

The Well-posedness of the Null-Timelike Boundary Problem for Quasilinear Waves

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The null-timelike initial-boundary value problem for a hyperbolic system of equations consists of the evolution of data given on an initial characteristic surface and on a timelike worldtube to produce a solution in the exterior of the worldtube. We establish the well-posedness of this problem for the evolution of a quasilinear scalar wave by means of energy estimates. The treatment is given in characteristic coordinates and thus provides a guide for developing stable finite difference algorithms. A new technique underlying the approach has potential application to other characteristic initial-boundary value problems.

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I. INTRODUCTION

The use of null hypersurfaces as coordinates to describe gravitational waves, as introduced by Bondi [1], was key to the understanding and geometric treatment of gravitational waves in the full nonlinear context of general relativity [2, 3]. In one version of the associated characteristic initial-boundary value problem for Einstein's equations, boundary data is given on a timelike worldtube and on an initial outgoing null hypersurface [4]. The physical picture underlying this null-timelike problem is that the worldtube data represent the outgoing gravitational radiation emanating from interior matter sources, while ingoing radiation incident on the system is represented by the initial null data. This problem has been developed into a Cauchy-characteristic matching scheme in which the worldtube data is supplied by a Cauchy evolution of the interior sources [5]. See [6] for a review. Cauchy-characteristic matching has been implemented as a numerical evolution code in which the Bondi news function describing the radiation is calculated at future null infinity using a finite numerical grid obtained by Penrose compactification [3]. Although characteristic evolution codes have successfully simulated many null-timelike problems [6] and have recently been applied to extract the radiation from the inspiral and merger of a binary black hole [7], the well-posedness of the null-timelike problem for the Einstein equations has not yet been established. The characteristic formulation of the Einstein equations implies that certain variables associated with the radiation satisfy a wave equation. Consequently, a necessary condition for the well-posedness of the gravitational problem is that the corresponding problem for the quasilinear wave equation be well-posed. In this paper, as a first step toward treating the gravitational case, we show that the quasilinear null-timelike problem for a scalar wave propagating on a curved space background is well posed.

The characteristic initial value problem did not receive much attention before its importance in general relativity was recognized. Historically, the development of computational physics has focused on hydrodynamics, where the characteristics typically do not define useful coordinate surfaces and there is no generic outer boundary behavior comparable to null infinity. The simplest problem for which the characteristic approach is useful is the Minkowski space wave equation, which is satisfied by the components of the fundamental special relativistic fields. Progress on the null-timelike problem traces back to Duff [8], where existence and uniqueness was shown for the linear wave equation with analytic coefficients and analytic data. Existence and uniqueness was later extended to the C^∞ case of the linear wave equation on an asymptotically flat curved space background by Friedlander [9, 10].

The demonstration of well-posedness of the quasilinear boundary problem, i.e. the continuous dependence of the solution on the data, depends upon establishing estimates on the derivatives for the linearized problem. This requires considering generic lower differential order terms [11]. Well-posedness depends crucially on the stability of the problem against such lower order perturbations. Otherwise, one cannot localize the problem and use the principle of frozen coefficients.

Partial results estimating the derivatives for characteristic boundary problems were first obtained by Müller zum Hagen and Seifert [12]. Later Balean carried out a comprehensive study of the differentiability of solutions of the null-timelike problem for the flat space wave equation [13, 14]. He was able to establish estimates for the derivatives tangential to the outgoing null cones but weaker estimates for the time derivatives transverse to the cones had to be obtained from a direct integration of the wave equation. The derivatives tangential to the null cone were controlled by the derivatives of the data but control of the transverse time derivative required two derivatives of the data. Balean

concentrated on the differentiability order of the solution and did not discuss the implications for well-posedness of the quasilinear problem.

Frittelli [15] made the first explicit consideration of well-posedness of the null-timelike problem for the wave equation. She adopted the approach of Duff, in which the characteristic formulation of the wave equation is reduced to a canonical first order differential form, in close analogue to the symmetric hyperbolic formulation of the Cauchy problem. The energy associated with this first order reduction gives estimates for the derivatives of the field tangential to the null hypersurfaces. As in Baean's treatment, weaker estimates for the time derivatives were obtained indirectly so that well-posedness is not ensured when lower order differential order terms or source terms are included as required for the quasilinear case, as she was careful to point out.

A difficulty underlying the problem can be illustrated in terms of the 1(spatial)-dimensional wave equation

$$(\partial_{\tilde{t}}^2 - \partial_{\tilde{x}}^2)\Phi = 0, \quad (1.1)$$

where (\tilde{t}, \tilde{x}) are standard space-time coordinates. The conserved energy

$$\tilde{E}(\tilde{t}) = \frac{1}{2} \int d\tilde{x} \left((\partial_{\tilde{t}}\Phi)^2 + (\partial_{\tilde{x}}\Phi)^2 \right) \quad (1.2)$$

leads to the well-posedness of the Cauchy problem. In characteristic coordinates $(t = \tilde{t} - \tilde{x}, x = \tilde{t} + \tilde{x})$, the wave equation transforms into

$$\partial_t \partial_x \Phi = 0. \quad (1.3)$$

The conserved energy on the characteristics $t = \text{const.}$,

$$\tilde{E}(t) = \int dx (\partial_x \Phi)^2, \quad (1.4)$$

no longer controls the derivative $\partial_t \Phi$.

Up to now, the only treatment of well-posedness of the characteristic initial value problem valid for the quasilinear wave equation has been the work of Rendall [16], who considered the double null problem where data is given on a pair of intersecting characteristic hypersurfaces. Rendall did not treat the characteristic problem head-on but reduced it to a standard Cauchy problem with data on a spacelike hypersurface passing through the intersection of the characteristic hypersurfaces. Well-posedness then follows from the classic result for the Cauchy problem. He extended his treatment to establish the well-posedness of the double-null formulation of the Einstein gravitational problem. The double null problem treated by Rendall is a limiting case of the null-timelike problem considered in this paper. However, Rendall's approach cannot be applied to the null-timelike problem. Also, the reduction to a Cauchy problem does not provide guidance for the development of a stable finite-difference approximation based upon characteristic coordinates.

Here we consider the null-timelike problem for the quasi-linear wave equation in second differential form in terms of characteristic coordinates.. The usual technique for showing that the initial-boundary value problem for a hyperbolic system of partial differential equations is well posed is to split the problem into a Cauchy problem and local halfplane problems and show that these individual problems are well posed. This works for hyperbolic systems based upon a spacelike foliation, in which case signals propagate with finite velocity. Besides the existence and uniqueness of a solution, well-posedness implies that the solution depend continuously on the data with respect to an appropriate norm. For (1.1), the solutions to the Cauchy problem with compact initial data on $\tilde{t} = 0$ are square integrable and well-posedness can be established using the L_2 norm (1.2).

However, In characteristic coordinates the 1-dimensional wave equation (1.3) admits signals traveling in the $+x$ -direction with infinite coordinate velocity. In particular, initial data of compact support $\Phi(0, x) = f(x)$ on the characteristic $t = 0$ admits the solution $\Phi = g(t) + f(x)$, provided that $g(0) = 0$. Here $g(t)$ represents the profile of a wave which travels from past null infinity ($x \rightarrow -\infty$) to future null infinity ($x \rightarrow +\infty$). Thus, without a boundary condition at past null infinity, there is no unique solution and the Cauchy problem is ill posed. Even with the boundary condition $\Phi(t, -\infty) = 0$, a source of compact support $S(t, x)$ added to (1.3), i.e.

$$\partial_t \partial_x \Phi = S, \quad (1.5)$$

produces waves propagating to $x = +\infty$ so that although the solution is unique it is still not square integrable.

On the other hand, consider the modified problem obtained by setting $\Phi = e^{ax}\Psi$,

$$\partial_t (\partial_x + a)\Psi = F, \quad \Psi(0, x) = e^{-ax} f(x), \quad a > 0 \quad (1.6)$$

where $F = e^{-ax}S$. With the boundary condition $\Psi(t, -\infty) = 0$, the solutions to (1.6) vanish at $x = +\infty$ and are square integrable. As a result, the Cauchy problem (1.6) is well posed with respect to an L_2 norm. For the simple example where $F = 0$, multiplication of (1.6) by $(2a\Psi + \partial_x\Psi + \frac{1}{2}\partial_t\Psi)$ and integration by parts gives

$$\frac{1}{2}\partial_t \int dx \left((\partial_x\Psi)^2 + 2a^2\Psi^2 \right) = \frac{a}{2} \int dx \left(2(\partial_t\Psi)\partial_x\Psi - (\partial_t\Psi)^2 \right) \leq \frac{a}{2} \int dx (\partial_x\Psi)^2. \quad (1.7)$$

The resulting inequality

$$\partial_t E \leq \text{const.} E \quad (1.8)$$

for the energy

$$E = \frac{1}{2} \int dx \left((\partial_x\Psi)^2 + 2a^2\Psi^2 \right) \quad (1.9)$$

provides the estimates for $\partial_x\Psi$ and Ψ which are necessary for well-posedness. Estimates for $\partial_t\Psi$, and other higher derivatives, follow from applying this approach to the derivatives of (1.6). The approach can be extended to include the source term F and other generic lower differential order terms. This allows well-posedness to be extended to the case of variable coefficients and, locally in time, to the quasilinear case.

The 2(spatial)-dimensional model problems considered in Sec. II illustrate how this approach generalizes to the multi-dimensional case. We consider the model problems in the modified form analogous to (1.6). By means of this technique, the characteristic initial-boundary value problem can again be treated by first considering Cauchy and half-plane problems. The demonstration of well-posedness of these model problems presents the underlying ideas in a transparent form.

Our main technique is the use of energy estimates. Although the model problems are treated in the modified form, the results can be translated back to the original problem. For example, the modification in going from (1.5) to (1.6) leads to an effective modification of the standard energy for the problem. Rewritten in terms of the original variable $\Phi = e^{ax}\Psi$, (1.9) corresponds to the energy

$$E = \frac{1}{2} \int dx e^{-2ax} \left((\partial_x\Phi)^2 + a^2\Phi^2 \right). \quad (1.10)$$

Thus while the Cauchy problem for (1.6) is ill posed with respect to the L_2 norm it is well posed with respect to the exponentially weighted norm (1.10). However, rather than modifying the norm, for technical simplicity we deal with the modified variable Ψ .

The general arguments presented for our model problems can be applied to a wide range of quasilinear characteristic problems. Our motivation for the work here is the application to the null-timelike problem for the quasilinear wave equation for a scalar field Φ in an asymptotically flat curved space background with source S ,

$$g^{ab}\nabla_a\nabla_b\Phi = S(\Phi, \partial_c\Phi, x^c), \quad (1.11)$$

where the metric g^{ab} and its associated covariant derivative ∇_a are explicitly prescribed functions of (Φ, x^c) .

The corresponding flat space wave equation,

$$(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\Phi = S, \quad (1.12)$$

takes the form

$$\frac{1}{r}(-2\partial_u\partial_r + \partial_r^2)(r\Phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin^2\theta\partial_\theta\Phi) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\Phi = S \quad (1.13)$$

in null-spherical coordinates (u, r, θ, ϕ) consisting of a retarded time $u = \tilde{t} - r$ and standard spherical coordinates (r, θ, ϕ) . In these coordinates, the Minkowski metric is

$$ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.14)$$

The null-timelike problem consists of determining Φ in the region $(r > R, u > 0)$ given data $\Phi(u, R, \theta, \phi)$ on the timelike worldtube $r = R$ and $\Phi(0, r, \theta, \phi)$ on the initial null hypersurface $u = 0$.

In an asymptotically flat background, the metric (1.14) generalizes to the Bondi-Sachs form

$$g_{ab}dx^a dx^b = -(e^{2\beta}W - r^{-2}h_{AB}W^A W^B)du^2 - 2e^{2\beta}dudr - 2h_{AB}W^B dudx^A + r^2 h_{AB}dx^A dx^B, \quad (1.15)$$

where x^A are angular coordinates such that $(u, x^A) = \text{const.}$ along the outgoing null rays. Here the radial coordinate r is a surface area coordinate so that the area of the topological spheres $(u, r) = \text{const.}$ is 4π as measured by h_{AB} . In the curved space version of angular coordinates analogous to (1.14), $\det(h_{AB}) = \sin^2 \theta$.

In Sec. III, we treat the null-timelike problem for the quasilinear wave equation (1.11) with Lorentzian metric (1.15),

$$\begin{aligned} \frac{1}{r}(-2\partial_u\partial_r + W\partial_r^2)(r\Phi) + (\partial_r W)\partial_r\Phi - \frac{1}{r^2}D_A(W^A\partial_r\Phi) - \frac{1}{r^2}\partial_r(W^AD_A\Phi) + \frac{1}{r^2}D_A(e^{2\beta}D^A\Phi) \\ = e^{2\beta}S(\Phi, \partial_c\Phi, x^c), \end{aligned} \quad (1.16)$$

with initial data $\Phi(0, r, x^A)$ and boundary data $\Phi(u, R, x^A)$. Here D_A is the 2-dimensional covariant derivative with respect to h_{AB} and the metric coefficients (W, β, W^A, h_{AB}) depend smoothly upon (Φ, u, r, x^A) and the source S depends smoothly upon $(\Phi, \partial_a\Phi, u, r, x^A)$.

An essential part of any initial-boundary value problem is the compatibility between the data at the intersection between the initial hypersurface and the boundary, i.e. at $(u = 0, r = R)$ in the above case. This compatibility affects the differentiability of the resulting solution. In order to avoid difficult issues of analysis, we only give a rigorous treatment for the case of smooth initial and boundary data with compact support bounded away from the intersection, in which case the solution is C^∞ locally in time. See the work of Baean [13, 14] for a discussion of the differentiability of the solution in the general case.

We assume that as $r \rightarrow \infty$ (the approach to null infinity) that the problem reduces to the flat space problem (1.13), so that the coefficients have the asymptotic behavior $W = 1 + O(1/r)$, $\beta = 0 + O(1/r)$, $W^A = O(1)$ and $h_{AB} = q_{AB} + O(1/r)$, where q_{AB} is the unit sphere metric. The results of Friedlander [10] then imply that the scalar wave falls off as $\Phi \sim \Phi_0(u, x^A)/r$ where Φ_0 is the asymptotic radiation field.

In Sec. III, we show that the nullcone-worldtube problem for (1.16) is well posed subject to the condition that $S = O(r^{-3})$ and a positivity condition that the principal part of the wave operator reduces to an elliptic operator in the stationary case. Our results are based upon energy estimates obtained by integration by parts with respect to the characteristic coordinates. As a result, the analogous finite difference estimates obtained by *summation by parts* provide guidance for the development of a stable numerical evolution algorithm for (1.16).

II. WELL-POSEDNESS OF MODEL CHARACTERISTIC PROBLEMS

We consider here several model 2(spatial)-dimensional problems which reveal the essential features underlying a well posed characteristic initial-boundary value problem. For simplicity of notation, we indicate partial derivatives by subscripts, e.g. $\Phi_t(t, x, y) = \partial_t\Phi(t, x, y)$. Also, we denote the L_2 scalar product and norm over the x, y domain by (Φ_1, Φ_2) and $\|\Phi\|^2 = (\Phi, \Phi)$.

We consider model linear problems with constant coefficients but show that the problems are stable against lower order perturbations. We also obtain estimates for arbitrarily high derivatives. Thus we can use standard techniques [11] to establish the well-posedness of the corresponding problem with smooth variable coefficients. For the extension to the quasilinear case, we require that the coefficients depend smoothly upon the field Φ with nonsingular behavior in the neighborhood of the initial data. Then well-posedness, locally in time, of the quasilinear problem also follows from standard techniques [11]. (See the Appendix of [17] for details concerning how these standard techniques apply to hyperbolic systems in second differential order form.)

Our goal is to show the well-posedness of the strip problem

$$2\Phi_{tx} = ((1-x)^2\Phi_x)_x + \Phi_{yy} + b \left(((1-x)^2\Phi_x)_y + ((1-x)^2\Phi_y)_x \right)$$

in the domain

$$0 \leq x \leq 1, \quad -\infty < y < \infty, \quad t \geq 0$$

with initial and boundary conditions

$$\Phi(0, x, y) = f(x, y), \quad \Phi(t, 0, y) = q(t, y),$$

respectively. The method used to show that this problem is well posed applies to the compactified version of the null-timelike boundary problem for the wave equation (1.16) treated in Sec. III. As explained in the Introduction, we treat the problem in the modified form obtained by the change of variable $\Phi = e^{ax}\Psi$, $a > 0$.

A. The Cauchy problem

We first consider the Cauchy problem

$$(\Psi_x + a\Psi)_t = \Psi_{yy} - 2b\Psi_y, \quad \bar{x} = (x, y) \in R^2, \quad t \geq 0, \quad (2.1)$$

$$\Psi(0, x, y) = f(x, y), \quad (2.2)$$

where x and t are both characteristic coordinates. Here a, b are real constants and $f(x, y) \in C_0^\infty$ (a smooth function with compact support). As explained in the Introduction, we investigate the behavior of square integrable solutions, so that $\Psi(t, \pm\infty, y) = 0$.

1. The Fourier method

We first solve the problem by Fourier transform. Let

$$\hat{f}(\bar{\omega}) = \frac{1}{2\pi} \int_{R^2} e^{-i\bar{\omega} \cdot \bar{x}} f(\bar{x}) dx dy, \quad \bar{\omega} = (\omega_1, \omega_2) \text{ real,}$$

denote the Fourier transform of f and $\hat{\Psi}(t, \bar{\omega})$ the Fourier transform of Ψ . Then $\hat{\Psi}(t, \bar{\omega})$ is the solution of

$$(i\omega_1 + a)\hat{\Psi}_t = -(\omega_2^2 + 2bi\omega_2)\hat{\Psi}, \quad (2.3)$$

$$\hat{\Psi}(0, \bar{\omega}) = \hat{f}(\bar{\omega}), \quad (2.4)$$

i.e.

$$\hat{\Psi}_t = s\hat{\Psi},$$

where

$$s = -\frac{\omega_2^2 + 2bi\omega_2}{i\omega_1 + a} = -\frac{(\omega_2^2 + 2bi\omega_2)(a - i\omega_1)}{a^2 + \omega_1^2}. \quad (2.5)$$

Therefore

$$\begin{aligned} \Re s &= -\frac{a\omega_2^2 + 2b\omega_1\omega_2}{a^2 + \omega_1^2}, \\ \Im s &= \frac{(\omega_2^2\omega_1 - 2ab\omega_1)}{a^2 + \omega_1^2}. \end{aligned} \quad (2.6)$$

We now discuss the dependence of the solutions on a, b in detail.

1) $b = 0, a > 0$. By (2.6),

$$\Re s = -\frac{a\omega_2^2}{\omega_1^2 + a^2} \leq 0.$$

There are no exponentially growing solutions.

2) $b = 0, a = 0$. By (2.6),

$$\Re s = 0, \quad |\Im s| \rightarrow \infty \quad \text{for} \quad |\omega_1| \rightarrow 0, \quad \omega_2 \neq 0.$$

Therefore the solution of (2.2) loses all smoothness in time if $\hat{f}(0, \omega_2) \neq 0$.

3) $b = 0, a < 0$. By (2.6),

$$\Re s \rightarrow +\infty \quad \text{for} \quad \omega_1 \rightarrow 0, \quad \omega_2^2 > 0.$$

Thus the problem is ill posed.

4) $b \neq 0$, $a > 0$. By (2.6),

$$\Re s = -\frac{a(\omega_2 + \frac{b}{a}\omega_1)^2}{\omega_1^2 + a^2} + \frac{b^2\omega_1^2}{a(\omega_1^2 + a^2)} \leq \frac{b^2}{a}.$$

There is exponential growth but the growth is bounded independently of $\bar{\omega}$.

5) $b \neq 0$, $a = 0$. By (2.6),

$$\Re s = -\frac{2b\omega_2}{\omega_1}.$$

Thus there is unbounded exponential growth as $\omega_1 \rightarrow 0$. The same is true if $a < 0$.

We now express our results in a more general setting.

Definition 2.1. *We call the Cauchy problem well posed if, for every $f \in C_0^\infty$, there is a unique, smooth, square integrable solution and if there is a constant α which does not depend on $\bar{\omega}$ such that*

$$\Re s \leq \alpha.$$

The problem is ill posed if there is no upper bound α , i.e., there is a sequence $\bar{\omega}^{(j)}$ such that

$$\lim_{j \rightarrow \infty} \Re s_j = \infty.$$

Theorem 2.1. *The Cauchy problem (2.2) is well posed if $a > 0$. But it is ill posed if $a < 0$ or $a = 0$, $b \neq 0$.*

2. The energy method

For the generalization to variable coefficients it is necessary to show that the differential equation (2.2) is stable against lower order perturbations. For this purpose we first apply the energy method to the doubly-characteristic Cauchy problem

$$\begin{aligned} (\Psi_x + a\Psi)_t &= \Psi_{yy} - 2b\Psi_y - c\Psi_x + d\Psi_t + e\Psi + F(t, x, y), \\ \Psi(0, x, y) &= f(x, y), \quad -\infty < x, y < \infty, \quad t \geq 0. \end{aligned} \quad (2.7)$$

Here $a > 0$, b , c , d , e are real constants and F is a forcing (source) term of compact spatial support.

The term $d\Psi_t$ can be absorbed into the left hand side and we obtain $(\Psi_x + (a-d)\Psi)_t$. Therefore we neglect this term and assume that a is sufficiently large so that $a-d > 0$. We also neglect the term $e\Psi$ because it has no influence on the required energy estimates.. Therefore we consider the corresponding Cauchy problem for

$$(\Psi_x + a\Psi)_t = \Psi_{yy} - 2b\Psi_y - c\Psi_x + F. \quad (2.8)$$

We now derive an energy estimate. By (2.8),

$$(\Psi, \Psi_{xt}) + a(\Psi, \Psi_t) = -(\Psi_x \Psi_t) + \frac{a}{2} \partial_t \|\Psi\|^2 = (\Psi, \Psi_{yy}) - (\Psi, 2b\Psi_y + c\Psi_x) + (\Psi, F).$$

Since

$$(\Psi_x, \Psi_t) = \left(\frac{2}{\sqrt{a}} \Psi_x, \frac{\sqrt{a}}{2} \Psi_t \right) \leq \frac{2}{a} \|\Psi_x\|^2 + \frac{a}{8} \|\Psi_t\|^2,$$

integration by parts gives

$$\frac{a}{2} \partial_t \|\Psi\|^2 + \|\Psi_y\|^2 = (\Psi_x, \Psi_t) + (\Psi, F) \leq \frac{2}{a} \|\Psi_x\|^2 + \frac{a}{8} \|\Psi_t\|^2 + \frac{1}{2} (\|\Psi\|^2 + \|F\|^2). \quad (2.9)$$

Next,

$$(\Psi_t, \Psi_{xt}) + a\|\Psi_t\|^2 = -\frac{1}{2} \partial_t \|\Psi_y\|^2 - 2b(\Psi_t, \Psi_y) - c(\Psi_t, \Psi_x) + (\Psi_t, F).$$

Since

$$c(\Psi_t, \Psi_x) = \left(\frac{\sqrt{a}}{2} \Psi_t, \frac{2c}{\sqrt{a}} \Psi_x \right) \leq \frac{a}{8} \|\Psi_t\|^2 + \frac{2c^2}{a} \|\Psi_x\|^2,$$

$$(\Psi_t, F) \leq \frac{a}{8} \|\Psi_t\|^2 + \frac{2}{a} \|F\|^2,$$

$$2b(\Psi_t, \Psi_y) = \left(\frac{\sqrt{a}}{2} \Psi_t, \frac{4b}{\sqrt{a}} \Psi_y \right) \leq \frac{a}{8} \|\Psi_t\|^2 + \frac{8b^2}{a} \|\Psi_y\|^2,$$

we obtain

$$\frac{5a}{8} \|\Psi_t\|^2 + \frac{1}{2} \partial_t \|\Psi_y\|^2 \leq \text{const.} (\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|F\|^2). \quad (2.10)$$

Next,

$$(\Psi_x, \Psi_{xt}) + a(\Psi_x, \Psi_t) = (\Psi_x, \Psi_{yy}) - (\Psi_x, 2b\Psi_y + c\Psi_x) + (\Psi_x, F).$$

Since $(\Psi_x, \Psi_{yy}) = -(\Psi_{xy}, \Psi_y) = 0$, we obtain

$$\frac{1}{2} \partial_t \|\Psi_x\|^2 \leq \text{const.} (\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|F\|^2) + \frac{a}{8} \|\Psi_t\|^2. \quad (2.11)$$

Adding (2.9)–(2.11) gives the energy estimate

$$\frac{3a}{8} \|\Psi_t\|^2 + \|\Psi_y\|^2 + \frac{1}{2} \partial_t (\|\Psi_x\|^2 + \|\Psi_y\|^2 + a\|\Psi\|^2) \leq \text{const.} (\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|\Psi\|^2 + \|F\|^2). \quad (2.12)$$

We have proved:

Theorem 2.2. *The Cauchy problem (2.7) is well posed with respect to the L_2 norm if $(a - d) > 0$. There is an energy estimate. Also, the problem is stable against lower order perturbations. In addition, estimates for the higher derivatives of Ψ follow from the equations obtained by differentiating (2.7).*

We now consider the Cauchy problem

$$\begin{aligned} (\Psi_x + a\Psi)_t &= \Psi_{xx} + \Psi_{yy} + F(t, x, y), \\ \Psi(0, x, y) &= f(x, y), \quad -\infty < x, y < \infty, \quad t \geq 0, \end{aligned} \quad (2.13)$$

where x is a characteristic coordinate but t is timelike. We again derive an energy estimate.

We have

$$\begin{aligned} (\Psi, \Psi_{xt} + a\Psi_t) &= -(\Psi_x, \Psi_t) + \frac{a}{2} \partial_t \|\Psi\|^2 \\ &= -(\|\Psi_x\|^2 + \|\Psi_y\|^2) + (\Psi, F), \end{aligned}$$

i.e.

$$\frac{a}{2} \partial_t \|\Psi\|^2 + \|\Psi_x\|^2 + \|\Psi_y\|^2 = (\Psi_x, \Psi_t) + (\Psi, F) \leq \frac{a}{8} \|\Psi_t\|^2 + \frac{2}{a} \|\Psi_x\|^2 + (\Psi, F). \quad (2.14)$$

Next,

$$\begin{aligned} (\Psi_t, \Psi_{xt} + a\Psi_t) &= (\Psi_t, \Psi_{xt}) + a\|\Psi_t\|^2 = a\|\Psi_t\|^2 \\ &= -\frac{1}{2} \partial_t (\|\Psi_x\|^2 + \|\Psi_y\|^2) + (\Psi_t, F), \end{aligned}$$

i.e.

$$a\|\Psi_t\|^2 + \frac{1}{2} \partial_t (\|\Psi_x\|^2 + \|\Psi_y\|^2) = (\Psi_t, F) \leq \frac{a}{8} \|\Psi_t\|^2 + \frac{2}{a} \|F\|^2. \quad (2.15)$$

Combining (2.14) and (2.15) as before, we obtain the desired estimate

$$\frac{3a}{4} \|\Psi_t\|^2 + \|\Psi_x\|^2 + \|\Psi_y\|^2 + \frac{1}{2} \partial_t (\|\Psi_x\|^2 + \|\Psi_y\|^2 + a\|\Psi\|^2) \leq \text{const.} (\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|\Psi\|^2 + \|F\|^2). \quad (2.16)$$

Remark . *As before, we can add a general lower order expression and still obtain the estimate. Also, we can estimate all derivatives.*

B. The half-plane problem

We now apply the energy method to the double-null halfplane problem for (2.8),

$$(\Psi_x + a\Psi)_t = \Psi_{yy} + 2b\Psi_y - c\Psi_x + F, \quad 0 \leq x < \infty, \quad -\infty < y < \infty, \quad t \geq 0, \quad (2.17)$$

with initial and boundary data

$$\Psi(0, x, y) = f(x, y), \quad \Psi(t, 0, y) = 0 \quad (2.18)$$

and source $F(t, x, y)$ of compact support.

There are no difficulties to derive the basic estimate (2.12) because for the estimates (2.9)–(2.11) we require only that $\Psi(t, 0, y) = 0$. To obtain estimates for higher derivatives we have to proceed in the following way.

We differentiate (2.17) with respect to y . Since $\Psi_y(t, 0, y) = 0$, we obtain the same problem for Ψ_y and therefore we obtain estimates for

$$\|\Psi_{yy}\|^2, \quad \|\Psi_{xy}\|^2.$$

If we differentiate (2.17) two times with respect to y , we obtain estimates for the third derivatives. The corresponding results hold for t -derivatives, e.g.

$$\|\Psi_t\|^2, \quad \|\Psi_{yt}\|^2, \quad \|\Psi_{xt}\|^2.$$

Now we differentiate (2.17) with respect to x .

$$(\Psi_{xx} + a\Psi_x)_t = \Psi_{yyx} + R. \quad (2.19)$$

Here R consists of source terms and terms which we have already estimated. (2.19) gives us

$$(\Psi_{xx}, \Psi_{xxt}) + a(\Psi_{xx}, \Psi_{xt}) = (\Psi_{xx}, \Psi_{yyx}) + (\Psi_{xx}, R). \quad (2.20)$$

We obtain

$$\frac{1}{2}\partial_t \|\Psi_{xx}\|^2 \leq \frac{1}{2} \left((1 + a^2) \|\Psi_{xx}\|^2 + \|\Psi_{xt}\|^2 + \|\Psi_{yyx}\|^2 + \|R\|^2 \right),$$

where we already have estimates for $\|\Psi_{xt}\|^2$ and $\|\Psi_{yyx}\|^2$. The process can be continued.

Remark . *Inhomogeneous boundary data $\Psi(t, 0, y) = q(t, y)$ may be treated in the same way through the transformation $\Psi \rightarrow \Psi - qe^{-x}$ and absorbing the boundary data in the source term F . We can also treat the timelike-null halfplane problem for (2.13) in the same way.*

C. The strip problem

As a prototype of the compactified wave equation considered in Sec. III, we consider the strip problem

$$2(\Psi_x + a\Psi)_t = ((1-x)^2\Psi_x)_x + \Psi_{yy} + b \left(((1-x)^2\Psi_x)_y + ((1-x)^2\Psi_y)_x \right) + F(t, x, y) \quad (2.21)$$

for

$$0 \leq x \leq 1, \quad -\infty < y < \infty, \quad t \geq 0$$

with initial and boundary conditions

$$\Psi(0, x, y) = f(x, y), \quad \Psi(t, 0, y) = q(t, y).$$

Here $a > 0$ and b , with $|b| < 1$, are real constants and F is a smooth function. The outer boundary Γ_1 at $x = 1$ is an ingoing characteristic so that no boundary condition is allowed.

Since the boundary data at Γ_0 can be absorbed into the source F , we treat the case $q = 0$ (see the remark in Sec. II B). We denote the L_2 norm over Γ_1 by

$$\|\Psi\|_{\Gamma_1}^2 = \int dy \Psi^2(t, 1, y)$$

and the L_2 norm over the boundary Γ_0 at $x = 0$ by

$$\|\Psi\|_{\Gamma_0}^2 = \int dy \Psi^2(t, 0, y).$$

We want to show that there is an energy estimate and that the problem is stable against lower order perturbations. We derive the necessary estimates. First,

$$\begin{aligned} & 2(\Psi, \Psi_{xt}) + 2a(\Psi, \Psi_t) = -2(\Psi_x, \Psi_t) + \partial_t \|\Psi\|_{\Gamma_1}^2 + a\partial_t \|\Psi\|^2 \\ & = -\left((1-x)\Psi_x, (1-x)\Psi_x\right) - \|\Psi_y\|^2 - 2b\left((1-x)\Psi_x, (1-x)\Psi_y\right) + (\Psi, F), \end{aligned}$$

i.e.

$$\begin{aligned} & \partial_t \|\Psi\|_{\Gamma_1}^2 + a\partial_t \|\Psi\|^2 + \|(1-x)\Psi_x\|^2 + \|\Psi_y\|^2 \\ & + 2b\left((1-x)\Psi_x, (1-x)\Psi_y\right) = 2(\Psi_x, \Psi_t) + (\Psi, F). \end{aligned} \quad (2.22)$$

Next,

$$\begin{aligned} & 2(\Psi_t, \Psi_{xt}) + 2a\|\Psi_t\|^2 = \|\Psi_t\|_{\Gamma_1}^2 + 2a\|\Psi_t\|^2 \\ & = -\frac{1}{2}\partial_t \left(\|(1-x)\Psi_x\|^2 + \|\Psi_y\|^2 + 2b((1-x)\Psi_x, (1-x)\Psi_y) \right) + (\Psi_t, F). \end{aligned} \quad (2.23)$$

Next,

$$\begin{aligned} & 2(\Psi_x, \Psi_{xt}) + 2a(\Psi_x, \Psi_t) = \partial_t \|\Psi_x\|^2 + 2a(\Psi_x, \Psi_t) = \left(\Psi_x, ((1-x)^2\Psi_x)_x\right) + (\Psi_x, \Psi_{yy}) \\ & + b\left(\Psi_x, ((1-x)^2\Psi_x)_y\right) + b\left(\Psi_x, ((1-x)^2\Psi_y)_x\right) + (\Psi_x, F). \end{aligned} \quad (2.24)$$

Now,

$$\begin{aligned} \left(\Psi_x, ((1-x)^2\Psi_x)_x\right) &= -\left(\Psi_x, 2(1-x)\Psi_x\right) + \left(\Psi_x, (1-x)^2\Psi_{xx}\right) \\ &= -(\Psi_x, 2(1-x)\Psi_x) - \left(\left((1-x)^2\Psi_x\right)_x, \Psi_x\right) - \|\Psi_x\|_{\Gamma_0}^2, \end{aligned}$$

i.e.

$$\left(\Psi_x, ((1-x)^2\Psi_x)_x\right) = -\left(\Psi_x, (1-x)\Psi_x\right) - \frac{1}{2}\|\Psi_x\|_{\Gamma_0}^2.$$

Also,

$$\begin{aligned} (\Psi_x, \Psi_{yy}) &= -(\Psi_{xy}, \Psi_y) = -\frac{1}{2}\|\Psi_y\|_{\Gamma_1}^2, \\ b\left(\Psi_x, ((1-x)^2\Psi_x)_y\right) &= -b\left(\left((1-x)^2\Psi_x\right)_y, \Psi_x\right) = 0, \\ b\left(\Psi_x, ((1-x)^2\Psi_y)_x\right) &= -2b\left(\Psi_x, (1-x)\Psi_y\right) + b\left(\Psi_x, (1-x)^2\Psi_{xy}\right) \\ &= -2b\left((1-x)\Psi_x, \Psi_y\right). \end{aligned}$$

Therefore (2.24) becomes

$$\partial_t \|\Psi_x\|^2 + \left(\Psi_x, (1-x)\Psi_x \right) + \frac{1}{2} \|\Psi_x\|_{\Gamma_0}^2 + \frac{1}{2} \|\Psi_y\|_{\Gamma_1}^2 + 2b \left((1-x)\Psi_x, \Psi_y \right) = -2a(\Psi_x, \Psi_t) + (\Psi_x, F). \quad (2.25)$$

All the boundary terms have the right sign to enhance the estimates. Therefore we ignore them. Adding the simplified estimates (2.22), (2.23), (2.25) gives

$$\begin{aligned} & \partial_t \left(a\|\Psi\|^2 + \|\Psi_x\|^2 + \frac{1}{2}Q \right) + Q + \left(\Psi_x, (1-x)\Psi_x \right) \\ &= -2b \left((1-x)\Psi_x, \Psi_y \right) + 2(1-a)(\Psi_x, \Psi_t) - 2a\|\Psi_t\|^2 + (\Psi + \Psi_t + \Psi_x, F) \\ &\leq \text{const.} \left(\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|\Psi\|^2 + \|F\|^2 \right), \end{aligned} \quad (2.26)$$

where

$$Q = \|(1-x)\Psi_x\|^2 + \|\Psi_y\|^2 + 2b \left((1-x)\Psi_x, (1-x)\Psi_y \right).$$

Since $|b| < 1$, there is a $\delta > 0$ such that

$$Q \geq \delta \left(\|(1-x)\Psi_x\|^2 + \|\Psi_y\|^2 \right).$$

Therefore (2.26) gives us an energy estimate. We shall now prove that the problem is stable against lower order perturbations. We add an expression

$$P = A\Psi_x + B\Psi_y + C\Psi_t + D\Psi$$

to (2.21). Then the estimates for (2.22), (2.23) and (2.24) will be changed by lower order terms

$$\begin{aligned} & (\Psi, A\Psi_x) + (\Psi, B\Psi_y) + (\Psi, C\Psi_t) + (\Psi, D\Psi) \\ & (\Psi_t, A\Psi_x) + (\Psi_t, B\Psi_y) + (\Psi_t, C\Psi_t) + (\Psi_t, D\Psi) \\ & (\Psi_x, A\Psi_x) + (\Psi_x, B\Psi_y) + (\Psi_x, C\Psi_t) + (\Psi_x, D\Psi). \end{aligned}$$

Clearly, there is an energy estimate, provided we choose $2a > |C|$. Thus the strip problem (2.21) is well posed.

Now we start with

$$2\Phi_{xt} = \left((1-x)^2\Phi_x \right)_x + \Phi_{yy} + b \left((1-x)^2\Phi_x \right)_y + b \left((1-x)^2\Phi_y \right)_x + S(t, x, y). \quad (2.27)$$

We make the change of variables

$$\Phi = e^{ax}\Psi,$$

i.e.,

$$\Phi_x = e^{ax}\Psi_x + ae^{ax}\Psi, \quad \Phi_{xx} = e^{ax}\Psi_{xx} + 2ae^{ax}\Psi_x + a^2e^{ax}\Psi,$$

and set $F = e^{-ax}S$. Then we obtain (2.21) which is modified by \mathcal{R}

$$2(\Psi_x + a\Psi)_t = \left((1-x)^2\Psi_x \right)_x + \Psi_{yy} + b \left(\left((1-x)^2\Psi_x \right)_y + \left((1-x)^2\Psi_y \right)_x \right) + F + \mathcal{R}. \quad (2.28)$$

Here \mathcal{R} consists of lower order terms,

$$\mathcal{R} = 2a(1-x)^2\Psi_x + (a^2(1-x)^2 - 2a(1-x))\Psi + 2ab(1-x)^2\Psi_y.$$

Since (2.21) is stable against lower order terms there is an energy estimate for (2.28). In the same way as in Sec. II B we can estimate the higher derivatives. This allows us to extend well-posedness to the variable coefficient problem and, locally in time, to the quasilinear problem.

III. THE QUASILINEAR WAVE EQUATION ON AN ASYMPTOTICALLY FLAT BACKGROUND

We now treat the null-timelike initial-boundary problem (1.16) for the quasilinear wave equation. We compactify the domain $R \leq r \leq \infty$ by the transformation $x = 1 - R/r$ to obtain a strip problem $0 \leq x \leq 1$ with future null infinity \mathcal{I}^+ at the boundary $x = 1$. In terms of the rescaled field $\hat{\Phi} = r\Phi$, the wave equation transforms into

$$\begin{aligned} & 2\partial_u\partial_x\hat{\Phi} - R^{-1}\partial_x(W(1-x)^2\partial_x\hat{\Phi}) + (1-x)R^{-1}(\partial_xW)\hat{\Phi} \\ & + R^{-2}D_A((1-x)^2W^A\partial_x\hat{\Phi}) + R^{-2}\partial_x((1-x)^2W^AD_A\hat{\Phi}) - R^{-2}(1-x)(D_AW^A)\hat{\Phi} \\ & - R^{-1}D_A(e^{2\beta}D^A\hat{\Phi}) = -r^3R^{-1}e^{2\beta}S. \end{aligned} \quad (3.1)$$

Here we use the 2-metric h_{AB} and its inverse h^{AB} to raise and lower indices of tensor fields on the spacelike ($u = \text{const.}, r = \text{const.}$) spherical cross-sections. Up to lower order terms, (3.1) is a 3(spatial)-dimensional version of (2.27) where the y -coordinate has been replaced by the x^A -coordinate on the spherical cross-sections and the t -coordinate has been replaced by the u -coordinate. In order for our treatment to apply to the quasilinear case, we assume that the metric coefficients (W, β, W^A, h_{AB}) depend smoothly upon (Φ, u, r, x^A) and that the source S depends smoothly upon $(\Phi, \partial_a\Phi, u, r, x^A)$, with non-singular Lorentzian geometry in the neighborhood of the initial data.

We treat the modified problem resulting from the transformation $\hat{\Phi} = e^{ax}\Psi$. The same argument used in Sec. II C shows that this problem is stable with respect to lower order terms. We ignore these terms and thus obtain the strip problem

$$\begin{aligned} 2\partial_u(\partial_x\Psi + a\Psi) &= R^{-1}\partial_x(W(1-x)^2\partial_x\Psi) - R^{-2}D_A((1-x)^2W^A\partial_x\Psi) - R^{-2}\partial_x((1-x)^2W^AD_A\Psi) \\ &+ R^{-1}D_A(e^{2\beta}D^A\Psi) + F, \quad \Psi(0, x, x^A) = f, \quad \Psi(u, 0, x^A) = q, \end{aligned} \quad (3.2)$$

where $F = -r^3R^{-1}e^{2\beta}e^{ax}S$. In order to treat (3.2), we require that the physical space source has asymptotic behavior $S = O(r^{-3})$ so that F is square integrable over the strip. No boundary condition is allowed at the outer boundary Γ_1 at $x = 1$ since \mathcal{I}^+ is an ingoing characteristic surface.

We obtain the required estimates for (3.2) by the same method used in Sec. II C. The data at the inner boundary Γ_0 at $x = 0$ may be absorbed into F so it suffices to treat the case $q = 0$. We define the inner product

$$(\Psi_1, \Psi_2) = \int_0^1 dx \oint d\omega \Psi_1 \Psi_2$$

and L_2 norm $\|\Psi\|^2 = (\Psi, \Psi)$, where $d\omega$ is the area element on the unit sphere. We write

$$\|V_A\|^2 = (V_A, V^A) = (h^{AB}V_A, V_B).$$

Since the spherical cross-sections are spacelike, their intrinsic 2-metric h_{AB} is positive definite so that $\|V_A\|$ serves as an L_2 norm for the angular components. We also need a metric norm for spacelike 3-vectors. In the standard Cauchy problem this is supplied by the intrinsic 3-metric of the spacelike Cauchy hypersurfaces. Since the characteristic hypersurfaces have a degenerate 3-metric, we take a different approach. We use the projection operator $\pi_b^a = \delta_b^a - t^a\partial_b u$, where $t^a\partial_a = \partial_u$, to define a 3-metric $\gamma^{ab} = \pi_c^a\pi_d^b g^{cd}$. For the Bondi-Sachs metric (1.15), the resulting components in the (u, r, x^A) coordinates are

$$\gamma^{au} = 0, \quad \gamma^{rr} = e^{-2\beta}W, \quad \gamma^{rA} = -e^{-2\beta}r^{-2}W^A, \quad \gamma^{AB} = r^{-2}h^{AB}.$$

Denoting $x^i = (r, X^A)$, this implies that $\gamma^{ij} \rightarrow e^{ij}$ as $r \rightarrow \infty$, where e^{ij} is the Euclidean 3-metric expressed in standard spherical coordinates. In the compactified coordinates, $\tilde{x}^i = (x, x^A)$ it is more useful to deal with the rescaled 3-metric $\tilde{\gamma}^{ab} = e^{2\beta}r^2\gamma^{ab}$ which has components

$$\tilde{\gamma}^{au} = 0, \quad \tilde{\gamma}^{xx} = (1-x)^2W, \quad \tilde{\gamma}^{xA} = -R^{-1}(1-x)^2W^A, \quad \tilde{\gamma}^{AB} = e^{2\beta}h^{AB}.$$

We then define

$$\|V_i\|^2 = (\tilde{\gamma}^{ij}V_i, V_j)$$

which serves as an L_2 norm for the \tilde{x}^i components. Thus

$$\|\partial_i\Psi\|^2 = \|W^{1/2}(1-x)\partial_x\Psi\|^2 + \|e^\beta D_A\Psi\|^2 - 2R^{-1}\left((1-x)\partial_x\Psi, (1-x)W^AD_A\Psi\right). \quad (3.3)$$

We also define the corresponding inner products and norms on the boundaries, e.g.

$$(\Psi_1, \Psi_2)_\Gamma = \oint_\Gamma d\omega \Psi_1 \Psi_2, \quad \|\Psi\|_\Gamma^2 = (\Psi, \Psi)_\Gamma.$$

Because the original radial coordinate r was a surface area coordinate, the 2-metric h_{AB} of the spherical cross-sections has determinant $\det(h_{AB}) = \det(q_{AB})$, where q_{AB} is the unit sphere metric. Consequently,

$$(\Psi_1, D_A D^A \Psi_2) = -(D_A \Psi_1, D^A \Psi_2) \quad (3.4)$$

and

$$(V^A, D_A \Psi) = -(D_A V^A, \Psi) \quad (3.5)$$

where $V^A(u, x, x^A)$ is any smooth vector field on the spherical cross-sections. These identities allow the necessary integration by parts.

We derive the required estimates by freezing the dependence of the metric coefficients on (Ψ, u, x) but we retain their dependence on x^A so that W^A and h_{AB} remain smooth vector and tensor fields on the spherical cross-sections. We follow the procedure in Sec. II C. First,

$$\begin{aligned} & 2(\Psi, \partial_u \partial_x \Psi) + 2a(\Psi, \partial_u \Psi) = -2(\partial_x \Psi, \partial_u \Psi) + \partial_u \|\Psi\|_{\Gamma_1}^2 + a \partial_u \|\Psi\|^2 \\ & = -R^{-1} \left(W(1-x) \partial_x \Psi, (1-x) \partial_x \Psi \right) - R^{-1} \|e^\beta D_A \Psi\|^2 + 2R^{-2} \left((1-x) W^A D_A \Psi, (1-x) \partial_x \Psi \right) + (\Psi, F), \end{aligned}$$

i.e.

$$\partial_u \|\Psi\|_{\Gamma_1}^2 + a \partial_u \|\Psi\|^2 + R^{-1} \|\partial_i \Psi\|^2 = 2(\partial_x \Psi, \partial_u \Psi) + (\Psi, F). \quad (3.6)$$

Next,

$$\begin{aligned} & 2(\partial_u \Psi, \partial_u \partial_x \Psi) + 2a \|\partial_u \Psi\|^2 = \|\partial_u \Psi\|_{\Gamma_1}^2 + 2a \|\partial_u \Psi\|^2 \\ & = -\frac{1}{2} R^{-1} \partial_u \left(\|W^{1/2} (1-x) \partial_x \Psi\|^2 + \|e^\beta D_A \Psi\|^2 - 2R^{-1} ((1-x) \partial_x \Psi, (1-x) W^A D_A \Psi) \right) + (\partial_u \Psi, F), \end{aligned}$$

so that

$$\|\partial_u \Psi\|_{\Gamma_1}^2 + 2a \|\partial_u \Psi\|^2 + \frac{1}{2} R^{-1} \partial_u \|\partial_i \Psi\|^2 = (\partial_u \Psi, F). \quad (3.7)$$

Next,

$$\begin{aligned} & 2(\partial_x \Psi, \partial_u \partial_x \Psi) + 2a(\partial_x \Psi, \partial_u \Psi) = \partial_u \|\partial_x \Psi\|^2 + 2a(\partial_x \Psi, \partial_u \Psi) \\ & = R^{-1} \left(\partial_x \Psi, \partial_x (W(1-x)^2 \partial_x \Psi) \right) + R^{-1} \left(\partial_x \Psi, D_A (e^{2\beta} D^A \Psi) \right) \\ & \quad - R^{-2} \left(\Psi_x, D_A ((1-x)^2 W^A \partial_x \Psi) \right) - R^{-2} \left(\partial_x \Psi, \partial_x ((1-x)^2 W^A D_A \Psi) \right) + (\partial_x \Psi, F). \end{aligned} \quad (3.8)$$

As shown in in Sec. II C,

$$\left(\partial_x \Psi, \partial_x (W(1-x)^2 \partial_x \Psi) \right) = - \left(\partial_x \Psi, W(1-x) \partial_x \Psi \right) - \frac{1}{2} \|W^{1/2} \partial_x \Psi\|_{\Gamma_0}^2.$$

Also,

$$(\partial_x \Psi, D_A (e^{2\beta} D^A \Psi)) = -\frac{1}{2} \|e^\beta D_A \Psi\|_{\Gamma_1}^2 \quad (3.9)$$

and

$$\left(\partial_x \Psi, D_A ((1-x)^2 W^A \partial_x \Psi) \right) + \left(\partial_x \Psi, \partial_x ((1-x)^2 W^A D_A \Psi) \right) = -2 \left((1-x) \partial_x \Psi, W^A D_A \Psi \right).$$

Therefore (3.8) becomes

$$\begin{aligned} \partial_u \|\partial_x \Psi\|^2 + R^{-1} \left(\partial_x \Psi, W(1-x) \partial_x \Psi \right) + \frac{1}{2} R^{-1} \|W^{1/2} \partial_x \Psi\|_{\Gamma_0}^2 + \frac{1}{2} R^{-1} \|e^\beta D_A \Psi\|_{\Gamma_1}^2 \\ - 2R^{-2} \left((1-x) \partial_x \Psi, W^A D_A \Psi \right) = -2a(\partial_x \Psi, \partial_u \Psi) + (\partial_x \Psi, F). \end{aligned} \quad (3.10)$$

As before, the boundary terms have the right sign to enhance the estimates so that we can ignore them. Adding the simplified estimates (3.6), (3.7), (3.10) gives

$$\begin{aligned} \partial_u \left(a \|\Psi\|^2 + \|\partial_x \Psi\|^2 + \frac{1}{2} R^{-1} \|\partial_i \Psi\|^2 \right) + R^{-1} \|\partial_i \Psi\|^2 + R^{-1} \left(\partial_x \Psi, W(1-x) \partial_x \Psi \right) \\ = 2R^{-2} \left((1-x) \partial_x \Psi, W^A D_A \Psi \right) + 2(1-a)(\partial_x \Psi, \partial_u \Psi) - 2a \|\partial_u \Psi\|^2 + (\Psi + \partial_u \Psi + \partial_x \Psi, F) \\ \leq \text{const.} \left(\|\partial_x \Psi\|^2 + \|D_A \Psi\|^2 + \|\Psi\|^2 + \|F\|^2 \right). \end{aligned} \quad (3.11)$$

Therefore (3.11) gives us an energy estimate provided that the 3-metric γ^{ij} has $(+++)$ signature, so that $\|\partial_i \Psi\|$ is a norm for the gradient $\partial_i \Psi = (\partial_r \Psi, \partial_A \Psi)$. This is equivalent to the requirement that the principle part of the wave operator reduce to an elliptic operator in the stationary case where the u -derivatives vanish. Since γ^{ij} is asymptotic to the Euclidean metric as $r \rightarrow \infty$, this positive-definite condition is satisfied throughout some exterior domain.

Estimates for the higher derivatives of Ψ and stability against lower order perturbations follow from the same arguments given in Sec. II. This establishes the well-posedness of the worldtube-nullcone problem for the case of smooth variable coefficients. The extension of well-posedness, locally in time, for the quasilinear case then follows from the standard techniques referred to in Sec. II .

For a mass M Schwarzschild geometry, $\gamma^{rr} = e^{-2\beta} W = 1 - 2M/r$ so that positive-definiteness of the 3-metric γ^{ij} breaks down at $r = 2M$ where the worldtube becomes null. In this limiting case of the double-null problem, the $\partial_u \|\partial_x \Psi\|^2$ term in (3.11) suffices to provide the required estimate. However, for $R < 2M$ the ‘‘worldtube’’ is spacelike and the problem must be treated differently.

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The Well-posedness of the Null-Timelike Boundary Problem for Quasilinear Waves

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The null-timelike initial-boundary value problem for a hyperbolic system of equations consists of the evolution of data given on an initial characteristic surface and on a timelike worldtube to produce a solution in the exterior of the worldtube. We establish the well-posedness of this problem for the evolution of a quasilinear scalar wave by means of energy estimates. The treatment is given in characteristic coordinates and thus provides a guide for developing stable finite difference algorithms. A new technique underlying the approach has potential application to other characteristic initial-boundary value problems.

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I. INTRODUCTION

The use of null hypersurfaces as coordinates to describe gravitational waves, as introduced by Bondi [1], was key to the understanding and geometric treatment of gravitational waves in the full nonlinear context of general relativity [2, 3]. In one version of the associated characteristic initial-boundary value problem for Einstein's equations, boundary data is given on a timelike worldtube and on an initial outgoing null hypersurface [4]. The physical picture underlying this null-timelike problem is that the worldtube data represent the outgoing gravitational radiation emanating from interior matter sources, while ingoing radiation incident on the system is represented by the initial null data. This problem has been developed into a Cauchy-characteristic matching scheme in which the worldtube data is supplied by a Cauchy evolution of the interior sources [5]. See [6] for a review. Cauchy-characteristic matching has been implemented as a numerical evolution code in which the Bondi news function describing the radiation is calculated at future null infinity using a finite numerical grid obtained by Penrose compactification [3]. Although characteristic evolution codes have successfully simulated many null-timelike problems [6] and have recently been applied to extract the radiation from the inspiral and merger of a binary black hole [7], the well-posedness of the null-timelike problem for the Einstein equations has not yet been established. The characteristic formulation of the Einstein equations implies that certain variables associated with the radiation satisfy a wave equation. Consequently, a necessary condition for the well-posedness of the gravitational problem is that the corresponding problem for the quasilinear wave equation be well-posed. In this paper, as a first step toward treating the gravitational case, we show that the quasilinear null-timelike problem for a scalar wave propagating on a curved space background is well posed.

The characteristic initial value problem did not receive much attention before its importance in general relativity was recognized. Historically, the development of computational physics has focused on hydrodynamics, where the characteristics typically do not define useful coordinate surfaces and there is no generic outer boundary behavior comparable to null infinity. The simplest problem for which the characteristic approach is useful is the Minkowski space wave equation, which is satisfied by the components of the fundamental special relativistic fields. Progress on the null-timelike problem traces back to Duff [8], where existence and uniqueness was shown for the linear wave equation with analytic coefficients and analytic data. Existence and uniqueness was later extended to the C^∞ case of the linear wave equation on an asymptotically flat curved space background by Friedlander [9, 10].

The demonstration of well-posedness of the quasilinear boundary problem, i.e. the continuous dependence of the solution on the data, depends upon establishing estimates on the derivatives for the linearized problem. This requires considering generic lower differential order terms [11]. Well-posedness depends crucially on the stability of the problem against such lower order perturbations. Otherwise, one cannot localize the problem and use the principle of frozen coefficients.

Partial results estimating the derivatives for characteristic boundary problems were first obtained by Müller zum Hagen and Seifert [12]. Later Balean carried out a comprehensive study of the differentiability of solutions of the null-timelike problem for the flat space wave equation [13, 14]. He was able to establish estimates for the derivatives tangential to the outgoing null cones but weaker estimates for the time derivatives transverse to the cones had to be obtained from a direct integration of the wave equation. The derivatives tangential to the null cone were controlled by the derivatives of the data but control of the transverse time derivative required two derivatives of the data. Balean

concentrated on the differentiability order of the solution and did not discuss the implications for well-posedness of the quasilinear problem.

Frittelli [15] made the first explicit consideration of well-posedness of the null-timelike problem for the wave equation. She adopted the approach of Duff, in which the characteristic formulation of the wave equation is reduced to a canonical first order differential form, in close analogue to the symmetric hyperbolic formulation of the Cauchy problem. The energy associated with this first order reduction gives estimates for the derivatives of the field tangential to the null hypersurfaces. As in Baean's treatment, weaker estimates for the time derivatives were obtained indirectly so that well-posedness is not ensured when lower order differential order terms or source terms are included as required for the quasilinear case, as she was careful to point out.

A difficulty underlying the problem can be illustrated in terms of the 1(spatial)-dimensional wave equation

$$(\partial_{\tilde{t}}^2 - \partial_{\tilde{x}}^2)\Phi = 0, \quad (1.1)$$

where (\tilde{t}, \tilde{x}) are standard space-time coordinates. The conserved energy

$$\tilde{E}(\tilde{t}) = \frac{1}{2} \int d\tilde{x} \left((\partial_{\tilde{t}}\Phi)^2 + (\partial_{\tilde{x}}\Phi)^2 \right) \quad (1.2)$$

leads to the well-posedness of the Cauchy problem. In characteristic coordinates $(t = \tilde{t} - \tilde{x}, x = \tilde{t} + \tilde{x})$, the wave equation transforms into

$$\partial_t \partial_x \Phi = 0. \quad (1.3)$$

The conserved energy on the characteristics $t = \text{const.}$,

$$\tilde{E}(t) = \int dx (\partial_x \Phi)^2, \quad (1.4)$$

no longer controls the derivative $\partial_t \Phi$.

The first proof of well-posedness of the characteristic initial value problem valid for the quasilinear wave equation has been the work of Rendall [16], who considered the double null problem where data is given on a pair of intersecting characteristic hypersurfaces. Rendall did not treat the characteristic problem head-on but reduced it to a standard Cauchy problem with data on a spacelike hypersurface passing through the intersection of the characteristic hypersurfaces. Well-posedness then follows from the classic result for the Cauchy problem. He extended his treatment to establish the well-posedness of the double-null formulation of the Einstein gravitational problem. The double null problem treated by Rendall is a limiting case of the null-timelike problem considered in this paper. However, Rendall's approach cannot be applied to the null-timelike problem. Also, the reduction to a Cauchy problem does not provide guidance for the development of a stable finite-difference approximation based upon characteristic coordinates.

Another limiting case of the null-timelike problem is the Cauchy problem on a characteristic cone, corresponding to the limit in which the timelike worldtube has shrunk to a nonsingular worldline. This problem is difficult to treat in characteristic coordinates because of their singular nature at the vertex of the cone. However, Choquet-Bruhat, Chruściel and Martín-García have been able to establish the existence of solutions to this problem, for both the scalar and gravitational case, by treating it in harmonic coordinates adapted to the null cones [17, 18].

Here we consider the null-timelike problem for the quasi-linear wave equation in second differential form in terms of characteristic coordinates. The usual technique for showing that the initial-boundary value problem for a hyperbolic system of partial differential equations is well posed is to split the problem into a Cauchy problem and local halfplane problems and show that these individual problems are well posed. This works for hyperbolic systems based upon a spacelike foliation, in which case signals propagate with finite velocity. Besides the existence and uniqueness of a solution, well-posedness implies that the solution depend continuously on the data with respect to an appropriate norm. For (1.1), the solutions to the Cauchy problem with compact initial data on $\tilde{t} = 0$ are square integrable and well-posedness can be established using the L_2 norm (1.2).

However, In characteristic coordinates the 1-dimensional wave equation (1.3) admits signals traveling in the $+x$ -direction with infinite coordinate velocity. In particular, initial data of compact support $\Phi(0, x) = f(x)$ on the characteristic $t = 0$ admits the solution $\Phi = g(t) + f(x)$, provided that $g(0) = 0$. Here $g(t)$ represents the profile of a wave which travels from past null infinity ($x \rightarrow -\infty$) to future null infinity ($x \rightarrow +\infty$). Thus, without a boundary condition at past null infinity, there is no unique solution and the Cauchy problem is ill posed. Even with the boundary condition $\Phi(t, -\infty) = 0$, a source of compact support $S(t, x)$ added to (1.3), i.e.

$$\partial_t \partial_x \Phi = S, \quad (1.5)$$

produces waves propagating to $x = +\infty$ so that although the solution is unique it is still not square integrable.

On the other hand, consider the modified problem obtained by setting $\Phi = e^{ax}\Psi$,

$$\partial_t(\partial_x + a)\Psi = F, \quad \Psi(0, x) = e^{-ax}f(x), \quad a > 0 \quad (1.6)$$

where $F = e^{-ax}S$. With the boundary condition $\Psi(t, -\infty) = 0$, the solutions to (1.6) vanish at $x = +\infty$ and are square integrable. As a result, the Cauchy problem (1.6) is well posed with respect to an L_2 norm. For the simple example where $F = 0$, multiplication of (1.6) by $(2a\Psi + \partial_x\Psi + \frac{1}{2}\partial_t\Psi)$ and integration by parts gives

$$\frac{1}{2}\partial_t \int dx \left((\partial_x\Psi)^2 + 2a^2\Psi^2 \right) = \frac{a}{2} \int dx \left(2(\partial_t\Psi)\partial_x\Psi - (\partial_t\Psi)^2 \right) \leq \frac{a}{2} \int dx (\partial_x\Psi)^2. \quad (1.7)$$

The resulting inequality

$$\partial_t E \leq \text{const.} E \quad (1.8)$$

for the energy

$$E = \frac{1}{2} \int dx \left((\partial_x\Psi)^2 + 2a^2\Psi^2 \right) \quad (1.9)$$

provides the estimates for $\partial_x\Psi$ and Ψ which are necessary for well-posedness. Estimates for $\partial_t\Psi$, and other higher derivatives, follow from applying this approach to the derivatives of (1.6). The approach can be extended to include the source term F and other generic lower differential order terms. This allows well-posedness to be extended to the case of variable coefficients and, locally in time, to the quasilinear case.

The 2(spatial)-dimensional model problems considered in Sec. II illustrate how this approach generalizes to the multi-dimensional case. We consider the model problems in the modified form analogous to (1.6). By means of this technique, the characteristic initial-boundary value problem can again be treated by first considering Cauchy and half-plane problems. The demonstration of well-posedness of these model problems presents the underlying ideas in a transparent form.

Our main technique is the use of energy estimates. Although the model problems are treated in the modified form, the results can be translated back to the original problem. For example, the modification in going from (1.5) to (1.6) leads to an effective modification of the standard energy for the problem. Rewritten in terms of the original variable $\Phi = e^{ax}\Psi$, (1.9) corresponds to the energy

$$E = \frac{1}{2} \int dx e^{-2ax} \left((\partial_x\Phi)^2 + a^2\Phi^2 \right). \quad (1.10)$$

Thus while the Cauchy problem for (1.6) is ill posed with respect to the L_2 norm it is well posed with respect to the exponentially weighted norm (1.10). However, rather than modifying the norm, for technical simplicity we deal with the modified variable Ψ .

The general arguments presented for our model problems can be applied to a wide range of quasilinear characteristic problems. Our motivation for the work here is the application to the null-timelike problem for the quasilinear wave equation for a scalar field Φ in an asymptotically flat curved space background with source S ,

$$g^{ab}\nabla_a\nabla_b\Phi = S(\Phi, \partial_c\Phi, x^c), \quad (1.11)$$

where the metric g^{ab} and its associated covariant derivative ∇_a are explicitly prescribed functions of (Φ, x^c) .

The corresponding flat space wave equation,

$$(-\partial_{\tilde{t}}^2 + \partial_{\tilde{x}}^2 + \partial_{\tilde{y}}^2 + \partial_{\tilde{z}}^2)\Phi = S, \quad (1.12)$$

takes the form

$$\frac{1}{r}(-2\partial_u\partial_r + \partial_r^2)(r\Phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\Phi = S \quad (1.13)$$

in null-spherical coordinates (u, r, θ, ϕ) consisting of a retarded time $u = \tilde{t} - r$ and standard spherical coordinates (r, θ, ϕ) . In these coordinates, the Minkowski metric is

$$ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.14)$$

The null-timelike problem consists of determining Φ in the region ($r > R, u > 0$) given data $\Phi(u, R, \theta, \phi)$ on the timelike worldtube $r = R$ and $\Phi(0, r, \theta, \phi)$ on the initial null hypersurface $u = 0$.

In an asymptotically flat background, the metric (1.14) generalizes to the Bondi-Sachs form

$$g_{ab}dx^a dx^b = -(e^{2\beta}W - r^{-2}h_{AB}W^A W^B)du^2 - 2e^{2\beta}dudr - 2h_{AB}W^B dudx^A + r^2 h_{AB}dx^A dx^B, \quad (1.15)$$

where x^A are angular coordinates such that $(u, x^A) = \text{const.}$ along the outgoing null rays. Here the radial coordinate r is a surface area coordinate so that the area of the topological spheres $(u, r) = \text{const.}$ is 4π as measured by the conformal 2-metric h_{AB} . In the curved space version of angular coordinates analogous to (1.14), $\det(h_{AB}) = \sin^2 \theta$.

In Sec. III, we treat the null-timelike problem for the quasilinear wave equation (1.11) with asymptotically flat Lorentzian metric (1.15),

$$\begin{aligned} \frac{1}{r}(-2\partial_u \partial_r + W \partial_r^2)(r\Phi) + (\partial_r W) \partial_r \Phi - \frac{1}{r^2} D_A(W^A \partial_r \Phi) - \frac{1}{r^2} \partial_r(W^A D_A \Phi) + \frac{1}{r^2} D_A(e^{2\beta} D^A \Phi) \\ = e^{2\beta} S(\Phi, \partial_c \Phi, x^c), \\ \Phi(0, r, x^A) = f(r, x^A), \quad \Phi(u, R, x^A) = q(u, x^A), \quad R \leq r < \infty, \quad u \geq 0. \end{aligned} \quad (1.16)$$

Here D_A is the 2-dimensional covariant derivative with respect to h_{AB} and the metric coefficients (W, β, W^A, h_{AB}) depend smoothly upon (Φ, u, r, x^A) and the source S depends smoothly upon $(\Phi, \partial_a \Phi, u, r, x^A)$.

An essential part of any initial-boundary value problem is the compatibility between the data at the intersection between the initial hypersurface and the boundary, i.e. at $(u = 0, r = R)$ in the above case. This compatibility affects the differentiability of the resulting solution. In order to avoid difficult issues of analysis, we only give a rigorous treatment for the case of smooth initial and boundary data with compact support bounded away from the intersection, in which case the solution is C^∞ locally in time. See the work of Baean [13, 14] for a discussion of the differentiability of the solution in the general case.

We assume that as $r \rightarrow \infty$ (the approach to null infinity) that the problem reduces to the flat space problem (1.13), so that the coefficients have the asymptotic behavior $W = 1 + O(1/r)$, $\beta = 0 + O(1/r)$, $W^A = O(1)$ and $h_{AB} = q_{AB} + O(1/r)$, where q_{AB} is the unit sphere metric. The results of Friedlander [10] then imply that the scalar wave falls off as $\Phi \sim \Phi_0(u, x^A)/r$ where Φ_0 is the asymptotic radiation field.

In Sec. III, we establish our **Main Theorem**:

The nullcone-worldtube problem (1.16) is well posed for smooth, compatible initial data $f(r, x^A)$ and boundary data $q(u, x^A)$ subject to the conditions that $f = O(r^{-1})$, $S = O(r^{-3})$ and a positivity condition that the principal part of the wave operator reduces to an elliptic operator in the stationary case.

Our treatment is based upon energy estimates obtained by integration by parts with respect to the characteristic coordinates. As a result, the analogous finite difference estimates obtained by *summation by parts* provide guidance for the development of a stable numerical evolution algorithm for (1.16).

II. WELL-POSEDNESS OF MODEL CHARACTERISTIC PROBLEMS

We consider here several model 2(spatial)-dimensional problems which reveal the essential features underlying a well posed characteristic initial-boundary value problem. For simplicity of notation, we indicate partial derivatives by subscripts, e.g. $\Phi_t(t, x, y) = \partial_t \Phi(t, x, y)$. Also, we denote the L_2 scalar product and norm over the x, y domain by (Φ_1, Φ_2) and $\|\Phi\|^2 = (\Phi, \Phi)$.

We consider model linear problems with constant coefficients but show that the problems are stable against lower order perturbations. We also obtain estimates for arbitrarily high derivatives. Thus we can use standard techniques [11] to establish the well-posedness of the corresponding problem with smooth variable coefficients. For the extension to the quasilinear case, we require that the coefficients depend smoothly upon the field Φ with nonsingular behavior in the neighborhood of the initial data. Then well-posedness, locally in time, of the quasilinear problem also follows from standard techniques [11]. (See the Appendix of [19] for details concerning how these standard techniques apply to hyperbolic systems in second differential order form.)

Our goal is to show the well-posedness of the strip problem

$$2\Phi_{tx} = ((1-x)^2 \Phi_x)_x + \Phi_{yy} + b \left(((1-x)^2 \Phi_x)_y + ((1-x)^2 \Phi_y)_x \right)$$

in the domain

$$0 \leq x \leq 1, \quad -\infty < y < \infty, \quad t \geq 0$$

with initial and boundary conditions

$$\Phi(0, x, y) = f(x, y), \quad \Phi(t, 0, y) = q(t, y),$$

respectively. The method used to show that this problem is well posed applies to the compactified version of the null-timelike boundary problem for the wave equation (1.16) treated in Sec. III. As explained in the Introduction, we treat the problem in the modified form obtained by the change of variable $\Phi = e^{ax}\Psi$, $a > 0$.

A. The Cauchy problem

We first consider the Cauchy problem

$$(\Psi_x + a\Psi)_t = \Psi_{yy} - 2b\Psi_y, \quad \bar{x} = (x, y) \in R^2, \quad t \geq 0, \quad (2.1)$$

$$\Psi(0, x, y) = f(x, y), \quad (2.2)$$

where x and t are both characteristic coordinates. Here a , b are real constants and $f(x, y) \in C_0^\infty$ (a smooth function with compact support). As explained in the Introduction, we investigate the behavior of square integrable solutions, so that $\Psi(t, \pm\infty, y) = 0$.

1. The Fourier method

We first solve the problem by Fourier transform. Let

$$\hat{f}(\bar{\omega}) = \frac{1}{2\pi} \int_{R^2} e^{-i\bar{\omega} \cdot \bar{x}} f(\bar{x}) dx dy, \quad \bar{\omega} = (\omega_1, \omega_2) \text{ real},$$

denote the Fourier transform of f and $\hat{\Psi}(t, \bar{\omega})$ the Fourier transform of Ψ . Then $\hat{\Psi}(t, \bar{\omega})$ is the solution of

$$(i\omega_1 + a)\hat{\Psi}_t = -(\omega_2^2 + 2bi\omega_2)\hat{\Psi}, \quad (2.3)$$

$$\hat{\Psi}(0, \bar{\omega}) = \hat{f}(\bar{\omega}), \quad (2.4)$$

i.e.

$$\hat{\Psi}_t = s\hat{\Psi},$$

where

$$s = -\frac{\omega_2^2 + 2bi\omega_2}{i\omega_1 + a} = -\frac{(\omega_2^2 + 2bi\omega_2)(a - i\omega_1)}{a^2 + \omega_1^2}. \quad (2.5)$$

Therefore

$$\begin{aligned} \Re s &= -\frac{a\omega_2^2 + 2b\omega_1\omega_2}{a^2 + \omega_1^2}, \\ \Im s &= \frac{(\omega_2^2\omega_1 - 2ab\omega_1)}{a^2 + \omega_1^2}. \end{aligned} \quad (2.6)$$

We now discuss the dependence of the solutions on a, b in detail.

1) $b = 0$, $a > 0$. By (2.6),

$$\Re s = -\frac{a\omega_2^2}{\omega_1^2 + a^2} \leq 0.$$

There are no exponentially growing solutions.

2) $b = 0$, $a = 0$. By (2.6),

$$\Re s = 0, \quad |\Im s| \rightarrow \infty \quad \text{for} \quad |\omega_1| \rightarrow 0, \quad \omega_2 \neq 0.$$

Therefore the solution of (2.2) loses all smoothness in time if $\hat{f}(0, \omega_2) \neq 0$.

3) $b = 0$, $a < 0$. By (2.6),

$$\Re s \rightarrow +\infty \quad \text{for} \quad \omega_2 \rightarrow \infty.$$

Thus there is unbounded exponential growth and the problem is ill posed.

4) $b \neq 0$, $a > 0$. By (2.6),

$$\Re s = -\frac{a(\omega_2 + \frac{b}{a}\omega_1)^2}{\omega_1^2 + a^2} + \frac{b^2\omega_1^2}{a(\omega_1^2 + a^2)} \leq \frac{b^2}{a}.$$

There is exponential growth but the growth is bounded independently of $\bar{\omega}$.

5) $b \neq 0$, $a = 0$. By (2.6),

$$\Re s = -\frac{2b\omega_2}{\omega_1}.$$

Thus there is unbounded exponential growth as $\omega_1 \rightarrow 0$. The same is true if $a < 0$.

We now express our results in a more general setting.

Definition 2.1. We call the Cauchy problem well posed if, for every $f \in C_0^\infty$, there is a unique, smooth, square integrable solution and if there is a constant α which does not depend on $\bar{\omega}$ such that

$$\Re s \leq \alpha.$$

The problem is ill posed if there is no upper bound α , i.e., there is a sequence $\bar{\omega}^{(j)}$ such that

$$\lim_{j \rightarrow \infty} \Re s_j = \infty.$$

Theorem 2.1. The Cauchy problem (2.2) is well posed if $a > 0$. But it is ill posed if $a < 0$ or $a = 0$, $b \neq 0$.

2. The energy method

For the generalization to variable coefficients it is necessary to show that the differential equation (2.2) is stable against lower order perturbations. For this purpose we first apply the energy method to the doubly-characteristic Cauchy problem

$$\begin{aligned} (\Psi_x + a\Psi)_t &= \Psi_{yy} - 2b\Psi_y - c\Psi_x + d\Psi_t + e\Psi + F(t, x, y), \\ \Psi(0, x, y) &= f(x, y), \quad -\infty < x, y < \infty, \quad t \geq 0. \end{aligned} \quad (2.7)$$

Here $a > 0$, b , c , d , e are real constants and F is a forcing (source) term of compact spatial support.

The term $d\Psi_t$ can be absorbed into the left hand side and we obtain $(\Psi_x + (a-d)\Psi)_t$. Therefore we neglect this term and assume that a is sufficiently large so that $a-d > 0$. We also neglect the term $e\Psi$ because it has no influence on the required energy estimates.. Therefore we consider the corresponding Cauchy problem for

$$(\Psi_x + a\Psi)_t = \Psi_{yy} - 2b\Psi_y - c\Psi_x + F. \quad (2.8)$$

We now derive an energy estimate. By (2.8),

$$(\Psi, \Psi_{xt}) + a(\Psi, \Psi_t) = -(\Psi_x, \Psi_t) + \frac{a}{2}\partial_t \|\Psi\|^2 = (\Psi, \Psi_{yy}) - (\Psi, 2b\Psi_y + c\Psi_x) + (\Psi, F).$$

Since

$$(\Psi_x, \Psi_t) = \left(\frac{2}{\sqrt{a}} \Psi_x, \frac{\sqrt{a}}{2} \Psi_t \right) \leq \frac{2}{a} \|\Psi_x\|^2 + \frac{a}{8} \|\Psi_t\|^2,$$

integration by parts gives

$$\frac{a}{2} \partial_t \|\Psi\|^2 + \|\Psi_y\|^2 = (\Psi_x, \Psi_t) + (\Psi, F) \leq \frac{2}{a} \|\Psi_x\|^2 + \frac{a}{8} \|\Psi_t\|^2 + \frac{1}{2} (\|\Psi\|^2 + \|F\|^2). \quad (2.9)$$

Next,

$$(\Psi_t, \Psi_{xt}) + a \|\Psi_t\|^2 = -\frac{1}{2} \partial_t \|\Psi_y\|^2 - 2b(\Psi_t, \Psi_y) - c(\Psi_t, \Psi_x) + (\Psi_t, F).$$

Since

$$c(\Psi_t, \Psi_x) = \left(\frac{\sqrt{a}}{2} \Psi_t, \frac{2c}{\sqrt{a}} \Psi_x \right) \leq \frac{a}{8} \|\Psi_t\|^2 + \frac{2c^2}{a} \|\Psi_x\|^2,$$

$$(\Psi_t, F) \leq \frac{a}{8} \|\Psi_t\|^2 + \frac{2}{a} \|F\|^2,$$

$$2b(\Psi_t, \Psi_y) = \left(\frac{\sqrt{a}}{2} \Psi_t, \frac{4b}{\sqrt{a}} \Psi_y \right) \leq \frac{a}{8} \|\Psi_t\|^2 + \frac{8b^2}{a} \|\Psi_y\|^2,$$

we obtain

$$\frac{5a}{8} \|\Psi_t\|^2 + \frac{1}{2} \partial_t \|\Psi_y\|^2 \leq \text{const.} (\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|F\|^2). \quad (2.10)$$

Next,

$$(\Psi_x, \Psi_{xt}) + a(\Psi_x, \Psi_t) = (\Psi_x, \Psi_{yy}) - (\Psi_x, 2b\Psi_y + c\Psi_x) + (\Psi_x, F).$$

Since $(\Psi_x, \Psi_{yy}) = -(\Psi_{xy}, \Psi_y) = 0$, we obtain

$$\frac{1}{2} \partial_t \|\Psi_x\|^2 \leq \text{const.} (\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|F\|^2) + \frac{a}{8} \|\Psi_t\|^2. \quad (2.11)$$

Adding (2.9)–(2.11) gives the energy estimate

$$\frac{3a}{8} \|\Psi_t\|^2 + \|\Psi_y\|^2 + \frac{1}{2} \partial_t (\|\Psi_x\|^2 + \|\Psi_y\|^2 + a\|\Psi\|^2) \leq \text{const.} (\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|\Psi\|^2 + \|F\|^2). \quad (2.12)$$

We have proved:

Theorem 2.2. *The Cauchy problem (2.7) is well posed with respect to the L_2 norm if $(a - d) > 0$. There is an energy estimate. Also, the problem is stable against lower order perturbations. In addition, estimates for the higher derivatives of Ψ follow from the equations obtained by differentiating (2.7).*

We now consider the Cauchy problem

$$\begin{aligned} (\Psi_x + a\Psi)_t &= \Psi_{xx} + \Psi_{yy} + F(t, x, y), \\ \Psi(0, x, y) &= f(x, y), \quad -\infty < x, y < \infty, \quad t \geq 0, \end{aligned} \quad (2.13)$$

where x is a characteristic coordinate but t is timelike. We again derive an energy estimate.

We have

$$\begin{aligned} (\Psi, \Psi_{xt} + a\Psi_t) &= -(\Psi_x, \Psi_t) + \frac{a}{2} \partial_t \|\Psi\|^2 \\ &= -(\|\Psi_x\|^2 + \|\Psi_y\|^2) + (\Psi, F), \end{aligned}$$

i.e.

$$\frac{a}{2}\partial_t\|\Psi\|^2 + \|\Psi_x\|^2 + \|\Psi_y\|^2 = (\Psi_x, \Psi_t) + (\Psi, F) \leq \frac{a}{8}\|\Psi_t\|^2 + \frac{2}{a}\|\Psi_x\|^2 + (\Psi, F). \quad (2.14)$$

Next,

$$\begin{aligned} (\Psi_t, \Psi_{xt} + a\Psi_t) &= (\Psi_t, \Psi_{xt}) + a\|\Psi_t\|^2 = a\|\Psi_t\|^2 \\ &= -\frac{1}{2}\partial_t(\|\Psi_x\|^2 + \|\Psi_y\|^2) + (\Psi_t, F), \end{aligned}$$

i.e.

$$a\|\Psi_t\|^2 + \frac{1}{2}\partial_t(\|\Psi_x\|^2 + \|\Psi_y\|^2) = (\Psi_t, F) \leq \frac{a}{8}\|\Psi_t\|^2 + \frac{2}{a}\|F\|^2. \quad (2.15)$$

Combining (2.14) and (2.15) as before, we obtain the desired estimate

$$\frac{3a}{4}\|\Psi_t\|^2 + \|\Psi_x\|^2 + \|\Psi_y\|^2 + \frac{1}{2}\partial_t(\|\Psi_x\|^2 + \|\Psi_y\|^2 + a\|\Psi\|^2) \leq \text{const.} (\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|\Psi\|^2 + \|F\|^2). \quad (2.16)$$

Remark . *As before, we can add a general lower order expression and still obtain the estimate. Also, we can estimate all derivatives.*

B. The half-plane problem

We now apply the energy method to the double-null halfplane problem for (2.8),

$$(\Psi_x + a\Psi)_t = \Psi_{yy} + 2b\Psi_y - c\Psi_x + F, \quad 0 \leq x < \infty, \quad -\infty < y < \infty, \quad t \geq 0, \quad (2.17)$$

with initial and boundary data

$$\Psi(0, x, y) = f(x, y), \quad \Psi(t, 0, y) = 0 \quad (2.18)$$

and source $F(t, x, y)$ of compact support.

There are no difficulties to derive the basic estimate (2.12) because for the estimates (2.9)–(2.11) we require only that $\Psi(t, 0, y) = 0$. To obtain estimates for higher derivatives we have to proceed in the following way.

We differentiate (2.17) with respect to y . Since $\Psi_y(t, 0, y) = 0$, we obtain the same problem for Ψ_y and therefore we obtain estimates for

$$\|\Psi_{yy}\|^2, \quad \|\Psi_{xy}\|^2.$$

If we differentiate (2.17) two times with respect to y , we obtain estimates for the third derivatives. The corresponding results hold for t -derivatives, e.g.

$$\|\Psi_t\|^2, \quad \|\Psi_{yt}\|^2, \quad \|\Psi_{xt}\|^2.$$

Now we differentiate (2.17) with respect to x .

$$(\Psi_{xx} + a\Psi_x)_t = \Psi_{yyx} + R. \quad (2.19)$$

Here R consists of source terms and terms which we have already estimated. (2.19) gives us

$$(\Psi_{xx}, \Psi_{xxt}) + a(\Psi_{xx}, \Psi_{xt}) = (\Psi_{xx}, \Psi_{yyx}) + (\Psi_{xx}, R). \quad (2.20)$$

We obtain

$$\frac{1}{2}\partial_t\|\Psi_{xx}\|^2 \leq \frac{1}{2}\left((1+a^2)\|\Psi_{xx}\|^2 + \|\Psi_{xt}\|^2 + \|\Psi_{yyx}\|^2 + \|R\|^2\right),$$

where we already have estimates for $\|\Psi_{xt}\|^2$ and $\|\Psi_{yyx}\|^2$. The process can be continued.

Remark . *Inhomogeneous boundary data $\Psi(t, 0, y) = q(t, y)$ may be treated in the same way through the transformation $\Psi \rightarrow \Psi - qe^{-x}$ and absorbing the boundary data in the source term F . We can also treat the timelike-null halfplane problem for (2.13) in the same way.*

C. The strip problem

As a prototype of the compactified wave equation considered in Sec. III, we consider the strip problem

$$2(\Psi_x + a\Psi)_t = ((1-x)^2\Psi_x)_x + \Psi_{yy} + b\left(\left((1-x)^2\Psi_x\right)_y + \left((1-x)^2\Psi_y\right)_x\right) + F(t, x, y) \quad (2.21)$$

for

$$0 \leq x \leq 1, \quad -\infty < y < \infty, \quad t \geq 0$$

with initial and boundary conditions

$$\Psi(0, x, y) = f(x, y), \quad \Psi(t, 0, y) = q(t, y).$$

Here $a > 0$ and b , with $|b| < 1$, are real constants and F is a smooth function. The outer boundary Γ_1 at $x = 1$ is an ingoing characteristic so that no boundary condition is allowed.

Since the boundary data at Γ_0 can be absorbed into the source F , we treat the case $q = 0$ (see the remark in Sec. II B). We denote the L_2 norm over Γ_1 by

$$\|\Psi\|_{\Gamma_1}^2 = \int dy \Psi^2(t, 1, y)$$

and the L_2 norm over the boundary Γ_0 at $x = 0$ by

$$\|\Psi\|_{\Gamma_0}^2 = \int dy \Psi^2(t, 0, y).$$

We want to show that there is an energy estimate and that the problem is stable against lower order perturbations. We derive the necessary estimates. First,

$$\begin{aligned} & 2(\Psi, \Psi_{xt}) + 2a(\Psi, \Psi_t) = -2(\Psi_x, \Psi_t) + \partial_t \|\Psi\|_{\Gamma_1}^2 + a\partial_t \|\Psi\|^2 \\ & = -\left((1-x)\Psi_x, (1-x)\Psi_x\right) - \|\Psi_y\|^2 - 2b\left((1-x)\Psi_x, (1-x)\Psi_y\right) + (\Psi, F), \end{aligned}$$

i.e.

$$\begin{aligned} & \partial_t \|\Psi\|_{\Gamma_1}^2 + a\partial_t \|\Psi\|^2 + \|(1-x)\Psi_x\|^2 + \|\Psi_y\|^2 \\ & + 2b\left((1-x)\Psi_x, (1-x)\Psi_y\right) = 2(\Psi_x, \Psi_t) + (\Psi, F). \end{aligned} \quad (2.22)$$

Next,

$$\begin{aligned} & 2(\Psi_t, \Psi_{xt}) + 2a\|\Psi_t\|^2 = \|\Psi_t\|_{\Gamma_1}^2 + 2a\|\Psi_t\|^2 \\ & = -\frac{1}{2}\partial_t \left(\|(1-x)\Psi_x\|^2 + \|\Psi_y\|^2 + 2b((1-x)\Psi_x, (1-x)\Psi_y) \right) + (\Psi_t, F). \end{aligned} \quad (2.23)$$

Next,

$$\begin{aligned} & 2(\Psi_x, \Psi_{xt}) + 2a(\Psi_x, \Psi_t) = \partial_t \|\Psi_x\|^2 + 2a(\Psi_x, \Psi_t) = \left(\Psi_x, ((1-x)^2\Psi_x)_x\right) + (\Psi_x, \Psi_{yy}) \\ & + b\left(\Psi_x, ((1-x)^2\Psi_x)_y\right) + b\left(\Psi_x, ((1-x)^2\Psi_y)_x\right) + (\Psi_x, F). \end{aligned} \quad (2.24)$$

Now,

$$\begin{aligned} & \left(\Psi_x, ((1-x)^2\Psi_x)_x\right) = -\left(\Psi_x, 2(1-x)\Psi_x\right) + \left(\Psi_x, (1-x)^2\Psi_{xx}\right) \\ & = -(\Psi_x, 2(1-x)\Psi_x) - \left(\left((1-x)^2\Psi_x\right)_x, \Psi_x\right) - \|\Psi_x\|_{\Gamma_0}^2, \end{aligned}$$

i.e.

$$\left(\Psi_x, ((1-x)^2\Psi_x)_x\right) = -\left(\Psi_x, (1-x)\Psi_x\right) - \frac{1}{2}\|\Psi_x\|_{\Gamma_0}^2.$$

Also,

$$\begin{aligned} (\Psi_x, \Psi_{yy}) &= -(\Psi_{xy}, \Psi_y) = -\frac{1}{2}\|\Psi_y\|_{\Gamma_1}^2, \\ b\left(\Psi_x, ((1-x)^2\Psi_x)_y\right) &= -b\left(((1-x)^2\Psi_x)_y, \Psi_x\right) = 0, \\ b\left(\Psi_x, ((1-x)^2\Psi_y)_x\right) &= -2b\left(\Psi_x, (1-x)\Psi_y\right) + b\left(\Psi_x, (1-x)^2\Psi_{xy}\right) \\ &= -2b\left((1-x)\Psi_x, \Psi_y\right). \end{aligned}$$

Therefore (2.24) becomes

$$\partial_t\|\Psi_x\|^2 + \left(\Psi_x, (1-x)\Psi_x\right) + \frac{1}{2}\|\Psi_x\|_{\Gamma_0}^2 + \frac{1}{2}\|\Psi_y\|_{\Gamma_1}^2 + 2b\left((1-x)\Psi_x, \Psi_y\right) = -2a(\Psi_x, \Psi_t) + (\Psi_x, F). \quad (2.25)$$

All the boundary terms have the right sign to enhance the estimates. Therefore we ignore them. (See the Remark below.) Adding the simplified estimates (2.22), (2.23), (2.25) gives

$$\begin{aligned} &\partial_t\left(a\|\Psi\|^2 + \|\Psi_x\|^2 + \frac{1}{2}Q\right) + Q + \left(\Psi_x, (1-x)\Psi_x\right) \\ &= -2b\left((1-x)\Psi_x, \Psi_y\right) + 2(1-a)(\Psi_x, \Psi_t) - 2a\|\Psi_t\|^2 + (\Psi + \Psi_t + \Psi_x, F) \\ &\leq \text{const.}\left(\|\Psi_x\|^2 + \|\Psi_y\|^2 + \|\Psi\|^2 + \|F\|^2\right), \end{aligned} \quad (2.26)$$

where

$$Q = \|(1-x)\Psi_x\|^2 + \|\Psi_y\|^2 + 2b\left((1-x)\Psi_x, (1-x)\Psi_y\right).$$

Since $|b| < 1$, there is a $\delta > 0$ such that

$$Q \geq \delta\left(\|(1-x)\Psi_x\|^2 + \|\Psi_y\|^2\right).$$

Therefore (2.26) gives us the required estimate for the energy norm

$$E = a\|\Psi\|^2 + \|\Psi_x\|^2 + \frac{1}{2}Q.$$

Remark: If the boundary term in (2.22) had not been ignored then we would have obtained a stronger estimate for the energy

$$\hat{E} = E + \|\Psi\|_{\Gamma_1}^2$$

which also controls the growth of Ψ on the boundary Γ_1 .

We shall now prove that the problem is stable against lower order perturbations. We add an expression

$$P = A\Psi_x + B\Psi_y + C\Psi_t + D\Psi$$

to (2.21). Then the estimates for (2.22), (2.23) and (2.24) will be changed by lower order terms

$$\begin{aligned} &(\Psi, A\Psi_x) + (\Psi, B\Psi_y) + (\Psi, C\Psi_t) + (\Psi, D\Psi) \\ &(\Psi_t, A\Psi_x) + (\Psi_t, B\Psi_y) + (\Psi_t, C\Psi_t) + (\Psi_t, D\Psi) \\ &(\Psi_x, A\Psi_x) + (\Psi_x, B\Psi_y) + (\Psi_x, C\Psi_t) + (\Psi_x, D\Psi). \end{aligned}$$

Clearly, there is an energy estimate, provided we choose $2a > |C|$. Thus the strip problem (2.21) is well posed.

Now we start with

$$2\Phi_{xt} = ((1-x)^2\Phi_x)_x + \Phi_{yy} + b((1-x)^2\Phi_x)_y + b((1-x)^2\Phi_y)_x + S(t, x, y). \quad (2.27)$$

We make the change of variables

$$\Phi = e^{ax}\Psi,$$

i.e.,

$$\Phi_x = e^{ax}\Psi_x + ae^{ax}\Psi, \quad \Phi_{xx} = e^{ax}\Psi_{xx} + 2ae^{ax}\Psi_x + a^2e^{ax}\Psi,$$

and set $F = e^{-ax}S$. Then we obtain (2.21) which is modified by \mathcal{R}

$$2(\Psi_x + a\Psi)_t = ((1-x)^2\Psi_x)_x + \Psi_{yy} + b(((1-x)^2\Psi_x)_y + ((1-x)^2\Psi_y)_x) + F + \mathcal{R}. \quad (2.28)$$

Here \mathcal{R} consists of lower order terms,

$$\mathcal{R} = 2a(1-x)^2\Psi_x + (a^2(1-x)^2 - 2a(1-x))\Psi + 2ab(1-x)^2\Psi_y.$$

Since (2.21) is stable against lower order terms there is an energy estimate for (2.28). In the same way as in Sec. II B we can estimate the higher derivatives. This allows us to extend well-posedness to the variable coefficient problem and, locally in time, to the quasilinear problem.

III. THE QUASILINEAR WAVE EQUATION ON AN ASYMPTOTICALLY FLAT BACKGROUND

We now treat the null-timelike initial-boundary problem (1.16) for the quasilinear wave equation. We compactify the domain $R \leq r \leq \infty$ by the transformation $x = 1 - R/r$ to obtain a strip problem $0 \leq x \leq 1$ with future null infinity \mathcal{I}^+ at the boundary $x = 1$. In terms of the rescaled field $\hat{\Phi} = r\Phi$, the wave equation transforms into

$$\begin{aligned} & 2\partial_u\partial_x\hat{\Phi} - R^{-1}\partial_x(W(1-x)^2\partial_x\hat{\Phi}) + (1-x)R^{-1}(\partial_xW)\hat{\Phi} \\ & + R^{-2}D_A((1-x)^2W^A\partial_x\hat{\Phi}) + R^{-2}\partial_x((1-x)^2W^AD_A\hat{\Phi}) - R^{-2}(1-x)(D_AW^A)\hat{\Phi} \\ & - R^{-1}D_A(e^{2\beta}D^A\hat{\Phi}) = -r^3R^{-1}e^{2\beta}S. \end{aligned} \quad (3.1)$$

Here we use the conformal 2-metric h_{AB} and its inverse h^{AB} to raise and lower indices of tensor fields on the spacelike ($u = \text{const.}, r = \text{const.}$) spherical cross-sections. Up to lower order terms, (3.1) is a 3(spatial)-dimensional version of (2.27) where the y -coordinate has been replaced by the x^A -coordinate on the spherical cross-sections and the t -coordinate has been replaced by the u -coordinate. In order for our treatment to apply to the quasilinear case, we assume that the metric coefficients (W, β, W^A, h_{AB}) depend smoothly upon (Φ, u, r, x^A) and that the source S depends smoothly upon $(\Phi, \partial_a\Phi, u, r, x^A)$, with non-singular Lorentzian geometry in the neighborhood of the initial data.

We treat the modified problem resulting from the transformation $\hat{\Phi} = e^{ax}\Psi$. The same argument used in Sec. II C shows that this problem is stable with respect to lower order terms. We ignore these terms and thus obtain the strip problem

$$\begin{aligned} 2\partial_u(\partial_x\Psi + a\Psi) &= R^{-1}\partial_x(W(1-x)^2\partial_x\Psi) - R^{-2}D_A((1-x)^2W^A\partial_x\Psi) - R^{-2}\partial_x((1-x)^2W^AD_A\Psi) \\ &+ R^{-1}D_A(e^{2\beta}D^A\Psi) + F, \quad \Psi(0, x, x^A) = f, \quad \Psi(u, 0, x^A) = q, \end{aligned} \quad (3.2)$$

where $F = -r^3R^{-1}e^{2\beta}e^{ax}S$. In order to treat (3.2), we require that the physical space source has asymptotic behavior $S = O(r^{-3})$ so that F is square integrable over the strip. No boundary condition is allowed at the outer boundary Γ_1 at $x = 1$ since \mathcal{I}^+ is an ingoing characteristic surface.

We obtain the required estimates for (3.2) by the same method used in Sec. II C. The data at the inner boundary Γ_0 at $x = 0$ may be absorbed into F so it suffices to treat the case $q = 0$. We define the inner product

$$(\Psi_1, \Psi_2) = \int_0^1 dx \oint d\omega \Psi_1 \Psi_2$$

and L_2 norm $\|\Psi\|^2 = (\Psi, \Psi)$, where $d\omega$ is the area element on the unit sphere. We write

$$\|V_A\|^2 = (V_A, V^A) = (h^{AB}V_A, V_B).$$

Since the spherical cross-sections are spacelike, their intrinsic 2-metric h_{AB} is positive definite so that $\|V_A\|$ serves as an L_2 norm for the angular components. We also need a metric norm for spacelike 3-vectors. In the standard Cauchy problem this is supplied by the intrinsic 3-metric of the spacelike Cauchy hypersurfaces. Since the characteristic hypersurfaces have a degenerate 3-metric, we take a different approach. We use the projection operator $\pi_b^a = \delta_b^a - t^a \partial_b u$, where $t^a \partial_a = \partial_u$, to define a 3-metric $\gamma^{ab} = \pi_c^a \pi_d^b g^{cd}$. For the Bondi-Sachs metric (1.15), the resulting components in the (u, r, x^A) coordinates are

$$\gamma^{au} = 0, \quad \gamma^{rr} = e^{-2\beta} W, \quad \gamma^{rA} = -e^{-2\beta} r^{-2} W^A, \quad \gamma^{AB} = r^{-2} h^{AB}.$$

Denoting $x^i = (r, X^A)$, this implies that $\gamma^{ij} \rightarrow e^{ij}$ as $r \rightarrow \infty$, where e^{ij} is the Euclidean 3-metric expressed in standard spherical coordinates. In the compactified coordinates, $\tilde{x}^i = (x, x^A)$ it is more useful to deal with the rescaled 3-metric $\tilde{\gamma}^{ab} = e^{2\beta} r^2 \gamma^{ab}$ which has components

$$\tilde{\gamma}^{au} = 0, \quad \tilde{\gamma}^{xx} = (1-x)^2 W, \quad \tilde{\gamma}^{xA} = -R^{-1} (1-x)^2 W^A, \quad \tilde{\gamma}^{AB} = e^{2\beta} h^{AB}.$$

We then define

$$\|V_i\|^2 = (\tilde{\gamma}^{ij} V_i, V_j)$$

which serves as an L_2 norm for the \tilde{x}^i components. Thus

$$\|\partial_i \Psi\|^2 = \|W^{1/2} (1-x) \partial_x \Psi\|^2 + \|e^\beta D_A \Psi\|^2 - 2R^{-1} \left((1-x) \partial_x \Psi, (1-x) W^A D_A \Psi \right). \quad (3.3)$$

We also define the corresponding inner products and norms on the boundaries, e.g.

$$(\Psi_1, \Psi_2)_\Gamma = \oint_\Gamma d\omega \Psi_1 \Psi_2, \quad \|\Psi\|_\Gamma^2 = (\Psi, \Psi)_\Gamma.$$

Because the radial coordinate r used in the Bondi-Sachs metric (1.15) is a surface area coordinate, the conformally rescaled 2-metric $h_{AB} = r^{-2} g_{AB}$ of the spherical cross-sections has determinant $\det(h_{AB}) = \det(q_{AB})$, where q_{AB} is the unit sphere metric. Consequently,

$$(\Psi_1, D_A D^A \Psi_2) = -(D_A \Psi_1, D^A \Psi_2) \quad (3.4)$$

and

$$(V^A, D_A \Psi) = -(D_A V^A, \Psi) \quad (3.5)$$

where $V^A(u, x, x^A)$ is any smooth vector field on the spherical cross-sections. These identities allow the necessary integration by parts.

We derive the required estimates by freezing the dependence of the metric coefficients on (Ψ, u, x) but we retain their dependence on x^A so that W^A and h_{AB} remain smooth vector and tensor fields on the spherical cross-sections. We follow the procedure in Sec. II C. First,

$$\begin{aligned} & 2(\Psi, \partial_u \partial_x \Psi) + 2a(\Psi, \partial_u \Psi) = -2(\partial_x \Psi, \partial_u \Psi) + \partial_u \|\Psi\|_{\Gamma_1}^2 + a \partial_u \|\Psi\|^2 \\ & = -R^{-1} \left(W(1-x) \partial_x \Psi, (1-x) \partial_x \Psi \right) - R^{-1} \|e^\beta D_A \Psi\|^2 + 2R^{-2} \left((1-x) W^A D_A \Psi, (1-x) \partial_x \Psi \right) + (\Psi, F), \end{aligned}$$

i.e.

$$\partial_u \|\Psi\|_{\Gamma_1}^2 + a \partial_u \|\Psi\|^2 + R^{-1} \|\partial_i \Psi\|^2 = 2(\partial_x \Psi, \partial_u \Psi) + (\Psi, F). \quad (3.6)$$

Next,

$$\begin{aligned} & 2(\partial_u \Psi, \partial_u \partial_x \Psi) + 2a \|\partial_u \Psi\|^2 = \|\partial_u \Psi\|_{\Gamma_1}^2 + 2a \|\partial_u \Psi\|^2 \\ & = -\frac{1}{2} R^{-1} \partial_u \left(\|W^{1/2} (1-x) \partial_x \Psi\|^2 + \|e^\beta D_A \Psi\|^2 - 2R^{-1} \left((1-x) \partial_x \Psi, (1-x) W^A D_A \Psi \right) \right) + (\partial_u \Psi, F), \end{aligned}$$

so that

$$\|\partial_u \Psi\|_{\Gamma_1}^2 + 2a\|\partial_u \Psi\|^2 + \frac{1}{2}R^{-1}\partial_u\|\partial_i \Psi\|^2 = (\partial_u \Psi, F). \quad (3.7)$$

Next,

$$\begin{aligned} & 2(\partial_x \Psi, \partial_u \partial_x \Psi) + 2a(\partial_x \Psi, \partial_u \Psi) = \partial_u \|\partial_x \Psi\|^2 + 2a(\partial_x \Psi, \partial_u \Psi) \\ & = R^{-1} \left(\partial_x \Psi, \partial_x (W(1-x)^2 \partial_x \Psi) \right) + R^{-1} \left(\partial_x \Psi, D_A(e^{2\beta} D_A \Psi) \right) \\ & \quad - R^{-2} \left(\Psi_x, D_A((1-x)^2 W^A \partial_x \Psi) \right) - R^{-2} \left(\partial_x \Psi, \partial_x((1-x)^2 W^A D_A \Psi) \right) + (\partial_x \Psi, F). \end{aligned} \quad (3.8)$$

As shown in in Sec. II C,

$$\left(\partial_x \Psi, \partial_x (W(1-x)^2 \partial_x \Psi) \right) = - \left(\partial_x \Psi, W(1-x) \partial_x \Psi \right) - \frac{1}{2} \|W^{1/2} \partial_x \Psi\|_{\Gamma_0}^2.$$

Also,

$$(\partial_x \Psi, D_A(e^{2\beta} D_A \Psi)) = -\frac{1}{2} \|e^\beta D_A \Psi\|_{\Gamma_1}^2 \quad (3.9)$$

and

$$\left(\partial_x \Psi, D_A((1-x)^2 W^A \partial_x \Psi) \right) + \left(\partial_x \Psi, \partial_x((1-x)^2 W^A D_A \Psi) \right) = -2 \left((1-x) \partial_x \Psi, W^A D_A \Psi \right).$$

Therefore (3.8) becomes

$$\begin{aligned} & \partial_u \|\partial_x \Psi\|^2 + R^{-1} \left(\partial_x \Psi, W(1-x) \partial_x \Psi \right) + \frac{1}{2} R^{-1} \|W^{1/2} \partial_x \Psi\|_{\Gamma_0}^2 + \frac{1}{2} R^{-1} \|e^\beta D_A \Psi\|_{\Gamma_1}^2 \\ & \quad - 2R^{-2} \left((1-x) \partial_x \Psi, W^A D_A \Psi \right) = -2a(\partial_x \Psi, \partial_u \Psi) + (\partial_x \Psi, F). \end{aligned} \quad (3.10)$$

As before, the boundary terms have the right sign to enhance the estimates so that we can ignore them. Adding the simplified estimates (3.6), (3.7), (3.10) gives

$$\begin{aligned} & \partial_u \left(a\|\Psi\|^2 + \|\partial_x \Psi\|^2 + \frac{1}{2}R^{-1}\|\partial_i \Psi\|^2 \right) + R^{-1}|\partial_i \Psi|^2 + R^{-1} \left(\partial_x \Psi, W(1-x) \partial_x \Psi \right) \\ & = 2R^{-2} \left((1-x) \partial_x \Psi, W^A D_A \Psi \right) + 2(1-a)(\partial_x \Psi, \partial_u \Psi) - 2a\|\partial_u \Psi\|^2 + (\Psi + \partial_u \Psi + \partial_x \Psi, F) \\ & \leq \text{const.} \left(\|\partial_x \Psi\|^2 + \|D_A \Psi\|^2 + \|\Psi\|^2 + \|F\|^2 \right). \end{aligned} \quad (3.11)$$

Therefore (3.11) gives us an energy estimate provided that the 3-metric γ^{ij} has $(+++)$ signature, so that $\|\partial_i \Psi\|$ is a norm for the gradient $\partial_i \Psi = (\partial_r \Psi, \partial_A \Psi)$. This is equivalent to the requirement that the principal part of the wave operator reduce to an elliptic operator in the stationary case where the u -derivatives vanish. Since γ^{ij} is asymptotic to the Euclidean metric as $r \rightarrow \infty$, this positive-definite condition is satisfied throughout some exterior domain.

Estimates for the higher derivatives of Ψ and stability against lower order perturbations follow from the same arguments given in Sec. II. This establishes the well-posedness of the worldtube-nullcone problem for the case of smooth variable coefficients. The extension of well-posedness, locally in time, for the quasilinear case then follows from the standard techniques referred to in Sec. II.

For a mass M Schwarzschild geometry, $\gamma^{rr} = e^{-2\beta} W = 1 - 2M/r$ so that positive-definiteness of the 3-metric γ^{ij} breaks down at $r = 2M$ where the worldtube becomes null. In this limiting case of the double-null problem, the $\partial_u \|\partial_x \Psi\|^2$ term in (3.11) suffices to provide the required estimate. However, for $R < 2M$ the ‘‘worldtube’’ is spacelike and the problem must be treated differently.

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