sqrt{30}\; m_{\widetilde{27}}\lambda_1^{2/3}\lambda_2^{1/3}}{m_{351'}\kappa_2}\,\overline{\widetilde{D}}_1 + \frac{\sqrt{30}\; m_{27}\lambda_1^{2/3}\lambda_2^{1/3}\kappa_4}{c_1\lambda_3\lambda_4\kappa_2}\,\overline{\widetilde{D}}_2 + \overline{\widetilde{D}}_3. \tag{391}$$

Notice that the two MSSM Higgses have non-zero components in all the tilde doublets and antidoublets: H_u has components in $\widetilde{D}_1$, $\widetilde{D}_2$ and $\widetilde{D}_3$, while H_d has components in $\overline{\widetilde{D}}_1$, $\overline{\widetilde{D}}_2$ and $\overline{\widetilde{D}}_3$. This means that all of the VEVs v_i and $\bar{v}_i$ are nonzero, where

$$v_u^2 = {v_1}^2 + {v_2}^2 + {v_3}^2, \tag{392}$$
$$v_d^2 = {\bar{v}_1}^2 + {\bar{v}_2}^2 + {\bar{v}_3}^2. \tag{393}$$

In particular, v_1, $\bar{v}_2$ and $\bar{v}_3$, which are the EW VEVs of the $\widetilde{27}$, are all non-vanishing. This will be important for the viability of Yukawa sector fermion masses.

4.4.3 Yukawa sector

The Yukawa terms of the superpotential are denoted by W_{Yukawa} in equation (373). Schematically, we have the terms

$$27_F^i 27_F^j\,\left(Y_{27}^{ij}\;27 + Y_{\overline{351'}}^{ij}\;\overline{351'} + Y_{\widetilde{27}}^{ij}\;\widetilde{27}\right). \tag{394}$$

Note that since the representations 27 and $\overline{351'}$ of E_6 couple to the symmetric product of two 27's, the matrices Y_{27}, $Y_{\overline{351'}}$ and $Y_{\widetilde{27}}$ can be taken to be symmetric.

Excluding the tilde part, this is analogous to the Yukawa sector in the minimal renormalizable SO(10) model [32, 33]:

$$W_{\text{Yukawa -SO(10)}} = \tfrac{1}{2}\;16_F^i\;16_F^j\;\left(Y_{10}^{ij}\;10 + Y_{\overline{126}}^{ij}\;\overline{126}\right). \tag{395}$$

In our model, the role of the 27 is analogous to the 16 of SO(10), while the role of $\overline{351'}$ is analogous to the role of $\overline{126}$ of SO(10). This is not a coincidence, since $10 \subset 27$ and $\overline{126} \subset \overline{351'}$. Furthermore, since also $16_F^i \subset 27_F^i$, our model contains all the terms from the renormalizable SO(10) model, but also some additional ones, such as $16_i^F 10_i^F (Y^{27}\;16 + Y^{\overline{351'}}\;144)$ and $10_i^F 10_i^F (Y^{27}\;1 + Y^{\overline{351'}}\;54)$ and some others involving the SO(10) singlets.

The mechanism of achieving flavor-mixing, however, is completely different in the two models. In the SO(10) model, we have the usual GUT case with the MSSM

Higgses present in both 10 and $\overline{126}$. The two generic matrices Y_{10} and $Y_{\overline{126}}$ cannot be diagonalized simultaneously, so we get flavor mixing.

In our model, however, the mechanism is more subtle. Remember that the MSSM Higgses are present only in the tilde fields, so we have only one Higgs terms $\widetilde{27}$. But after the breaking at the GUT scale, the breaking sector representations 27 and $\overline{351'}$ acquire VEVs. These VEVs mix the two $\bar{5}$'s of SU(5) in the 27_F, and the SU(5) breaking VEVs (e_4, f_4, e_5, f_5) ensure that this mixing is different for different SM representations in the $\bar{5}$'s. Although there will still be a heavy vector-like pair $5 \oplus \bar{5}$ of SU(5) of exotic fermions, the mixing of the $\bar{5}$'s causes a mixing between the 16 and the 10, so the Standard Model fields are not contained just in the 16 of SO(10). Flavor mixing therefore arises due to mixing to produce vector-like heavy states at the GUT scale, and not from the simultaneous presence of the Higgs in two different representations at the EW scale. This situation is analogous to [8].

We now compute the mass matrices, taking all the relevant terms of W_{Yukawa} into account. They are (skipping the hermitian conjugate terms)

$$
\begin{aligned}
u^T(-v_1)Y_{\widetilde{27}}u^c &+ \begin{pmatrix} d^{cT} & d'^{cT} \end{pmatrix} \begin{pmatrix} \bar{v}_2 Y_{\widetilde{27}} & c_2 Y_{27} + \frac{f_5}{\sqrt{15}} Y_{\overline{351}'} \\ -\bar{v}_3 Y_{\widetilde{27}} & -c_1 Y_{27} + \frac{f_4}{\sqrt{15}} Y_{\overline{351}'} \end{pmatrix} \begin{pmatrix} d \\ d' \end{pmatrix} \\
&+ \begin{pmatrix} e^T & e'^T \end{pmatrix} \begin{pmatrix} -\bar{v}_2 Y_{\widetilde{27}} & c_2 Y_{27} - \frac{3}{2}\frac{f_5}{\sqrt{15}} Y_{\overline{351}'} \\ \bar{v}_3 Y_{\widetilde{27}} & -c_1 Y_{27} - \frac{3}{2}\frac{f_4}{\sqrt{15}} Y_{\overline{351}'} \end{pmatrix} \begin{pmatrix} e^c \\ e'^c \end{pmatrix} \\
&+ \begin{pmatrix} \nu^T & \nu'^T \end{pmatrix} \begin{pmatrix} v_1 Y_{\widetilde{27}} & 0 & c_2 Y_{27} - \frac{3}{2}\frac{f_5}{\sqrt{15}} Y_{\overline{351}'} \\ 0 & -v_1 Y_{\widetilde{27}} & -c_1 Y_{27} - \frac{3}{2}\frac{f_4}{\sqrt{15}} Y_{\overline{351}'} \end{pmatrix} \begin{pmatrix} \nu^c \\ s \\ \nu'^c \end{pmatrix} \\
&+ \frac{1}{2} \begin{pmatrix} \nu^{cT} & s^T & \nu'^{cT} \end{pmatrix} \begin{pmatrix} f_1 Y_{\overline{351}'} & \frac{f_2}{\sqrt{2}} Y_{\overline{351}'} & -\bar{v}_3 Y_{\widetilde{27}} \\ \frac{f_2}{\sqrt{2}} Y_{\overline{351}'} & f_3 Y_{\overline{351}'} & \bar{v}_2 Y_{\widetilde{27}} \\ -\bar{v}_3 Y_{\widetilde{27}} & \bar{v}_2 Y_{\widetilde{27}} & 0 \end{pmatrix} \begin{pmatrix} \nu^c \\ s \\ \nu'^c \end{pmatrix} \\
&+ \frac{1}{2} \begin{pmatrix} \nu^T & \nu'^T \end{pmatrix} \begin{pmatrix} \Delta_1 Y_{\overline{351}'} & \frac{1}{\sqrt{2}} \Delta_2 Y_{\overline{351}'} \\ \frac{1}{\sqrt{2}} \Delta_2 Y_{\overline{351}'} & \Delta_3 Y_{\overline{351}'} \end{pmatrix} \begin{pmatrix} \nu \\ \nu' \end{pmatrix}.
\end{aligned} \tag{396}
$$

For greater clarity, flavor indices are suppressed; they are present in the Yukawa matrices Y_{27}, $Y_{\overline{351}'}$ and $Y_{\widetilde{27}}$, as well as in every field written to the left or right of the matrices.

Notice that the mass matrix contributions are both from GUT scale VEVs (red), and EW scale VEV from the Higgses (blue). Each entry can be traced back to an invariant part in the SU(5) and SO(10) languages with the help of Tables 6 and 9, so they can also be checked, except for the numeric coefficients, manually. For example, the second term with the f_5 in the 12-entry of the down-quark matrix comes from the $\bar{5}_F \cdot 5_F \cdot \langle 24 \rangle$ term in SU(5) language, which is part of the terms $16_F \cdot 10_F \cdot \langle 144 \rangle$ in SO(10) language, which in turn is part of the term $27_F \cdot 27_F \cdot \langle \overline{351'} \rangle$ of E_6.

Also note the forms of matrices of the down-quark sector and charged lepton sector. They are of very similar form, since the doublet and the triplet are part of the same representation $\bar{5}$ (or 5) of SU(5). For GUT scale VEVs, the coefficients are the same, except for the $-3/2$ factors in front of f_4 and f_5, which come directly from the $\langle 24 \rangle$ of the the SU(5). The EW scale VEVs are also equal up to a minus sign, which is due to the definitions of the fields in Figure 5.

In addition to the terms with GUT and EW VEVs, we also have to pay special attention to the masses of the neutrino sector; for now, note only that $\bar{\Delta} \sim (1, 3, +1)$

and $\Delta \sim (1,3,-1)$ weak triplets, which are defined via Table 18, will be important for type II seesaw. Since there are no such weak triplet states in the 27 of E_6, they all come from the representations $351'$ and $\overline{351'}$.

Table 18: Weak triplet scalars $(1,3,\pm1)$ relevant for seesaw type II.

label	$E_6 \supseteq \mathrm{SO}(10) \supseteq \mathrm{SU}(5)$	p.n.	label	$E_6 \supseteq \mathrm{SO}(10) \supseteq \mathrm{SU}(5)$	p.n.
$\overline{\Delta}_1$	$351' \supseteq 126 \supseteq \overline{15}$	$L\,L$	Δ_1	$\overline{351'} \supseteq \overline{126} \supseteq 15$	$\bar{L}\,\bar{L}$
$\overline{\Delta}_2$	$351' \supseteq \overline{144} \supseteq \overline{15}$	$L\,L'$	Δ_2	$\overline{351'} \supseteq 144 \supseteq 15$	$\bar{L}\,\bar{L}'$
$\overline{\Delta}_3$	$351' \supseteq 54 \supseteq \overline{15}$	$L'L'$	Δ_3	$\overline{351'} \supseteq 54 \supseteq 15$	$\bar{L}'\bar{L}'$
$\overline{\Delta}_4$	$351' \supseteq 54 \supseteq 15$	$L'^cL'^c$	Δ_4	$\overline{351'} \supseteq 54 \supseteq \overline{15}$	$\bar{L}'^c\bar{L}'^c$

The triplets Δ and $\overline{\Delta}$ get non-zero VEVs, which we can determine by writing all the terms with these triplets that we get, when turning on both GUT-scale VEVs and EW VEVs, coming from W_{SSB} and W_{DT}, respectively. The terms are computed to be

$$\begin{aligned} W\big|_\Delta = {}& \begin{pmatrix}\overline{\Delta}_1 & \overline{\Delta}_2 & \overline{\Delta}_3 & \overline{\Delta}_4\end{pmatrix} \begin{pmatrix} m_{351'} & 0 & 0 & 6\lambda_1 e_1 \\ 0 & m_{351'} & 0 & -6\lambda_1 e_2 \\ 0 & 0 & m_{351'} & 6\lambda_1 e_3 \\ 6\lambda_2 f_1 & -6\lambda_2 f_2 & 6\lambda_2 f_3 & m_{351'} \end{pmatrix} \begin{pmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\\ \Delta_4\end{pmatrix} \\ &+ \begin{pmatrix}\overline{\Delta}_1 & \overline{\Delta}_2 & \overline{\Delta}_3 & \overline{\Delta}_4\end{pmatrix} \begin{pmatrix} \kappa_2 v_3{}^2 \\ \kappa_2\sqrt{2} v_2 v_3 \\ \kappa_2 v_2{}^2 \\ \kappa_1 v_1{}^2 \end{pmatrix} \\ &+ \begin{pmatrix}\kappa_1 \overline{v}_3{}^2 & \kappa_1\sqrt{2}\,\overline{v}_3\overline{v}_2 & \kappa_1 \overline{v}_2{}^2 & \kappa_2 \overline{v}_1{}^2\end{pmatrix} \begin{pmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\\ \Delta_4\end{pmatrix}. \end{aligned} \tag{397}$$

Wee see these Δ-triplets are heavy, so they can be integrated out of the theory by the conditions $\partial W/\partial\Delta_i = 0$ and $\partial W/\partial\overline{\Delta}_i = 0$ to determine how they effectively alter the low-energy theory. These conditions yield

$$\begin{pmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\\ \Delta_4\end{pmatrix} = \begin{pmatrix} m_{351'} & 0 & 0 & 6\lambda_1 e_1 \\ 0 & m_{351'} & 0 & -6\lambda_1 e_2 \\ 0 & 0 & m_{351'} & 6\lambda_1 e_3 \\ 6\lambda_2 f_1 & -6\lambda_2 f_2 & 6\lambda_2 f_3 & m_{351'} \end{pmatrix}^{-1} \begin{pmatrix} \kappa_2 v_3{}^2 \\ \kappa_2\sqrt{2} v_2 v_3 \\ \kappa_2 v_2{}^2 \\ \kappa_1 v_1{}^2 \end{pmatrix}. \tag{398}$$

Although equation (397) gives the full matrices, the EW scale and GUT scale masses are all entangled, so it is hard to see what is going on. The first thing we notice, however, is that there is a heavy vector-like pair of down-quarks and leptons, which receive their mass through the GUT scale VEVs c_1, c_2, f_4 and f_5: the particle d' becomes heavy by coupling to a combination of d^c and d'^c, while the particle e'^c becomes heavy by coupling to a combination of e and e'. The possibility of light vector-states is phenomenologically very intriguing, but although E_6 models have vector-like quarks and leptons automatically included, they generically seem to predict them to be at the GUT scale.

We would ultimately like to compute the masses in the low energy limit $E \ll M_{\mathrm{GUT}}$, so we need to integrate out the heavy vector-like pairs and neutrinos, with methods described in [75, 76] and also outlined below.

Suppose a matrix M has the block form

$$\mathcal{M} = \begin{pmatrix} M_1 & A \\ M_2 & B \end{pmatrix}, \tag{399}$$

where $M_{1,2}$ are $n\times n$ matrices with entries of order $\mathcal{O}(m_W)$, while A, B are $n\times n$ matrices with entries of order $\mathcal{O}(M_{GUT})$. In our case, $n = 3$ (due to having 3 generations of fermions).

We define a rotation matrix $\mathcal{U}$ by

$$\mathcal{U} := \begin{pmatrix} \Lambda & -\Lambda X \\ X^\dagger \Lambda & \bar{\Lambda} \end{pmatrix}, \tag{400}$$

where we used the definitions

$$X := A\,B^{-1}, \tag{401}$$

$$\Lambda := \left(1 + X X^\dagger\right)^{-1/2}, \tag{402}$$

$$\bar{\Lambda} := \left(1 + X^\dagger X\right)^{-1/2}. \tag{403}$$

In the above, we use 1 to denote the $n \times n$ identity matrix. The matrix power $-1/2$ is defined and can be computationally handled by the Taylor series for this function.

One can check that $\mathcal{U}$ is a unitary matrix and thus indeed a rotation: $\mathcal{U}\mathcal{U}^\dagger = \mathcal{U}^\dagger \mathcal{U} = I$. Also, the following identities hold:

$$X^\dagger \Lambda = \bar{\Lambda} X^\dagger, \tag{404}$$

$$X\bar{\Lambda} = \Lambda X. \tag{405}$$

If we multiply the matrix $\mathcal{M}$ with the rotation matrix $\mathcal{U}$ from the left, we get

$$\mathcal{U}\,\mathcal{M} = \begin{pmatrix} \Lambda(M_1 - X\,M_2) & 0 \\ X^\dagger\,\Lambda\,M_1 + \bar{\Lambda}\,M_2 & X^\dagger\,\Lambda\,A + \bar{\Lambda}\,B \end{pmatrix} = \begin{pmatrix} \mathcal{O}(M_{\mathrm{EW}}) & 0 \\ \mathcal{O}(M_{\mathrm{EW}}) & \mathcal{O}(M_{\mathrm{GUT}}) \end{pmatrix}. \tag{406}$$

We see that we managed to separate the light and heavy parts of the states, as seen by the order of the masses in the diagonal blocks: the first half of the states is light and the second half of the states is heavy. Due to the presence of the off-diagonal block, the separation is valid only up to relative corrections of order $M_{\mathrm{EW}}/M_{\mathrm{GUT}}$, which is sufficient for our purposes.

With the separation of the light and heavy states in the $\mathcal{U}\mathcal{M}$ matrix, we can now unambiguously integrate out the heavy states in the lower right part. We are left, in leading order of $M_{\mathrm{EW}}/M_{\mathrm{GUT}}$, with the matrix M for the light states:

$$M = \Lambda\,(M_1 - X\,M_2). \tag{407}$$

The neutrino sector is a bit more complicated to unravel, since alongside a vector-like pair of neutrinos, it also contains heavy right-handed states. The light states are all left-handed, so the right-handed states need to be integrated out. Suppose we denote the column of left handed states ν and ν' simply by ν, and the right-handed states ν^c, s and ν'^c by n. We can then write the Yukawa terms of the neutrino sector as

$$W\Big|_{\text{neutrino}} = \nu^T M_\nu n + \tfrac{1}{2} n^T M_n n + \tfrac{1}{2}\nu^T M_\Delta \nu. \tag{408}$$

We denoted the matrix of Dirac type masses by M_ν, the matrix of Majorana type masses by M_n, and the matrix from type II seesaw contributions with M_Δ. We first integrate out the right-handed states by $\partial W/\partial n = 0$:

$$\nu^T M_\nu + n^T M_n = 0, \tag{409}$$
$$n = -(M_n^T)^{-1}\, M_\nu^T\, \nu. \tag{410}$$

Plugging this into the Yukawa terms, we get

$$W\big|_{\text{neutrino}} = \tfrac{1}{2}\nu^T \left(M_\Delta - M_\nu (M_n^T)^{-1} M_\nu^T\right) \nu. \tag{411}$$

We now take care of the projection due to vector-like pairs. If $\mathcal{U}$ denotes the matrix, which rotates to light states in the matrix ($(\mathcal{U}M)_{1,1}$ is light), we have

$$W = \tfrac{1}{2}\nu_{\text{light}}^T \left(\mathcal{U}\left(M_\Delta - M_\nu (M_n^T)^{-1} M_\nu^T\right)\mathcal{U}^T\right) \nu_{\text{light}}, \tag{412}$$

where only the $(1,1)$ block entry in the matrix in the middle is important.

We now have all the tools to compute the masses of the light states. We use the described procedures on the matrices in equation (397), using also the ansatz $c_2 = f_2 = f_4 = 0$ due to the vacuum solution. We define

$$X_0 := \sqrt{\frac{3}{20}}\, \frac{f_5}{c_1}\, Y_{\overline{351}'} Y_{27}^{-1}, \tag{413}$$

and the light fermion masses become

$$M_D^T = \left(1 + (4/9) X_0 X_0^\dagger\right)^{-1/2} \left(\bar{v}_2 - (2/3)\bar{v}_3 X_0\right) Y_{\widetilde{27}}, \tag{414}$$
$$M_E = -\left(1 + X_0 X_0^\dagger\right)^{-1/2} \left(\bar{v}_2 + \bar{v}_3 X_0\right) Y_{\widetilde{27}}, \tag{415}$$
$$M_U = -v_1 Y_{\widetilde{27}}, \tag{416}$$
$$M_N = \frac{1}{2}\left(1 + X_0 X_0^\dagger\right)^{-1/2} \times \Bigg(\Delta_1 Y_{\overline{351}'} - \frac{\Delta_2}{\sqrt{2}}\left(X_0 Y_{\overline{351}'} + Y_{\overline{351}'} X_0^T\right) + \Delta_3 X_0 Y_{\overline{351}'} X_0^T$$
$$- \frac{{v_1}^2}{f_1} Y_{\widetilde{27}} Y_{\overline{351}'}^{-1} Y_{\widetilde{27}} - \frac{{v_1}^2}{f_3} X_0 Y_{\widetilde{27}} Y_{\overline{351}'}^{-1} Y_{\widetilde{27}} X_0^T\Bigg) \times \left(1 + X_0^* X_0^T\right)^{-1/2}. \tag{417}$$

Notice that there both type I [41, 42, 43, 44, 45] and type II [46, 47, 48, 49] seesaw contributions to neutrino masses in M_N. The type I terms are proportional to v_1^2, while the type II terms are proportional to $\Delta_{1,2,3}$. As one expects from the seesaw mechanism, both contributions are of the scale $\mathcal{O}(M_{\text{EW}}^2/M_{GUT})$: factors $\frac{v_1^2}{f_1}$ and $\frac{v_1^2}{f_3}$ for type I contributions, while type II contributions have factors $\Delta_i \sim \mathcal{O}(m_W^2/M_{GUT})$, with the scale of Δ's easily seen from equation (398).

There are no other contributions to the neutrino matrix at tree-level. Although type III [50] seesaw with (fermionic) weak triplets $(1,3,0)$ exists, our model does not have any seesaw type III contributions, which can be deduced by group-theoretic arguments. First notice that the weak triplets $\chi \sim (1,3,0)$ are only in the $\overline{351'}$ and $351'$ (in the 24 of §(5)). To have type III seesaw, one needs a vertex $HL\chi$ (see section 2.5), which combines the SM Higgs, the light SM lepton doublet, and the weak triplet. The Higgses are in the tilde fields, the SM fermions are in 27_F while the weak triplets are in the non-tilde part of the breaking sector. But since we forbade such mixed terms with matter parity, there are no type III interaction vertices in our model.

The general conclusions on the fermion masses in this model are the following:

- In addition to the SM fermions, we have additional degrees of freedom in each generation of 27_F: a vector-like pair of quarks and leptons, and two SM singlets, which have the role of right-handed neutrinos. The additional degrees of freedom are all automatically massive (of the order $M_{\rm GUT}$): we have no light exotics, or in particular, no light vector-like states.

- The light states are not purely in the 16 of SO(10). In fact, the state in the light states in the $\bar{5}$ of SU(5) are a mixture of the $\bar{5}$'s in the 16 and 10 of SO(10). This mixing manifests itself as flavor mixing at low energies.

- The masses of the light neutrinos have type I and type II seesaw contributions.

We conclude the section on the Yukawa sector by some remarks on the prospects of an explicit fit of the mass matrices to the experimental values of the masses and mixing angles. This numerics themselves will not be performed within this PhD thesis.

Our model has the following parameters :

- 3 mass parameters: m_{27}, $m_{351'}$ and $m_{\widetilde{27}}$.

- 10 couplings: 6 λ's and 4 κ's. The parameters λ_5 and λ_6 are not involved in the low-energy mass matrices of fermions.

- 3 symmetric Yukawa matrices. (Not all parameters here are physical though, since one of the matrices can be diagonalized by family rotations).

The fit is complicated by the non-linear way in which the Yukawa matrices Y_{27}, $Y_{\overline{351}'}$ and $Y_{\widetilde{27}}$ enter into the low energy matrices of equations (414)–(417), which is a typical feature when vector-like families are present. Since there are 3 Yukawa matrices in this model instead of the typical 2 (such as in minimal renormalizable SO(10) [32, 33]), it seems very likely that a fit can be performed; in fact, one will have many degrees of freedom in the parameters still left-over, so we conclude with a high degree of certainty that the Yukawa sector of this model is viable, but not predictive.

4.4.4 Proton decay

As a final phenomenological feature of model I, we study $D = 5$ proton decay [37, 38, 39, 40] in this model (see also section 2.4.3). It is mediated by the color triplet-antitriplet pairs $(3, 1, -1/3)$–$(\overline{3}, 1, +1/3)$. Although there are also triplet–antitriplet pairs of the type $(3, 1, -4/3)$–$(\overline{3}, 1 + 4/3)$ present in the theory, they give no direct contribution, since the triplets $(3, 1, -4/3)$ are present only in the representation $\overline{351'}$, but their corresponding antitriplets are in the $351'$, which does not couple to the fermionic 27_F's.

All such triplets in our model have been identified in section 4.4.2 and their labels can be found in Tables 9 and 17: there are 15 triplet–antitriplet pairs in the breaking sector altogether, with 12 pairs coming from the non-tilde fields of the breaking sector (27, $\overline{27}$, $351'$,$\overline{351'}$), while 3 pairs are in the tilde fields ($\widetilde{27}$, $\overline{\widetilde{27}}$).The triplets and antitriplets in the fermionic sector 27^i_F do not mediate proton decay, since the $\mathbb{Z}_2$ matter parity forbids cubic vertices 27^3_F.

Noting the full superpotential of our model I in equations (370),(371),(372) and (373), we can compute the relevant couplings for proton decay. In terms of SM representations in the fermionic 27_F's, they are

$$W\big|_{\text{proton}} = T_A\,(\mathcal{M}_T)^{AB}\,\overline{T}_B + C_1^{ijA}\,Q_i\,Q_j\,T_A + C_2^{ijA}\,u_i^c\,e_j^c\,T_A \\ + \overline{C}_1^{ijA}\,Q_i\,L_j\,\overline{T}_A + \overline{C}_1^{\prime ijA} Q_i\,L'_j\,\overline{T}_A + \overline{C}_2^{ijA}\,d_i^c\,u_j^c\,\overline{T}_A + \overline{C}_2^{\prime ijA}\,d_i^{\prime c}\,u_j^c\,\overline{T}_A, \quad (418)$$

where i,j are generation indices and $A,B = 1,\ldots,15$ are indices over all the color triplets/antitriplets, with sums over repeated indices meant implicitly; we have defined $T_{12+A} := \widetilde{T}_A$ and $\overline{T}_{12+A} := \overline{\widetilde{T}}_A$ (with $A = 1,2,3$). We suppressed the $\mathrm{SU}(3)_C$ and $\mathrm{SU}(2)_L$ indices in our notation; it is understood that these indices are contracted with the epsilon tensors in the order the fields themselves are written, with the convention $\varepsilon_{123} = \varepsilon_{12} = 1$.

The relevant terms consists of the mass terms of the triplets, along with the cubic couplings, written with the C coefficients. The triplet mass matrix $\mathcal{M}_T$ has contributions from the mass terms m_{27}, $m_{351'}$ and $m_{\widetilde{27}}$, the λ-terms and the κ-terms. The tilde and non-tilde fields do not mix in the mass terms (the tilde fields have vanishing VEVs), so $\mathcal{M}_T$ has the following block form:

$$\mathcal{M}_T = \begin{pmatrix} (\mathcal{M}_{\text{triplets}})_{12\times 12} & 0 \\ 0 & (\widetilde{\mathcal{M}}_{\text{triplets}})_{3\times 3} \end{pmatrix}. \quad (419)$$

The two block matrices can be found in equations (361) (with appropriate α and β) and (384), respectively. We also need to plug-in the vacuum solution of equations (375)-(381) and the DT fine-tuning from equation (389).

We now focus on the C-coefficient terms in equation (418), which come from the three Yukawa terms Y_{27}^{ij}, $Y_{351'}^{ij}$ and $Y_{\widetilde{27}}^{ij}$ in equation (373). We distinguish between the unbarred C's which couple to the triplets T, and the barred $\overline{C}$'s, which couple to the antitriplets $\overline{T}$. Furthermore, the barred C coefficients come in pairs, e.g. $\overline{C}_1$ and $\overline{C}'_1$, since the light leptons, denoted by $\hat{L}$, are a linear combination of L and L', and similarly the light down-type quarks $\hat{d}^c$ are a combination of d^c and d'^c. Since we are interested in the low-energy effective theory, we are interested in diagrams containing only light states (MSSM), so the terms of interest contain $\hat{L}$ and $\hat{d}^c$. The coefficients C are computed to have the explicit form

$$2\,C_1^{ijA} = -Y_{27}^{ij}\,\delta^A{}_1 - Y_{\widetilde{27}}^{ij}\,\delta^A{}_{1+12} + \tfrac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_5 - \tfrac{1}{2\sqrt{6}}\,Y_{351'}^{ij}\,\delta^A{}_7 - \tfrac{1}{2\sqrt{3}}\,Y_{351'}^{ij}\,\delta^A{}_{12}, \quad (420)$$

$$2\,C_2^{ijA} = -Y_{27}^{ij}\,\delta^A{}_1 - Y_{\widetilde{27}}^{ij}\,\delta^A{}_{1+12} + \tfrac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_5 - \tfrac{1}{2\sqrt{6}}\,Y_{351'}^{ij}\,\delta^A{}_7 + \tfrac{2}{2\sqrt{3}}\,Y_{351'}^{ij}\,\delta^A{}_{12}, \quad (421)$$

$$2\,\overline{C}_1^{ijA} = -Y_{27}^{ij}\,\delta^A{}_2 - Y_{\widetilde{27}}^{ij}\,\delta^A{}_{2+12} + \tfrac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_4 + \tfrac{1}{2\sqrt{2}}\,Y_{351'}^{ij}\,\delta^A{}_8, \quad (422)$$

$$2\,\overline{C}_1^{\prime ijA} = Y_{27}^{ij}\,\delta^A{}_3 + Y_{\widetilde{27}}^{ij}\,\delta^A{}_{3+12} - \tfrac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_9 - \tfrac{1}{2\sqrt{2}}\,Y_{351'}^{ij}\,\delta^A{}_{11}, \quad (423)$$

$$2\,\overline{C}_2^{ijA} = -Y_{27}^{ij}\,\delta^A{}_2 - Y_{\widetilde{27}}^{ij}\,\delta^A{}_{2+12} + \tfrac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_4 - \tfrac{1}{2\sqrt{2}}\,Y_{351'}^{ij}\,\delta^A{}_8, \quad (424)$$

$$2\,\overline{C}_2^{\prime ijA} = Y_{27}^{ij}\,\delta^A{}_3 + Y_{\widetilde{27}}^{ij}\,\delta^A{}_{3+12} - \tfrac{1}{2\sqrt{10}}\,Y_{351'}^{ij}\,\delta^A{}_9 + \tfrac{1}{2\sqrt{2}}\,Y_{351'}^{ij}\,\delta^A{}_{11}. \quad (425)$$

Note that the different coefficients in front of $\delta^A{}_{12}$ are a consequence of different Clebsch-Gordan coefficients.

We obtain the effective superpotential operators by integrating out the triplets T_A and antitriplets $\overline{T}_A$ from the relevant terms. We obtain (to lowest order in the operators)

$$W = -\Big(\overline{C}_1^{ijA}\, Q_i\, L_j + \overline{C}_1^{\prime ijA}\, Q_i\, L_j' + \overline{C}_2^{ijA}\, d_i^c\, u_j^c + \overline{C}_2^{\prime ijA}\, d_i^{\prime c}\, u_j^c\Big)\big(\hat{\mathcal{M}}_T^{-1}\big)_{AB}\Big(C_1^{klB}\, Q_k\, Q_l + C_2^{klB}\, u_k^c\, e_l^c\Big). \tag{426}$$

There is an important detail concerning the mass matrix $\mathcal{M}_T$, so we wrote the inverse matrix $\hat{\mathcal{M}}_T^{-1}$ with a hat. Note that a triplet–antitriplet mode is massless (due to the Higgs mechanism, which start operating once we plug-in the vacuum solution). This means the block $\mathcal{M}_{\text{triplets}}$ cannot be inverted, and the massless would-be Goldstone modes somehow need to be removed, since they represent unphysical degrees of freedom (we know these fields can be rotated out of the Yukawa terms by a gauge transformation, which is equivalent to plugging a zero for their field value, thus removing them). The removal of the would-be Goldstone modes is formally equivalent to introducing a mass term for these modes, integrating them out, and then pushing the introduced mass to infinity, so they decouple from the theory. With this idea, one can write a basis independent ansatz for the computation of the inverse matrix of the physical degrees of freedom, with the would-be Goldstone bosons automatically decoupled:

$$\hat{\mathcal{M}}_T^{-1} = \lim_{M\to\infty} \big(\mathcal{M}^{AB} + M\, f^A e^B\big)^{-1}, \tag{427}$$

where e^A and f^A are components of right and left null-mass eigenvectors of $\mathcal{M}_T$, respectively. We need not normalize them, since the normalization factors can be absorbed into the parameter M of the added mass term. In our basis of triplets, we take for example (the normalization)

$$e^A = \frac{3\sqrt{2}c_1\lambda_1^{2/3}\lambda_2^{1/3}}{m_{351'}}\,\delta^A{}_3 + \frac{\lambda_1^{1/3}}{\sqrt{6}\lambda_2^{1/3}}\,\delta^A{}_4 + \frac{6{c_1}^2\lambda_1^{2/3}\lambda_2^{1/3}\lambda_3}{m_{351'}^2}\,\delta^A{}_6 + \frac{\sqrt{5}\lambda_1^{1/3}}{\sqrt{6}\lambda_2^{1/3}}\,\delta^A{}_8 + \delta^A{}_{10}, \tag{428}$$

$$f^A = \frac{3m_{27}\lambda_1^{1/3}\lambda_2^{2/3}}{\sqrt{2}c_1\lambda_3\lambda_4}\,\delta^A{}_3 + \frac{\lambda_2^{1/3}}{\sqrt{6}\lambda_1^{1/3}}\,\delta^A{}_4 + \frac{3m_{27}^2\lambda_1^{1/3}\lambda_2^{2/3}}{2{c_1}^2\lambda_3^2\lambda_4}\,\delta^A{}_6 + \frac{\sqrt{5}\lambda_2^{1/3}}{\sqrt{6}\lambda_1^{1/3}}\,\delta^A{}_8 + \delta^A{}_{10}. \tag{429}$$

Although the procedure of adding the mass term is formally elegant, the method is hard to implement, since it requires an explicit inversion of a large matrix, and only then taking the limit $M\to\infty$ to remove the unphysical degrees of freedom. We now present an equivalent but computationally less troublesome procedure: we consider a general $(N+1)\times(N+1)$ matrix with one left and one right null eigenmode, and rotate it into a physical basis, where the basis vectors, in addition to the zero-mass vector, are the N states orthogonal to the Nambu-Goldstone mode. Suppose we write the normalized right and left Nambu-Goldstone eigenstates respectively by

$$\frac{e}{|e|} \equiv \begin{pmatrix}\sqrt{1-\alpha^\dagger\alpha}\\ \alpha\end{pmatrix}, \qquad \frac{f}{|f|} \equiv \begin{pmatrix}\sqrt{1-\bar{\alpha}^\dagger\bar{\alpha}}\\ \bar{\alpha}\end{pmatrix}, \tag{430}$$

with the columns $\alpha = \alpha^a$ and $\bar{\alpha} = \bar{\alpha}^a$, $a = 1,\ldots,N$. The unitary $(N+1)\times(N+1)$ matrix

$$U(\alpha) = \begin{pmatrix}\sqrt{1-\alpha^\dagger\alpha} & -\alpha^\dagger\\ \alpha & 1-\frac{\alpha\alpha^\dagger}{1+\sqrt{1-\alpha^\dagger\alpha}}\end{pmatrix} \tag{431}$$

then transforms the old basis T_A by

$$T_A \to U_A{}^B(\alpha) T_B, \tag{432}$$

where now the new basis is $T_B = (T_0, T_a)$ with T_0 the would-be Nambu-Goldstone triplet. The old basis antitriplets $\overline{T}_A$ are analogously transformed by $U(\bar{\alpha})$. The choice of U represents just one simple possibility of choosing the transformations matrix; the transformation is not unique and is defined up to an arbitrary rotation in the orthogonal complement of the zero mode (space of T_a's). Dropping the unphysical zero modes $T_0, \overline{T}_0$, equation (418) can now be written in the new basis of physical states as

$$\begin{aligned} W\Big|_{\text{proton}} &= T_a\,(U^T)^a{}_A(\alpha)\,(\mathcal{M}_T)^{AB}\,U_B{}^b(\bar{\alpha})\,\overline{T}_b + T_a\,(U^T)^a{}_A(\alpha)\left(C_1^{ijA}\,Q_i\,Q_j + C_2^{ijA}\,u_i^c\,e_j^c\right) \\ &+ \left(\overline{C}_1^{ijB}\,Q_i\,L_j + \overline{C}_1^{\prime ijB} Q_i\,L_j' + \overline{C}_2^{ijB}\,d_i^c\,u_j^c + \overline{C}_2^{\prime ijB}\,d_i^{\prime c}\,u_j^c\right)\,U_B{}^b(\bar{\alpha})\,\overline{T}_b. \end{aligned} \tag{433}$$

We now define the $N \times N$ invertible matrix

$$(m_T)^{ab} \equiv (U^T)^a{}_A\,(\alpha)(\mathcal{M}_T)^{AB}\,U_B{}^b(\bar{\alpha}), \tag{434}$$

and obtain the form of the proton decay superpotential written in equation (433) with the inverse of the $\mathcal{M}_{\text{triplets}}$ block given by

$$\left(\hat{\mathcal{M}}_T^{-1}\right)_{AB} = U_A{}^a(\bar{\alpha})\ \left(m_T^{-1}\right)_{ab}\,(U^T)^b{}_B(\alpha). \tag{435}$$

Finally, we also need to project the states onto the light matter superfields, i.e.onto $\hat{L}$ and $\hat{d}^c$. This projection involves finding the light combination of the two $\overline{5}$'s of SU(5) in the 27_F's, and has been computed in from the fermion masses in the Yukawa sector section 4.4.3, so d^c mixes with d'^c and L mixes with L'. This old basis is rotated into the basis of light and heavy states with the help of the matrix $\mathcal{U}$ in equation (400). Writing generically, the particles q and q' can be decomposed into light and heavy states q_l and q_H, respectively, with

$$\begin{pmatrix} q & q' \end{pmatrix} = \begin{pmatrix} q_l & q_H \end{pmatrix}\,\mathcal{U}. \tag{436}$$

This implies the following projections to the light states $\hat{d}^c$ and $\hat{L}$:

$$d_i^c = \left[(1 + \tfrac{4}{9} X_0^*\,X_0^T)^{-1/2}\right]_i{}^j\,\hat{d}_j^c + \dots, \tag{437}$$

$$d_i'^c = \left[\tfrac{2}{3} X_0^T\ (1 + \tfrac{4}{9} X_0^*\,X_0^T)^{-1/2}\right]_i{}^j\,\hat{d}_j^c + \dots, \tag{438}$$

$$L_i = \left[(1 + X_0^*\,X_0^T)^{-1/2}\right]_i{}^j\,\hat{L}_j + \dots, \tag{439}$$

$$L_i' = \left[-\,X_0^T\ (1 + X_0^*\,X_0^T)^{-1/2}\right]_i{}^j\,\hat{L}_j + \dots, \tag{440}$$

with X_0 defined in equation (413).

Writing only the terms of the light states (those at the scale m_W) for the lepton and baryon number violating operators, we get the following low-energy effective operators for $D = 5$ proton decay:

$$\begin{aligned} W\Big|_{\text{proton}} = &-\left[\left(\overline{C}_1^{inA} - \overline{C}_1^{\prime imA}(X_0^T)_m{}^n\right)\left[(1 + X_0^* X_0^T)^{-1/2}\right]_n{}^j\,(\hat{\mathcal{M}}_T^{-1})_{AB}\,C_1^{klB}\right] Q_i \hat{L}_j Q_k Q_l \\ &- \left[\left(\overline{C}_2^{njA} + \tfrac{2}{3}\overline{C}_2^{\prime mjA}(X_0^T)_m{}^n\right)\left[(1 + \tfrac{4}{9} X_0^* X_0^T)^{-1/2}\right]_n{}^i\,(\hat{\mathcal{M}}_T^{-1})_{AB}\,C_2^{klB}\right] \hat{d}_i^c u_j^c u_k^c e_l^c. \end{aligned} \tag{441}$$

Although the final expression is rather complicated, we can still draw some general conclusions without doing a detailed numerical analysis.

- Since we have a E_6, with many heavy particles and thus the possibility of several possible heavy thresholds, coupling unification does not necessarily require a light color triplet as in the minimal renormalizable SU(5) with low-scale supersymmetry [77, 34, 35].

- Similar to the minimal SO(10) case, which also has contributions from multiple triplets, only some elements of the inverse matrix need to be small. This is due to the fact that the triplets in $351'$, $\overline{27}$ and $\overline{\widetilde{27}}$ do not couple to the matter fields in the 27_F's, and neither do some triplets in $\overline{351'}$ (seen from the C-coefficients).

- The final expressions for proton decay are functions of the following parameters: the masses, the λ and κ parameters, as well as three Yukawa matrices (one λ is not arbitrary, and is determined by fine-tuning). Since the constraints on these parameters come from the fit to a smaller number of parameters of the SM Yukawas, there will likely be some residual freedom present in the parameter space; it is conceivable that this freedom can then be further used for proton decay suppression if necessary.

For the reasons above, we very likely have sufficient suppression of proton decay not to violate the phenomenological bounds. Finally, if all this fails, we can still use some version of a (moderately) split supersymmetric spectrum.

4.4.5 Summary

The model $3 \cdot 27_F \oplus 27 \oplus \overline{27} \oplus 351' \oplus \overline{351'} \oplus \widetilde{27} \oplus \overline{\widetilde{27}}$, as far as a non-numeric study shows, is realistic. The model was found to contain the following features:

- There exist vacuum solutions which break E_6 directly to the Standard Model, all of them equivalent. We obtained the full classification of solutions, proving that the solutions breaking to the Standard Model are equivalent, while others break either to Pati-Salam, SU(5), or higher. We only considered solutions where the non-tilde part of the breaking sector acquires non-vanishing VEVs. This does not seem a major limitation, and we likely get an analogous result even if we allow for the tilde fields to get VEVs and write the full superpotential without any distinction between the tilde and non-tilde sectors.

- Doublet-triplet splitting can only occur in the tilde sector, while the non-tilde part of the breaking sector contains would-be Goldstone bosons of the doublets and triplets. All the doublets in the tilde block get EW VEVs.

- The Yukawa sector gives realistic masses: we get the correct number of light SM degrees of freedom. There are 3 Yukawa matrices, with 2 involved in high energy mixing between the two $\overline{5}$'s in the 27_F, while the tilde Yukawa matrix is relevant for EW masses. Flavor mixing occurs due to different parts of the $\overline{5}$'s being mixed differently at the GUT scale. Neutrino masses are light and get type I and II seesaw contributions. Vector-like states are at the GUT scale.

- We computed the contributions to $D = 5$ proton decay and argued why the proton decay rate can be made sufficiently small.

- The β function in the RG running of the coupling for this model is -153.

- Although the model is realistic, it is likely not very predictive due to the 3 Yukawa matrices.

4.5 Model II: $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27} \oplus 78$

This model is the second possible minimalistic extension of the prototype model of section 4.3. We add an adjoint 78 chiral supermultiplet, so that the breaking sector consists of

$$351' \oplus \overline{351'} \;\oplus\; 27 \oplus \overline{27} \;\oplus\; 78. \tag{442}$$

In contrast to model I in section 4.4, we shall not impose any further restrictions than the usual matter parity (under which the fermionic fields 27_F are odd, while all the multiplets in the breaking sector are even). Due to the representation 78 not coupling to two symmetric 27_F's, the Yukawa sector will be a simpler than in model I, but the presence of the new representation 78 in the breaking sector will make the old symmetry breaking solution invalid, so we will need to solve the EOM from scratch. In this model, there is no further division of the breaking sector, and fields can and will acquire both GUT and EW scale VEVs.

The superpotential of this model is the following:

$$W = W_{\text{SSB}} + W_{\text{Yukawa}}, \tag{443}$$

with

$$\begin{aligned} W_{\text{SSB}} = \;& m_{351'}\, I_{351'\otimes\overline{351'}} + m_{27}\, I_{27\otimes\overline{27}} + m_{78}\, I_{78^2} \\ & + \lambda_1\, I_{351'^3} + \lambda_2\, I_{\overline{351'}^3} + \lambda_3\, I_{27^2\otimes\overline{351'}} + \lambda_4\, I_{\overline{27}^2\otimes 351'} \\ & + \lambda_5\, I_{27^3} + \lambda_6\, I_{\overline{27}^3} + \lambda_7\, I_{27\otimes 78\otimes\overline{78}} + \lambda_8\, I_{351'\otimes 78\otimes\overline{351'}}, \end{aligned} \tag{444}$$

$$W_{\text{Yukawa}} = \sum_{i,j=1}^{3} \tfrac{1}{2}\left(Y_{27}^{ij}\, I_{27^i_F\otimes 27^j_F\otimes 27} + Y_{\overline{351'}}^{ij}\, I_{27^i_F\otimes 27^j_F\otimes\overline{351'}}\right). \tag{445}$$

Compared to the superpotential of the prototype model in equation (247), we identify the Yukawa sector to be identical, while the breaking sector has an extra mass term and two extra λ terms.

4.5.1 Symmetry breaking

4.5.1.1 Equations of motion Since we will be considering solutions with $\langle 78\rangle \neq 0$, the equations of motion will differ from those of the prototype model. We still take the ansatz $\langle 27^i_F\rangle = 0$, so the fermion sector is not involved in symmetry breaking and can be ignored in the equations of motion (due it being odd under matter parity).

In this model, the number of SM singlets is $19 = 14 + 5$, with the 14 being familiar from the prototype model, while there are 5 new ones in the representation 78, which we denote by u_1, u_2, v, w and y, as is shown in Table 6.

The F-terms will not be given in explicit form here, since they are quite complicated, but are easily computed by considering the breaking part of the superpotential W_{SSB} in equation (444), and plugging in the explicit expressions for the invariants from Table 11 and equations (174)–(190). One computes the F terms by taking derivatives over the all the singlet fields, so we have 19 F-terms:

$$F_{c_i}, \qquad F_{d_i}, \qquad F_{e_j}, \qquad F_{f_j}, \qquad F_{u_i}, \qquad F_v, \qquad F_w, \qquad F_y, \tag{446}$$

where $i = 1, 2$ and $j = 1, 2, 3, 4, 5$.

The D-terms are also different and have to be computed. They have the schematic form

$$\begin{aligned} D^a = \quad & (27^\dagger)_i\, (\hat{t}^a\, 27)^i + (\overline{27}^\dagger)^i\, (\hat{t}^a\, \overline{27})_i + (78^\dagger)^j{}_i\, (\hat{t}^a\, 78)^i{}_j \\ & + (351'^\dagger)_{ij}\, (\hat{t}^a\, 351')^{ij} + (\overline{351'}^\dagger)^{ij}\, (\hat{t}^a\, \overline{351'})_{ij}, \end{aligned} \tag{447}$$

where $\hat{t}^a$ is the action of the a-th generator, where a runs from 1 to 78. As in the prototype model, only 5 D-terms do not vanish trivially, and they correspond to the generators, which are singlets under the Standard Model: t^8_L, t^3_R. t^6_R, t^7_R and t^8_R. They are computed to be

$$\begin{aligned} D^8_L = \tfrac{1}{\sqrt{3}} & \left(|c_1|^2 + |c_2|^2 + 2|e_1|^2 + 2|e_2|^2 + 2|e_3|^2 - |e_4|^2 - |e_5|^2\right. \\ & \left. -|d_1|^2 - |d_2|^2 - 2|f_1|^2 - 2|f_2|^2 - 2|f_3|^2 + |f_4|^2 + |f_5|^2\right), \end{aligned} \tag{448}$$

$$\begin{aligned} D^3_R = \tfrac{1}{6} & \left(-3|c_2|^2 - 6|e_1|^2 - 3|e_2|^2 + 3|e_5|^2 - |u_1|^2\right. \\ & \left. +3|d_2|^2 + 6|f_1|^2 + 3|f_2|^2 - 3|f_5|^2 + |u_2|^2\right), \end{aligned} \tag{449}$$

$$\begin{aligned} D^6_R = \tfrac{1}{12} & \Big(6c_2{c_1}^* + 6c_1{c_2}^* - \sqrt{3}u_1w^* - \sqrt{3}w{u_1}^* + \sqrt{5}u_1v^* + \sqrt{5}v{u_1}^* \\ & -6d_2{d_1}^* - 6d_1{d_2}^* + \sqrt{3}u_2w^* + \sqrt{3}w{u_2}^* - \sqrt{5}u_2v^* - \sqrt{5}v{u_2}^* \\ & +6\sqrt{2}e_1{e_2}^* + 6\sqrt{2}e_2{e_1}^* + 6\sqrt{2}e_2{e_3}^* + 6\sqrt{2}e_3{e_2}^* + 6e_4{e_5}^* + 6e_5{e_4}^* \\ & -6\sqrt{2}f_1{f_2}^* - 6\sqrt{2}f_2{f_1}^* - 6\sqrt{2}f_2{f_3}^* - 6\sqrt{2}f_3{f_2}^* - 6f_4{f_5}^* - 6f_5{f_4}^*\Big), \end{aligned} \tag{450}$$

$$\begin{aligned} D^7_R = \tfrac{i}{12} & \Big(6c_2{c_1}^* - 6c_1{c_2}^* - \sqrt{3}u_1w^* + \sqrt{3}w{u_1}^* + \sqrt{5}u_1v^* - \sqrt{5}v{u_1}^* \\ & +6d_2{d_1}^* - 6d_1{d_2}^* - \sqrt{3}u_2w^* + \sqrt{3}w{u_2}^* + \sqrt{5}u_2v^* - \sqrt{5}v{u_2}^* \\ & +6\sqrt{2}e_1{e_2}^* - 6\sqrt{2}e_2{e_1}^* + 6\sqrt{2}e_2{e_3}^* - 6\sqrt{2}e_3{e_2}^* + 6e_4{e_5}^* - 6e_5{e_4}^* \\ & +6\sqrt{2}f_1{f_2}^* - 6\sqrt{2}f_2{f_1}^* + 6\sqrt{2}f_2{f_3}^* - 6\sqrt{2}f_3{f_2}^* + 6f_4{f_5}^* - 6f_5{f_4}^*\Big), \end{aligned} \tag{451}$$

$$\begin{aligned} D^8_R = \tfrac{1}{2\sqrt{3}} & \left(-2|c_1|^2 + |c_2|^2 + 2|e_1|^2 - |e_2|^2 - 4|e_3|^2 + 2|e_4|^2 - |e_5|^2 + |u_1|^2\right. \\ & \left. +2|d_1|^2 - |d_2|^2 - 2|f_1|^2 + |f_2|^2 + 4|f_3|^2 - 2|f_4|^2 + |f_5|^2 - |u_2|^2.\right) \end{aligned} \tag{452}$$

As in the prototype mode, the combination corresponding to the hypercharge generator $Y/2$ has to be trivially zero. There are therefore 4 real constraints from the D-terms, 2 of which can be written with a single complex equation. Using the same combinations $D^I = \sqrt{3}D^8_L + 2D^3_R$, $D^{II} = -2D^3_R$, $D^{III} = D^6_R + iD^7_R$ as in the prototype model, the new D-terms are

$$D^I = |c_1|^2 - |d_1|^2 + |e_2|^2 - |f_2|^2 + 2|e_3|^2 - 2|f_3|^2 - |e_4|^2 + |f_4|^2 - \tfrac{1}{3}|u_1|^2 + \tfrac{1}{3}|u_2|^2, \tag{453}$$

$$D^{II} = |c_2|^2 - |d_2|^2 + |e_2|^2 - |f_2|^2 + 2|e_1|^2 - 2|f_1|^2 - |e_5|^2 + |f_5|^2 + \tfrac{1}{3}|u_1|^2 - \tfrac{1}{3}|u_2|^2, \tag{454}$$

$$\begin{aligned} D^{III} = & +c_1{c_2}^* - \tfrac{\sqrt{3}}{6}w{u_1}^* + \tfrac{\sqrt{5}}{6}v{u_1}^* + \sqrt{2}e_2{e_1}^* + \sqrt{2}e_3{e_2}^* + e_5{e_4}^* \\ & - d_2{d_1}^* + \tfrac{\sqrt{3}}{6}u_2w^* - \tfrac{\sqrt{5}}{6}u_2v^* - \sqrt{2}f_1{f_2}^* - \sqrt{2}f_2{f_3}^* - f_4{f_5}^*. \end{aligned} \tag{455}$$

The new D-terms are of course similar to the old D-terms from equations (268)–(270), except for the extra terms with the new SM singlet VEVs u_1, u_2, v, w and y.

One can immediately notice that y is not present in the D-terms, while it is present only in the mass term of W (see equations (174)–(190) and also the group-theoretic discussion in section 4.2.4), so we automatically have $y = 0$.

4.5.1.2 Symmetries of EOM In the prototype model in section 4.3.1.2, we looked at two symmetries of the EOM. We now look at those symmetries again in light of the presence of a new representation 78, which plays a part through the 3 new invariants in the model.

1. *Conjugation symmetry*: we extend the symmetry operation by defining its action on the 78. For complex representations, conjugation symmetry exchanges the representation and its conjugate. But 78 is a real representation: its conjugate is again 78, so we expect conjugation symmetry to operate by exchanging VEVs within the 78. By looking at U(1)$'$ and U(1)$''$ quantum numbers of the SM singlets in Table 6, we can deduce that u_1 and u_2 need to be exchanged, since they have opposite quantum numbers, while the other SM singlets in the 78 can remain the same. Indeed, we can define the conjugation symmetry operation by

$$c_i \leftrightarrow d_i, \tag{456}$$
$$e_i \leftrightarrow f_i, \tag{457}$$
$$u_1 \leftrightarrow u_2, \tag{458}$$
$$\lambda_1 \leftrightarrow \lambda_2, \tag{459}$$
$$\lambda_3 \leftrightarrow \lambda_4, \tag{460}$$
$$\lambda_5 \leftrightarrow \lambda_6. \tag{461}$$

   Under this symmetry, each of the 3 new invariants I_{78^2}, $I_{27\otimes78\otimes\overline{27}}$ and $I_{351'\otimes78\otimes\overline{351'}}$ remains the same, so the superpotential W in equation (443) stays the same, and hence the system of equations of F-terms stays the same. Note that the mass m_{78} and the two new couplings λ_7 and λ_8 do not need to change, since each new invariant is conjugate to itself. The D-terms in equations (453)–(455), change the same way as in the prototype model: $D^I \to -D^I$, $D^{II} \to -D^{II}$ and $D^{III} \to -D^{III*}$. The transformed D-terms represent equivalent conditions to the untransformed versions, so conjugation symmetry also preserved the D-term part.

   Due to the presence of conjugation symmetry, we again approach solving the EOM by first solving the F-terms, and worrying about the D-terms after.

2. *Alignment symmetry*: **no longer present**. This symmetry operated within each representation and each of the old invariants is symmetric under the alignment operation. Since the new invariants are constructed from different representations, they too should each be symmetric. We could again define the exchange $u_1 \leftrightarrow u_2$, and we take care of the mass term I_{78^2}. But the new cubic invariants are not alignment symmetric and the singlets v and w pose problems. y is not present in the cubic terms, so it can remain the same. If we keep v and w, $v \to v$ and $w \to w$, the new cubic's change under the alignment

transformation. Suppose now that we want to change them non-trivially with a linear transformation. If we want this to be a parity transformation, its square needs to be the identity ($A^2 = I$), so it needs to have one of the following forms:

$$\begin{pmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{pmatrix}, \qquad \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \tag{462}$$

where the $\pm$ signs are independent of each other. We can check by explicit computation that all these transformations fail to impose symmetry on the new cubic invariants for any α and β independent from the other VEVs. This means the alignment symmetry operation cannot be extended as a parity transformation when we include the 78.

4.5.1.3 A solution of EOM Due to conjugation symmetry, our strategy for solving the EOM will again be to first solve the F-terms, and then deal with the D-terms. With the inclusion of the 78, the EOM now become much more complicated, and there is little hope we will be able to make a full analysis of the solutions; knowing alternative solutions in this case is not that crucial, however, since the solution we will present here already seem to be viable. We proceed in the following steps:

- First, we see that F_y leads directly to $y = 0$. Taking the ansatz

$$c_1 = d_1 = f_5 = e_5 = 0, \tag{463}$$

and further

$$u_1 = u_2 = e_2 = f_2 = 0, \tag{464}$$

the system simplifies quite a bit: $F_{c_1}, F_{d_1}, F_{e_2}, F_{f_2}, F_{e_5}, F_{f_5}, F_{u_1}, F_{u_2}, F_y$ and D^{III} are solved automatically.

- Solve F_{e_3} and F_{f_3} for f_3 and e_3, respectively.

$$e_3 = -\frac{9{f_4}^2\lambda_2}{3m_{27} - \sqrt{2}w\lambda_8}, \qquad f_3 = -\frac{9{e_4}^2\lambda_1}{3m_{27} - \sqrt{2}w\lambda_8}. \tag{465}$$

- Solve F_{c_2} and Fd_2 for f_1 and e_1, respectively.

$$e_1 = \frac{-12c_2 m_{351'} + \sqrt{2}wc_2\lambda_7 + \sqrt{30}vc_2\lambda_7}{24d_2\lambda_4}, \tag{466}$$

$$f_1 = \frac{-12d_2 m_{351'} + \sqrt{2}wd_2\lambda_7 + \sqrt{30}vd_2\lambda_7}{24c_2\lambda_3}. \tag{467}$$

- Simultaneously solve F_{e_4} and F_{f_4} for f_4:

$$f_4 = \frac{18m_{27}^2 - 3\sqrt{2}w\lambda_8 m_{27} - 2w^2\lambda_8^2}{324e_4\lambda_1\lambda_2}. \tag{468}$$

- Simultaneously solve F_{e_1} and F_{f_1} for d_2:

$$\begin{aligned} d_2 = \frac{1}{144c_2\lambda_3\lambda_4}\Big(&\lambda_7\lambda_8 w^2 - 6\sqrt{2}m_{27}\lambda_7 w - 6\sqrt{2}m_{351'}\lambda_8 w + 2\sqrt{15}wv\lambda_7\lambda_8 \\ &+72m_{27}m_{351'} - 6\sqrt{30}vm_{27}\lambda_7 - 6\sqrt{30}vm_{351'}\lambda_8 + 15v^2\lambda_7\lambda_8\Big). \end{aligned} \tag{469}$$

- For our convenience, we shall define the VEV quantity A through $w \equiv \frac{A-15v}{\sqrt{15}}$, which explicitly gives

$$A := 15v + \sqrt{15}w. \tag{470}$$

We can now solve F_v for v:

$$v = \frac{1}{34560 m_{78}\lambda_3\lambda_4}\Big(360\sqrt{30}\lambda_8 m_{351'}^2 + 720\sqrt{30} m_{27}\lambda_7 m_{351'} \\ -240A\lambda_7\lambda_8 m_{351'} - 120A m_{27}\lambda_7^2 + \sqrt{30}A^2\lambda_7^2\lambda_8\Big). \tag{471}$$

- Three variables remain to be determined: A, c_2 and e_4. We are left with only one unsolved F-term F_w, which is a polynomial in A:

$$0 = P_0 + P_1 A + P_2 A^2 + P_3 A^3 + P_4 A^4, \tag{472}$$

$$P_0 = -\frac{m_{351'}\left(2m_{27}\lambda_7 + m_{351'}\lambda_8\right)}{165888\sqrt{2}m_{78}^2\lambda_1\lambda_2\lambda_3^2\lambda_4^2}\Big(25 m_{351'}\left(2m_{27}\lambda_7 + m_{351'}\lambda_8\right)\lambda_8^3 \\ - 480 m_{27}m_{78}\lambda_3\lambda_4\lambda_8^2 + 110592 m_{78}^2\lambda_1\lambda_2\lambda_3\lambda_4\Big), \tag{473}$$

$$P_1 = \frac{1}{497664\sqrt{15}m_{78}^2\lambda_1\lambda_2\lambda_3^2\lambda_4^2}\Big(995328\lambda_1\lambda_2\lambda_3^2\lambda_4^2 m_{78}^3 \\ + 4608\lambda_3\lambda_4\left(24 m_{351'}\lambda_1\lambda_2\lambda_7\lambda_8 + m_{27}\left(12\lambda_1\lambda_2\lambda_7^2 - \lambda_3\lambda_4\lambda_8^2\right)\right)m_{78}^2 \\ + 240\lambda_3\lambda_4\lambda_8^2\left(-m_{27}^2\lambda_7^2 + 2m_{27}m_{351'}\lambda_8\lambda_7 + 2m_{351'}^2\lambda_8^2\right)m_{78} \\ + 25 m_{351'}\lambda_7\lambda_8^3\left(2m_{27}^2\lambda_7^2 + 5m_{27}m_{351'}\lambda_8\lambda_7 + 2m_{351'}^2\lambda_8^2\right)\Big), \tag{474}$$

$$P_2 = -\frac{1}{89579520\sqrt{2}m_{78}^2\lambda_1\lambda_2\lambda_3^2\lambda_4^2}\Big(\lambda_8\left(18432\lambda_3\lambda_4\left(9\lambda_1\lambda_2\lambda_7^2 + \lambda_3\lambda_4\lambda_8^2\right)m_{78}^2 \\ +240\lambda_3\lambda_4\lambda_7\lambda_8^2\left(5m_{27}\lambda_7 + 16 m_{351'}\lambda_8\right)m_{78} \\ +25\lambda_7^2\lambda_8^2\left(2m_{27}^2\lambda_7^2 + 14 m_{27}m_{351'}\lambda_8\lambda_7 + 11 m_{351'}^2\lambda_8^2\right)\right)\Big), \tag{475}$$

$$P_3 = \frac{\lambda_7^2\lambda_8^4\left(96 m_{78}\lambda_3\lambda_4 + 5\lambda_7\left(m_{27}\lambda_7 + 2m_{351'}\lambda_8\right)\right)}{35831808\sqrt{15}m_{78}^2\lambda_1\lambda_2\lambda_3^2\lambda_4^2}, \tag{476}$$

$$P_4 = -\frac{\lambda_7^4\lambda_8^5}{859963392\sqrt{2}m_{78}^2\lambda_1\lambda_2\lambda_3^2\lambda_4^2}. \tag{477}$$

Note that the coefficients P_i depend only on the Lagrangian parameters; choosing those, we can determine A numerically.

The remaining D-terms are

$$0 = D^I = 2|e_3|^2 - |e_4|^2 - 2|f_3|^2 + |f_4|^2, \tag{478}$$
$$0 = D^{II} = |c_2|^2 - |d_2|^2 + 2|e_1|^2 - 2|f_1|^2. \tag{479}$$

Note that w and v are determined, once A is. Fixing w and v and examining the obtained expressions so far, we can see that $f_4 \propto 1/e_4$, $e_3 \propto f_4^2 \propto 1/e_4^2$ and $f_3 \propto e_4^2$. Therefore D^I can be written as a quartic polynomial in $|e_4|^2$; the constant term has a positive coefficient (coefficient in front of the e_3 term), while the highest order term in $|e_4|^2$ has a negative coefficient: a solution for a real $e_4 > 0$ will always exist. Similarly, $d_2 \propto 1/c_2$, $e_1 \propto c_2/d_2 \propto c_2^2$

and $f_1 \propto d_2/c_2 \propto 1/c_2^2$, D^{II} is a quartic polynomial in c_2 independent of e_4; the constant coefficient will be negative (the f_1 term), while the highest order coefficient in $|c_2|^2$ is positive (the e_1 term), which again guarantees a real solution $c_2 > 0$.

The solution is then as follows: A is determined through the polynomial in equations (472)–(477), e_4 and c_2 are determined at the end through the polynomials in D^I and D^{II}, respectively, while the other VEVs are:

$$c_1 = 0, \qquad d_1 = 0, \tag{480}$$
$$e_2 = 0, \qquad f_2 = 0, \tag{481}$$
$$e_5 = 0, \qquad f_5 = 0, \tag{482}$$
$$u_1 = 0, \qquad u_2 = 0, \tag{483}$$
$$y = 0, \tag{484}$$

$$w = \frac{1}{2304\sqrt{15}m_{78}\lambda_3\lambda_4}\Bigg(-\sqrt{30}A^2\lambda_8\lambda_7^2 - 360\sqrt{30}m_{351'}\left(2m_{27}\lambda_7 + m_{351'}\lambda_8\right) + 24A\left(96m_{78}\lambda_3\lambda_4 + 5\lambda_7\left(m_{27}\lambda_7 + 2m_{351'}\lambda_8\right)\right)\Bigg), \tag{485}$$

$$v = \frac{1}{34560m_{78}\lambda_3\lambda_4}\left(\left(360\sqrt{30}m_{351'}^2 - 240A\lambda_7 m_{351'} + \sqrt{30}A^2\lambda_7^2\right)\lambda_8 - 120m_{27}\lambda_7\left(A\lambda_7 - 6\sqrt{30}m_{351'}\right)\right). \tag{486}$$

Then in terms of A, w, v, c_2 and e_4:

$$d_2 = \frac{6m_{27}\left(180m_{351'} - \sqrt{30}A\lambda_7\right) + A\left(A\lambda_7 - 6\sqrt{30}m_{351'}\right)\lambda_8}{2160c_2\lambda_3\lambda_4}, \tag{487}$$

$$e_1 = \frac{6{c_2}^2\lambda_3\left(180m_{351'} - \sqrt{30}A\lambda_7\right)}{m_{27}\left(6\sqrt{30}A\lambda_7 - 1080m_{351'}\right) + A\left(6\sqrt{30}m_{351'} - A\lambda_7\right)\lambda_8}, \tag{488}$$

$$f_1 = \frac{1}{777600{c_2}^2\lambda_3^2\lambda_4}\Bigg(A\left(1080\sqrt{30}m_{351'}^2 - 360A\lambda_7 m_{351'} + \sqrt{30}A^2\lambda_7^2\right)\lambda_8 - 180m_{27}\left(1080m_{351'}^2 - 12\sqrt{30}A\lambda_7 m_{351'} + A^2\lambda_7^2\right)\Bigg), \tag{489}$$

$$e_3 = -\frac{\left(-18m_{27}^2 + 3\sqrt{2}w\lambda_8 m_{27} + 2w^2\lambda_8^2\right)^2}{11664{e_4}^2\lambda_1^2\lambda_2\left(3m_{27} - \sqrt{2}w\lambda_8\right)}, \tag{490}$$

$$f_3 = -\frac{9{e_4}^2\lambda_1}{3m_{27} - \sqrt{2}w\lambda_8}, \tag{491}$$

$$f_4 = \frac{18m_{27}^2 - 3\sqrt{2}w\lambda_8 m_{27} - 2w^2\lambda_8^2}{324e_4\lambda_1\lambda_2}. \tag{492}$$

As we can see, the solution is very complicated and has to be computed in steps; finding all the solutions of the EOM in model II would be prohibitively complicated, if analytically possible at all.

4.5.1.4 Properties of the solution There are two things we need to check:

1. We have to confirm that the solution from subsection 4.5.1.3 does indeed break E_6 into the SM group. We do that by computing gauge boson masses, similar to what we did for the solution in the prototype model in section 4.3.1.4. Using the ansatz of vanishing VEVs

$$0 = c_1 = d_1 = e_5 = f_5 = e_2 = f_2 = u_1 = u_2 = y, \tag{493}$$

we get 12 massless gauge bosons (exactly those of the SM) as shown in Table 19, assuming the remaining VEVs are non-vanishing. It is hard to analytically check that other VEVs do not vanish, but we did check numerically, by taking generic values for the breaking parameters of the model: the masses and the λ's.

2. We also need to check that there are no flat directions in the F-terms, so that all singlet fields are massive. In this model, there are 19 SM singlets in the breaking sector, so their mass matrix is a 19×19 matrix. Since the matrix is prohibitively large to be written on A4 paper, we will not write it here explicitly. But assisted by a computer and doing it numerically, it is indeed possible to see that generic values of parameters give 4 massless singlets, which are exactly the 4 would-be Goldstone bosons in the $E_6 \to$ SM breaking, as discussed in section 4.3.1.4.

Both conditions check out proving that the solution is viable.

4.5.2 Doublet-triplet splitting

We shall now perform doublet triplet splitting in this model. Compared to the prototype model of section 4.3, we now have an additional doublet-antidoublet pair and an additional triplet-antitriplet pair, so the mass matrices now become 12×12 for the doublets and 13×13 for the triplets. The doublet D and triplet T states are defined in Table 9, with the new states in the 78 having the index 0. The mass terms are written as

$$\begin{pmatrix} D_0 & \cdots & D_{11} \end{pmatrix} \mathcal{M}_{\text{doublets}} \begin{pmatrix} \overline{D}_1 \\ \vdots \\ \overline{D}_{11} \end{pmatrix} + \begin{pmatrix} T_0 & \cdots & T_{12} \end{pmatrix} \mathcal{M}_{\text{triplets}} \begin{pmatrix} \overline{T}_1 \\ \vdots \\ \overline{T}_{12} \end{pmatrix}. \tag{494}$$

The two mass matrices $\mathcal{M}_{\text{doublets}}$ and $\mathcal{M}_{\text{triplets}}$ can be compactly written with equation (495), where we remove the last row and column and take $\alpha = -3$ and $\beta = -\sqrt{3}$ for the doublet matrix $\mathcal{M}_{\text{doublets}}$, while we keep the full size matrix and take $\alpha = \beta = 2$ for the triplet matrix $\mathcal{M}_{\text{triplets}}$. Also, the ansatz of vanishing VEVs in equation (493) is already applied. Note that the compact matrix has the block form $(1+3+9) \times (1+3+9)$, where the numbers correspond to states in the representations 78, $27 \oplus \overline{27}$ and $351' \oplus \overline{351'}$, respectively. The compact matrix is written in block form

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{495}$$

where the diagonal blocks are

$$M_{11} = \begin{pmatrix} m_{78} & 0 & \frac{\lambda_7 d_2}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & m_{27}+\frac{\lambda_7(\sqrt{3}v+\sqrt{5}w)}{3\sqrt{10}} & \frac{\alpha\lambda_3 f_4}{\sqrt{15}} & 6\lambda_5 c_2 & 0 & 0 \\ \frac{\lambda_7 c_2}{2\sqrt{3}} & \frac{\alpha\lambda_4 e_4}{\sqrt{15}} & m_{27}+\frac{\lambda_7(\sqrt{5}w-\sqrt{3}v)}{3\sqrt{10}} & 0 & 0 & 0 \\ 0 & 6\lambda_6 d_2 & 0 & m_{27}+\frac{\lambda_7(3\sqrt{3}v-\sqrt{5}w)}{6\sqrt{10}} & 0 & -\frac{\lambda_4 d_2}{\sqrt{10}} \\ 0 & 0 & 0 & 0 & m_{351'}+\frac{\lambda_8 v}{2\sqrt{30}}-\frac{\lambda_8 w}{6\sqrt{2}} & \sqrt{\frac{3}{5}}\alpha\lambda_1 e_4 \\ 0 & 0 & 0 & -\frac{\lambda_3 c_2}{\sqrt{10}} & \sqrt{\frac{3}{5}}\alpha\lambda_2 f_4 & m_{351'}-\frac{\lambda_8 v}{2\sqrt{30}}-\frac{\lambda_8 w}{6\sqrt{2}} \end{pmatrix}, \tag{496}$$

$$M_{22} = \begin{pmatrix} m_{351'}+\frac{\sqrt{3}}{4\sqrt{10}}\lambda_8 v-\frac{5\lambda_8 w}{12\sqrt{2}} & 0 & 0 & \frac{1}{2}\sqrt{3}\alpha\lambda_2 f_4 & 0 & \frac{1}{2}\sqrt{15}\beta\lambda_2 f_4 & 0 \\ 0 & m_{351'}-\frac{\lambda_8 v}{2\sqrt{30}}-\frac{\lambda_8 w}{6\sqrt{2}} & \sqrt{5}\beta\lambda_2 f_4 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{5}\beta\lambda_1 e_4 & m_{351'}+\frac{\lambda_8 v}{2\sqrt{30}}-\frac{\lambda_8 w}{6\sqrt{2}} & 0 & 0 & 0 & 2\sqrt{10}\lambda_1 e_4 \\ \frac{1}{2}\sqrt{3}\alpha\lambda_1 e_4 & 0 & 0 & m_{351'}-\frac{\sqrt{3}}{4\sqrt{10}}\lambda_8 v+\frac{\lambda_8 w}{12\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{351'}-\frac{7\lambda_8 v}{4\sqrt{30}}+\frac{\lambda_8 w}{12\sqrt{2}} & 0 & 0 \\ \frac{1}{2}\sqrt{15}\beta\lambda_1 e_4 & 0 & 0 & 0 & 0 & m_{351'}-\frac{\sqrt{3}}{4\sqrt{10}}\lambda_8 v+\frac{\lambda_8 w}{12\sqrt{2}} & 0 \\ 0 & 0 & 2\sqrt{10}\lambda_2 f_4 & 0 & 0 & 0 & m_{351'}-\frac{\lambda_8 v}{2\sqrt{30}}-\frac{\lambda_8 w}{6\sqrt{2}} \end{pmatrix}, \tag{497}$$

and the off-diagonal blocks are

$$M_{12} = \begin{pmatrix} -\frac{\lambda_8 f_3}{2\sqrt{6}} & 0 & 0 & \frac{\alpha\lambda_8 e_4}{24\sqrt{2}} & -\frac{\lambda_8 f_1}{2\sqrt{6}} & \frac{\sqrt{5}\beta\lambda_8 e_4}{24\sqrt{2}} & 0 \\ 0 & 0 & 0 & 2\sqrt{\frac{2}{5}}\lambda_3 c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2}\lambda_4 d_2 & 0 & 0 \\ 0 & -\sqrt{\frac{3}{2}}\lambda_4 d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{498}$$

$$M_{21} = \begin{pmatrix} -\frac{\lambda_8 e_3}{2\sqrt{6}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{3}{2}}\lambda_3 c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha\lambda_8 f_4}{24\sqrt{2}} & 2\sqrt{\frac{2}{5}}\lambda_4 d_2 & 0 & 0 & 0 & 0 \\ -\frac{\lambda_8 e_1}{2\sqrt{6}} & 0 & -\sqrt{2}\lambda_3 c_2 & 0 & 0 & 0 \\ \frac{\sqrt{5}\beta\lambda_8 f_4}{24\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{499}$$

We note that taking the full vacuum solution from subsection 4.5.1.3, the determinants of both matrices are zero:

$$\det(\mathcal{M}_{\text{doublets}}) = 0, \tag{500}$$
$$\det(\mathcal{M}_{\text{triplets}}) = 0. \tag{501}$$

This implies that there is a massless doublet-antidoublet pair and a massless triplet-antitriplet pair, which are would-be Goldstone modes corresponding to $E_6 \to$ SM breaking, as discussed in section 4.3.2. Fine tuning now needs to be performed, so that the doublet matrix gets an extra massless mode, while keeping the triplets massive. The condition on the mass matrix has already been discussed in the section of the prototype model:

$$\text{Cond}(\mathcal{M}) := \frac{\lim_{\epsilon\to 0}\det(\mathcal{M}+\epsilon I)/\epsilon}{\langle f|e\rangle} = 0, \tag{502}$$

where $|e\rangle$ and $|f\rangle$ are the right and left zero-mass eigenmodes of $\mathcal{M}$.

We now want to perform a fine-tuning, such that

$$\text{Cond}(\mathcal{M}_{\text{doublets}}) = 0, \tag{503}$$
$$\text{Cond}(\mathcal{M}_{\text{triplets}}) \neq 0. \tag{504}$$
$$\tag{505}$$

The best parameters in which to perform the fine-tuning are λ_5 and λ_6 due to the following two reasons:

- These are the coupling constants in front of the operators 27^3 and $\overline{27}^3$. These invariants do not contain singlet-only terms, so they do not appear in the EOM or the vacuum solution. Choosing values for them has no impact on the vacuum solution or its properties.

- There is only one instance of λ_5 and one instance of λ_6 in the mass matrix of equation (495). This means they will be present linearly in conditions (503) and (504). Since they are located at entries which exchange under transposing the mass matrix, it turns out they are always present in the form of the product, so that the fine-tuning condition have the form

$$K_1 - K_2\,\lambda_5\,\lambda_6 = 0, \tag{506}$$
$$K_1' - K_2'\,\lambda_5\,\lambda_6 \neq 0, \tag{507}$$

  where K_1, K_1', K_2 and K_2' are expressions of the other parameters of the breaking sector $(m_{351'}, m_{78}, m_{27}, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_8)$.

Computing the expressions K_1, K_1', K_2 and K_2' analytically is very hard, since even the left and right zero-mode eigenvectors of the matrices have a very complicated form. But we checked numerically that taking generic values for the other parameters and computing the values of K_1, K_1', K_2 and K_2', it is possible to satisfy both conditions. For the fine-tuning, we take for example

$$\lambda_5 = \frac{K_1}{K_2\,\lambda_6}, \tag{508}$$

and plugging this into the second condition, we get

$$K_1' K_2 \neq K_2' K_1. \tag{509}$$

For generic values of parameters, we indeed do not get an equality, so the doublet-triplet splitting is successful, although computationally not particularly simple. Numerically, we can check, after identifying which zero-mode is the would-be Goldstone, that the new Higgs zero-modes (the MSSM fields H_u ad H_d) have components in all the doublets and anti-doublets. We denote the EW VEVs by v_i and $\bar{v}_i$ with $i = 0, 1, \ldots, 11$. The Higgses having components in all the doublets implies all the EW VEVs v_i and $\bar{v}_i$ are nonzero. Also, we have the relations

$$v_u = \sum_{i=0}^{11} v_i^2, \tag{510}$$
$$v_d = \sum_{i=0}^{11} \bar{v}_i^2, \tag{511}$$

where v_u and v_d are EW VEVs of H_u and H_d (choosing a basis, which has these two among the basis vectors).

Finally, we compare the DT-splitting scenario in the prototype model and model II. The current model has the mass matrices of the doublets and triplets enlarged by one (both in rows and in columns). In the prototype model, we also had a single appearance of the same parameters λ_5 and λ_6, but when computing the $\lim_{\epsilon\to 0} \det(\mathcal{M} + \epsilon I)/\epsilon$ part of Cond, the product term $\lambda_5\lambda_6$ disappeared once the vacuum solution was inserted (essentially $K_2 = K_2' = 0$ in the prototype model). In the current model, however, it appears that the addition of the new row and column, as well as a new vacuum solution, now do not make the $\lambda_5\lambda_6$ term vanish. The specific of why this happens are not clear and we are not aware of any theoretical arguments, which would predict such a result *a priori*: it is only after explicit computation that we discover this fact.

4.5.3 Yukawa sector

The Yukawa terms of this model are the in the W_{Yukawa} part of the superpotential in equation (445). Schematically, we have the terms

$$27^i_F 27^j_F \left(Y^{ij}_{27}\, 27 + Y^{ij}_{\overline{351'}}\, \overline{351'}\right). \tag{512}$$

We therefore have two symmetric Yukawa matrices in our model: Y_{27} and $Y_{\overline{351'}}$. The Yukawa sector in this model is analogous to the $10 \oplus \overline{126}$ Yukawa sector of minimal SO(10).

In model II, flavor mixing will come about in the standard way: the Higgs fields need to be both in the 27 and in the $\overline{351'}$. In fact, both H_u and H_d are part of $351'$ and $\overline{27}$ also, so the fermions only see part of both Higgs fields.

Computing the mass terms explicitly, we get (skipping the hermitian conjugate terms)

$$\begin{aligned}
&u^T\left(-v_1 Y_{27} + \tfrac{1}{2\sqrt{10}} v_5 Y_{\overline{351'}} - \tfrac{1}{2\sqrt{6}} v_7 Y_{\overline{351'}}\right) u^c \\
&+ \begin{pmatrix} d^{cT} & d'^{cT}\end{pmatrix} \begin{pmatrix} \bar v_2 Y_{27} + \frac{1}{2\sqrt{10}}\bar v_4 Y_{\overline{351'}} + \frac{1}{2\sqrt{6}}\bar v_8 Y_{\overline{351'}} & c_2 Y_{27} + \frac{1}{\sqrt{15}} f_5 Y_{\overline{351'}} \\ -\bar v_3 Y_{27} - \frac{1}{2\sqrt{10}}\bar v_9 Y_{\overline{351'}} - \frac{1}{2\sqrt{6}}\bar v_{11} Y_{\overline{351'}} & -c_1 Y_{27} + \frac{1}{\sqrt{15}} f_4 Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} d \\ d' \end{pmatrix} \\
&+ \begin{pmatrix} e^T & e'^T\end{pmatrix} \begin{pmatrix} -\bar v_2 Y_{27} - \frac{1}{2\sqrt{10}}\bar v_4 Y_{\overline{351'}} + \sqrt{\frac{3}{8}}\bar v_8 Y_{\overline{351'}} & c_2 Y_{27} - \frac{3}{2}\frac{1}{\sqrt{15}} f_5 Y_{\overline{351'}} \\ \bar v_3 Y_{27} + \frac{1}{2\sqrt{10}}\bar v_9 Y_{\overline{351'}} - \sqrt{\frac{3}{8}}\bar v_{11} Y_{\overline{351'}} & -c_1 Y_{27} - \frac{3}{2}\frac{1}{\sqrt{15}} f_4 Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} e^c \\ e'^c \end{pmatrix} \\
&+ \begin{pmatrix} \nu^T & \nu'^T\end{pmatrix} \begin{pmatrix} v_1 Y_{27} - \frac{1}{2\sqrt{10}} v_5 Y_{\overline{351'}} - \sqrt{\frac{3}{8}} v_7 Y_{\overline{351'}} & -\frac{1}{\sqrt{2}} v_6 Y_{\overline{351'}} & c_2 Y_{27} - \frac{3}{2}\frac{1}{\sqrt{15}} f_5 Y_{\overline{351'}} \\ -\frac{1}{\sqrt{2}} v_{10} Y_{\overline{351'}} & -v_1 Y_{27} - \sqrt{\frac{2}{5}} v_5 Y_{\overline{351'}} & -c_1 Y_{27} - \frac{3}{2}\frac{1}{\sqrt{15}} f_4 Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} \nu^c \\ s \\ \nu'^c \end{pmatrix} \\
&+ \frac{1}{2}\begin{pmatrix} \nu^{cT} & s^T & \nu'^{cT}\end{pmatrix} \begin{pmatrix} f_1 Y_{\overline{351'}} & \frac{1}{\sqrt{2}} f_2 Y_{\overline{351'}} & -\bar v_3 Y_{27} + \sqrt{\frac{2}{5}}\bar v_9 Y_{\overline{351'}} \\ \frac{1}{\sqrt{2}} f_2 Y_{\overline{351'}} & f_3 Y_{\overline{351'}} & \bar v_2 Y_{27} - \sqrt{\frac{2}{5}}\bar v_4 Y_{\overline{351'}} \\ -\bar v_3 Y_{27} + \sqrt{\frac{2}{5}}\bar v_9 Y_{\overline{351'}} & \bar v_2 Y_{27} - \sqrt{\frac{2}{5}}\bar v_4 Y_{\overline{351'}} & 0 \end{pmatrix} \begin{pmatrix} \nu^c \\ s \\ \nu'^c \end{pmatrix} \\
&+ \frac{1}{2}\begin{pmatrix} \nu^T & \nu'^T\end{pmatrix} \begin{pmatrix} \Delta_1 Y_{\overline{351'}} & \frac{1}{\sqrt{2}}\Delta_2 Y_{\overline{351'}} \\ \frac{1}{\sqrt{2}}\Delta_2 Y_{\overline{351'}} & \Delta_3 Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} \nu \\ \nu' \end{pmatrix}.
\end{aligned} \tag{513}$$

The notation used above suppresses flavor indices. We again color coded GUT scale VEVs and EW scale VEVs. The ansatz of vanishing VEVs in equation (493) is not yet taken into account. The EW VEVs correspond to the following doublet/antidoublets: v_i is the EW VEV of the doublet D_i, while $\bar v_i$ is the VEV of the antidoublet $\overline{D}_i$ (see). Due to the structure of the Yukawa sector, only the VEVs (both EW, but also GUT scale) from the representations $\overline{351'}$ and 27 are present.

As in model I, the matrices in the down-quark sector and the charged lepton sector (and also the appropriate part of the neutrino matrices) have very similar structure due to these field being part of the same SU(5) representations $\bar{5}$ or 5. Comparing these matrices of different sectors, the GUT scale VEVs are positioned in the same places. The coefficients in front of GUT scale VEVs are the same, if the VEV is an SU(5) singlet, or are in the ratio of $-3/2$ if the GUT scale VEV comes from the 24 of SU(5). The situation is very similar for EW VEVs: the down and charged lepton sectors have them in the same places, with also the same coefficients (up to a minus sign coming from the definition of the fermion states in Figure 5) in front of most of the EW VEVs. The exceptions are the EW VEVs $\bar{v}_8$ and $\bar{v}_{11}$, which are part of the $\overline{45}$ of SU(5), so they couple differently to doublets and triplets, unlike the more standard $\bar{5}$'s of SU(5) (for Higgs doublet definitions see Table 9).

Notice the following peculiarity: if a solution has for example an ansatz $c_2 = f_5 = 0$, we eliminate the upper-right entry in the down-quark and charged lepton matrices, which means that up to leading order in $M_{\mathrm{EW}/M_{\mathrm{GUT}}}$, the heavy states are in the remaining heavy block in the lower-right part of the matrices, and the light states are the ones in the upper-left corner: the states in the 16 are light and the states in the 10 are heavy, so there is no mixing between the two $\bar{5}$'s in the 27_F. A similar situation would occur if we instead chose $c_1 = f_4 = 0$, when the upper-right part would become heavy and the lower-left would become light. If we had these situations, and if we further had $\bar{v}_8 = \bar{v}_{11} = 0$ (the $\langle\overline{45}\rangle_{\mathrm{EW}}$'s of SU(5)), all the Higgses would be in the $\bar{5}$'s in either the lower- or the upper-left part, and we would have the bad SU(5) prediction $M_D^T = M_E$ at the GUT scale (see section 2.4.1). This presence of the 45's is even more crucial in SO(10) GUT theories, where there are no vector-like quarks to alleviate the problem; for this reason, the minimal SO(10) models needs to have a $\overline{126}$ representation of the Higgs (which has a $\overline{45}$ of SU(5)) in addition to the 10 (with the $\bar{5}$ of SU(5)). If we just had multiple 10's of Higgses, they would all couple to the down-quarks and to the charged leptons equally, again recreating the Yukawa couplings $M_D^T = M_E$.

In our solution, we have the ansatz $c_1 = f_5 = 0$. This means that both the upper- and lower-left part of the matrices in the down sector and the charged-lepton sector are non-vanishing. This means that in our case, we indeed do have a mixing of 16_F and 10_F, so that the light states are present in both. Since the coefficients in front of the f_4 VEV differ, the 16_F–10_F mixing for the light states in the down sector is different than in the charged lepton sector. Furthermore, we determined in section 4.5.2 that all EW VEVs are nonzero, so the difference in the down sector and lepton sector is also due to the different coefficients in front of the $\langle\overline{45}\rangle_{\mathrm{EW}}$'s of SU(5) (the $\bar{v}_8$ and $\bar{v}_{11}$). Ultimately, the important fact is that our model contains mechanisms to have $M_E \neq M_D^T$ for the light states.

Some comments on the masses of the neutrino sector. Again, as in model I, we have similar terms, where ν^c and s are Majorana neutrinos due to upper-left 2×2 block in the second neutrino matrix; ν^c and s are thus heavy. We also have an additional heavy pair of ν'^c and a combination of ν and ν' due to the last column in the first neutrino matrix.

The third neutrino matrix represents type II seesaw contributions from $\bar{\Delta} \sim (1,3,+1)$ and $\Delta \sim (1,3,-1)$ weak triplet scalars, as discussed in section 4.4.3. Note that the matrix has the same form as in model I, but the values of Δ's are not the same. The triplets Δ and $\overline{\Delta}$ are still only in $351'$ and $\overline{351'}$, with their definitions already given in Table 18. There are, however, new terms with the 78 in W_{SSB}, so the Δ's are integrated out differently: writing out all the terms from equation (444), we get

$$
\begin{aligned}
W\big|_{\Delta} = &\begin{pmatrix} \overline{\Delta}_1 & \overline{\Delta}_2 & \overline{\Delta}_3 & \overline{\Delta}_4 \end{pmatrix} M_{\Delta} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{pmatrix} \\
&+ \begin{pmatrix} \overline{\Delta}_1 & \overline{\Delta}_2 & \overline{\Delta}_3 & \overline{\Delta}_4 \end{pmatrix} \begin{pmatrix} \lambda_4 {v_3}^2 \\ \lambda_4 \sqrt{2} v_2 v_3 \\ \lambda_4 {v_2}^2 \\ \lambda_3 {v_1}^2 \end{pmatrix} \\
&+ \begin{pmatrix} \lambda_3 {\overline{v}_3}^2 & \lambda_3 \sqrt{2} \overline{v}_3 \overline{v}_2 & \lambda_3 {\overline{v}_2}^2 & \lambda_4 {\overline{v}_1}^2 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{pmatrix} .
\end{aligned}
\tag{514}
$$

The mass matrix M_{Δ} gets contributions from both the $351' \cdot \overline{351'}$ term, as well as from $351' \cdot \langle 78 \rangle \cdot \overline{351'}$. It has the explicit form

$$
M_{\Delta} = \begin{pmatrix}
m_{351'} - \lambda_8 \left(\frac{w}{6\sqrt{2}} - \frac{1}{2}\sqrt{\frac{3}{10}} v + \frac{y}{2\sqrt{5}} \right) & \frac{u_1 \lambda_8}{2\sqrt{3}} & 0 & 6\lambda_1 e_1 \\
\frac{\lambda_8}{2\sqrt{3}} u_2 & m_{351'} + \lambda_8 \left(\frac{w}{12\sqrt{2}} + \frac{v}{4\sqrt{30}} - \frac{y}{2\sqrt{5}} \right) & \frac{\lambda_8}{2\sqrt{3}} u_1 & -6\lambda_1 e_2 \\
0 & \frac{\lambda_8}{2\sqrt{3}} u_2 & m_{351'} + \lambda_8 \left(\frac{w}{3\sqrt{2}} - \frac{v}{\sqrt{30}} - \frac{y}{2\sqrt{5}} \right) & 6\lambda_1 e_3 \\
6\lambda_2 f_1 & -6\lambda_2 f_2 & 6\lambda_2 f_3 & m_{351'} + \lambda_8 \left(\frac{w}{3\sqrt{2}} + \frac{v}{\sqrt{30}} + \frac{y}{2\sqrt{5}} \right)
\end{pmatrix} .
\tag{515}
$$

Integrating out the heavy triplets and inserting our F-term ansatz of vanishing VEVs in the 78 ($u_1 = u_2 = y = 0$),) we get

$$
\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{pmatrix} = \begin{pmatrix}
m_{351'} - \lambda_8 \left(\frac{w}{6\sqrt{2}} - \frac{1}{2}\sqrt{\frac{3}{10}} v \right) & 0 & 0 & 6\lambda_1 e_1 \\
0 & m_{351'} + \lambda_8 \left(\frac{w}{12\sqrt{2}} + \frac{v}{4\sqrt{30}} \right) & 0 & 0 \\
0 & 0 & m_{351'} + \lambda_8 \left(\frac{w}{3\sqrt{2}} - \frac{v}{\sqrt{30}} \right) & 6\lambda_1 e_3 \\
6\lambda_2 f_1 & 0 & 6\lambda_2 f_3 & m_{351'} + \lambda_8 \left(\frac{w}{3\sqrt{2}} + \frac{v}{\sqrt{30}} \right)
\end{pmatrix}^{-1} \begin{pmatrix} \lambda_4 {v_3}^2 \\ \lambda_4 \sqrt{2} v_2 v_3 \\ \lambda_4 {v_2}^2 \\ \lambda_3 {v_1}^2 \end{pmatrix} .
\tag{516}
$$

To get the mass matrices of the light states, we integrate the heavy states out of the matrices in equation (513), as discussed for the Yukawa sector in model I in section 4.4.3. Using $c_1 = f_5 = 0$ from the vacuum solution, we define the "ratio matrix" X_0 by

$$
X_0 = -2\sqrt{\tfrac{5}{3}} \, \frac{c_2}{f_4} \, Y_{27} \, Y_{351'}^{-1} .
\tag{517}
$$

After some computation, which is a bit more involved especially in the neutrino sector, we get the low energy matrices:

$$M_U = -v_1 Y_{27} + \left(\tfrac{1}{2\sqrt{10}} v_5 - \tfrac{1}{2\sqrt{6}} v_7\right) Y_{\overline{351'}}, \tag{518}$$

$$M_D^T = \left(1 + (9/4) X_0 X_0^\dagger\right)^{-1/2} \left(\left(\bar{v}_2 - \tfrac{3}{2}\bar{v}_3 X_0\right) Y_{27} + \left(\tfrac{1}{2\sqrt{10}}(\bar{v}_4 - \tfrac{3}{2}\bar{v}_9 X_0) + \tfrac{1}{2\sqrt{6}}(\bar{v}_8 - \tfrac{3}{2}\bar{v}_{11} X_0)\right) Y_{\overline{351'}}\right), \tag{519}$$

$$M_E = \left(1 + X_0 X_0^\dagger\right)^{-1/2} \left(\left(-\bar{v}_2 - \bar{v}_3 X_0\right) Y_{27} + \left(-\tfrac{1}{2\sqrt{10}}(\bar{v}_4 + \bar{v}_9 X_0) + \sqrt{\tfrac{3}{8}}(\bar{v}_8 + \bar{v}_{11} X_0)\right) Y_{\overline{351'}}\right), \tag{520}$$

$$\begin{aligned} M_N = &- \left(1 + X_0 X_0^\dagger\right)^{-1/2} \\ &\Bigg(\left(-\tfrac{1}{\sqrt{10}}\tfrac{v_1 v_5}{f_1} - \sqrt{\tfrac{3}{2}}\tfrac{v_1 v_7}{f_1} + \tfrac{1}{\sqrt{3}}\tfrac{v_5 v_{10}}{f_1}\tfrac{c_2}{f_4} + \sqrt{5}\,\tfrac{v_7 v_{10}}{f_1}\tfrac{c_2}{f_4} + \tfrac{4}{\sqrt{3}}\tfrac{v_5 v_6}{f_3}\tfrac{c_2}{f_4} - 2\sqrt{\tfrac{10}{3}}\,\Delta_2\,\tfrac{c_2}{f_4}\right) Y_{27} \\ &+ \left(\frac{1}{40}\frac{{v_5}^2}{f_1} + \sqrt{\frac{3}{80}}\frac{v_7 v_5}{f_1} + \frac{3}{8}\frac{{v_7}^2}{f_1} + \frac{1}{2}\frac{{v_6}^2}{f_3} - \Delta_1\right) Y_{\overline{351'}} \\ &+ \left(\tfrac{{v_1}^2}{f_1} - 2\sqrt{\tfrac{10}{3}}\tfrac{v_1 v_{10}}{f_1}\tfrac{c_2}{f_4} + \tfrac{10}{3}\tfrac{{v_{10}}^2}{f_1}\tfrac{{c_2}^2}{{f_4}^2} + 2\sqrt{\tfrac{10}{3}}\tfrac{v_1 v_6}{f_3}\tfrac{c_2}{f_4} + \tfrac{8}{3}\tfrac{{v_5}^2}{f_3}\tfrac{{c_2}^2}{{f_4}^2} - \tfrac{20}{3}\,\Delta_3\,\tfrac{{c_2}^2}{{f_4}^2}\right) Y_{27} Y_{\overline{351'}}^{-1} Y_{27} \\ &+ \left(\frac{8\sqrt{10}}{3}\frac{v_1 v_5}{f_3}\frac{{c_2}^2}{{f_4}^2}\right) Y_{27} Y_{\overline{351'}}^{-1} Y_{27} Y_{\overline{351'}}^{-1} Y_{27} \\ &+ \left(\frac{20}{3}\frac{{v_1}^2}{f_3}\frac{{c_2}^2}{{f_4}^2}\right) Y_{27} Y_{\overline{351'}}^{-1} Y_{27} Y_{\overline{351'}}^{-1} Y_{27} Y_{\overline{351'}}^{-1} Y_{27}\Bigg) \\ &\left(1 + X_0^* X_0^T\right)^{-1/2}. \end{aligned} \tag{521}$$

Analogous to model I, the neutrinos have seesaw type I (the non-Δ terms) and type II (the Δ terms) contributions, but no type III. All contributions to the neutrinos are $\mathcal{O}(M_{\mathrm{EW}}^2/M_{GUT})$: type I contributions manifestly have this structure (with possible extra factors $c_2/f_4 \sim \mathcal{O}(1)$), while we see the scale of Δ's from equation (516).

The general conclusions on the fermion masses in model II are very similar to model I: all the exotic degrees of freedom are heavy (vector-like states are thus at the GUT scale), the light states in the $\bar{5}$ of SU(5) are a mixture of the $\bar{5}$'s in the 16_F and 10_F of SO(10) (mixing controlled by X_0), and neutrinos have type I and type II seesaw contributions.

The main difference between the two models, however, is in the number of Yukawa matrices and the mechanism of flavor mixing. In this model, we get flavor mixing in the usual way of GUTs: the Higgs is simultaneously present in two representations, which couple to the fermions. Furthermore, having only two Yukawa matrices decreases the number of parameters compared to model I, and makes the parameter situation similar to minimal SO(10) models [32, 33]. Therefore, this Yukawa structure seems viable, but it is not a forgone conclusion that the fit can work out. Due to the smaller number of parameters in the Yukawa sector, model II is more predictive, so a numeric fit would be instructive.

We leave the numeric fit to be done in the future.

Our model has the following parameters :

- 3 mass parameters: m_{27}, $m_{351'}$ and m_{78}.
- 10 couplings: 8 λ's. The parameters λ_5 and λ_6 are not involved in the low-energy mass matrices of fermions, but one is determined by the fine-tuning condition.

- 2 symmetric Yukawa matrices. (Not all parameters here are physical though, since one matrix can be diagonalized by a rotation in the 3 families).

4.5.4 Proton decay

We now perform the same analysis of $D = 5$ proton decay in model II as was done in section 4.4.4 for model I. The results are completely analogous, we only need to compute the new triplet mass matrix and the new C coefficients. The proton decay operators in the superpotential are

$$\begin{aligned} W\big|_{\text{proton}} = &-\Big[\big(\overline{C}_1^{inA} - \overline{C}_1^{\prime imA}(X_0^T)_m{}^n\big)\big[(1 + X_0^* X_0^T)^{-1/2}\big]_n{}^j \,(\hat{\mathcal{M}}_T^{-1})_{AB}\, C_1^{klB}\Big]\, Q_i \hat{L}_j Q_k Q_l \\ &- \Big[\big(\overline{C}_2^{njA} + \tfrac{2}{3}\overline{C}_2^{\prime mjA}(X_0^T)_m{}^n\big)\big[(1 + \tfrac{4}{9} X_0^* X_0^T)^{-1/2}\big]_n{}^i \,(\hat{\mathcal{M}}_T^{-1})_{AB}\, C_2^{klB}\Big]\, \hat{d}_i^c u_j^c u_k^c e_l^c, \end{aligned} \tag{522}$$

with the X_0 in model II now defined by equation (517), the mass matrix $\mathcal{T}$ is now from equation (495) (with the solution from equations (480)–(492) and a numeric fine-tuning of the product $\lambda_5\lambda_6$), and the C coefficients now computed to be

$$2\, C_1^{ijA} = -Y_{27}^{ij}\,\delta^A{}_1 + \tfrac{1}{2\sqrt{10}}\, Y_{351'}^{ij}\,\delta^A{}_5 - \tfrac{1}{2\sqrt{6}}\, Y_{351'}^{ij}\,\delta^A{}_7 - \tfrac{1}{2\sqrt{3}}\, Y_{351'}^{ij}\,\delta^A{}_{12}, \tag{523}$$

$$2\, C_2^{ijA} = -Y_{27}^{ij}\,\delta^A{}_1 + \tfrac{1}{2\sqrt{10}}\, Y_{351'}^{ij}\,\delta^A{}_5 - \tfrac{1}{2\sqrt{6}}\, Y_{351'}^{ij}\,\delta^A{}_7 + \tfrac{2}{2\sqrt{3}}\, Y_{351'}^{ij}\,\delta^A{}_{12}, \tag{524}$$

$$2\, \overline{C}_1^{ijA} = -Y_{27}^{ij}\,\delta^A{}_2 + \tfrac{1}{2\sqrt{10}}\, Y_{351'}^{ij}\,\delta^A{}_4 + \tfrac{1}{2\sqrt{2}}\, Y_{351'}^{ij}\,\delta^A{}_8, \tag{525}$$

$$2\, \overline{C}_1^{\prime ijA} = Y_{27}^{ij}\,\delta^A{}_3 - \tfrac{1}{2\sqrt{10}}\, Y_{351'}^{ij}\,\delta^A{}_9 - \tfrac{1}{2\sqrt{2}}\, Y_{351'}^{ij}\,\delta^A{}_{11}, \tag{526}$$

$$2\, \overline{C}_2^{ijA} = -Y_{27}^{ij}\,\delta^A{}_2 + \tfrac{1}{2\sqrt{10}}\, Y_{351'}^{ij}\,\delta^A{}_4 - \tfrac{1}{2\sqrt{2}}\, Y_{351'}^{ij}\,\delta^A{}_8, \tag{527}$$

$$2\, \overline{C}_2^{\prime ijA} = Y_{27}^{ij}\,\delta^A{}_3 - \tfrac{1}{2\sqrt{10}}\, Y_{351'}^{ij}\,\delta^A{}_9 + \tfrac{1}{2\sqrt{2}}\, Y_{351'}^{ij}\,\delta^A{}_{11}. \tag{528}$$

We see that the C-coefficients are the same as the coefficient in equations (420)–(425) of model I, when we remove the tilde fields of model I by $Y_{\widetilde{27}} = 0$. In particular, there are no $A = 0$ contributions from triplet/antitriplets in the new representation 78. No $A = 0$ triplets/antitriplets is expected due the fact that the 78 does not couple to two 27_F's (no 78 term in the Yukawa sector), which is a requirement due to the definition of the C terms.

The general remarks on why proton decay rate can be very likely made small are the same as in section 4.4.4 for model I: not all values of the inverse $(\hat{\mathcal{M}}_T)^{-1}$ need to be small (since not all triplets contribute to proton decay), no necessary lighter triplet (multiple possible thresholds), and some possible redundancy in the parameter space. In this model, however, the number of parameters is smaller than in model I, so less redundancy in parameter space is expected, and similar to the Yukawa section of this model, a numeric analysis would be needed to fully confirm that the proton decay rate can be made small enough.

4.5.5 Summary

The model $3 \cdot 27_F \oplus 27 \oplus \overline{27} \oplus 351' \oplus \overline{351'} \oplus 78$, as far as a non-numeric study shows, is realistic. The model was found to contain the following features:

- We found a solution, which break E_6 to the Standard Model. A full classification of solutions, however, was not obtained.

- Doublet-triplet splitting occurs by fine-tuning in the breaking part, where all representations contribute with doublets and triplets. The triplet matrix is a 13×13 matrix, allowing for some features only to be checked numerically. The big matrices contain would-be Goldstone bosons of the vacuum solution. The massless doublet/antidoublet are present in all the doublet/antidoublet components — all components get a nonzero EW VEV.

- The Yukawa sector gives realistic masses: we get the correct number of light SM degrees of freedom. There are 2 Yukawa matrices, which allow for flavor mixing in the standard way in GUT. The $\overline{5}$'s in the 27_F mix. Neutrino masses are light and get type I and II seesaw contributions. Vector-like states cannot be light and are at the GUT scale.

- We computed the contributions to $D = 5$ proton decay and argued why the proton decay rate can be made sufficiently small in the same way as in model I.

- The β function in the RG running of the coupling for this model is -159.

- Since this model contains only 2 Yukawa matrices, it is more predictive than model I, and a numeric fit of the results is left for the future.

Table 19: Masses-squared of gauge bosons in model II using the ansatz $c_1 = d_1 = e_2 = f_2 = e_5 = f_5 = u_1 = u_2 = y = 0$.

SO(10) ⊃	SU(5) ⊃	SM ⊃	$(\text{mass})^2/g^2$
45	24	$(8,1,0)$	0
45	24	$(1,3,0)$	0
45	24	$(1,1,0)$	0
45	24	$(3,2,+\frac{5}{6})$ $(\overline{3},2,-\frac{5}{6})$	$\frac{5}{6}\left(\lvert e_4\rvert^2+\lvert f_4\rvert^2\right)$
45	10 $\overline{10}$	$(3,2,+\frac{1}{6})$ $(\overline{3},2,-\frac{1}{6})$	$\frac{4}{15}\lvert v\rvert^2+\frac{1}{2}\lvert c_2\rvert^2+\frac{1}{2}\lvert d_2\rvert^2$ $+\lvert e_1\rvert^2+\lvert f_1\rvert^2+\frac{5}{6}\lvert e_4\rvert^2+\frac{5}{6}\lvert f_4\rvert^2$
45	10 $\overline{10}$	$(\overline{3},1,-\frac{2}{3})$ $(3,1,+\frac{2}{3})$	$\frac{4}{15}\lvert v\rvert^2+\frac{1}{2}\lvert c_2\rvert^2+\frac{1}{2}\lvert d_2\rvert^2+\lvert e_1\rvert^2+\lvert f_1\rvert^2$
45	10 $\overline{10}$	$(1,1,+1)$ $(1,1,-1)$	$\frac{4}{15}\lvert v\rvert^2+\frac{1}{2}\lvert c_2\rvert^2+\frac{1}{2}\lvert d_2\rvert^2+\lvert e_1\rvert^2+\lvert f_1\rvert^2$
16 $\overline{16}$	10 $\overline{10}$	$(3,2,+\frac{1}{6})$ $(\overline{3},2,-\frac{1}{6})$	$\frac{1}{60}\lvert\sqrt{15}w-v\rvert^2+\lvert e_3\rvert^2+\lvert f_3\rvert^2+\frac{1}{2}\lvert e_4\rvert^2+\frac{1}{2}\lvert f_4\rvert^2$
16 $\overline{16}$	10 $\overline{10}$	$(\overline{3},1,-\frac{2}{3})$ $(3,1,+\frac{2}{3})$	$\frac{1}{60}\lvert\sqrt{15}w-v\rvert^2+\lvert e_3\rvert^2+\lvert f_3\rvert^2+\frac{1}{2}\lvert e_4\rvert^2+\frac{1}{2}\lvert f_4\rvert^2$
16 $\overline{16}$	10 $\overline{10}$	$(1,1,+1)$ $(1,1,-1)$	$\frac{1}{60}\lvert\sqrt{15}w-v\rvert^2+\lvert e_3\rvert^2+\lvert f_3\rvert^2+\frac{1}{2}\lvert e_4\rvert^2+\frac{1}{2}\lvert f_4\rvert^2$
16 $\overline{16}$	$\overline{5}$ 5	$(\overline{3},1,+\frac{1}{3})$ $(3,1,-\frac{1}{3})$	$\frac{1}{4}\lvert w+\sqrt{3/5}\,v\rvert^2+\frac{1}{2}\lvert c_2\rvert^2+\frac{1}{2}\lvert d_2\rvert^2+\lvert e_1\rvert^2+\lvert f_1\rvert^2+$ $+\lvert e_3\rvert^2+\lvert f_3\rvert^2+\frac{1}{2}\lvert e_4\rvert^2+\frac{1}{2}\lvert f_4\rvert^2$
16 $\overline{16}$	$\overline{5}$ 5	$(1,2,-\frac{1}{2})$ $(1,2,+\frac{1}{2})$	$\frac{1}{4}\lvert w+\sqrt{3/5}\,v\rvert^2+\frac{1}{2}\lvert c_2\rvert^2+\frac{1}{2}\lvert d_2\rvert^2+\lvert e_1\rvert^2+\lvert f_1\rvert^2+$ $+\lvert e_3\rvert^2+\lvert f_3\rvert^2+\frac{1}{2}\lvert e_4\rvert^2+\frac{1}{2}\lvert f_4\rvert^2$
45 1	1 1	$(1,1,0)$ $(1,1,0)$	They mix: $\frac{2}{3}\left((A+B)\pm\sqrt{(A+B)^2-\frac{15}{4}AB}\right)$, $A\equiv 4\lvert e_3\rvert^2+4\lvert f_3\rvert^2+\lvert e_4\rvert^2+\lvert f_4\rvert^2$ $B\equiv 4\lvert e_1\rvert^2+4\lvert f_1\rvert^2+\lvert c_2\rvert^2+\lvert d_2\rvert^2$
16 $\overline{16}$	1 1	$(1,1,0)$ $(1,1,0)$	They mix: $\frac{1}{2}\left((C+D+\lvert F\rvert^2)\pm\sqrt{(C-D)^2+16\lvert E\rvert^2}\right)$, $C\equiv\lvert c_2\rvert^2+2\lvert e_1\rvert^2+2\lvert f_3\rvert^2+\lvert e_4\rvert^2$ $D\equiv\lvert d_2\rvert^2+2\lvert f_1\rvert^2+2\lvert e_3\rvert^2+\lvert f_4\rvert^2$ $E\equiv e_1{e_3}^*+{f_1}^*f_3$ $F\equiv\sqrt{\frac{5}{6}}v-\sqrt{\frac{1}{2}}w$

5 Conclusion

In this PhD thesis, we studied various renormalizable supersymmetric E_6 models, constructing the breaking sector with representations 27, 78, 351, 351′ and 650 (and their conjugates, where applicable) in various combinations.

We concluded that some of the simplest models are not viable. The following models were found not to be able to break to the Standard Model group:

- Renormalizable models with only representations 27, $\overline{27}$ and 78, with arbitrary many copies of each representation.
- A model with $351 \oplus \overline{351}$, and arbitrary many copies of 27 and $\overline{27}$.
- The $351' \oplus \overline{351'}$ model.
- The 650 model.

The model $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}$, which we called the prototype model, was found to have a solution breaking E_6 to the Standard Model group, but it failed due to its inability to perform doublet-triplet splitting. Building on the partial success of this model, we found two realistic extensions:

- Model I: $351' \oplus \overline{351'} \;\oplus\; 27 \oplus \overline{27} \;\oplus\; \widetilde{27} \oplus \overline{\widetilde{27}}$.
- Model II: $351' \oplus \overline{351'} \;\oplus\; 27 \oplus \overline{27} \;\oplus\; 78$.

For both models, we found a vacuum breaking E_6 to the Standard Model group, successfully performed doublet-triplet splitting, analyzed the Yukawa sector, computed the low-energy mass matrices and computed contributions to $D = 5$ proton decay. At the analytic level, both models are realistic. Numerically, however, we expect model II to be more predictive, since it has only 2 Yukawa matrices compared to 3 Yukawa terms of model I. To really check whether model II can predict the correct values of the masses and mixing angles, a numeric fit would need to be performed.

We did not consider, however, the details of the RG flow of the gauge couplings in these models. Since both in model I and II the low-energy theory is MSSM, with all other degrees of freedom approximately at the GUT scale, we assumed that unification of gauge couplings does indeed take place (since heavy extra degrees of freedom have little effect on the running at that late stage). Another concern in the running are the large negative β functions -153 and -159, which can lead to a Landau pole relatively quickly once all the degrees of freedom are present (though a similar problem is also present in SO(10); the minimal SUSY model has the β function -109 [21, 22, 23]). To study the details of the gauge coupling unification, and to study the RG flow of the single coupling after unification, one would tediously need to compute the masses of all degrees of freedom in the models. A spread in the masses of the heavy states could alleviate the Landau pole problem.

We argue that models I and II are likely the minimal realistic E_6 models, which are supersymmetric and renormalizable. We studied the models from simpler to more complicated in a roughly systematic manner, and models I and II were the simplest realistic ones we found. We did for example omit models with breaking sectors $351 \oplus 78 \oplus \overline{351}$ or $351' \oplus 78 \oplus \overline{351'}$, but it does seem these models are incomplete in regards to the Yukawa sector; a minimal extension with $27 \oplus \overline{27}$ would lead to model II in the second case, while the first case would likely have problems with the antisymmetric Yukawa of the $\overline{351}$. Note that by minimal we can mean many things, for example the minimal number of degrees of freedom, the minimum number of Yukawa matrix parameters, the minimal β function of the model; whichever criteria we choose, models I and/or II are at least on the short-list of minimal models. It is interesting to point out, however, that the prototype model is even simpler, but alas not realistic. Naively, one would expect the prototype model to be the candidate for the minimal realistic model, but it failed due to its surprising inability to split the doublet and the triplet mass; this non-trivial result was discovered only after a detailed analysis was performed.

As a final point, let us reiterate the results obtained in the Yukawa sectors in model I and model II. Although the mechanisms of flavor mixing are different in the two models, the conclusions are the same. The low-energy spectrum of fermions is that of the MSSM, where the neutrinos get contributions from seesaw type I and type II. All other particles are heavy. Take special note that this includes the vector-like states of quarks and leptons; light vector-like states are an attractive option in phenomenology, but E_6 models seem to predict for the vector-like states to be generically at the GUT scale.

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6 Razširjeni povzetek

6.1 Uvod

Spoznanja raziskav v 20. in 21. stoletju na področju osnovnih delcev so uspešno združena v teoriji, ki jo imenujemo Standardni model. V tej teoriji opišemo vse znane delce in vse njihove medsebojne interakcije z izjemo gravitacijske: opis združuje močno, šibko in elektromagnetno interakcijo. Čeprav se napovedi Standardnega modela neverjetno dobro ujemajo z eksperimentalnimi opažanji, kljub temu obstajajo eksperimentalne in teoretične uganke, ki morebiti nakazujejo na novo fiziko onkraj Standardnega modela. Med zanimivi kandidati te fizike so Teorije poeotenja, v katerih poenotimo interakcije Standardnega modela, tako da vse postanejo del interakcije istega tipa na visokih energijskih skalah.

Med najpogosteje študiranimi kandidati v Teoriji poenotenja sta grupi SU(5) in SO(10). Še en obetaven kandidat je grupa E_6, ki se pogosto omenja že od začetkov Teorij poenotenja (glej [1]), a kljub temu v svetovni literaturi ne obstaja (vsaj po poznavanju avtorja) noben E_6 model, ki bi bil obravnavan v celoti in bi bil hkrati realističen. Z obravnavo v celoti imamo tu v mislih analizo modela s pristopom "od zgoraj navzdol", kjer hkrati obravnavamo tako spontani zlom simetrije, kot tudi Yukawin sektor in ostale podrobnosti, kot na primer razcep dublet-triplet. Z besedo realističen imamo tu v mislih predvsem ustrezen opis mas in mešalnih kotov Standardnega modela, kjer vsaj na analitičnem nivoju ni očitnih težav. Posamezne teme v modelih E_6 so bile občasno študirane, npr. Yukawin sektor v [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] ter zlom simetrije v nekaterih preprostih modelih v [14, 15, 16, 7].

Za manj aktivnosti na področju sestave modelov poenotenja, ki temeljijo na grupi E_6, sta verjetno dva razloga. Prvi je njena relativna kompliciranost v primerjavi z grupama SU(5) in SO(10), ki sta obe njeni podgrupi. Za razliko od ortogonalnih in unitarnih grup imamo pri predstavi in intuiciji z grupo E_6 veliko težav, hkrati je pa tudi njena matematična konstrukcija bolj komplicirana. Drugi razlog pa je odsotnost konkretne motivacije: SU(5) je na primer najmanša grupa primerna za poenotenje, medtem ko ima grupa SO(10) ugodne lastnosti, kot so npr. avtomatična prisotnost desnoročnega nevtrina, kar elegantno razloži nevtrinske mase. Grupa E_6 ima sicer fenomenološko zanimivo vsebino, npr. njena fundamentalna upodobitev vsebuje dodatne vektorske kvarke in leptone, vendar se določene prednosti SO(10) nad SU(5) pri prehodu na E_6 izgubijo. Primer je denimo R-parnost, ki jo moramo v modelih z E_6 postulirati, medtem ko je v SO(10) modelih, kjer rang grupe zlomi upodobitev 126, avtomatična (glej npr. [18, 19, 20, 21, 22, 23]. Situacija v modelih E_6 je nekoliko bolj komplicirana tudi zato, ker pri najpreprostejših modelih zlom do Standardnega modela ni možen, npr. pri renormalizabilnem supersimetričnem modelu $27 \oplus \overline{27} \oplus 78$ iz [14].

Namen te doktorske disertacije je, da zapolnimo praznino v modeliranju z grupo E_6, tako da študiramo najpreprostejše modele in skušamo najti realistične kandidate. V tej disertaciji so prav tako zbrane tudi mnoge tehnične podrobnosti v zvezi z grupo E_6, kar lahko služi kot referenca. Zaradi njihove sorazmerne preprostosti se bomo omejili na renormalizabilne in supersimetrične modele.

Razširjeni povzetek v slovenščini je razdeljen na naslednji način: v poglavju 6.2 so zbrane tehnične informacije v zvezi z grupo E_6, medtem ko v poglavju 6.3 obravnavamo konkretne modele: najprej določimo osnovno izhodišče za te modele, nato pa po predstavitvi nerealističnih kandidatov navedemo dva realistična, ki jih poimenujemo model I in model II. V obeh modelih spekter lahkih delcev ustreza MSSM, medtem ko so vektorski delci na skali poenotenja.

Ta disertacija temelji na članku [24], vsebuje pa tudi dodatne vsebine, ki še niso objavljene.

Barvni dogovor glede pričakovanih vrednosti: rdeče, če so na skali poenotenja, in modro, če so na elektrošibki skali.

6.2 Grupa E$_6$ in poenotenje

6.2.1 Motivacija

Grupa E$_6$ je preprosta Liejeva grupa, ki je dober kandidat za grupo v Teorijah poenotenja (prvič študirana v [1]). Izrek o klasifikaciji preprostih Liejevih algeber (glej npr. [56, 54]) nam ponudi naslednje kandidate Liejevih grup (glej tudi tabelo 2):

- *Specialne ortogonalne grupe*, v Dynkinovi notaciji so lihe in sode po vrsti označene z B_n in D_n, v klasični notaciji pa se uporablja oznake $\mathrm{SO}(2n+1)$ in $\mathrm{SO}(2n)$.

- *Specialne unitarne grupe*, v Dynkinovi notaciji označene z A_n, klasično pa s $\mathrm{SU}((n+1)^2-1)$.

- *Simplektične grupe*, v Dynkinovi notaciji označene s C_n, klasično pa s $\mathrm{Sp}(n)$.

- *Izjemne grupe*, ki jih je natanko 5: G_2, F_4, E_6, E_7 in E_8.

Za grupo poenotenja bi bili smiselni naslednji kriteriji:

- Grupa, ki vsebuje interakcije Standardnega modela, je preprosta, torej ena od zgoraj naštetih v klasifikaciji.

- Na skali poneotenja pride do spontanega zloma simetrije, kjer se simetrija grupe poenotenja zmanjša na grupo Standardnega modela $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Zato mora grupa poenotenja vsebovati grupo SM kot podgrupo.

- Grupa poenotenja ima kompleksne upodobitve. Stanadardni model je namreč kiralna teorija (levoročni delci so v drugačnih upodobitvah kot desnoročni delci), in ima vse delce v kompleksnih upodobitvah. V kolikor bi grupa poenotenja ne imela kompleksnih upodobitev, bi vse njene upodobitve poleg delcev Standardnega modela vsebovale tudi konjugirane upodobitve, in bi se s tem število prostostnih stopenj (vsaj) podvojilo, kjer bi za konjugirane delce morali nekako poskrbeti, da so težki (recimo nekje blizu skale poenotenja M_{GUT}).

- Kjer je več podobnih kandidatov, naj bo grupa čim manjša. V neskončni družini $\mathrm{SU}(n)$ bi lahko npr. vzeli poljubno velik $n \geq 5$, a bolj zanimive so razširitve Standardnega modela, kjer je potrebno predpostaviti le manjše število novih prostostnih stopenj.

Na podlagi kriterija o kompleksnih reprezentacijah so izločene lihe ortogonalne grupe $\mathrm{SO}(2n+1)$ in simplektične grupe $\mathrm{Sp}(n)$, ter izjemni grupi E_7 in E_8. Grupi G_2 in F_4 ne vsebujeta grupe Standardnega modela kot podgrupe, zato sta prav tako izločeni. Na podlagi zgornjih meril ostanejo kot najbolj primerni trije kandidati: SU(5), SO(10) in E$_6$. Spodaj si bomo na kratko ogledali vsakega od kandidatov, kjer bomo podatke o upodobitvah črpali iz [57].

- SU(5) je najmanjša unitarna grupa, ki vsebuje standardni model. Na njeni podlagi je bil zasnovan tudi prvi model poenotenja Georgi-Glashow [2], ki je bil sicer nesupersimetričen. V SU(5) teorijah običajno fermione Standardnega modela (ene družine) združimo v naslednje upodobitve: $\overline{5}_F = d^c \oplus L$, $10_F = Q \oplus u^c \oplus e^c$. V supersimetričnih teorijah nato dodamo še Higgsovi polji H_u in H_d, ki sta del 5 in $\overline{5}$, za zlom SU(5) pa naprimer uporabimo adjungirano upodobitev 24.

- SO(10) je najmnajša ortogonalna grupa, ki vsebuje Standardni model. Prav tako kot podgrupo vsebuje tudi SU(5). Fermioni Standardnega modela se poenotijo v eno samo upodobitev 16_F, ki je v SO(10) spinorska in kompleksna. Njena dekompozicija v ireducibilne upodobitve SU(5) se glasi $16 = 10 \oplus \overline{5} \oplus 1$, kjer 1 predstavlja desnoročni nevtrino ν^c. Za Yukawine člene uporabimo tiste SO(10) upodobitve, ki se sklapljajo z dvema fermionskima 16_F, za kar sta primerni npr. 10 in $\overline{126}$. Za zlom SO(10) pa so primerne upodobitve, ki vsebujejo singlete Standardnega modela, npr. 126 in $\overline{126}$, 210 ali 54 (glej npr. [32]).

- E_6 je edina med izjemnimi grupami, ki ima hkrati kompleksne upodobitve in vsebuje Standardni model kot podgrupo. Grupi SU(5) in SO(10) sta njeni podgrupi, zato so E_6 teorije poenotenja običajno bolj komplicirane in z večjimi upodobitvami, kot ostale grupe poenotenja. Fermioni ene družine so v E_6 poenoteni v upodobotvi 27_F, ki ima naslednjo SO(10) dekompozicijo:

  $$27 = 16 \oplus 10 \oplus 1. \tag{529}$$

  V 10 in 1 grupe SO(10) se skrivajo eksotični delci: 10 vsebuje vektorske kvarke $d' \oplus d'^c$ in vekorske leptone $L' \oplus L'^c$, medtem ko 1 predstavlja dodatni desnoročni nevtrino, ki ga označimo s s, in je singlet ne samo pod SU(5) ampak celo pod SO(10). Za opis Yukawine interakcije potrebujemo take upodobitve, ki se sklapljajo z dvema fermionskima upodobitvama 27_F; to sta na primer upodobitvi 27 in $\overline{351'}$. Za zlom E_6 v grupo Standardnega modela uporabimo take upodobitve E_6, ki vsebujejo singlete Standardnega modela; ker so upodobitve E_6 dokaj velike, tipično vsebujejo večje število teh singletov, tako da za zlom na skali poenotenja lahko v principu pridejo v poštev katerekoli upodobitve E_6. Katere kombinacije teh upodobitev omogočijo želeni zlom simetrije študiramo v tem doktorskem delu.

6.2.2 Obravnva grupe E_6

Za razliko od klasičnih grup, npr. ortogonalnih in unitarnih, generatorjev E_6 ne moremo zapisati v preprosti in pregledni obliki; prav tako se grupa E_6 ne podreja najbolje kaki intuitivni predstavi, zato moramo k njej pristopiti nekoliko formalno.

Najlažje je grupo E_6 obravnavati skozi dekompozicijo njene adjungirane upodobitve 78, in s tem generatorjev, v upodobitve njene (maksimalne) podgrupe $SU(3)_C \times SU(3)_L \times SU(3)_R$:

$$78 = (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \oplus (3,\overline{3},\overline{3}) \oplus (\overline{3},3,3), \tag{530}$$

kjer generatorje, ki po vrsti pripadajo tem upodobitvam, označimo s t^A_C, t^A_L, t^A_R, $t^\alpha{}_{aa'}$ in $\bar{t}_{\alpha aa'}$. Tu je $A = 1,\ldots,8$ adjungirani indeks grupe SU(3), medtem ko (anti)fundamentalne indekse grup $SU(3)_C$, $SU(3)_L$ in $SU(3)_R$ označimo po vrsti z α,

a in a'. Algebra grupe E_6 je definirana prek (povzeto po [59] in prirejeno, tako da je faktor $SU(3)_C$ konjugiran glede na $SU(3)_L$ in $SU(3)_R$, kar nudi za fiziko bolj primerno vložitev podgrupe $SU(3)^3$)

$$\left[t_C^A, t_R^B\right] = \left[t_R^A, t_L^B\right] = \left[t_L^A, t_C^B\right] = 0, \tag{531}$$

$$\left[t_C^A, t_C^B\right] = i f^{ABC}\, t_C^C, \tag{532}$$
$$\left[t_L^A, t_L^B\right] = i f^{ABC}\, t_L^C, \tag{533}$$
$$\left[t_R^A, t_R^B\right] = i f^{ABC}\, t_R^C, \tag{534}$$

$$\left[t_C^A, t^\alpha{}_{aa'}\right] = -\tfrac{1}{2}(\lambda_A)^\alpha{}_\beta\, t^\beta{}_{aa'}, \tag{535}$$
$$\left[t_L^A, t^\alpha{}_{aa'}\right] = \tfrac{1}{2}(\lambda_A)^b{}_a\, t^\alpha{}_{ba'}, \tag{536}$$
$$\left[t_R^A, t^\alpha{}_{aa'}\right] = \tfrac{1}{2}(\lambda_A)^{b'}{}_{a'}\, t^\alpha{}_{ab'}, \tag{537}$$
$$\left[t_C^A, \bar{t}_\alpha{}^{aa'}\right] = \tfrac{1}{2}(\lambda_A)^\beta{}_\alpha\, \bar{t}_\beta{}^{aa'}, \tag{538}$$
$$\left[t_L^A, \bar{t}_\alpha{}^{aa'}\right] = -\tfrac{1}{2}(\lambda_A)^a{}_b\, \bar{t}_\alpha{}^{ba'}, \tag{539}$$
$$\left[t_R^A, \bar{t}_\alpha{}^{aa'}\right] = -\tfrac{1}{2}(\lambda_A)^{a'}{}_{b'}\, \bar{t}_\alpha{}^{ab'}, \tag{540}$$

$$\left[t^\alpha{}_{aa'}, t^\beta{}_{bb'}\right] = -\varepsilon^{\alpha\beta\gamma}\, \varepsilon_{abc}\, \varepsilon_{a'b'c'}\, \bar{t}_\gamma{}^{cc'}, \tag{541}$$
$$\left[\bar{t}_\alpha{}^{aa'}, \bar{t}_\beta{}^{bb'}\right] = \varepsilon_{\alpha\beta\gamma}\, \varepsilon^{abc}\, \varepsilon^{a'b'c'}\, t^\gamma{}_{cc'}, \tag{542}$$

$$\left[\bar{t}_\alpha{}^{aa'}, t^\beta{}_{bb'}\right] = (\lambda_A)^\beta{}_\alpha\, \delta^a{}_b\, \delta^{a'}{}_{b'}\, t_C^A - \delta^\beta{}_\alpha\, (\lambda_A)^a{}_b\, \delta^{a'}{}_{b'}\, t_L^A - \delta^\beta{}_\alpha\, \delta^a{}_b\, (\lambda_A)^{a'}{}_{b'}\, t_R^A. \tag{543}$$

Tu smo z f^{ABC} označili strukturne konstante grupe SU(3), λ^A so Gell-Mannove matrike (zapisane v poglavju o konvencijah), medtem ko je δ Kroneckerjev simbol in ε predstavlja Levi-Civita tenzor (dogovor $\varepsilon_{123} = \varepsilon^{123} = 1$).

Generatorje lahko zapišemo v fundamenalni upodobitvi kot matrike 27×27, če poznamo njihovo delovanje na stolpec dolžine 27. Fundamentalna upodobitev grupe E_6 ima naslednjo $SU(3)^3$ dekompozicijo:

$$27 = (3,3,1) \oplus (1,\overline{3},3) \oplus (\overline{3},1,\overline{3}). \tag{544}$$

Zgornje tri upodobitve $SU(3)^3$ lahko zapišemo v obliki treh matrik 3×3, ki jih po vrsti označimo z L, M in N. Delovanje generatorjev na te matrike lahko nato zapišemo kot

$$(t_C^A\, L)^{\alpha a} = \tfrac{1}{2}(\lambda_A)^\alpha{}_\beta\, L^{\beta a}, \tag{545}$$
$$(t_C^A\, M)_a{}^{a'} = 0, \tag{546}$$
$$(t_C^A\, N)_{a'\alpha} = -\tfrac{1}{2}(\lambda_A^*)_\alpha{}^\beta\, N_{a'\beta}, \tag{547}$$

$$(t_L^A\, L)^{\alpha a} = \tfrac{1}{2}(\lambda_A)^a{}_b\, L^{\alpha b}, \tag{548}$$
$$(t_L^A\, M)_a{}^{a'} = -\tfrac{1}{2}(\lambda_A^*)_a{}^b\, M_b{}^{a'}, \tag{549}$$
$$(t_L^A\, N)_{a'\alpha} = 0, \tag{550}$$

$$(t_R^A\, L)^{\alpha a} = 0, \tag{551}$$
$$(t_R^A\, M)_a{}^{a'} = \tfrac{1}{2}(\lambda_A)^{a'}{}_{b'}\, M_a{}^{b'}, \tag{552}$$
$$(t_R^A\, N)_{a'\alpha} = -\tfrac{1}{2}(\lambda_A^*)_{a'}{}^{b'} N_{b'\alpha}, \tag{553}$$

$$(t^\alpha{}_{aa'}\, L)^{\beta b} = \ \ \varepsilon^{\alpha\beta\gamma}\, \delta^b{}_a\, N_{a'\gamma}, \tag{554}$$
$$(t^\alpha{}_{aa'}\, M)_b{}^{b'} = -\varepsilon_{abc}\, \delta^{b'}{}_{a'}\, L^{\alpha c}, \tag{555}$$
$$(t^\alpha{}_{aa'}\, N)_{b'\beta} = -\varepsilon_{a'b'c'}\, \delta^\alpha{}_\beta\, M_a{}^{c'}, \tag{556}$$
$$(\bar{t}_\alpha{}^{aa'}\, L)^{\beta b} = \ \ \varepsilon^{abc}\, \delta^\beta{}_\alpha\, M_c{}^{a'}, \tag{557}$$
$$(\bar{t}_\alpha{}^{aa'}\, M)_b{}^{b'} = \ \ \varepsilon^{a'b'c'}\, \delta^a{}_b\, N_{c'\alpha}, \tag{558}$$
$$(\bar{t}_\alpha{}^{aa'}\, N)_{b'\beta} = -\varepsilon_{\alpha\beta\gamma}\, \delta^{a'}{}_{b'}\, L^{\gamma a}. \tag{559}$$

S pomočjo teh relacij lahko sestavimo generatorje E$_6$ v eksplicitni obliki, kar uporabljamo v nadaljnih računih, opravljenih na računalniku. Kateri od teh generatorjev so del zanimivih podgrup E$_6$, je prikazano na sliki 7.

Za modeliranje teorij poenotenja z grupo E$_6$ bomo uporabljali njene upodobitve. S pomočjo tenzorskih metod v E$_6$ (splošno o teh metodah glej [56], za E$_6$ glej [59]) lahko ireducibilne upodobitve E$_6$ dimenzije manj od 1000 zapišemo na naslednji način:

- Fundamentalno upodobitev 27 lahko zapišemo kot stolpec višine 27, torej kot ψ^i, kjer $i = 1, \dots, 27$. Če gre za upodobitev fermionskega tipa, bomo stanja označevali z oznakami delcev iz Standardnega modela, kot prikazuje slika 5. Konjugirano upodobitev $\overline{27}$ lahko zapišemo v obliki vrstice $\overline{\psi}_i$.

- Adjungirana upodobitev 78 se lahko zapiše kot linearna kombinacija generatorjev, torej v obliki 27×27 matrike. Upodobitev označimo s $\phi = \phi^a\, t^a$, kjer $a = 1, \dots, 78$ in ϕ^a stanja v tej upodobitvi. Napisano eksplicitno z (anti)fundamentalnimi indeksi se 78 zapiše kot $\phi^i{}_j$.

- Upodobitev 351 lahko zapišemo v obliki antisimetrične matrike 27×27 z dvema zgornjima (fundamentalnima) indeksoma: Ξ^{ij}. Konjugirana upodobitev $\overline{351}$ lahko zapišemo z antisimetrično matriko $\overline{\Xi}_{ij}$.

- Upodobitev $351'$ lahko zapišemo v obliki simetrične matrike 27×27 z dvema indeksoma zgoraj, ki zadošča dodatni relaciji: Θ^{ij}, z relacijo $d_{ijk}\Theta^{jk} = 0$, kjer je d_{ijk} invarianten tenzor grupe E$_6$ (definiramo ga kasneje). Konjugirano upodobitev $\overline{351'}$ zapišemo v obliki simetrične matrike $\overline{\Theta}_{ij}$, ki zadošča analogni relaciji $d^{ijk}\overline{\Theta}_{jk} = 0$.

- Upodobitev 650, ki je realna, zapišemo v obliki matrike 27×27 $X^i{}_j$, ki zadošča relacijam $\mathrm{Tr}(X) = \mathrm{Tr}(t^a\, X) = 0$.

Nekaj več informacij o teh upodobitvah je možno razbrati iz tabele 5. Dekompozicije teh upodobitev v ireducibilne upodobitve grupe SO(10) in nadalje SU(5), so predstavljene na sliki 8.

Invariante sestavljene iz teh upodobitev tvorimo, tako da sestavimo izraze, kjer pri vseh indeksih pride do kontrakcije in nastali objekt nima več prostih indeksov. Pri tem si lahko pomagamo s tenzorjema Kroneckerjev delta $\delta^i{}_j$ (kjer $i, j = 1, \ldots, 27$) in d-tenzorjema d_{ijk} in d^{ijk}, ki sta posebnost E$_6$. Tenzor d_{ijk} je definiran z relacijo

$$\tfrac{1}{6} d_{ijk} \psi^i \psi^j \psi^k = -\det L + \det M - \det N - \mathrm{Tr}(LMN). \tag{560}$$

Tenzor z zgornjimi indeksi d^{ijk} ima numerično enake vrednosti kot d_{ijk}, zanju pa veljajo naslednje lastnosti (naštete najpomembnejše):

- Tenzorja d^{ijk} in d_{ijk} sta popolnoma simetrična na zamenjavo indeksov.
- Čim sta dva indeksa enaka, je njuna vrednost enaka 0. Edine neničelne vrednosti v teh tenzorjih so 1 in -1.
- Velja normalizacija
$$d^{ikl}\, d_{jkl} = 10\, \delta^i{}_j. \tag{561}$$
- Kontrakcija le enega indeksa definira nov tenzor, ki je neodvisen od produktov in vsot, ki jih lahko sestavimo s Kroneckerjevimi delta simboli:
$$d^{ijm}\, d_{klm} =: \; D^{ij}_{kl}. \tag{562}$$

Ker bomo študirali renormalizabilne supersimetrične modele, kjer bo pomemben superpotencial W masne dimenzije 3, nas bodo zanimale invariante največ reda 3. V tabelah 11, 12 in 14 so navedene vse take invariante za kombinacije upodobitev 27, $\overline{27}$, 78, 351, $\overline{351}$, $351'$ in $\overline{351'}$.

V upodobitvah E$_6$ bodo poseben pomen imela tista stanja, ki se pod Standardnim modelom transformirajo na enega od naslednjih načinov:

- Singleti $(1, 1, 0)$, ki so pomembni pri spontanem zlomu simetrije iz E$_6$ v Standardni model. Z oznakami so definirani v tabelah 6 in 8. Nekaj teh informacij bo v nadaljevanju posebej pomembnih. Upodobitvi 27 in $\overline{27}$ imata vsaka po dva singleta, označimo pa jih s c_k in d_k, $k = 1, 2$. Upodobitve $351'$, $\overline{351'}$, 351 in $\overline{351}$ pa imajo vsaka po 5 singletov, ki jih označimo po vrsti z e_k, f_k, g_k in h_k, kjer $k = 1, \ldots, 5$. Adjungirana upodobitev 78 ima prav tako 5 singletov, označimo jih z u_1, u_2, v, w, y.
- Dubleti $(1, 2, +1/2)$ in "antidubleti" $(1, 2, -1/2)$, med katerimi se morata nahajati polji H_u in H_d minimalnega supersimetričnega Standardnega modela (MSSM). Označimo jih s črkami D in $\overline{D}$, z bolj podrobnimi definicijami v tabeli 9. V naših modelih morajo biti vsi (anti)dubleti težki (na skali poenotenja), medtem ko mora biti en par dublet-antidublet lahek (ustrezata Higgsom H_u in H_d).
- Tripleti $(3, 1, -1/3)$ in antitripleti $(\overline{3}, 1, +1/3)$, ki so v naših modelih mediatorji protonskega razpada. Za njih mora veljati, da imajo vsi maso na skali poenotenja. Označimo jih s T in $\overline{T}$, bolj podrobne definicije pa so podane v tabeli 9.

Eksplicitni izračuni invariant, kjer so neničelne le pričakovane vrednosti singletov Standardnega modela, so podane v seznamu enačb (174)–(190) in (191)–(195).

6.3 Renormalizabilni supersimetrični modeli E_6

6.3.1 O izbiri modelov

Modeli, ki jih bomo študirali, bodo renormalizabilni supersimetrični modeli zgrajeni na naslednjih načelih:

- Iščemo renormalizabilne supersimetrične modele, ki v enem koraku zlomijo E_6 do grupe Standardnega modela. Naša nizko-energijska teorija bo v resnici MSSM; zloma supersimetrije ne bomo obravnavali, saj predstavlja vprašanje, ki je ortogonalno na vprašanje zloma umeritvene simetrije E_6 (obstajajo izjeme, npr. [66, 67]).

- Fermione Standardnega modela bomo vključili prek treh družin upodobitve 27. Označimo jih s 27^i_F, kjer $i = 1, 2, 3$. Te upodobitve sestavljajo "fermionski sektor".

- Ostale upodobitve v modelu sestavljajo "Higgsov sektor". Te upodobitve bodo dobile pričakovane vrednosti na skali poenotenja M_{GUT}, kar zlomi E_6 v grupo Standardnega modela. Prav tako bodo vsebovala Higgsova polja H_u in H_d iz MSSM, zato bodo upodobitve Higgsovega sektorja dobila pričakovane vrednosti tudi na elektrošibki skali (kar bo pomembno v Yukawinem sektorju).

- Predpostavili bomo dodatno $\mathbb{Z}_2$ simetrijo, pod katero je fermionski sektor lih, Higgsov sektor pa sod. To pomeni, da morajo fermionske upodobitve v superpotencialu nastopati v parih. S tem v teorijo uvedemo R-parnost, hkrati pa omenjena simetrija omogoči nastavek, kjer so pričakovane vrednosti v fermionskem sektorju enake 0.

Za realistične modele bomo preverili, ali je zlom simetrije do grupe SM možen, ali je možen razcep dublet-triplet (en par tipa $(1, 2, \pm 1/2)$ mora biti lahek, medtem ko morajo biti vsi tripleti tipa $(3, 1, -1/3)$ in $(\overline{3}, 1, +1/3)$ na skali poenotenja, saj ti povzročajo protonski razpad) ter preverili, kakšno napoved dobimo za mase v fermionskem sektorju. Izbira modelov v tem kontekstu pomeni izbiro upodobitev v Higgsovem sektorju.

Spomnimo, da ima Lagrangeva funkcija v supersimetričnih modelih obliko, kot je navedena v enačbah (20)–(24), kjer je W holomorfna funkcija skalarnih polj, ki jo imenujemo superpotencial. Enačbe gibanja se delijo na F-člene in D-člene (podrobnosti zloma v supersimetričnih modelih npr. v [69, 70, 71]): F-členi imajo obliko $F_s = \partial W/\partial s = 0$, kjer so s vsa skalarna polja, medtem ko imajo D-členi obliko $D^a = 0$, kjer smo definirali (seštejemo prispevke vseh upodobitev ϕ za skalarje)

$$D^a = -g \sum_{\phi} \phi^\dagger_i \, (t^a)^i{}_j \, \phi^j. \tag{563}$$

6.3.2 Nerealistični modeli

V sistematičnem študiju najpreprostejših E_6 modelov, ki ustrezajo opisanim načelom, lahko mnoge relativno hitro zavržemo kot nerealistične. Spodaj je podan seznam teh modelov in opisi, kje v fenomenologiji teh modelov se zalomi.

- Modeli tipa $n_1 \cdot 27 \oplus n_2 \cdot 78 \oplus n_3 \cdot \overline{27}$: v teh modelih se zalomi pri zlomu simetrije. Singleti v 27 in $\overline{27}$ so tudi SU(5) singleti, le 78 vsebuje singlet y, ki je del 24 grupe SU(5). Če je $y = 0$, potem zlomimo lahko največ v SU(5). Edine kubične invariante, ki vsebujejo upodobitev 78, so $27 \cdot 78 \cdot \overline{27}$ in 78^3. Singlet y v prvi invarianti ne nastopa, ker se bi v jeziku SU(5) moral povezati z ostalimi členi v obliki $1 \cdot 24 \cdot 1$, kar pa ni invarianta. Kubični člen 78^3 pa je antisimetričen v faktorjih, a členi $24 \cdot 24 \cdot 1$ in $24 \cdot 24 \cdot 24$ v jeziku SU(5) tvorijo invariante simetrične v faktorjih 24. Singlet y je tako prisoten le v masnem členu 78^2, zato je njegova pričakovana vrednost enaka 0, s tem pa imamo možnost zloma kvečjemu do SU(5).

- Modeli tipa $n_1 \cdot 27 \oplus n_2 \cdot \overline{27} \oplus 351 \oplus \overline{351}$: tu se ponovno zalomi pri zlomu simetrije. Singleti SM, ki niso hrkati tudi singleti SU(5), se nahajajo le v 351 in $\overline{351}$ kot 24 grupe SU(5). Kubične invariante s tema dvema upodobitvama, ki bodo prisotne v superpotencialu, bodo $27^2 \cdot \overline{351}$, $\overline{27} \cdot 351$. Ker v SU(5) jeziku $1 \cdot 1 \cdot 24$ ne tvori invariante, singleti $\langle 24 \rangle$ ne bodo prisotni drugje kot v masnem členu $351 \cdot \overline{351}$, torej bodo njihove pričakovane vrednosti 0, s tem pa bomo zlomili E_6 le do SU(5). Kubični invarianti 351^3 in $\overline{351}^3$ v tem modelu nista prisotni, ker sta antisimetrični v svojih faktorjih, obravnavani model ima pa le po eno različico upodobitev 351 in $\overline{351}$.

- Model $351' \oplus \overline{351}$: ta model lahko zlomi kvečjemu do grupe Pati-Salam, tj. $SU(4)_C \times SU(2)_L \times SU(2)_R$. To se izkaže pri obravnavi enačb gibanja za SM singlete, kar je možno preveriti kot poseben primer pri obravnavi bolj kompliciranega modela, $351' \oplus \overline{351} \oplus 27 \oplus \overline{27}$, ki ga obravnavamo v poglavju o modelu I.

6.3.3 Model I

V tem modelu se odločimo za Higgsov sektor z upodobitvami

$$351' \oplus \overline{351'} \;\oplus\; 27 \oplus \overline{27} \;\oplus\; \widetilde{27} \oplus \overline{\widetilde{27}}. \tag{564}$$

V tem modelu Higgsov sektor razdelimo na dva dela: upodobitve brez tilde bodo dobile pričakovane vrednosti na skali poenotenja, medtem ko se bodo elektrošibke pričakovane vrednosti polj H_u in H_d nahajale v upodobitvah s tildo. Da je taka ureditev lahko konsistentna, izpustimo nekatere člene s tilda upodobitvami iz superpotentciala. Za superpotencial vzamemo

$$W = W_{\mathrm{SSB}} + W_{DT} + W_{\mathrm{Yukawa}}, \tag{565}$$

kjer so posamezni kosi definirani kot

$$\begin{aligned} W_{\mathrm{SSB}} = \;& m_{351'}\, I_{351' \otimes \overline{351'}} + m_{27}\, I_{27 \otimes \overline{27}} \\ & + \lambda_1\, I_{351'^3} + \lambda_2\, I_{\overline{351'}^3} + \lambda_3\, I_{27^2 \otimes \overline{351'}} + \lambda_4\, I_{\overline{27}^2 \otimes 351'} + \lambda_5\, I_{27^3} + \lambda_6\, I_{\overline{27}^3}, \end{aligned} \tag{566}$$

$$W_{DT} = m_{\widetilde{27}}\, I_{\widetilde{27} \otimes \overline{\widetilde{27}}} + \kappa_1\, I_{\widetilde{27}^2 \otimes \overline{351'}} + \kappa_2\, I_{\overline{\widetilde{27}}^2 \otimes 351'} + \kappa_3\, I_{\widetilde{27}^2 \otimes 27} + \kappa_4\, I_{\overline{\widetilde{27}}^2 \otimes \overline{27}}, \tag{567}$$

$$W_{\mathrm{Yukawa}} = \sum_{i,j=1}^{3} \tfrac{1}{2} \left(Y_{27}^{ij}\, I_{27_F^i \otimes 27_F^j \otimes 27} + Y_{\overline{351'}}^{ij}\, I_{27_F^i \otimes 27_F^j \otimes \overline{351'}} + Y_{\widetilde{27}}^{ij}\, I_{27_F^i \otimes 27_F^j \otimes \widetilde{27}} \right). \tag{568}$$

Členi v W_{SSB} so odgovorni za zlom simetrije, W_{DT} za razcep dublet-triplet, medtem ko so členi W_{Yukawa} del Yukawinega sektorja, od koder pridejo mase fermionov Standardnega modela.

Najprej si oglejmo zlom simetrije. Higgsov sektor brez tilde ima 14 singletov: c_i, d_i, e_j, f_j, kjer $i = 1,2$ in $j = 1,\ldots,5$. Posledično imamo v enačbah gibanja 14 F-členov, ki jih dobimo z odvajanjem superpotenciala po teh singletih (enačbe se trivialno izpeljejo, zato jih tu ne bomo zapisali; eksplicitno obliko superpotenciala preberemo s pomočjo izračunanih invariant v enačbah(174)–(190) in tabelo 11). Med D-členi pa moramo zadostiti trem neodvisnim enačbam (dve sta realni, ena je pa kompleksna, torej imamo 4 realne pogoje):

$$D^I \equiv \sqrt{3}D^8_L + 2D^3_R = |c_1|^2 - |d_1|^2 + |e_2|^2 - |f_2|^2 + 2|e_3|^2 - 2|f_3|^2 - |e_4|^2 + |f_4|^2, \tag{569}$$

$$D^{II} \equiv \quad - 2D^3_R = |c_2|^2 - |d_2|^2 + |e_2|^2 - |f_2|^2 + 2|e_1|^2 - 2|f_1|^2 - |e_5|^2 + |f_5|^2, \tag{570}$$

$$\begin{aligned} D^{III} \equiv \quad D^6_R + iD^7_R = \; & c_1{c_2}^* - {d_1}^*d_2 + \sqrt{2}{e_1}^*e_2 - \sqrt{2}f_1{f_2}^* \\ & + \sqrt{2}{e_2}^*e_3 - \sqrt{2}f_2{f_3}^* + {e_4}^*e_5 - f_4{f_5}^*. \end{aligned} \tag{571}$$

Enačbe gibanja reši npr. naslednja rešitev:

$$c_2 = 0, \qquad d_2 = 0, \tag{572}$$

$$e_2 = 0, \qquad f_2 = 0, \tag{573}$$

$$e_4 = 0, \qquad f_4 = 0, \tag{574}$$

$$d_1 = \frac{m_{351'}m_{27}}{2\lambda_3\lambda_4 c_1}, \tag{575}$$

$$e_1 = -\frac{m_{351'}}{6\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_1 = -\frac{m_{351'}}{6\lambda_1^{1/3}\lambda_2^{2/3}}, \tag{576}$$

$$e_3 = -\lambda_3{c_1}^2/m_{351'}, \qquad f_3 = -\frac{m_{351'}m_{27}^2}{4\lambda_3^2\lambda_4{c_1}^2}, \tag{577}$$

$$e_5 = \frac{m_{351'}}{3\sqrt{2}\lambda_1^{2/3}\lambda_2^{1/3}}, \qquad f_5 = \frac{m_{351'}}{3\sqrt{2}\lambda_1^{1/3}\lambda_2^{2/3}}, \tag{578}$$

kjer je edina preostala količina c_1 določena prek polinoma

$$\begin{aligned} 0 = \; & |m_{351'}|^4|m_{27}|^4 + 2|m_{351'}|^4|m_{27}|^2|\lambda_3|^2|c_1|^2 \\ & - 8|m_{351'}|^2|\lambda_3|^4|\lambda_4|^2|c_1|^6 - 16|\lambda_3|^6|\lambda_4|^2|c_1|^8. \end{aligned} \tag{579}$$

Ta rešitev zlomi E_6 v Standardni model, kar lahko preverimo s pomočjo mas umeritvenih bozonov, izračunanih v tabeli 16.

Nato skušamo izvesti razcep dublet-triplet. Higgsov sektor brez tilde vsebuje skupaj 11 parov dublet-antidublet in 12 parov triplet-antitriplet (preštejemo s pomočjo slike 8 in definicijami v tabeli 9), medtem ko Higgsov sektor s tildo vsebuje 3 pare dublet-antidublet in 3 pare triplet-antitriplet (oznake definirane v tabeli 17). Bloka stanj s tildo in brez tilde sta v masni matriki ločena zaradi odsotnosti določenih členov v superpotencialu tega modela. Analiza pokaže, da dublet-triplet za zgornjo vakuumsko rešitev v bloku brez tilde ni možen, prav tako pa tudi ne za nobeno drugo rešitev, ki zlomi v Standardni model, kar pokaže komplicirana analiza rešitev enačb gibanja. Razcep zato izvedemo v bloku stanj s tildo. Masna matrika tega bloka za dublete in triplete se glasi

$$\widetilde{\mathcal{M}} = \begin{pmatrix} m_{\widetilde{27}} & -2\kappa_3 c_1 + \alpha\kappa_1 \frac{f_4}{\sqrt{15}} & 2\kappa_3 c_2 + \alpha\kappa_1 \frac{f_5}{\sqrt{15}} \\ -2\kappa_4 d_1 + \alpha\kappa_2 \frac{e_4}{\sqrt{15}} & m_{\widetilde{27}} & 0 \\ 2\kappa_4 d_2 + \alpha\kappa_2 \frac{e_5}{\sqrt{15}} & 0 & m_{\widetilde{27}} \end{pmatrix}, \tag{580}$$

kjer postavimo $\alpha = -3$ za dublete in $\alpha = 2$ za triplete. Z natančno nastavitvijo parametra κ_1 na

$$\kappa_1 \approx 30\,(m_{\widetilde{27}}^2 \lambda_3\lambda_4 - 2m_{351'}m_{27}\kappa_3\kappa_4)\frac{\lambda_1\lambda_2}{m_{351'}^2\lambda_3\lambda_4\kappa_2}, \tag{581}$$

postane en par dublet-antidublet brezmasen, medtem ko ostanejo vsi tripleti težki. Pri zgornjem izboru κ_1 izračun levega in desnega lastnega vektorja z maso 0 za dublete pokaže, da so prisotne vse komponente dubletov s tildo, zato vse te komponente dobijo elektrošibko pričakovano vrednost.

Za konec izračunamo še mase fermionov v tem modelu. Eksplicitni računalniški izračun členov v Yukawinem sektorju superpotenciala nam za mase poda naslednje izraze:

$$\begin{aligned} u^T(-v_1)Y_{\widetilde{27}}u^c &+ \begin{pmatrix} d^{cT} & d'^{cT} \end{pmatrix} \begin{pmatrix} \bar{v}_2 Y_{\widetilde{27}} & c_2 Y_{27} + \frac{f_5}{\sqrt{15}} Y_{\overline{351'}} \\ -\bar{v}_3 Y_{\widetilde{27}} & -c_1 Y_{27} + \frac{f_4}{\sqrt{15}} Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} d \\ d' \end{pmatrix} \\ &+ \begin{pmatrix} e^T & e'^T \end{pmatrix} \begin{pmatrix} -\bar{v}_2 Y_{\widetilde{27}} & c_2 Y_{27} - \frac{3}{2}\frac{f_5}{\sqrt{15}} Y_{\overline{351'}} \\ \bar{v}_3 Y_{\widetilde{27}} & -c_1 Y_{27} - \frac{3}{2}\frac{f_4}{\sqrt{15}} Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} e^c \\ e'^c \end{pmatrix} \\ &+ \begin{pmatrix} \nu^T & \nu'^T \end{pmatrix} \begin{pmatrix} v_1 Y_{\widetilde{27}} & 0 & c_2 Y_{27} - \frac{3}{2}\frac{f_5}{\sqrt{15}} Y_{\overline{351'}} \\ 0 & -v_1 Y_{\widetilde{27}} & -c_1 Y_{27} - \frac{3}{2}\frac{f_4}{\sqrt{15}} Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} \nu^c \\ s \\ \nu'^c \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} \nu^{cT} & s^T & \nu'^{cT} \end{pmatrix} \begin{pmatrix} f_1 Y_{\overline{351'}} & \frac{f_2}{\sqrt{2}} Y_{\overline{351'}} & -\bar{v}_3 Y_{\widetilde{27}} \\ \frac{f_2}{\sqrt{2}} Y_{\overline{351'}} & f_3 Y_{\overline{351'}} & \bar{v}_2 Y_{\widetilde{27}} \\ -\bar{v}_3 Y_{\widetilde{27}} & \bar{v}_2 Y_{\widetilde{27}} & 0 \end{pmatrix} \begin{pmatrix} \nu^c \\ s \\ \nu'^c \end{pmatrix} \\ &+ \frac{1}{2} \begin{pmatrix} \nu^T & \nu'^T \end{pmatrix} \begin{pmatrix} \Delta_1 Y_{\overline{351'}} & \frac{1}{\sqrt{2}}\Delta_2 Y_{\overline{351'}} \\ \frac{1}{\sqrt{2}}\Delta_2 Y_{\overline{351'}} & \Delta_3 Y_{\overline{351'}} \end{pmatrix} \begin{pmatrix} \nu \\ \nu' \end{pmatrix}. \end{aligned} \tag{582}$$

Družinskih indeksov nismo pisali. V sektorju kvarkov tipa d in v sektorju nabitih leptonov imamo mešanje med 16_F in 10_F deli upodobitve 27_F: nas bo zanimala le tista linearna kombinacija, ki je lahka. V nevtrinskem sektorju je kombinacija še bolj komplicirana, saj je kombinacija levoročnih nevtrinov ν in ν' težka skupaj z ν'^c, desnoročna nevtrina ν^c in s imata prispevek Majorananih mas in prispevata h gugalničnemu mehanizmu tipa I, medtem ko so polja Δ šibki tripleti $(1, 3, \pm 1)$ definirani v tabeli 18 in predstavljajo prispevek gugalničnega mehanizma tipa II.

Po daljšem tehničnem postopku lahko težka stanja integriramo ven iz teorije in dobimo sledeče matrike za lahke fermione Standardnega modela:

$$M_D^T = \left(1 + (4/9) X_0 X_0^\dagger\right)^{-1/2} \left(\bar{v}_2 - (2/3)\bar{v}_3 X_0\right) Y_{\widetilde{27}}, \tag{583}$$

$$M_E = -\left(1 + X_0 X_0^\dagger\right)^{-1/2} \left(\bar{v}_2 + \bar{v}_3 X_0\right) Y_{\widetilde{27}}, \tag{584}$$

$$M_U = -v_1 Y_{\widetilde{27}}, \tag{585}$$

$$\begin{aligned} M_N = \frac{1}{2}\left(1 + X_0 X_0^\dagger\right)^{-1/2} \times \Bigg(& \Delta_1 Y_{\overline{351}'} - \frac{\Delta_2}{\sqrt{2}}\left(X_0 Y_{\overline{351}'} + Y_{\overline{351}'} X_0^T\right) + \Delta_3 X_0 Y_{\overline{351}'} X_0^T \\ & - \frac{{v_1}^2}{f_1} Y_{\widetilde{27}} Y_{\overline{351}'}^{-1} Y_{\widetilde{27}} - \frac{{v_1}^2}{f_3} X_0 Y_{\widetilde{27}} Y_{\overline{351}'}^{-1} Y_{\widetilde{27}} X_0^T \Bigg) \times \left(1 + X_0^* X_0^T\right)^{-1/2}, \end{aligned} \tag{586}$$

kjer je matrika X_0 definirana prek

$$X_0 := \sqrt{\frac{3}{20}}\, \frac{f_5}{c_1}\, Y_{\overline{351}'} Y_{27}^{-1}, \tag{587}$$

in pričakovane vrednosti tripletov Δ enake

$$\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{pmatrix} = \begin{pmatrix} m_{351'} & 0 & 0 & 6\lambda_1 e_1 \\ 0 & m_{351'} & 0 & -6\lambda_1 e_2 \\ 0 & 0 & m_{351'} & 6\lambda_1 e_3 \\ 6\lambda_2 f_1 & -6\lambda_2 f_2 & 6\lambda_2 f_3 & m_{351'} \end{pmatrix}^{-1} \begin{pmatrix} \kappa_2 {v_3}^2 \\ \kappa_2 \sqrt{2} v_2 v_3 \\ \kappa_2 {v_2}^2 \\ \kappa_1 {v_1}^2 \end{pmatrix}. \tag{588}$$

Med zgornjimi masnimi matrikami ni nobenih posebnih relacij (npr. ne velja $M_D^T = M_E$), prav tako nimajo nobenih neželenih lastnosti (npr. M_U in M_D nista hkrati diagonalni, torej imamo lahko ustrezen opis CKM matrike). Do mešanja okusov pride v tem modelu na neobičajen način: čeprav se Higgsovi polji H_u in H_d nahajata le v enem Yukawinem členu (torej na EW skali ni nujno eksplicitnega mešanja na nivoju vseh stanj v 27_F), pa preostala Yukawina člena mešata SU(5) dela $\overline{5}$ v 27_F; to mešanje na skali poenotenja se na nizki skali manifestira kot mešanje okusov.

Enačbe za masne matrike so nelinearne, kar otežuje analitično obravnavo. A glede na dovoljšnje število parametrov (imamo 3 Yukawine matrike, poleg tega imamo na voljo še 3 mase, 6 parametrov λ in 4 parametre κ) sklepamo, da je numerični fit na eksperimentalne vrednosti mas možen, zelo verjetno s kar nekaj svobode v prametrskem prostoru.

Model I je torej realističen: imamo zlom do grupe Standardnega modela, razcep dublet in triplet je uspešen (v poljih s tildo), prav tako pa napovemo tudi pravilno število lahkih fermionov. Zaradi 3 Yukawinih matrik ta model verjetno ni zelo prediktiven.

6.3.4 Model II

V tem modelu se odločimo za Higgsov sektor z upodobitvami

$$351' \oplus \overline{351'} \;\oplus\; 27 \oplus \overline{27} \;\oplus\; 78, \tag{589}$$

in sicer brez kakršnihkoli omejitev (z izjemo R-parnosti). Superpotencial shematsko napišemo kot

$$W = W_{\mathrm{SSB}} + W_{\mathrm{Yukawa}}, \tag{590}$$

kjer so posamezni kosi definirani kot

$$\begin{aligned} W_{\mathrm{SSB}} = \;& m_{351'}\; I_{351'\otimes\overline{351'}} + m_{27}\; I_{27\otimes\overline{27}} + m_{78}\; I_{78^2} \\ & + \lambda_1\; I_{351'^3} + \lambda_2\; I_{\overline{351'}^3} + \lambda_3\; I_{27^2\otimes\overline{351'}} + \lambda_4\; I_{\overline{27}^2\otimes 351'} \\ & + \lambda_5\; I_{27^3} + \lambda_6\; I_{\overline{27}^3} + \lambda_7\; I_{27\otimes 78\otimes\overline{78}} + \lambda_8\; I_{351'\otimes 78\otimes\overline{351'}}, \end{aligned} \tag{591}$$

$$W_{\mathrm{Yukawa}} = \sum_{i,j=1}^{3} \tfrac{1}{2}\left(Y_{27}^{ij}\; I_{27^i_F\otimes 27^j_F\otimes 27} + Y_{351'}^{ij}\; I_{27^i_F\otimes 27^j_F\otimes\overline{351'}}\right). \tag{592}$$

Členi v W_{SSB} bodo povzročili zlom simetrije, v W_{Yukawa} pa so členi Yukawinega sektorja.

Za zlom simetrije do Standardnega modela nam je na voljo 19 singletov standardnega modela: c_i, d_i, u_i, e_j, f_j, v, w in y, kjer $i = 1, 2$ in $j = 1, \ldots, 5$. Definicije teh singletov so v tabeli 6. Eksplicitno obliko členov s temi singleti lahko preberemo iz enačb (174)–(190) ter iz tabele 11. V tem modelu imamo torej 19 F-členov, ki jih ne bomo zapisali eksplicitno, ter naslednje neodvisne D-člene:

$$D^I = |c_1|^2 - |d_1|^2 + |e_2|^2 - |f_2|^2 + 2|e_3|^2 - 2|f_3|^2 - |e_4|^2 + |f_4|^2 - \tfrac{1}{3}|u_1|^2 + \tfrac{1}{3}|u_2|^2, \tag{593}$$

$$D^{II} = |c_2|^2 - |d_2|^2 + |e_2|^2 - |f_2|^2 + 2|e_1|^2 - 2|f_1|^2 - |e_5|^2 + |f_5|^2 + \tfrac{1}{3}|u_1|^2 - \tfrac{1}{3}|u_2|^2, \tag{594}$$

$$\begin{aligned} D^{III} = & +c_1 {c_2}^* - \tfrac{\sqrt{3}}{6} w {u_1}^* + \tfrac{\sqrt{5}}{6} v {u_1}^* + \sqrt{2} e_2 {e_1}^* + \sqrt{2} e_3 {e_2}^* + e_5 {e_4}^* \\ & - d_2 {d_1}^* + \tfrac{\sqrt{3}}{6} u_2 w^* - \tfrac{\sqrt{5}}{6} u_2 v^* - \sqrt{2} f_1 {f_2}^* - \sqrt{2} f_2 {f_3}^* - f_4 {f_5}^*. \end{aligned} \tag{595}$$

V tem modelu smo našli naslednjo rešitev enačb gibanja: ko uvedemo pomožno polje A, lahko zapišemo

$$c_1 = 0, \qquad d_1 = 0, \tag{596}$$
$$e_2 = 0, \qquad f_2 = 0, \tag{597}$$
$$e_5 = 0, \qquad f_5 = 0, \tag{598}$$
$$u_1 = 0, \qquad u_2 = 0, \tag{599}$$
$$y = 0, \tag{600}$$

$$\begin{aligned} w = & \frac{1}{2304\sqrt{15} m_{78}\lambda_3\lambda_4}\Bigg(- \sqrt{30} A^2 \lambda_8 \lambda_7^2 - 360\sqrt{30} m_{351'}\left(2m_{27}\lambda_7 + m_{351'}\lambda_8\right) \\ & + 24A\left(96 m_{78}\lambda_3\lambda_4 + 5\lambda_7\left(m_{27}\lambda_7 + 2m_{351'}\lambda_8\right)\right)\Bigg), \end{aligned} \tag{601}$$

$$\begin{aligned} v = & \frac{1}{34560 m_{78}\lambda_3\lambda_4}\left(\left(360\sqrt{30} m_{351'}^2 - 240 A\lambda_7 m_{351'} + \sqrt{30} A^2 \lambda_7^2\right)\lambda_8 \right. \\ & \left. -120 m_{27}\lambda_7\left(A\lambda_7 - 6\sqrt{30} m_{351'}\right)\right). \end{aligned} \tag{602}$$

Od tod, izraženo s pričakovanimi vrednostmi A, w, v, c_2 in e_4, nato sledi

$$d_2 = \frac{6m_{27}\left(180m_{351'} - \sqrt{30}A\lambda_7\right) + A\left(A\lambda_7 - 6\sqrt{30}m_{351'}\right)\lambda_8}{2160c_2\lambda_3\lambda_4}, \tag{603}$$

$$e_1 = \frac{6{c_2}^2\lambda_3\left(180m_{351'} - \sqrt{30}A\lambda_7\right)}{m_{27}\left(6\sqrt{30}A\lambda_7 - 1080m_{351'}\right) + A\left(6\sqrt{30}m_{351'} - A\lambda_7\right)\lambda_8}, \tag{604}$$

$$f_1 = \frac{1}{777600{c_2}^2\lambda_3^2\lambda_4}\left(A\left(1080\sqrt{30}m_{351'}^2 - 360A\lambda_7 m_{351'} + \sqrt{30}A^2\lambda_7^2\right)\lambda_8\right.$$
$$\left. - 180m_{27}\left(1080m_{351'}^2 - 12\sqrt{30}A\lambda_7 m_{351'} + A^2\lambda_7^2\right)\right), \tag{605}$$

$$e_3 = -\frac{\left(-18m_{27}^2 + 3\sqrt{2}w\lambda_8 m_{27} + 2w^2\lambda_8^2\right)^2}{11664{e_4}^2\lambda_1^2\lambda_2\left(3m_{27} - \sqrt{2}w\lambda_8\right)}, \tag{606}$$

$$f_3 = -\frac{9{e_4}^2\lambda_1}{3m_{27} - \sqrt{2}w\lambda_8}, \tag{607}$$

$$f_4 = \frac{18m_{27}^2 - 3\sqrt{2}w\lambda_8 m_{27} - 2w^2\lambda_8^2}{324e_4\lambda_1\lambda_2}. \tag{608}$$

Vrednosti polj c_2 in e_4 sta določeni s pomočjo D-členov

$$0 = D^I = 2|e_3|^2 - |e_4|^2 - 2|f_3|^2 + |f_4|^2, \tag{609}$$
$$0 = D^{II} = |c_2|^2 - |d_2|^2 + 2|e_1|^2 - 2|f_1|^2, \tag{610}$$

medtem ko je pomožna vrednost A določena kot rešitev polinoma v enačbah (472)–(477). Rešitev res zlomi v grupo Standardnega modela, kar se lahko prepričamo s pomočjo izrazov za mase umeritvenih bozonov, ki so za ta model izračunane v tabeli 19. Nasploh lahko rečemo, da je dobljena rešitev dokaj komplicirana, zato je v tem modelu določene lastnosti potrebno preveriti numerično.

Razcep dublet-triplet je v tem modelu možen, kar lahko numerično preverimo na masni matriki za dublete in triplete (enačba (495)). Tu tesno prilagodimo vrednost produkta parametrov $\lambda_5\lambda_6$, ki v zlomu simetrije nista udeležena. Numerično prav tako lahko preverimo, da ima brezmasno stanje dubletov vse komponente neničelne, zato vsi dubleti D_i in $\overline{D_i}$ za $i = o, \ldots 11$ (za definicijo oznak glej tabelo 9) dobijo pričakovane vrednosti na elektrošibki skali.

V tem modelu je Yukawin sektor preprostejši kot v modelu I, saj imamo le dve Yukawini matriki, zato tudi način mešanja okusov poteka na običajen način za teorije poenotenja (glej npr. zgoraj opisani SO(10) model s Higgsi v 10 in $\overline{126}$), tako da se lahki Higgs nahaja hkrati v obeh členih Yukawinega sektorja (v 27 in v $\overline{351'}$). Eksplicitni račun pokaže, da so masne matrike v fermionskem sektorju naslednje:

$$
\begin{aligned}
&u^T\left(-v_1 Y_{27}+\tfrac{1}{2\sqrt{10}}v_5 Y_{\overline{351'}}-\tfrac{1}{2\sqrt{6}}v_7 Y_{\overline{351'}}\right)u^c\\
&+\begin{pmatrix}d^{cT} & d'^{cT}\end{pmatrix}\begin{pmatrix}\bar{v}_2 Y_{27}+\tfrac{1}{2\sqrt{10}}\bar{v}_4 Y_{\overline{351'}}+\tfrac{1}{2\sqrt{6}}\bar{v}_8 Y_{\overline{351'}} & c_2 Y_{27}+\tfrac{1}{\sqrt{15}}f_5 Y_{\overline{351'}}\\ -\bar{v}_3 Y_{27}-\tfrac{1}{2\sqrt{10}}\bar{v}_9 Y_{\overline{351'}}-\tfrac{1}{2\sqrt{6}}\bar{v}_{11} Y_{\overline{351'}} & -c_1 Y_{27}+\tfrac{1}{\sqrt{15}}f_4 Y_{\overline{351'}}\end{pmatrix}\begin{pmatrix}d\\ d'\end{pmatrix}\\
&+\begin{pmatrix}e^T & e'^T\end{pmatrix}\begin{pmatrix}-\bar{v}_2 Y_{27}-\tfrac{1}{2\sqrt{10}}\bar{v}_4 Y_{\overline{351'}}+\sqrt{\tfrac{3}{8}}\bar{v}_8 Y_{\overline{351'}} & c_2 Y_{27}-\tfrac{3}{2}\tfrac{1}{\sqrt{15}}f_5 Y_{\overline{351'}}\\ \bar{v}_3 Y_{27}+\tfrac{1}{2\sqrt{10}}\bar{v}_9 Y_{\overline{351'}}-\sqrt{\tfrac{3}{8}}\bar{v}_{11} Y_{\overline{351'}} & -c_1 Y_{27}-\tfrac{3}{2}\tfrac{1}{\sqrt{15}}f_4 Y_{\overline{351'}}\end{pmatrix}\begin{pmatrix}e^c\\ e'^c\end{pmatrix}\\
&+\begin{pmatrix}\nu^T & \nu'^T\end{pmatrix}\begin{pmatrix}v_1 Y_{27}-\tfrac{1}{2\sqrt{10}}v_5 Y_{\overline{351'}}-\sqrt{\tfrac{3}{8}}v_7 Y_{\overline{351'}} & -\tfrac{1}{\sqrt{2}}v_6 Y_{\overline{351'}} & c_2 Y_{27}-\tfrac{3}{2}\tfrac{1}{\sqrt{15}}f_5 Y_{\overline{351'}}\\ -\tfrac{1}{\sqrt{2}}v_{10} Y_{\overline{351'}} & -v_1 Y_{27}-\sqrt{\tfrac{2}{5}}v_5 Y_{\overline{351'}} & -c_1 Y_{27}-\tfrac{3}{2}\tfrac{1}{\sqrt{15}}f_4 Y_{\overline{351'}}\end{pmatrix}\begin{pmatrix}\nu^c\\ s\\ \nu'^c\end{pmatrix}\\
&+\frac{1}{2}\begin{pmatrix}\nu^{cT} & s^T & \nu'^{cT}\end{pmatrix}\begin{pmatrix}f_1 Y_{\overline{351'}} & \tfrac{1}{\sqrt{2}}f_2 Y_{\overline{351'}} & -\bar{v}_3 Y_{27}+\sqrt{\tfrac{2}{5}}\bar{v}_9 Y_{\overline{351'}}\\ \tfrac{1}{\sqrt{2}}f_2 Y_{\overline{351'}} & f_3 Y_{\overline{351'}} & \bar{v}_2 Y_{27}-\sqrt{\tfrac{2}{5}}\bar{v}_4 Y_{\overline{351'}}\\ -\bar{v}_3 Y_{27}+\sqrt{\tfrac{2}{5}}\bar{v}_9 Y_{\overline{351'}} & \bar{v}_2 Y_{27}-\sqrt{\tfrac{2}{5}}\bar{v}_4 Y_{\overline{351'}} & 0\end{pmatrix}\begin{pmatrix}\nu^c\\ s\\ \nu'^c\end{pmatrix}\\
&+\frac{1}{2}\begin{pmatrix}\nu^T & \nu'^T\end{pmatrix}\begin{pmatrix}\Delta_1 Y_{\overline{351'}} & \tfrac{1}{\sqrt{2}}\Delta_2 Y_{\overline{351'}}\\ \tfrac{1}{\sqrt{2}}\Delta_2 Y_{\overline{351'}} & \Delta_3 Y_{\overline{351'}}\end{pmatrix}\begin{pmatrix}\nu\\ \nu'\end{pmatrix}. \qquad (611)
\end{aligned}
$$

Z analogno integracijo težkih prostostnih stopenj kot v modelu I dobimo naslednje matrike za fermione Standardnega modela:

$$M_U = -v_1 Y_{27} + \left(\tfrac{1}{2\sqrt{10}}v_5 - \tfrac{1}{2\sqrt{6}}v_7\right) Y_{\overline{351'}}, \qquad (612)$$

$$M_D^T = \left(1+(9/4)X_0X_0^\dagger\right)^{-1/2}\left(\left(\bar{v}_2-\tfrac{3}{2}\bar{v}_3X_0\right)Y_{27}+\left(\tfrac{1}{2\sqrt{10}}(\bar{v}_4-\tfrac{3}{2}\bar{v}_9X_0)+\tfrac{1}{2\sqrt{6}}(\bar{v}_8-\tfrac{3}{2}\bar{v}_{11}X_0)\right)Y_{\overline{351'}}\right), \qquad (613)$$

$$M_E = \left(1+X_0X_0^\dagger\right)^{-1/2}\left(\left(-\bar{v}_2-\bar{v}_3X_0\right)Y_{27}+\left(-\tfrac{1}{2\sqrt{10}}(\bar{v}_4+\bar{v}_9X_0)+\sqrt{\tfrac{3}{8}}(\bar{v}_8+\bar{v}_{11}X_0)\right)Y_{\overline{351'}}\right), \qquad (614)$$

$$
\begin{aligned}
M_N = &-\left(1+X_0X_0^\dagger\right)^{-1/2}\\
&\Bigg(\left(-\tfrac{1}{\sqrt{10}}\tfrac{v_1v_5}{f_1}-\sqrt{\tfrac{3}{2}}\tfrac{v_1v_7}{f_1}+\tfrac{1}{\sqrt{3}}\tfrac{v_5v_{10}}{f_1}\tfrac{c_2}{f_4}+\sqrt{5}\,\tfrac{v_7v_{10}}{f_1}\tfrac{c_2}{f_4}+\tfrac{4}{\sqrt{3}}\tfrac{v_5v_6}{f_3}\tfrac{c_2}{f_4}-2\sqrt{\tfrac{10}{3}}\,\Delta_2\,\tfrac{c_2}{f_4}\right)Y_{27}\\
&\quad+\left(\frac{1}{40}\frac{{v_5}^2}{f_1}+\sqrt{\frac{3}{80}}\frac{v_7v_5}{f_1}+\frac{3}{8}\frac{{v_7}^2}{f_1}+\frac{1}{2}\frac{{v_6}^2}{f_3}-\Delta_1\right)Y_{\overline{351'}}\\
&\quad+\left(\tfrac{{v_1}^2}{f_1}-2\sqrt{\tfrac{10}{3}}\tfrac{v_1v_{10}}{f_1}\tfrac{c_2}{f_4}+\tfrac{10}{3}\tfrac{{v_{10}}^2}{f_1}\tfrac{{c_2}^2}{{f_4}^2}+2\sqrt{\tfrac{10}{3}}\tfrac{v_1v_6}{f_3}\tfrac{c_2}{f_4}+\tfrac{8}{3}\tfrac{{v_5}^2}{f_3}\tfrac{{c_2}^2}{{f_4}^2}-\tfrac{20}{3}\,\Delta_3\,\tfrac{{c_2}^2}{{f_4}^2}\right)Y_{27}Y_{\overline{351'}}^{-1}Y_{27}\\
&\quad+\left(\frac{8\sqrt{10}}{3}\frac{v_1v_5}{f_3}\frac{{c_2}^2}{{f_4}^2}\right)Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}\\
&\quad+\left(\frac{20}{3}\frac{{v_1}^2}{f_3}\frac{{c_2}^2}{{f_4}^2}\right)Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}Y_{\overline{351'}}^{-1}Y_{27}\Bigg)\\
&\left(1+X_0^*X_0^T\right)^{-1/2}, \qquad (615)
\end{aligned}
$$

kjer definiramo

$$X_0 = -2\sqrt{\tfrac{5}{3}}\,\frac{c_2}{f_4}\,Y_{27}\,Y^{-1}_{\overline{351}'}, \tag{616}$$

medtem ko so vrednosti šibkih tripletov Δ (definicije v tabeli 18, upoštevan nastavek $u_1 = u_2 = y = 0$ iz rešitve za vakuum) naslednje:

$$\begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{pmatrix} = \begin{pmatrix} m_{351'} - \lambda_8\left(\frac{w}{6\sqrt{2}} - \frac{1}{2}\sqrt{\frac{3}{10}}v\right) & 0 & 0 & 6\lambda_1 e_1 \\ 0 & m_{351'} + \lambda_8\left(\frac{w}{12\sqrt{2}} + \frac{v}{4\sqrt{30}}\right) & 0 & 0 \\ 0 & 0 & m_{351'} + \lambda_8\left(\frac{w}{3\sqrt{2}} - \frac{v}{\sqrt{30}}\right) & 6\lambda_1 e_3 \\ 6\lambda_2 f_1 & 0 & 6\lambda_2 f_3 & m_{351'} + \lambda_8\left(\frac{w}{3\sqrt{2}} + \frac{v}{\sqrt{30}}\right) \end{pmatrix}^{-1} \begin{pmatrix} \lambda_4 v_3{}^2 \\ \lambda_4\sqrt{2}v_2 v_3 \\ \lambda_4 v_2{}^2 \\ \lambda_3 v_1{}^2 \end{pmatrix}. \tag{617}$$

Analogno z modelom I imamo tudi tu mešanje med SU(5) upodobitvama $\overline{5}$, ki se nahajata v 27_F, kar se odraža v sektorju kvarkov d in leptonov. V nevtrinskem sektorju imamo prispevka gugalničnega mehanizma tipa I in tipa II. Izrazi za masne matrike nizkoenergijskih fermionov so nelinearni, a ne zadoščajo nobenim posebnim relacijam; s tega vidika so, vsaj kar se njihove analitične oblike tiče, mase fermionov realistične.

Model II je na analitičnem nivoju realističen: našli smo vakuumsko rešitev, ki zlomi v Standardni model, uspeli smo z razcepom dublet-triplet (tako da imamo en par dublet-antidublet lahek), ter dobili realistično obliko mas za fermione Standardnega modela, medtem ko so ostali delci težki. Model II ima naslednje parametre v superpotencialu: dve simetrični Yukawini matriki Y_{27} in $Y_{\overline{351}'}$, 3 mase, ter 8 parametrov λ. Za razliko od modela I smo v Yukawinem sektorju zdaj bolj prediktivni, saj sta prisotni le 2 Yukawini matriki. V tem primeru bi zato bilo zanimivo preveriti, ali je možno napovedati mase in mešalne kote Standardnge modela tudi numerično, kar pa ne bo del te doktorske disertacije.

6.4 Zaključek

V tej doktorski disertaciji smo študirali različne renormalizabilne supersimetrične modele poenotenja z grupo E_6. Njihove Higgsove sektorje smo sestavljali iz upodobitev 27, 78, 351, $351'$ in 650 (in njihovih konjugiranih slik) v različnih kombinacijah.

Ugotovili smo, da nekateri najpreprostejši modeli niso ustrezni. Probleme z zlomom v Standardni model imajo naslednji modeli:

- Model s poljubnim številom upodobitev 27, $\overline{27}$ in 78.
- Model $351 \oplus \overline{351}$, ki mu lahko dodamo poljubno število upodobitev 27 in $\overline{27}$.
- Model $351' \oplus \overline{351'}$.
- Model 650.

Za model $351' \oplus \overline{351'} \oplus 27 \oplus \overline{27}$, ki smo ga poimenovali "prototipni" model, smo našli ustrezno vakuumsko rešitev, a se je izkazalo, da v tem modelu razcep dublet-triplet ni možen. Na delnem uspehu prototip modela smo nato študirali dve realistični nadgradnji:

- Model I: $351' \oplus \overline{351'} \ \oplus\ 27 \oplus \overline{27} \ \oplus\ \widetilde{27} \oplus \overline{\widetilde{27}}$.
- Model II: $351' \oplus \overline{351'} \ \oplus\ 27 \oplus \overline{27} \ \oplus\ 78$.

Za oba modela smo našli ustrezno rešitev enačb gibanja, ki zlomi E_6 v grupo Standardnega modela, uspešno smo izvedli razcep dublet-triplet, analizirali Yukawin sektor, izračunali masne matrike nizkoenergijskih stanj in določili prispevke k protonskemu razpadu dimenzije 5. Oba modela sta realistična, vsaj kar se tiče analitičnih izrazov; numerično pričakujemo, da je model II bolj prediktiven, saj ima 2 Yukawini matriki, model I pa 3.

V naših modelih nismo preučevali podrobnosti drsenja sklopitvenih konstant. Poenotenje sklopitvenih konstant smo privzeli na podlagi tega, da se tako model I kot model II v nizkoenergijski limiti zreducirata na MSSM. Poleg tega se nevarnost lahko skriva tudi v koeficientih drsenja β, ki sta za model I in II po vrsti enaka -153 and -159, kar hitro vodi v Landauov pol nad energijami, ko so vse prostostne stopnje prisotne (podobne težave so tudi v SO(10); minimalni supersimetrični model ima npr. $\beta = -109$ [21, 22, 23]). Za podroben študij obeh vprašanj bi bil potreben izračun mas vseh prostostnih stopenj v modelih; ustrezna razpršenost teh mas bi lahko omilila problem Landauovega pola.

Model I in model II sta verjetno minimalna realistična modela v sklopu renormalizabilnih supersimetričnih modelov E_6. Z modeli, ki smo jih preučevali, smo namreč dokaj sistematično postopali od manj k bolj kompliciranim, in omenjena modela sta najpreprostejša najdena realistična modela. Preprostejša modela, ki ju nismo preučevali, sta le $351 \oplus 78 \oplus \overline{351}$ in $351' \oplus 78 \oplus \overline{351'}$, ki pa imata pomanjkljiv Yukawin sektor. Preprostejši od modelov I in II je tudi prototipni model: naivno bi pričakovali, da je ta model minimalen, a presenetljivo spodleti pri razcepu dublet-triplet. To je netrivialen rezultat, do katerega smo se dokopali šele s podrobno analizo modela.

Za konec pa še enkrat poudarimo rezultate v Yukawinih sektorjih modela I in modela II. Čeprav se mehanizma mešanja okusov v obeh modelih razlikujeta, pa so zaključki podobni. Spekter lahkih delcev ustreza stanjem iz MSSM, kjer nevtrini dobijo maso prek gugalničnega mehanizma tipa I in tipa II. Ostali delci so težki, in sicer se nahajajo na skali poenotenja; posebej je tu potrebno izpostaviti vektorske delce (kvarke in leptone), za katere se v naših E_6 modelih izkaže, da niso lahki.