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Flavour Symmetries and Models of Neutrino Mixing

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Abstract

In my Bachelor thesis the theory of finite discrete groups is reviewed with respect to the alternating group A_4. Neutrinos in the standard model and the implications of neutrino oscillations are discussed. The observation of massive neutrinos implies physics beyond the standard model. A popular mixing pattern for the leptonic mixing matrix the tri-bimaximal (TBM) mixing is presented. This mixing pattern causes a form of the neutrino mass matrix which can be specified by means of symmetries. These symmetries give rise to flavour symmetry models based on finite discrete groups. The special case of an A_4 flavour symmetry is examined. Finally a general numerical analysis of the form of the neutrino mass matrix induced by the 9 parameters in the neutrino sector is performed. As the possible experimental deviations from TBM are rather large there exist some mass ranges for normal and inverted ordering showing considerable asymmetries compared to the pure TBM form.

Zusammenfassung

In dieser Bachelorarbeit werden einige Ergebnisse der Theorie endlicher diskreter Gruppen zusammengefasst und auf die alternierende Gruppe A_4 angewendet. Des weiteren wird die Rolle von Neutrinos im Standardmodell der Teilchenphysik und die Bedeutung von Neutrinooszillationen diskutiert. Die Existenz von Neutrinomassen impliziert Physik jenseits des Standarmodells. Ein populärer Ansatz für die Beschreibung der Mischungsmatrix für Leptonen, die tribimaximale Mischung (TBM), wird vorgestellt. Mit dieser Beschreibung geht eine spezielle Form der Massenmatrix für Neutrinos einher, welche sich mittels Symmetrien festlegen lässt. Diese Symmetrien legen Modelle für “Flavoursymmetrien”, die auf endlichen diskreten Gruppen aufbauen, nahe. Als Fallstudie wir ein A_4-Flavourmodell betrachtet. Schließlich wird der Einfluss der 9 Parameter im Neutrinosektor auf die Form der Neutrinomassenmatrix numerisch untersucht. Da die experimentellen Abweichung von der TBM-Mischung relativ groß sind, ergeben sich für einige Massenbereiche, sowohl für ein normales wie invertiertes Neitrinomassenspektrum, erhebliche Abweichungen von der durch reine TBM-Mischung verlangten Form.

Preface

I thank Dr. Werner Rodejohann for his supervision and introducing me to the interesting topic of this Bachelor thesis. Further I thank my family and friends for their support.

Contents

1. Introduction

The observation of neutrino oscillations implies that neutrinos are massive. As the standard model predicts massless neutrinos one of the components, either renormalizability or the particle content, are incorrect. In contrast to the other fermions neutrinos have a tiny mass. This could be explained by the seesaw mechanism which introduces a heavy right-handed neutrino. However, the seesaw mechanism assumes a Majorana nature of neutrinos which is still not experimentally proven.

In general, neutrino mass models do not predict the mixing angles involved in neutrino oscillations. In the lepton sector of the standard model the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix U_{PMNS} which describes these oscillations has a very peculiar form close to the tri-bi-maximal mixing (TBM).

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

This differs strongly from the quark sector where the CKM-matrix is close to unity. The form of U_{PMNS} gives rise to flavour symmetry models. These flavour symmetries which act "horizontally" on the three known generations could explain the arising elements in U_{PMNS} as a consequence of group product coefficients. Assuming that the left- and right-handed charged leptons as well as the neutrinos transform as irreducible representations of the flavour group, these transformation properties should show up as invariances of the mass matrix.

In this thesis I focus on one special flavour symmetry model which predicts the mixing angles. Therefore, the nature of the invariances is further investigated. If neutrinos are treated as Majorana particles they have a symmetric mass matrix. Combined with the above mixing matrix the form of the neutrino mass matrix can be completely determined by the invariance under two transformations, notably a μ-τ-symmetry and an additional symmetry S with $S^2 = 1$ which is one of two generators of group A_4. Thus, the most general mass matrix leading to TBM is invariant under a generator of A_4. Together with a diagonal charged lepton mass matrix which is invariant under the second generator of A_4 this suggests to consider the alternating group A_4 as an appropriate flavour symmetry group. Compared to other finite discrete groups containing these two generators A_4 has the advantage of a quite economical structure. The A_4-model predicts the TBM-mixing angles but needs a proliferation of the scalar sector. This is needed in order to explain the μ-τ-symmetry which is in general not part of A_4. The μ-τ-symmetry is realized by a special vacuum alignment separating the spontaneous symmetry breaking of the charged lepton sector from the neutrino sector. Additionally, the A_4-model leaves the experimentally unknown neutrino mass spectrum and mass scale more or less undetermined. Some mass relations are explicitly calculated for a particular model [1] in order

to give an estimate of the predictive possibilities of an A_4-model.

The TBM-mixing assumes that θ_{13} in U_{PMNS} is zero which is not compatible with the experimental value in its 3σ-range. Hence, this deviation must be explained in the context of the A_4-model. In general this is possible because one expects the flavour symmetry group to be broken at a higher scale. Then renormalization group evolution could lead to deviations at a lower scale. Higher order corrections in the A_4-model could disturb the form of the mass matrix and thereby affect the mixing angles as well.

Independent from theoretical corrections, the symmetries of the neutrino mass matrix $\mathcal{M}_\nu$ predicted by TBM-mixing are analysed. Therefore, the impact of the experimental 3σ-errors of the six parameters in U_{PMNS} and the three neutrino masses on the form of $\mathcal{M}_\nu$ are numerically calculated. The symmetries describing $\mathcal{M}_\nu$ are translated into three asymmetry parameters containing entries of the mass matrix ($m_{ij} = (\mathcal{M}_\nu)_{ij}$).

$$X_1 \equiv \left|\frac{m_{12}-m_{13}}{m_{12}+m_{13}}\right| \quad X_2 \equiv \left|\frac{m_{22}-m_{33}}{m_{22}+m_{33}}\right| \quad X_3 \equiv \left|\frac{m_{11}+m_{13}-m_{22}-m_{23}}{m_{11}+m_{13}+m_{22}+m_{23}}\right|$$

I study the dependence of these asymmetry parameters on the lightest neutrino mass and the mixing angles. This is done in more detail with respect to the θ_{13}-dependence in order to find out if a non-vanishing value could lead to maximal violation of the symmetries. Due to the large number of free parameters the expressions for the asymmetry parameters are difficult to handle analytically. Consequently, the asymmetry parameters are maximised numerically using random numbers for the mixing parameters. The analysis with free mixing angles indicates stable mass ranges with nearly constant asymmetries for a hierarchical spectrum for X_1 and X_2 in a normal ordering and for X_2 and X_3 in an inverted ordering. For these stable mass ranges approximate analytical expressions describing the dependencies on the angles are derived. These are obtained by setting the smallest neutrino mass to zero and by relating the masses m_2 and m_3 in a normal spectrum to the rather small ratio of the solar and atmospheric mass squared differences. For a normal hierarchical spectrum X_1 depends strongly on θ_{13} and X_2 depends mostly on θ_{23}. For an inverted hierarchical spectrum X_2 depends largely on θ_{13} and X_3 depends equally on all three mixing angles. It is illustrated that for special configurations of the three complex phases the symmetry could be maximally violated. In an inverted spectrum for X_1 a small value of θ_{13} forbids a maximal asymmetry. For a normal hierarchical spectrum the minimum of $X_1^{\text{min}} \approx 0.2$ is considerably above the other minimal asymmetries which are approximately below 10^{-3}.

The thesis starts with a recapitulation of group theory of finite discrete groups in Section 2 . Results are applied to group A_4 and certain general properties of it are shown. In Section 3 neutrinos as a part of the standard model and the necessity of extensions of the standard model are discussed. A "prototypical" A_4-model [2, 3, 1] in an effective description using dimension-5 operators is examined as an example of a flavour symmetry model producing TBM-mixing in Section 4. This is followed by

the analysis of asymmetry parameters in Section 5 and completed by a discussion in Section 6.

2. Finite Groups

2.1. Basics

This section is intended to give a compact summary of theorems of finite groups which are used in section 2.2 to investigate group A_4. The discussion in this section is mostly taken from [4], [5] and a lecture script on group theory [6]. Most of the proofs of the presented theorems are given in [4].

Definition 1. *A group is a set together with an operation* $(G, \bullet)$ *satisfying four axioms*

1. *For all* $a, b \in G$*, the result of the operation,* $a \bullet b$*, is also in* G*. (closure)*
2. *For all* $a, b, c \in G$*,* $(a \bullet b) \bullet c = a \bullet (b \bullet c)$*. (associativity)*
3. *There exists a* $e \in G$*, such that for every* $a \in G$*, the equation* $e \bullet a = a \bullet e = a$ *holds. (identity element)*
4. *For each* $a \in G$ *exists an element* $b \in G$ *such that* $a \bullet b = b \bullet a = e$ *(inverse element)*

The multiplication is generally not commutative. If $ab = ba \ \forall \ a, b \in G$ the group is called abelian. The order of a group $[G]$ is the number of elements in the group. Finite groups are groups with a finite number of elements. They possess a number of convenient properties which are useful for building flavour symmetry models [5]. They have a finite number of inequivalent irreducible representations (irreps), all irreps are equivalent to unitary irreps and all numbers concerning properties of the group are finite. In addition, if discrete groups are used, there are only easy to handle sums.

An important group is the symmetric group S_n which contains all permutations of n objects. The notation

$$P = \begin{pmatrix} 1 & 2 & \dots & n \\ p_1 & p_2 & \dots & p_n \end{pmatrix} \tag{2.1}$$

describes on which object p_i the object i is mapped. The order of S_n can be derived by looking at the possible mappings. Element 1 can be replaced by any of the n objects 1 to n. For the second element 2 there are only $n-1$ objects left and so on. Thus $n!$ is the order of the symmetric group S_n. A more convenient notation for a permutation is the cycle notation which is based on the fact, that for each permutation there are independent groups of indices. A cycle containing two symbols (2-cycle) is called a transposition, it denotes the exchange of two objects. Any permutation r-cycle can be

expressed as a product of transpositions. For a r-cycle $r-1$ transpositions are needed. A permutation is even (odd) according to the number of transpositions. The product of two even permutations is even again. The inverse of a product of cycles is the product of their inverses. Evidently the identity is an even permutation. So the even permutations form a group, the alternating group A_n. The order of group A_n can be derived from the order of S_n. Each permutation $\sigma \in S_n$ is defined by the values of $\sigma(1), \ldots, \sigma(n)$. If the values of $\sigma(1), \ldots, \sigma(n-2)$ are already fixed, the choice of the last to permutations decide the sign of σ. So the order $[A_n]$ of A_n is the half of S_n.

$$[A_n] = \frac{n!}{2} \tag{2.2}$$

Each equivalence relation provides a partition of a set into disjunct classes. The conjugation of two elements gives a partition which is quite useful for the further analysis of group representations.

Definition 2. *The conjugate class of an element $a \in G$ contains all elements $b \in G$ which are conjugate to a.*

$$(a) = \left\{ b \in M \;\middle|\; b = gag^{-1},\ g \in M \right\} \tag{2.3}$$

Definition 3. *A representation is a homomorphism $D : G \to GL(n, \mathbb{C})$ from the group G to the invertible complex $n \times n$ matrices.*

A group homomorphism is preserving the group structure $D(g_1 g_2) = D(g_1)D(g_2)$. Two representations $D^{(1)}D^{(2)}$ satisfying the relation

$$D^{(1)}(g) = SD^{(2)}S^{-1} \;\forall\; g \in G \tag{2.4}$$

are called equivalent. For the matrices of the representation, S can be considered as a basis transformation. The character of a group can be used to distinguish inequivalent representations.

Definition 4. *Character χ of representation D of a group G is the set of the traces of the representation matrices $D(g)$*

$$\chi = \{\, \chi(g) \equiv \mathrm{tr}[D(g)] \mid g \in G \,\} \tag{2.5}$$

As the trace is similarity-invariant, equivalent representations have the same character. Most of the theorems in this section are based on the following Schur lemmas.

Lemma 1. *Each matrix B which commutes with all matrices D of an irreducible representation of a group G, must be a multiple of the unit matrix*

$$BD(g) = D(g)B \;\forall g \in G \Rightarrow B = \lambda \mathbf{1} \tag{2.6}$$

Lemma 2. *If D and D' are irreducible representations of the group G having the same dimensions, and if the matrix B satisfies*

$$BD(g) = D'(g)B \ \forall g \in G \tag{2.7}$$

then $B = 0$ or D and D' are equivalent

From the Schur lemmas one can deduce the orthogonality relation for the matrices of irreducible representations.

Theorem 1. *(Theorem of Orthogonality) For irreducible representations $D^{(\mu)}$ and $D^{(\nu)}$*

$$\sum_{g\in G} D^{(\mu)}_{ir}(g) D^{(\nu)}_{sj}(g^{-1}) = \frac{[G]}{n_\mu}\, \delta^{\mu\nu}\delta_{ij}\delta_{rs} \tag{2.8}$$

In general a representation is equivalent to an infinite number of representations. For finite groups one can show additionally that every given representation is equivalent to a unitary representation. Therefore by taking the trace in Eq. (2.8) a simple formula for the characters follows.

$$\frac{1}{[G]} \sum_{g\in G} \chi^{(\mu)}(g)\chi^{(\nu)}(g)^* = \delta_{\mu\nu} \tag{2.9}$$

Suppose G consists of k conjugate classes K_i having k_i number of elements and character χ_i. Then Eq. (2.9) becomes

$$\frac{1}{[G]} \sum_{i=1}^{k} k_i \chi^{(\mu)}_i \chi^{(\nu)*}_i = \delta_{\mu\nu} \tag{2.10}$$

A similar orthogonality relation for the sum over the irreps allows to prove the following theorem.

Theorem 2. *The number of inequivalent irreducible representations r equals the number of conjugate classes k.*

$$r = k \tag{2.11}$$

Reducible representations D of finite groups can be expressed as a direct sum of irreducible representations $D^{(\nu)}$.

$$D = \sum_{\oplus} a_\nu D^{(\nu)} \tag{2.12}$$

The coefficients $a_\nu \in \mathbb{N}$ indicate how many times an irreducible representation $D^{(\nu)}$ is contained in the sum. They can be calculated using the orthogonality relation and the character table of the group.

$$\chi(g) = \sum_\nu a_\nu \chi^{(\nu)}(g) \tag{2.13}$$

$$\Rightarrow a_\mu = \frac{1}{[G]} \sum_{g\in G} \chi(g)\chi^{(\mu)}(g^{-1}) \equiv \langle \chi^{(\mu)}, \chi \rangle \tag{2.14}$$

By applying this result to the regular representation, the number and dimension of irreducible representations can be related to the order of the group. The regular representation is a $[G]$ dimensional representation of G. The coefficients for the decomposition of the regular representation in irreducible representations are calculated using that only $\chi(e)$ is non-zero.

$$a_\mu = \frac{1}{[G]}\chi(e)\chi^{(\mu)}(e) = \frac{1}{[G]}[G]\chi^{(\mu)}(e) = \chi^{(\mu)}(e) = n_\mu \tag{2.15}$$

Setting $g = e$ in relation

$$\chi(g) = \sum_\nu a_\nu \chi^{(\nu)}(g) \tag{2.16}$$

leads to a formula between the order of a group and the dimension n_ν of irreducible representations

$$[G] = \sum_\nu n_\nu^2 \tag{2.17}$$

From a physics point of view it is interesting to consider two particles with fields ψ_a and ϕ_b transforming with two irreducible representations $D^{(\mu)}$ and $D^{(\nu)}$ in the following way

$$\psi'_\alpha = D^{(\mu)}_{\alpha\beta}(g)\psi_\beta \tag{2.18}$$

$$\phi'_\gamma = D^{(\nu)}_{\gamma\delta}(g)\phi_\delta \tag{2.19}$$

the product field $\Psi_{\alpha\gamma} = \psi_\alpha\phi_\gamma$ is transforming as

$$\Psi'_{\alpha\gamma} = \underbrace{D^{(\mu)}_{\alpha\beta}(g)D^{(\nu)}_{\gamma\delta}(g)}_{D^{(\mu\times\nu)}_{\alpha\gamma;\beta\delta}(g)}\Psi_{\beta\delta} \tag{2.20}$$

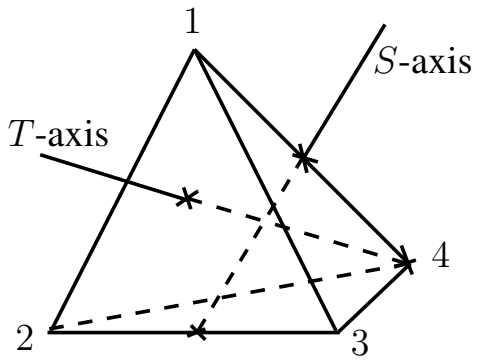

Figure 2.1: Group A_4 can be considered as a geometrical object, namely, the invariance group of a tetrahedron. The two generators of the group correspond to rotations around the indicated axes.

The representation $D^{(\mu\times\nu)}(g)$ preserves the group structure and is labelled by $D^{(\mu)}\otimes D^{(\nu)}$. A product representation of two irreducible representations is generally reducible for interacting particles and can therefore be decomposed into a Clebsch-Gordon series

$$D^{(\mu)} \otimes D^{(\nu)} = \sum_{\oplus} a_\sigma D^{(\sigma)} \tag{2.21}$$

$$a_\sigma = \langle \chi^{(\sigma)}, \chi^{(\mu\times\nu)} \rangle = \langle \chi^{(\sigma)}, \chi^{(\mu)}\chi^{(\nu)} \rangle \tag{2.22}$$

2.2. The Alternating Group A_4

A_4 is the group of even permutations of 4 objects.

$$G = A_4 = gp\{S, T\} \quad \text{with} \quad S^2 = T^3 = (ST)^3 = 1 \tag{2.23}$$

A_4 can as well be considered as the invariance group of a tetrahedron. After labelling the vertices of a tetrahedron by $1, \ldots, 4$ the elements of A_4 are all produced by rotations of the Tetrahedron as indicated in Fig. 2.1.

In cycle notation, where the cycles have to be read and multiplied from the right by convention the generators are given by

$$S \equiv (14)(23) \qquad T \equiv (123) \tag{2.24}$$

One can easily calculate that they fulfil Eq. (2.23). Group A_4 consists of 4 conjugate

A_4	C_1	$4C_2$	$4C_3$	$3C_4$
$\mathbf{1}$	1	1	1	1
$\mathbf{1'}$	1	ω	ω^2	1
$\mathbf{1''}$	1	ω^2	ω	1
$\mathbf{3}$	3	0	0	-1

Table 2.1: Charactertable of A_4

classes

$$
\begin{aligned}
C_1 &= \{e\} \\
C_2 &= \{T = (123), ST = (134), TS = (142), STS = (243)\} \\
C_3 &= \{T^2 = (132), ST^2 = (124), T^2S = (143), TST = (234)\} \\
C_4 &= \{S, T^2ST = ((12)(34), TST^2 = (13)(24)\}
\end{aligned}
$$

According to theorem 2 there are four irreducible representations. As group A_4 is of order 12, it is now possible to infer the dimensions of the four irreducible representations using Eq. (2.17).

$$1 + n_2^2 + n_3^2 + n_4^2 = 12 \tag{2.25}$$

This equation implies that their are three one-dimensional and one three-dimensional representations. These will be denoted by $\mathbf{1}, \mathbf{1'}, \mathbf{1''}$ and $\mathbf{3}$. These representations which solely have to fulfil Eq. (2.23) can be expressed in terms of $\omega = e^{\frac{2}{3}\pi i}$.

$$
\begin{aligned}
\mathbf{1} &: S = 1, \qquad T = 1 \\
\mathbf{1'} &: S = 1, \qquad T = \omega \\
\mathbf{1''} &: S = 1, \qquad T = \omega^2 \\
\mathbf{3} &: S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{2.26}
$$

The product of two A_4 triplets is a reducible representation. Its decomposition into a Clebsch-Gordon series can be calculated using Eq. (2.22) and the known character table of A_4. The decomposition of the product of two A_4 triplets is given explicitly by

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{1''} \oplus \mathbf{3} \oplus \mathbf{3} \tag{2.27}$$

and the following corresponding product coefficients satisfying the transformation prop-

erties in Eq. (2.26)

$$\mathbf{1} : (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) \tag{2.28}$$
$$\mathbf{1'} : (e_1 \otimes e_1 + \omega^2 e_2 \otimes e_2 + \omega e_3 \otimes e_3) \tag{2.29}$$
$$\mathbf{1''} : (e_1 \otimes e_1 + \omega e_2 \otimes e_2 + \omega^2 e_3 \otimes e_3) \tag{2.30}$$
$$\mathbf{3} : (e_2 \otimes e_3, e_3 \otimes e_1, e_1 \otimes e_2) \tag{2.31}$$
$$\mathbf{3} : (e_3 \otimes e_2, e_1 \otimes e_3, e_2 \otimes e_1) \tag{2.32}$$

The relevant product rules for the singlets which produce invariants are calculated in the same way.

$$\mathbf{1} = \mathbf{1} \otimes \mathbf{1} = \mathbf{1'} \otimes \mathbf{1''}, \quad \mathbf{3} \otimes \mathbf{1} = \mathbf{3} \otimes \mathbf{1'} = \mathbf{3} \otimes \mathbf{1''} = \mathbf{3} \tag{2.33}$$

Unlike the previous definition of the three-dimensional unitary representation where S is diagonal, it is possible to choose a basis where T is diagonal. This T-diagonal basis can be useful in flavour symmetry models where the charged lepton mass matrix is diagonal.

$$\mathbf{3} : T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \tag{2.34}$$

In the T-diagonal basis the product coefficients for the multiplication of two triplets as given in Eq. (2.33) change.

$$\mathbf{1} : (e_1 \otimes e_1 + e_2 \otimes e_3 + e_3 \otimes e_2) \tag{2.35}$$
$$\mathbf{1'} : (e_3 \otimes e_3 + e_1 \otimes e_2 + e_2 \otimes e_1) \tag{2.36}$$
$$\mathbf{1''} : (e_2 \otimes e_2 + e_1 \otimes e_3 + e_3 \otimes e_1) \tag{2.37}$$
$$\mathbf{3}_\mathrm{s} : \begin{pmatrix} 2e_1 \otimes e_1 - e_2 \otimes e_3 - e_3 \otimes e_2 \\ 2e_3 \otimes e_3 - e_1 \otimes e_2 - e_2 \otimes e_1 \\ 2e_2 \otimes e_2 - e_1 \otimes e_3 - e_3 \otimes e_1 \end{pmatrix}, \quad \mathbf{3}_{as} : \begin{pmatrix} e_2 \otimes e_3 - e_3 \otimes e_2 \\ e_1 \otimes e_2 - e_2 \otimes e_1 \\ e_3 \otimes e_1 - e_1 \otimes e_3 \end{pmatrix} \tag{2.38}$$

If two indices are changed one of the triplets is totally symmetric and the other one is totally antisymmetric. This basis allows to simplify the upcoming calculations because the neutrino mass matrix is known to be symmetric and consequently only the symmetric triplet has to be considered. Additionally this basis makes it easy to classify the impact of the one-dimensional representations. The $\mathbf{1}$ is invariant under $2-3$ exchange while $\mathbf{1'}$ is symmetric under $1-2$ exchange and $\mathbf{1''}$ under $1-3$ exchange.

Knowing the product rules for two triplets and the singlets one can construct further invariants involving the products of more than two triplets like $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ for example.

3. Neutrino Physics

3.1. Neutrinos in the Standard Model

In contrast to the other standard model fermions neutrinos are electrically neutral. There are at least three generations of light neutrinos, the ν_e, ν_μ and ν_τ, named after their charged lepton partner particle in the charged current process. Among others neutrinos participate in neutral current reactions including Z bosons via scattering and decay processes like $Z \to \nu_\alpha \overline{\nu}_\alpha$. Investigating this decay process, the number of light neutrino generations N_ν can be fixed. The neutrinos from the Z boson decay are not detected directly because they are neutral but they appear as an invisible part Γ_ν in the Z-boson decay width Γ_Z .

$$\Gamma_\mathrm{Z} = \Gamma_\mathrm{had} + 3\Gamma_\mathrm{l} + \underbrace{N_\nu \Gamma_\nu}_{\Gamma_\mathrm{inv}} \tag{3.1}$$

$$N_\nu = \frac{\Gamma_\mathrm{inv}}{\Gamma_{\nu,SM}} = \left(\frac{\Gamma_\mathrm{inv}}{\Gamma_\mathrm{l}}\right)_\mathrm{exp} \times \left(\frac{\Gamma_\mathrm{l}}{\Gamma_\nu}\right)_\mathrm{SM} \tag{3.2}$$

$$N_\nu = 2.984 \pm 0.008 \tag{3.3}$$

This number taken from Ref. [7] is very well consistent with the existence of three neutrino flavours. Possibly existing right-handed particles which are $SU(2)_\mathrm{L} \times U(1)_\mathrm{Y}$ singlets cannot be detected in the process described above.

Explicit mass terms $\overline{\psi}\psi$ for fermions are forbidden in the standard model because the $SU(2) \times U(1)$ symmetry with the quantum numbers of the fermions tabulated in Tab. 3.2 is not compatible with this term. The existence of $\overline{\psi}\psi$ would break the gauge invariance explicitly. This problem is solved by the Higgs mechanism. The masses in the Standard model are generated through the spontaneous symmetry breaking (SSB) of the $SU(2) \times U(1)$ group by the Higgs doublet ϕ. The Lagrangian for this complex scalar field is

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) \tag{3.4}$$

with the scalar potential V

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \frac{\lambda}{2}(\phi^\dagger \phi)^2 \tag{3.5}$$

The new term for the coupling of the Higgs field to the left- and right-handed fermions is called “Yukawa” Lagrangian.

$$g_\mathrm{Y} \underbrace{\left(\overline{L}\phi\right) R}_{(\mathbf{3}\oplus\mathbf{1})\otimes\mathbf{1}} = g_\mathrm{Y} v \overline{f}_\mathrm{L} f_\mathrm{R} = -m_\mathrm{f} \overline{f}_\mathrm{L} f_\mathrm{R} \tag{3.6}$$

with the vacuum expectation value v.

The standard model does not contain a right-handed neutrino. Thus the neutrino mass is zero in the standard model. However experiments have shown that neutrinos are massive but neutrinos are different from other standard model fermions as they have tiny mass and huge mixing [7]. This is difficult to explain in case they get their mass through the same mechanism as the other fermions.

3.2. Neutrino Oscillation

There are different types of natural neutrino sources like the sun or atmospheric processes and various neutrino experiments with reactors and accelerators as sources. The neutrinos in these sources are created in a charged current reaction with a charged lepton. The species of the charged lepton (e, μ, τ) involved in this process defines the "flavour" of the neutrino $(\nu_e, \nu_\mu, \nu_\tau)$. Generally, the mass matrix in this flavour basis is not diagonal. This means that the mass, or put in the words of quantum mechanics, the free propagation eigenstates ν_1, ν_2 and ν_3 do not coincide with the flavour states. However, neutrinos are created and detected as flavour eigenstates and therefore their probability to be in a different flavour state oscillates with time.

Suppose a basis change for the leptons from the mass to the flavour basis is given by the unitary matrices $U^{(l)}_{\mathrm{L(R)}}$ for the charged leptons and $U^{(\nu)}_{\mathrm{L(R)}}$ for the neutrinos with

$$l'_{L(R)} = U^{(l)}_{\mathrm{L(R)}} l_{\mathrm{L(R)}} \qquad \nu'_{\mathrm{L(R)}} = U^{(\nu)}_{\mathrm{L(R)}} \nu_{\mathrm{L(R)}} \tag{3.7}$$

$$\tag{3.8}$$

The fields ν_{R} denote hypothetical right-handed neutrinos. The mechanism for diagonalization of the derived mass matrices is described in appendix A. There are only a few parts of the standard model Lagrangian which are non-trivially affected by these transformations. Beside the mass Lagrangian this is the charged current reaction coming from the kinetic part of the fermion Lagrangian.

$$\mathcal{L}_{\mathrm{CC}} = \frac{g}{\sqrt{2}} W^\mu \overline{l'_L} \gamma_\mu \nu'_L + h.c. \tag{3.9}$$

$$= \frac{g}{\sqrt{2}} W^\mu \overline{(U^{(l)}_{\mathrm{L}} l_{\mathrm{L}})} \gamma_\mu (U^{(\nu)}_{\mathrm{L}} \nu_{\mathrm{L}}) + h.c. \tag{3.10}$$

$$= \frac{g}{\sqrt{2}} W^\mu (\overline{l_{\mathrm{L}}}) \gamma_\mu \left(\left(U^{(l)}_{\mathrm{L}} \right)^\dagger U^{(\nu)}_{\mathrm{L}} \right) \nu_{\mathrm{L}} + h.c. \tag{3.11}$$

$$U_{\mathrm{PMNS}} \equiv \left(U^{(l)}_{\mathrm{L}} \right)^\dagger U^{(\nu)}_{\mathrm{L}} \tag{3.12}$$

The matrix U_{PMNS} is the lepton mixing matrix which is the leptonic analogue to the CKM matrix in the quark sector. It is called Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. The mixing matrix for N generations is a complex $N \times N$ matrix with N^2

elements and $2N^2$ parameters. By requiring the matrix to be unitary this number gets reduced to N^2 free parameters. The number of physical parameters can be further restricted. Especially as not all of the complex phases in this matrix are physically relevant, they can be absorbed in redefinitions of the fermion fields. The number of complex phases can be calculated by comparison with a real matrix. For the real case there is an orthogonal matrix with $\frac{1}{2}N(N-1)$ parameters. From the difference to the number of free parameters of a unitary matrix we get $\frac{1}{2}N(N+1)$ complex phases. For $2N$ different fields only $2N-1$ phases can be absorbed because the Lagrangian still posses a global $U(1)$ symmetry. Finally there are $\frac{1}{2}(N-1)(N-2)$ complex physical relevant phases called Dirac phases. For the Majorana case, which will be discussed later on, two additional phases have to be considered in a three-dimensional case.

The propagation of the wave function in the mass eigenbasis can be described in a relativistic picture through

$$|\nu_\alpha(t,x)\rangle = U^*_{\alpha j} e^{i(E_j t - p_j x)} |\nu_j\rangle \tag{3.13}$$

The mass and flavour basis do not coincide, hence the probabilities for oscillations between various flavour states are given by the square of the transition amplitude.

$$A_{\alpha\to\beta} = \langle \nu_\beta \,|\, \nu_\alpha(t,x)\rangle \tag{3.14}$$

$$= \sum_{ij} U_{\beta j} U^*_{\alpha i} \left\langle \nu_j \,\middle|\, e^{i(E_i t - p_i x)} \,\middle|\, \nu_i \right\rangle \tag{3.15}$$

$$= \sum_{i} U_{\beta i} U^*_{\alpha i} e^{i(E_i t - p_i x)} \tag{3.16}$$

$$P_{\alpha\beta} = |A_{\alpha\to\beta}|^2 = \sum_{ik} U_{\beta i} U^*_{\beta k} U^*_{\alpha i} U_{\alpha k} e^{-i\phi_{ik}} \tag{3.17}$$

For the calculation of the oscillation amplitude $P_{\alpha\beta}$ the oscillation phase ϕ must be further evaluated. The oscillation phase is given by

$$\phi = (E_2 - E_1)t - (p_2 - p_1)x \tag{3.18}$$

this expression can be simplified by using the relativistic energy mass relation and the following definitions

$$\Delta E \equiv E_2 - E_1,\ \Delta E^2 = E_2^2 - E_1^2,\ \overline{E} = \frac{1}{2}(E_1 + E_2) \tag{3.19}$$

and analogue definitions for m and p

$$\Rightarrow \Delta E^2 = 2\overline{E}\Delta E, \quad \Delta p^2 = 2\overline{p}\Delta p, \quad \Delta m^2 = 2\overline{m}\Delta m \tag{3.20}$$

$$\phi = \Delta E t - \frac{\Delta p^2}{2\overline{p}} x = \Delta E t - \frac{\Delta E^2 - \Delta m^2}{2\overline{p}} x \tag{3.21}$$

$$= \Delta E t - \underbrace{\frac{\overline{E}}{\overline{p}}}_{\equiv v^{-1}} \Delta E x + \frac{\Delta m^2}{2\overline{p}} x \tag{3.22}$$

by assuming that the neutrino has an average velocity v and $x \approx vt$ this results in

$$\approx \frac{\Delta m^2}{2\overline{p}} x \approx \frac{\Delta m^2}{2\overline{E}} x \tag{3.23}$$

$$P_{\alpha\beta} = |A_{\alpha\to\beta}|^2 = \sum_{ik} U_{\beta i} U^*_{\beta k} U^*_{\alpha i} U_{\alpha k} \exp\left(-i \frac{\Delta m^2_{ik} L}{2E_\nu}\right) \tag{3.24}$$

with $\Delta m^2_{ik} = m^2_i - m^2_k$ and L the distance of a specific experiment. In order to illustrate some properties of $P_{\alpha\beta}$ the most simple case of two flavours $n = 2$ is discussed. Then U is a two-dimensional rotation matrix and the transition probability gets the form

$$P_{\alpha\beta} = \sin^2(2\theta_{\alpha\beta}) \sin^2\left(\frac{\Delta m^2_{\alpha\beta} L}{4E_\nu}\right) \quad \text{for } \alpha \neq \beta. \tag{3.25}$$

Now it is apparent that an appropriate value of $\frac{L}{E_\nu}$ is needed to observe the mass splitting. There are three ranges of the oscillation phase ϕ. For very small values of ϕ, which corresponds to $L/E \ll \Delta m^{-2}$, their are no sizeable oscillations and $P_{\alpha\beta} \approx \delta_{\alpha\beta}$. For very large values the oscillation is averaged out. As L and E are experimentally not exactly determined P contains several oscillations. The average $\langle \sin \Delta \rangle = 1/2$ allows the measurement of the amplitude in P. The best range to observe oscillations is $\phi \gtrsim \frac{\pi}{2}$.

The neutrino oscillation probability given by Eq. (3.24) is invariant under diagonal phase transformations of the leptonic mixing matrix and thus not sensitive to Majorana phases. To receive the oscillation probability for anti-neutrinos the replacement $U \to U^*$ has to be performed. The CP violating phase can be measured through the construction of the Jarlskog determinant as an analogue to the CKM matrix.

In the previous section it was shown that the elements of the PMNS matrix are related to physical observables in neutrino oscillation experiments. A principal oscillation for charged leptons cannot be measured as the mass squared differences for charged leptons would be too large. A standard parametrization of the PMNS matrix consists of expressing it by three rotation matrices around an angle θ_{ij}, a complex Dirac phase δ and

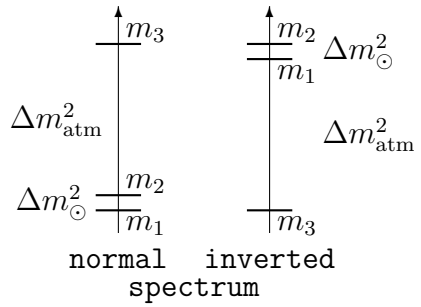

Figure 3.1: The normal and the inverted 3-neutrino mass spectrum, taken from Ref.[8]

two complex Majorana phases α and β.

$$U_{\text{PMNS}} = U_{23}U_{13}U_{12}P$$

with

$$\begin{aligned} U_{\text{PMNS}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & e^{-i\delta}s_{13} \\ 0 & 1 & 0 \\ -e^{i\delta}s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \operatorname{diag}(1, \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta}) \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \operatorname{diag}(1, \mathrm{e}^{i\alpha}, \mathrm{e}^{i\beta}) \end{aligned} \tag{3.26}$$

where c_{ij} and s_{ij} denotes $\cos\theta_{ij}$ and $\sin\theta_{ij}$. The Majorana phases are discussed in Sec. 3.3 in more detail.

The mass spectrum for the up- and down-quarks and for the charged leptons possess a large hierarchy [7]. From neutrino oscillation experiments which measure the mass squared differences it is well known that the neutrino spectrum is non-degenerated. Two different mass splittings are observed. By convention the smaller mass-squared difference is called "solar" mass-squared difference $\Delta m^2_{\odot} \equiv \Delta m^2_{21}$ and $\Delta m^2_{\text{atm}} \equiv \Delta m^2_{31}$ is the larger "atmospheric" mass squared difference. This notation is summarized in Figure 3.1. The current neutrino data allows two different scenarios, namely, the "normal" and the "inverted" ordering depending on the sign of Δm^2_{31}. The sign of Δm^2_{21} is known from coherent forward scattering effects in matter. These predict that the mass state containing most of the ν_e is the lighter one. In both orderings the spectrum could still be quasi-degenerated if $m_1 \approx m_2 \approx m_3$ and $m_i \gg \Delta m^2_{ij}$. The spectrum can be called hierarchical if the lightest mass state is much smaller than the mass squared differences ($m_1^2 \ll \Delta m^2_{\odot}$ or $m_3 \ll \Delta m^2_{\text{atm}}$). A normal hierarchical spectrum gives a relation for the

parameter	best-fit $\pm 1\sigma$	2σ	3σ
Δm^2_{21} $[10^{-5}\text{eV}^2]$	$7.59^{+0.20}_{-0.18}$	7.24–7.99	7.09–8.19
Δm^2_{31} $[10^{-3}\text{eV}^2]$	$2.50^{+0.09}_{-0.16}$	2.25–2.68	2.14–2.76
	$-(\ 2.40^{+0.08}_{-0.09})$	$-(\ 2.23$–$2.58)$	$-(\ 2.13$–$2.67)$
$\sin^2\theta_{12}$	$0.312^{+0.017}_{-0.015}$	0.28–0.35	0.27–0.36
$\sin^2\theta_{23}$	$0.52^{+0.06}_{-0.07}$	0.41–0.61	0.39–0.64
	$0.52^{+0.06}_{-0.06}$	0.42–0.61	0.39–0.64
$\sin^2\theta_{13}$	$0.013^{+0.007}_{-0.005}$	0.004–0.028	0.001–0.035
	$0.016^{+0.008}_{-0.006}$	0.005–0.031	0.001–0.039
δ	$(-\ 0.61^{+0.75}_{-0.65})\pi$	$0\ -2\pi$	$0\ -2\pi$
	$(-\ 0.41^{+0.65}_{-0.70})\pi$	$0\ -2\pi$	$0\ -2\pi$

Table 3.1: Neutrino oscillation parameters from global fits, taken from Ref. [9, 10]. The upper (lower) row corresponds to normal (inverted) neutrino mass hierarchy, with $\Delta m^2_{31} > 0$ ($\Delta m^2_{31} < 0$).

"strength" of the hierarchy

$$\frac{\Delta m^2_{\odot}}{\Delta m^2_{\text{atm}}} \simeq \frac{m_2^2}{m_3^2} \sim 0.03 \tag{3.27}$$

Compared to the hierarchy of the charged fermions this is a quite "weak" hierarchy. Until now the mass splitting Δm^2_{31} was only examined in $\nu_\mu \to \nu_\mu(\nu_\tau)$ oscillations where a matter-effect is not observable. For an inverted spectrum the masses m_1 and m_2 are always nearly degenerate, because $\Delta m_{\text{atm}} \gg \Delta m_{\odot}$.

3.3. Dirac- and Majorananeutrinos

A fermion mass term in a Lagrangian always has the form $-\frac{m}{2}\overline{\psi}\psi$ which leads to the Dirac equation. This can be examined further by looking at the chiral structure. So the chiral projectors are introduced.

$$\mathcal{P}_{L(R)} \equiv (\mathbf{1} \mp \gamma_5)/2 \tag{3.28}$$
$$\mathcal{P}^2_{L(R)} = \mathcal{P}_{L(R)} \quad \text{(indempotence)} \tag{3.29}$$
$$\mathcal{P}_L \mathcal{P}_R = 0 \quad \text{(orthogonality)} \tag{3.30}$$
$$\mathcal{P}_L + \mathcal{P}_R = \mathbf{1} \quad \text{(completeness)} \tag{3.31}$$

These projectors can produce either left-handed ($P_L\psi = \psi_L$) or right-handed ($P_R\psi = \psi_R$) spinors. Inserting the properties of the projection operators in the mass term shows

that only terms in which different chiralities mix lead to a mass.

$$\begin{aligned} -\frac{m}{2}\overline{\psi}\psi &= -\frac{m}{2}\overline{\psi}(\mathcal{P}_L^2+\mathcal{P}_R^2)\psi \\ &= -\frac{m}{2}\psi^\dagger\gamma_0(\mathcal{P}_L^2+\mathcal{P}_R^2)\psi \\ &= -\frac{m}{2}(\overline{\psi}_L\psi_R+\overline{\psi}_R\psi_L) \end{aligned}$$

If ψ_L and ψ_R are independent one has the usual Dirac mass term for the Lagrangian. Notably there is a second possibility for a field to mix left and right chirality, namely, by applying the charge conjugation operator.

$$\hat{\mathcal{C}} : \psi \to \psi^c \equiv C\overline{\psi}^T = C\left(\psi^\dagger\gamma_0\right)^T = C\gamma_0^T\psi^* \tag{3.32}$$

The following algebraic relations are useful for dealing with the charge conjugation.

$$\begin{aligned} C^{-1}\gamma_\mu C &= -\gamma_\mu^T \\ C^T &= -C = C^\dagger = C^{-1} \\ (\psi^c)^c = \psi \qquad \overline{\psi^c} &= \psi^T C, \qquad \overline{\psi_1}A\psi_2 = \overline{\psi_2^c}\left(CA^TC^{-1}\right)\psi_1^c \end{aligned} \tag{3.33}$$

Now it is easy to show that the charge conjugation operation changes a (right-) left-handed particle into a (left-) right-handed antiparticle.

$$(\psi_{L(R)})^c = (\psi^c)_{R(L)} \tag{3.34}$$

For Majorana particles defined as particles which are their own antiparticles ($\Psi^c = \Psi$), the relation above allows to decompose the Majorana field Ψ_M.

$$\Psi_\mathrm{M} = \psi_L + \psi_R = (\psi_R)^c + \psi_R \tag{3.35}$$

Now it is possible to write down a Majorana mass term in the Lagrangian where the field Ψ_R denotes a vector in flavour space.

$$\begin{aligned} \mathcal{L}_\mathrm{Majorana} &= -\frac{1}{2}\left(\overline{\Psi_R^c}M_\mathrm{M}\Psi_R + \overline{\Psi_R}M_\mathrm{M}^\dagger\Psi_R^c\right) \\ &= -\frac{1}{2}\left(\Psi_R^T C M_\mathrm{M}\Psi_R + h.c.\right) \end{aligned} \tag{3.36}$$

Eq. (3.36) reveals some further properties of the Majorana mass matrix M_{M}. By setting $\psi_1 \to \Psi^c_{R,i}$ and $\psi_2 \to \Psi_{R,j}$ with flavour indices i and j in Eq. (3.33)

$$\overline{\Psi^c_{R,i}}\Psi_{R,j} = \overline{\Psi^c_{R,j}}\Psi_{R,i} \tag{3.37}$$

it follows that the mass matrix M_{M} is symmetric. The diagonalization of symmetric matrices is discussed in appendix A.

$$U^T M_{\mathrm{M}} U = \hat{M} \tag{3.38}$$

where $\hat{M}$ is diagonal with nonnegative entries and U is unitary. Matrix U is not unique.

The Majorana mass term obviously does not have a $U(1)$ symmetry like a Dirac mass term. This $U(1)$ symmetry is responsible for the charge conservation. Therefore charged particles cannot have a Majorana mass term. For fermions there only remain the neutral neutrinos as candidates for Majorana particles. Nevertheless, a Majorana mass term for neutrinos would violate the lepton number. The Majorana phases α and β in Eq. (3.26) are physical and cannot be absorbed into a redefinition of fields. A redefinition of the mass eigenstates ν_2 and ν_3 would lead to a shift of masses m_2 and m_3 for example.

3.4. Upper Limits on the Neutrino Mass

The absolute neutrino mass has never been measured. Nonetheless there exist upper bounds from particle physics and cosmology. A famous bound for the absolute neutrino mass comes from β-decays like the decay ${}^3\mathrm{H} \to {}^3\mathrm{He} + e^- + \overline{\nu}_e$. The kinematics of this decay is a general 3-body decay. The endpoint of the β-spectrum can be described by a Kurie plot with a form given in Eq. 3.39. The ν_e state consists of several mass eigenstates. Thus the measured mass m_β is a sum of them $m_\beta = \sum_{i=1}^{3} |U_{ei}|^2 m_i$. The mismatch between the shape for massless neutrinos and massive neutrinos allows the determination of an upper limit. The β-spectrum vanishes for $p_\nu = 0$ at the energy $E_{\max} = E_0 - m_\beta$, where E_0 is the maximal energy for a massless neutrino.The upper limit is $m_\beta < 2.2\,\mathrm{eV}$ at 95%CL. This number is rather small compared to the absolute energy scale the measurement takes place ($E_0 \approx 18\,\mathrm{MeV}$) which is why experiments need enough precision and statistics to reduce this limit. The KATRIN experiment could once reach a sensitivity of about $\sim 0.2\,\mathrm{eV}$.

$$K(E) = \left[\sqrt{(E_0 - E)^2 - m_\beta^2}(E_0 - E)\right]^{\frac{1}{2}} \tag{3.39}$$

3.5. Extensions of the Standard Model

Neutrino masses are zero in the Standard model because of various reasons. First of all there are no right handed neutrinos. Thus there is no Dirac mass term. A Majorana mass term would require either scalars which are no $SU(2)$-doublets or nonrenormalizable

field	$SU(3)_{\mathrm{C}}$	$SU(2)_{\mathrm{L}}$	$U(1)_{\mathrm{Y}}$
$L_{\mathrm{Q}} = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	3	2	1/3
u_R	3	1	4/3
d_R	3	1	-2/3
$L_{\mathrm{L}} = \begin{pmatrix} \nu_L \\ l_L \end{pmatrix}$	1	2	-1
N_R	1	1	0
l_R	1	1	-2
ϕ	1	2	1
$\tilde{\phi}$	1	2	-1

Table 3.2: Quantum numbers of fermions and the Higgs particle

operators. Since neutrino oscillations observe neutrino masses and mixing, there has to be a modification of the standard model which creates the new neutrino properties. Beside nonrenormalizable interactions which do not need a specific model input, there are three main categories for mass models:

1. Extensions in the Higgs sector
2. Extensions in the lepton sector
3. Extensions in both sectors

In the seesaw model the Lagrangian is including the standard model particle content plus an arbitrary number r of right handed neutrinos N_R which are singlets under the standard model gauge group. The corresponding quantum numbers are tabulated in table 3.2. As the Lagrangian will contain a Majorana part one needs to account for violation of Lepton numbers as well. It is possible to write down a Majorana mass term for the right-handed neutrinos because they transform as singlets of the $SU(2) \times U(1)$ group in contrast to the left-handed neutrinos. A Majorana mass term for the left-handed neutrinos would require a triplet Higgs for example. In the standard model plus N_R the Higgs mechanism generates the neutrino Dirac masses.

$$\mathcal{L}_{\mathrm{Dirac}} = -\overline{N}_R M_D \nu_L + h.c. \tag{3.40}$$

where r was set to 3 so the fields N_R and ν_L denote three-dimensional vectors in flavour space and M_D is a generally complex 3×3 matrix. The right-handed neutrinos form a Majorana mass term as given in Eq. (3.36)

$$\mathcal{L}_{\mathrm{Majorana}} = -\frac{1}{2}\left(\Psi_R^T C M_{\mathrm{R}} \Psi_R + h.c.\right) \tag{3.41}$$

with the symmetric Majorana mass matrix M_R. The two mass matrices for the neutrinos can be combined into a 6×6 matrix M_{D+M}

$$\mathcal{L}_\nu = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Majorana}} \tag{3.42}$$

$$= \frac{1}{2}\overline{n^c} M_{D+M} n + h.c. \tag{3.43}$$

$$M_{D+M} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \text{ and } n = \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} \tag{3.44}$$

To arrive at this form of the neutrino mass Lagrangian, the Dirac part of the neutrinos in Eq. (3.40) has to be rewritten in terms of N_R^c using the transformation properties under charge conjugation as given in Eq. (3.33).

$$-\overline{N}_R M_D \nu_L = -\overline{\nu_L^c} M_D^T N_R^c \tag{3.45}$$

$$\mathcal{L}_{\text{Dirac}} = -\frac{1}{2}\left(\overline{\nu_L^c} M_D^T N_R^c + \overline{N}_R M_D \nu_L\right) \tag{3.46}$$

The 6×6 matrix M_{D+M} is obviously symmetric and can be brought into block diagonal form by applying a basis change for the left-handed field n and for the right-handed field n^c. This is achieved by a transformation matrix U with

$$U^T M U \approx \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \tag{3.47}$$

$$U \equiv \begin{pmatrix} 1 & \rho \\ -\rho^\dagger & 1 \end{pmatrix} \tag{3.48}$$

Matrices m_1, m_2 are symmetric and $\rho = \left(M_R^{-1} M_D\right)^\dagger$. The mass scales m_1 and m_2 are related to M_D and M_R

$$m_1 = -2 M_D^T M_R^{-1} M_D \tag{3.49}$$

$$m_2 = M_R \tag{3.50}$$

assuming that the elements of M_R are much larger than the elements of M_D this model leads to small masses m_1 which is approximately the mass scale of the left handed neutrinos ν_L because the basis change has only a small impact on the original state

$$n = \begin{pmatrix} \nu_L \\ N_R^c \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{3.51}$$

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \nu_L + \rho^\dagger N_R^c \\ N_R^c - \rho \nu_L \end{pmatrix} = n + \mathcal{O}(\rho) \tag{3.52}$$

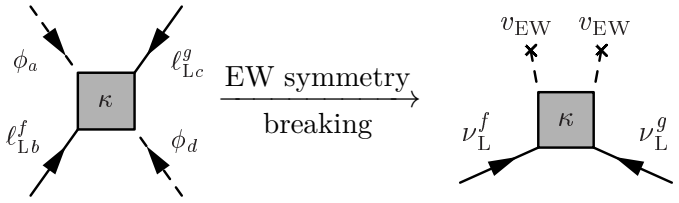

Figure 3.2: Vertex from the dimension 5 operator which yields a Majorana mass matrix for the light neutrinos, figure taken from Ref. [11].

The seesaw formula for the light neutrino mass in Eq. (3.49) allows to estimate the energy scale of the right-handed neutrinos. A rough estimation for the coupling strength of the Dirac terms gives a scale which would be above experimental reach.

$$M_D \sim m_\mu,\, v \quad m_1 \sim m_\beta,\, \sqrt{\Delta m^2_{\mathrm{atm}}} \tag{3.53}$$

$$\Rightarrow M_R \sim (10^6 - 10^{15})\,\mathrm{GeV} \tag{3.54}$$

For a fixed M_D, heavier right-handed neutrinos make the left-handed neutrinos lighter.

If one does not want to enlarge the field content of the standard model because their are no new particles at accessible energies, their still remains the possibility to explain neutrino oscillations by introducing non-renormalizable interactions. A prototype of a non-renormalizable interaction is the Fermi theory of weak interactions. In this theory the effect of virtual W and Z exchange is described by an effective theory containing four fermion interactions. The lowest dimension non-renormalizable operators in the standard model are of dimension 5. It can be shown that there is a unique dimension 5 operator consistent with the standard model particle content, gauge invariance and Lorentz invariance. There are seven dimension 5 expressions which can be constructed by scalars ϕ with mass dimension 1, fermions ψ with mass dimension $\frac{3}{2}$ and bosons expressed as covariant derivatives D with mass dimension 1.

$$\phi^5,\ D^1\phi^4,\ D^2\phi^3,\ D^3\phi^2,\ \overline{\psi}D^2\psi,\ \phi D\overline{\psi}\psi,\ \phi^2\overline{\psi}\psi \tag{3.55}$$

Only the last expression $\phi^2\overline{\psi}\psi$ is consistent with these requirements. The exact form of the fields has to be

$$\mathcal{L}_{\mathrm{dim5}} = \kappa\overline{L^c}L\phi^T\phi \xrightarrow{\mathrm{SSB}} \kappa v^2\overline{L^c}L \tag{3.56}$$

Surprisingly, this term describes a lepton number violating Majorana mass term. The dimension 5 vertex is displayed in Fig. 3.2.

4. Flavoursymmetry Models

Depending on whether neutrinos are Dirac or Majorana particles, the standard model consists of 26 or 28 parameters. The fermion sector of the standard model contains the majority of them. There are 9 masses for the quarks and charged leptons and 4 CKM mixing parameters. The fact that neutrinos are massiv enlarges this with the 4 or 6 PMNS parameters and the neutrino masses. Flavour symmetries give restrictions on the Yukawa couplings and hence might predict some parameters. In the quark sector the CKM matrix is close to the unit matrix which does not allow a huge playground for flavour symmetry models based on finite groups. But the up and down quark mass spectra are known to be hierarchical which suggests to relate the mixing angles to quark mass ratios. The Cabbibo angle for example is consistent with the following relation

$$\sin\theta_c \simeq \sqrt{\frac{m_d}{m_s}} \tag{4.1}$$

In the lepton sector the mixing matrix is completely different. A hierarchy as for charged fermions cannot be observed. Therefore it seems less effective to start building a flavour symmetry out of mass ratios. Instead there are several possibilities to identify the elements of the PMNS matrix with "pure numbers" which seem to be connected to product coefficients of an underlying flavour symmetry group

$$U_{\text{PMNS}} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) & \epsilon \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \qquad U_{\text{CKM}} = \begin{pmatrix} 1 & \epsilon & {}_{\epsilon} \\ \epsilon & 1 & \epsilon \\ {}_{\epsilon} & \epsilon & 1 \end{pmatrix} \tag{4.2}$$

4.1. Tri-Bimaximal Lepton Mixing

The central assumptions of the Tri-Bimaximal mixing scheme [12] are $U_{e3} = 0$, $|U_{\mu 3}|^2 = \frac{1}{2}$ and $|U_{e2}|^2 = \frac{1}{3}$ which is close to the experimental values in Table 3.1. Using these conditions for the mixing angles, together with the unitary condition

$$\sum_i U_{\alpha i} U^*_{\beta i} = \delta_{\alpha\beta}, \sum_\alpha U_{\alpha i} U^*_{\alpha j} = \delta_{ij} \ , \tag{4.3}$$

the mixing matrix takes the Harrison-Perkins-Scott form

$$U_{\text{HPS}} \equiv \left(|U_{li}|^2\right) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix} \tag{4.4}$$

This means that the μ- and τ-flavour contribute maximal in the ν_3 neutrino mass eigenstate and all three flavours contribute equally in the ν_2 neutrino mass eigenstate. This

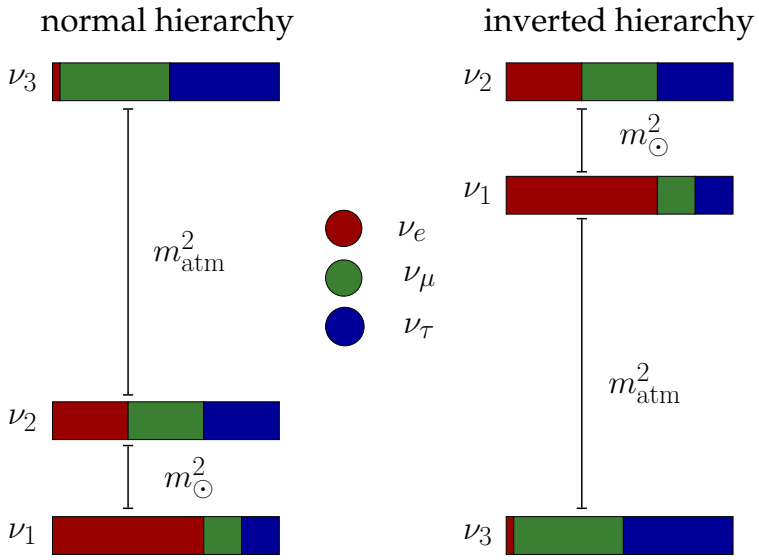

Figure 4.1: Normal and inverted 3-neutrino mass hierarchy. The coloured fractions of each mass state correspond schematically to different $|U_{li}|^2$ contributions.

feature is illustrated in Fig. 4.1. Thus the mixing pattern is sometimes called Tri-Bi-Maximal mixing. Since the mass matrix $\mathcal{M}_\nu$ is directly related to the mixing matrix

$$\mathcal{M}_\nu = U \text{diag}(m_1, m_2, m_3) U^T \tag{4.5}$$

the form of $\mathcal{M}_\nu$ can be calculated if the three assumptions of the TBM mixing are translated into specific values for the mixing parameters. One choice is $\theta_{13} = 0$, $\theta_{12} = \arcsin\sqrt{\frac{1}{3}}$ and $\theta_{23} = -\frac{\pi}{4}$ which corresponds to a mixing matrix U_{TBM} and to a neutrino mass matrix $\mathcal{M}_\nu^{\text{TBM}}$ of the form

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{4.6}$$

$$\mathcal{M}_\nu^{\text{TBM}} \equiv \begin{pmatrix} A & B & B \\ \cdots & \frac{1}{2}(A+B+D) & \frac{1}{2}(A+B-D) \\ \cdots & \cdots & \frac{1}{2}(A+B+D) \end{pmatrix} \tag{4.7}$$

The coefficients in the TBM case are given by

$$A = \frac{1}{3}(2m_1 + m_2 e^{i2\alpha}), \quad B = \frac{1}{3}(-m_1 + m_2 e^{i2\alpha}) \quad D = m_3 e^{i2\beta} \tag{4.8}$$

The physical non-distinguishable choice of the sign of the mixing angles has quite a large impact on the form of the mass matrix. For $\theta_{23} = \frac{\pi}{4}$ the corresponding mass matrix would be completely different with an opposite sign of the 12- and 13-element.

$$\mathcal{M}_\nu = \begin{pmatrix} \tilde{A} & \tilde{B} & -\tilde{B} \\ \cdots & \frac{1}{2}(\tilde{A}+\tilde{B}+\tilde{D}) & \frac{1}{2}(-\tilde{A}-\tilde{B}+\tilde{D}) \\ \cdots & \cdots & \frac{1}{2}(\tilde{A}+\tilde{B}+\tilde{D}) \end{pmatrix} \tag{4.9}$$

The matrix $\mathcal{M}_\nu^{\mathrm{TBM}}$ instead has a $\mu-\tau$ symmetry. This means that it is invariant under $A_{\mu\tau}$ with

$$A_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.10}$$

$$\mathcal{M}_\nu^{\mathrm{TBM}} = A_{\mu\tau}\mathcal{M}_\nu^{\mathrm{TBM}}A_{\mu\tau} \tag{4.11}$$

This invariance can be used together with the invariance under S_{TBM} to completely define the form of $\mathcal{M}_\nu^{\mathrm{TBM}}$ as presented in Eq. (4.7), where S_{TBM} is

$$S_{\mathrm{TBM}} = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \tag{4.12}$$

$$\mathcal{M}_\nu^{\mathrm{TBM}} = S_{\mathrm{TBM}}\mathcal{M}_\nu^{\mathrm{TBM}}S_{\mathrm{TBM}} \tag{4.13}$$

with $[A_{\mu\tau}, S_{\mathrm{TBM}}] = 0$.

A component of the above discussion is that it is worked out in a basis where the charged lepton mass matrix M_l is diagonal. It is unknown whether M_l, which is defined by the Dirac-Lagrangian for charged leptons, is symmetric like M_ν. However, the hermetian product $M_l^2 = M_l^\dagger M_l$ is diagonalizable by a unitary transformation $U_L^{(l)}$ as described in Appendix A. The matrix $U_L^{(l)}$ is the transformation matrix for the left-handed charged leptons. The most general diagonal matrix M_l^2 is invariant under a diagonal phase matrix T.

$$M_l^2 = T^\dagger M_l^2 T \tag{4.14}$$

If the matrix T fullfills $T^n = 1$ it generates the cyclic group C_n. For $n = 3$ and $\omega = \exp(2\pi i/3)$ a choice of T is given by

$$T_{\mathrm{TBM}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{4.15}$$

4.2. A_4-Model

In principle, a flavour symmetry group G_f can be continuous or discrete. However, a specific mixing pattern is supposed to arise from a spontaneous symmetry breaking of the flavour group. Thus, there could be unwanted Goldstone bosons in a continuous group. Next, the symmetry group should be as simple as possible in order to make testable predictions. This means, for example, that the flavour symmetry group is not gauged and affects only the mass Lagrangian and the scalar potential. Generally G_f enlarges the scalar sector. The symmetries which are not contained in G_f directly, are generated by an alignment of the vacuum expectation values of the scalars. Of course a specific mixing scenario like the HPS mixing still allows several different mixing matrices. Nevertheless the TBM matrix has the advantage that it can be described in terms of three symmetries. These symmetries are a starting point to find an appropriate flavour symmetry group. If particles are identified with group representations the transformation properties show up in the mass terms.

A flavour symmetry model which describes the HPS mixing should satisfy the three conditions for the mixing angles θ_{ij} and additionally describe the ratio for the solar and atmospheric mass squared differences to be compatible with neutrino observables as presented in Table 3.1.

1. $\theta_{13} = 0$
2. $\sin^2\theta_{23} = \frac{1}{2}$
3. $\sin^2\theta_{12} = \frac{1}{3}$
4. $r = \frac{\Delta m^2_{\odot}}{\Delta m^2_{\text{atm}}} \sim 0.03$

The three mixing angles are automatically obtained if the mass matrix is invariant under S_{TBM} and $A_{\mu\tau}$. The considered symmetry group is $G_{\text{SM}} \otimes G_f \otimes X$ with the standard model gauge group augmented by a flavour symmetry group G_f and an additional auxiliary group X which will be motivated later on. To reproduce $\mathcal{M}_\nu^{\text{TBM}}$ in a flavour model, which does not contain $A_{\mu\tau}$, the flavour group G_f has to be broken down to G_S in the neutrino sector and to G_T in the charged lepton sector.

$$G_f \xrightarrow[\nu\text{-sector}]{\text{SSB}} G_S \cong C_2 \tag{4.16}$$

$$G_f \xrightarrow[l\text{-sector}]{\text{SSB}} G_T \cong C_3 \tag{4.17}$$

The whole flavour group G_f is completely broken.

Group A_4 offers good possibilities to build a flavour symmetry model on it. Before writing down invariants in the Lagrangian, a suitable assignment of transformation properties for the fields has to be performed. The group A_4 has three one-dimensional irreducible representations and one three-dimensional irreducible representation which

have to be allocated to three known flavours. In addition A_4 is the smallest group with a three-dimensional representation. The matrices S_{TBM} and T_{TBM} are contained in A_4, whereas the μ-τ symmetry is explained by a special vacuum alignment. The left-handed lepton doublets form a triplet (L_e, L_μ, L_τ) under A_4. The right handed charged leptons transform according to the three singlets. In order to achieve an independent breaking pattern for the neutrino- and the charged lepton sector at least two A_4 scalar fields transforming as triplets are required. The observed "strong" mass hierarchy for the charged lepton masses can be realised by an additional Frogatt-Nielson $U(1)_{\mathrm{FN}}$ flavour symmetry. The charged leptons (e_R, μ_R, τ_R) get FN charges $2q$, q and 0 and a scalar field θ gets FN-charge -1. This produces a dependency between the vacuum expectation value of the scalar field and the Yukawa coupling matrices for the charged leptons $y_l(\theta)$. Λ' is the cutoff scale for the $U(1)_{\mathrm{FN}}$ symmetry.

$$\frac{\langle\theta\rangle}{\Lambda'} \equiv \lambda < 1 \xrightarrow{q=2} \quad m_e : m_\mu : m_\tau = \lambda^4 : \lambda^2 : 1 \tag{4.18}$$

For $q = 2$ the value $\lambda \sim 0.25$ is needed to explain the observed charged lepton mass hierarchy. For simplicity $\Lambda' = \Lambda$ is assumed with Λ describing the cut-off scale of the A_4-theory. The A_4 scalar fields φ_T, φ_S and ξ transform as singlets of the standard model gauge group. Therefore, only two Higgs doublets $h_{u,d}$, which are A_4 singlets, are needed. It is assumed that their vacuum expectation value is much smaller than Λ. The transformation properties of all fields are given in Table 4.1. The A_4 invariant Yukawa interactions for leptons expressed in an expansion in powers of $\frac{1}{\Lambda}$ are

$$\begin{aligned}\mathcal{L}_{\mathrm{lepton}} =& \mathcal{L}_{\mathrm{l}} + \mathcal{L}_\nu \\ =& \frac{y_e e_R \left(\varphi_T L\right)_{\mathbf{1}} h_d}{\Lambda} + \frac{y_\mu \mu_R \left(\varphi_T L\right)_{\mathbf{1}'} h_d}{\Lambda} + \frac{y_\tau \tau_R \left(\varphi_T L\right)_{\mathbf{1}''} h_d}{\Lambda} \\ &+ \frac{x_a \xi \left(L h_u L h_u\right)_{\mathbf{1}}}{\Lambda^2} + \frac{x_b \left(\varphi_S L h_u L h_u\right)_{\mathbf{1}}}{\Lambda^2} + h.c. + \ldots\end{aligned} \tag{4.19}$$

The dots stand for higher dimensional terms. The notation of Ref. [2] is partially used where the product of two triplets $(\mathbf{33})_{\mathbf{1}}$ transforms as $\mathbf{1}$, $(\mathbf{33})_{\mathbf{1}'}$ transforms as $\mathbf{1}'$ and $(\mathbf{33})_{\mathbf{1}''}$ transforms as $\mathbf{1}''$. The forbidden interchange of φ_T and φ_S will be explained by the auxiliary group X later on. There exist models with extra dimensions explaining this exclusion rule without auxiliary symmetries too. The aim of the model is to realise a μ-τ symmetric mass matrix. Thus it seems reasonable to use only one singlet ξ transforming as $\mathbf{1}$. This is the singlet with a 2-3 exchange symmetry (see Eq. (2.35)). The non-appearance of a term $(LL)_{\mathbf{1}}$ must also be explained by an additional symmetry. Here the discrete group C_3 is used as an appropriate auxiliary group. The transformation properties of the fields under C_3 are listed in Tab. 4.1 as well.

The A_4 lepton Lagrangian in Eq. (4.19) is evaluated using the product rules for the representations (2.35) to (2.38). The cut-off scale of the theory (Λ) and the Higgs

group	L	e_R	μ_R	τ_R	N_R^c	$h_{u,d}$	θ	φ_T	φ_S	ξ	ξ'	ξ''
A_4	**3**	**1**	**1″**	**1′**	**3**	**1**	**1**	**3**	**3**	**1**	**1′**	**1″**
C_3	ω	ω^2	ω^2	ω^2	$\cdots$	1	1	1	ω^2	ω	$\cdots$	$\cdots$
$U(1)_{\mathrm{FN}}$	0	4	2	0	0	0	-1	0	0	0	0	0
$SU(2)_{\mathrm{L}}$	2	1	1	1	1	2	1	1	1	1	1	1

Table 4.1: Transformation properties of the fields. If additional (gray) fields are added to the model an appropriate assignment of C_3 transformations has to be done for all fields

doublets $h_{u,d}$ are not written out explicitly for notational simplicity.

$$\begin{aligned}\mathcal{L}_{\mathrm{l}} = & \, y_e e_R \left(\varphi_{T_1} L_e + \varphi_{T_2} L_\tau + \varphi_{T_3} L_\mu\right) \\ & + y_\mu \mu_R \left(\varphi_{T_3} L_\tau + \varphi_{T_1} L_\mu + \varphi_{T_2} L_e\right) \\ & + y_\tau \tau_R \left(\varphi_{T_2} L_\mu + \varphi_{T_1} L_\tau + \varphi_{T_3} L_e\right)\end{aligned} \tag{4.20}$$

To realize the desired breakdown to G_T, the vacuum expectation values for φ_{T_2} and φ_{T_3} have to vanish. Looking at the product rules (2.35) to (2.38) a diagonal lepton mass matrix could be equally realized by different assignments of the singlets to the charged leptons in combination with exclusively non-vanishing $\langle\varphi_{T_2}\rangle$ or $\langle\varphi_{T_3}\rangle$ and appropriate changes in the Lagrangian. In the neutrino sector the products are explicitly given by

$$\begin{aligned}\mathcal{L}_\nu = & \, x_a \xi \left(L_e L_e + L_\mu L_\tau + L_\tau L_\mu\right) \\ & + \frac{x_b}{3}[\varphi_{S_1}(2L_e L_e - L_\mu L_\tau - L_\tau L_\mu) \\ & + \varphi_{S_2}(2L_\mu L_\mu - L_e L_\tau - L_\tau L_e) \\ & + \varphi_{S_3}(2L_\tau L_\tau - L_e L_\mu - L_\mu L_e)] \, .\end{aligned} \tag{4.21}$$

This requires the alignment $\varphi_S \equiv \varphi_{S_1} = \varphi_{S_2} = \varphi_{S_3}$ in order to break A_4 down to G_S. Now the vacuum alignment for all the scalar fields after symmetry breaking is given by

$$\begin{aligned}\langle h_{u,d}\rangle = v_{u,d} \ll \Lambda, \quad \langle\xi\rangle = u, \quad \frac{\langle\theta\rangle}{\Lambda} = \lambda \\ \langle(\varphi_{S_1}, \varphi_{S_2}, \varphi_{S_3})\rangle = (v_S, v_S, v_S), \quad \langle(\varphi_{T_1}, \varphi_{T_2}, \varphi_{T_3})\rangle = (v_T, 0, 0)\end{aligned} \tag{4.22}$$

It is assumed that the scalar potential $V(\varphi_T, \varphi_S, \xi, h_{u,d})$ is of the form which produces the above vacuum alignment. The task of the correct vacuum alignment turns out to be difficult to realize just by implying the A_4-symmetry on the scalar potential. The minimization of two triplets φ_S and φ_T results in six equations which should be satisfied by the two unknowns v_T and v_S. It turns out that this is not possible unless the scalar

potential is further restricted [1].

The vacuum alignment in Eq. (4.22) leads to the diagonal mass matrix for charged leptons

$$M_l = v_d \frac{v_T}{\Lambda} \mathrm{diag}(y_e, y_\mu, y_\tau) \tag{4.23}$$

and charged lepton masses

$$m_e = y_e(\lambda) v_d \frac{v_T}{\Lambda}, \quad m_\mu = y_\mu(\lambda) v_d \frac{v_T}{\Lambda}, \quad m_\tau = y_\tau(\lambda) v_d \frac{v_T}{\Lambda} \tag{4.24}$$

the neutrino mass matrix looks like

$$M_\nu = m_0 \begin{pmatrix} a + 2b/3 & -b/3 & -b/3 \\ -b/3 & 2b/3 & a - b/3 \\ -b/3 & a - b/3 & 2b/3 \end{pmatrix} \tag{4.25}$$

with the substitutions

$$m_0 = \frac{v_u^2}{\Lambda}, \quad b = x_b \frac{v_S}{\Lambda}, \quad a = x_a \frac{u}{\Lambda} \tag{4.26}$$

Indeed, this matrix has the form of $\mathcal{M}_\nu^{\mathrm{TBM}}$ in Eq. (4.7) with

$$A = a + \frac{2}{3}b, \quad B = -\frac{2}{3}b, \quad D = -a + b \tag{4.27}$$

This implies that this matrix is diagonalized by U_{TBM}

$$U_{\mathrm{TBM}}^T M_\nu U = m_0 \mathrm{diag}(a + b, a, -a + b) \tag{4.28}$$

with the resulting neutrino masses

$$m_1 = a + b, \quad m_2 = a, \quad m_3 = -a + b \tag{4.29}$$

The derived observables are just one possible example. The A_4 model allows many modifications leading to different results for the observables. Most of these modifications do not have a differing theoretical motivation and correspond only to different assignments of transformation properties or, for example, the adding of further scalar singlets. In contrast to the mixing angles the predictions for the observables involving neutrino masses are rather lax. The variables a and b in Eq. (4.26) are supposed to have the same order of magnitude. Hence, they can be chosen in order to create a small ratio

r of the solar and atmospheric mass squared difference.

$$r = \left| \frac{1}{2} + \frac{|b|}{4|a|\cos\Delta} \right|, \quad \Delta m^2_{\odot} = -(|b|^2 + 2|a||b|\cos\Delta), \tag{4.30}$$

$$\Delta m^2_{\text{atm}} = |m_3|^2 - |m_1|^2 = -4|a||b|\cos\Delta \tag{4.31}$$

with Δ being the phase difference between a and b. The A_4 flavour symmetry does in general not exclude a specific neutrino ordering. In order to produce a realistic value of r, $\cos\Delta$ should be chosen negative in this effective model. This implies a normal ordering. By expressing a and b in terms of r and Δm_{atm} the neutrino masses can be further investigated [1].

$$|a|\sqrt{2}m_0 = \frac{-\sqrt{\Delta m^2_{\text{atm}}}}{2\cos\Delta\sqrt{1-2r}} \tag{4.32}$$

$$|b|\sqrt{2}m_0 = \sqrt{1-2r}\sqrt{\Delta m^2_{\text{atm}}} \tag{4.33}$$

$$|m_1|^2 = \left[-r + \left(8(1-2r)\cos^2\Delta\right)^{-1}\right]\Delta m_{\text{atm}} \tag{4.34}$$

$$|m_2|^2 = \left(8(1-2r)\cos^2\Delta\right)^{-1}\Delta m_{\text{atm}} \tag{4.35}$$

$$|m_3|^2 = \left[1 - r + \left(8(1-2r)\cos^2\Delta\right)^{-1}\right]\Delta m_{\text{atm}} \tag{4.36}$$

In general the A_4-model makes no prediction of the neutrino mass spectrum. Depending on the choice of $\cos\Delta$ a hierarchical ($\cos\Delta \sim 1$) or a degenerated (small $\cos\Delta$) neutrino spectrum could be achieved. The scale Λ can be estimated [1] using Eq. (4.26) and Eq. (4.33) to be between

$$7.2 \times 10^{12}\,\text{GeV} < \Lambda < 1.8 \times 10^{15}\,\text{GeV} \tag{4.37}$$

A rather small value of Λ would suggest that higher dimensional operators have to be taken into account to derive the correct form of the mass matrix. Nevertheless, even on this scale ($\sim \mathcal{O}(10^{15}\,\text{GeV})$) higher dimensional operators lead to corrections of the mixing angles from their tri-bimaximal values. The next to leading order corrections (NLO) affect all three mixing angles with the same strength.

5. Analysis of Asymmetry Parameters

The motivation of the A_4 flavour symmetry model presented in the privious section was based on the assumption that TBM-mixing is a correct first order description. The TBM-mixing scheme is linked to a special form of the neutrino mass matrix which gives rise to symmetries. These symmetries should be provided by the flavour symmetry group. Even though deviations arise in any realistic flavour symmetry model due to higher

order corrections or renormalization group running, there the question whether TBM-mixing is a good starting point remains. Especially the value of θ_{13} listed in Tab. 3.1 is not compatible with zero in its 3σ-range. The task of this section is to determine how strongly the experimentally allowed deviations affect the form of the mass matrix in Eq. (4.7). All mixing parameters are varied in their 3σ-range as given in Table 3.1. The smallness of the ratio $r = \Delta m^2_{\odot}/\Delta m^2_{\text{atm}} \simeq 0.026 \ldots 0.038$ is used as well. In order to see how well TBM-mixing can be satisfied the value of θ_{23} is chosen to be in interval $[-\frac{\pi}{2}, 0]$.

$$\mathcal{M}_\nu^{\text{TBM}} = \begin{pmatrix} A & B & B \\ \cdots & \frac{1}{2}(A+B+D) & \frac{1}{2}(A+B-D) \\ \cdots & \cdots & \frac{1}{2}(A+B+D) \end{pmatrix} \tag{5.1}$$

$$m_{\alpha\beta} \equiv \left(\mathcal{M}_\nu^{\text{TBM}}\right)_{\alpha\beta} = \sum_i U_{\alpha i} U_{\beta i} m_i \tag{5.2}$$

the form of $\mathcal{M}_\nu^{\text{TBM}}$ can be obtained by imposing three conditions on the elements m_{ij}

$$m_{12} = m_{13}, \quad m_{22} = m_{33}, \quad m_{11} + m_{13} = m_{22} + m_{23} \tag{5.3}$$

These are used to define three asymmetry parameters.

$$X_1 \equiv \left|\frac{m_{12} - m_{13}}{m_{12} + m_{13}}\right| \quad X_2 \equiv \left|\frac{m_{22} - m_{33}}{m_{22} + m_{33}}\right| \quad X_3 \equiv \left|\frac{m_{11} + m_{13} - m_{22} - m_{23}}{m_{11} + m_{13} + m_{22} + m_{23}}\right| \tag{5.4}$$

In order to investigate the resulting asymmetry values, the following definitions for the deviations from TBM-mixing for the angles and from zero for the parameter r are used.

$$\epsilon_{12} \equiv \theta_{12} + \frac{\pi}{4} \quad \text{with} \quad |\epsilon_{12}| < 0.07, \tag{5.5}$$

$$\epsilon_{23} \equiv \theta_{23} - \arcsin\left(\frac{1}{\sqrt{3}}\right) \quad \text{with} \quad |\epsilon_{23}| < 0.14 \tag{5.6}$$

$$\epsilon_{13} \equiv \theta_{13} \quad \text{with} \quad 0.03 < \epsilon_{13} < 0.2, \quad \epsilon_r \equiv \sqrt{|r|} \quad \text{with} \quad 0.160 < \epsilon_r < 0.196, \tag{5.7}$$

$$\epsilon \equiv \max\left(\epsilon_{12}, \epsilon_{13}, \epsilon_{23}, \epsilon_r\right) \tag{5.8}$$

The symmetric neutrino mass matrix is expressed by the 9 free parameters in the PMNS mixing matrix in Eq. (3.26) and the diagonalization equation (4.5). A computer program is generating random numbers in the 3σ-range of the three mixing angles and the Dirac phase as listed in Tab. 3.1. The Majorana phases take values from 0 to 2π. The asymmetry parameters are plotted as functions of the smallest neutrino mass according to the respective mass spectrum. The other two masses are generated as random numbers respecting the 3σ-range of Δm^2_{atm} and $\Delta m^2_{\odot}$. If the minimal value is not plotted or given in the text it is supposedly smaller than $\mathcal{O}(\epsilon^2_{r,13})$. In the same way non-plotted maximal

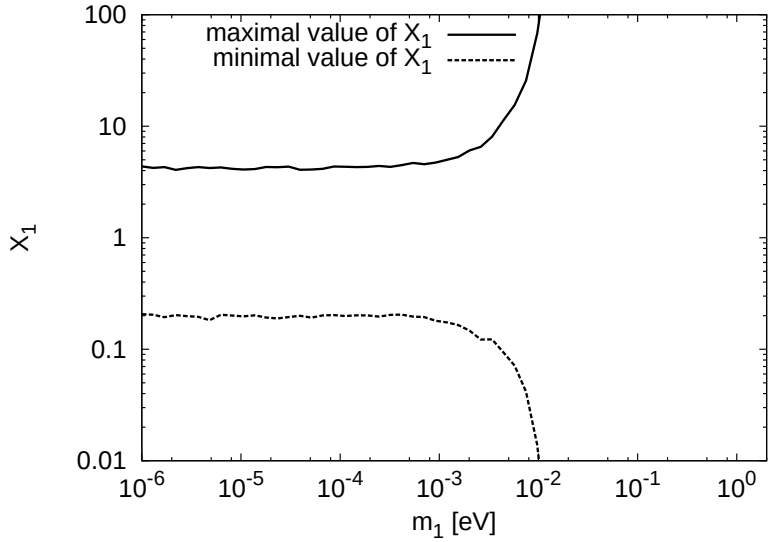

Figure 5.1: Minimal and maximal X_1 for fixed mass m_1 and a normal spectrum.

values correspond to large or possibly even infinite asymmetry parameters.

5.1. Normal Spectrum

The first asymmetry parameter X_1 has a stable range for small neutrino mass m_1 with a minimal value $X_1 \approx 0.2$ and a maximal value $X_1 \approx 4$ as displayed in Fig. 5.1. For $m_1 > 0.002$ the value X_1 becomes unstable. In the stable range no mass dependency on m_1 is observable. Hence, in order to investigate the θ_{13} dependence m_1 is set to zero and the mixing angles are evaluated by a Taylor expansion around their TBM values.

$$X_1 \approx \left| \theta_{13} \left(\frac{e^{i\delta}}{\sqrt{2}} - \frac{3e^{-2i(\alpha-\beta+\delta/2)}}{\sqrt{2}\epsilon_r} \right) + c_1\epsilon_{12} + c_2\epsilon_{23} + \mathcal{O}(\epsilon^2) \right| \tag{5.9}$$

The coefficients c_1 and c_2 are $\mathcal{O}(1)$. For a minimal asymmetry parameter X the deviations of θ_{12} and θ_{23} are set to zero. The term in parentheses involving the phases obviously cannot become zero. Therefore, the expression gets a minimal $X_1 > \mathcal{O}(\epsilon^2)$ which is achieved by small θ_{13} and large ϵ_r

$$X_1^{\min} \approx \left| \theta_{13} \frac{3e^{-2i(\alpha-\beta+\delta/2)}}{\sqrt{2}\epsilon_r} + \mathcal{O}(\epsilon) \right| \approx 10.8 \times \theta_{13} + \mathcal{O}(\epsilon) \approx 0.3 + \mathcal{O}(\epsilon) \tag{5.10}$$

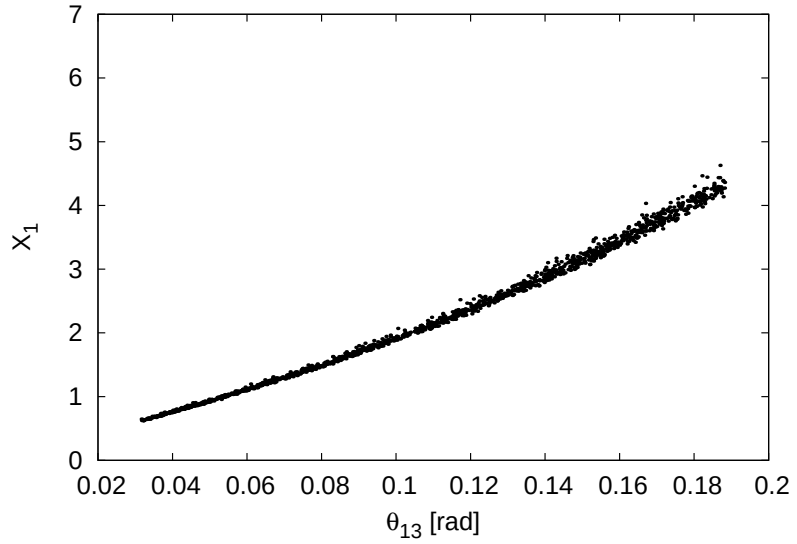

Figure 5.2: θ_{13}-dependence of the maximal X_1 for a normal spectrum plotted in the mass range $10^{-6}\,\mathrm{eV} \leq m_1 \leq 0.5 \times 10^{-3}\,\mathrm{eV}$

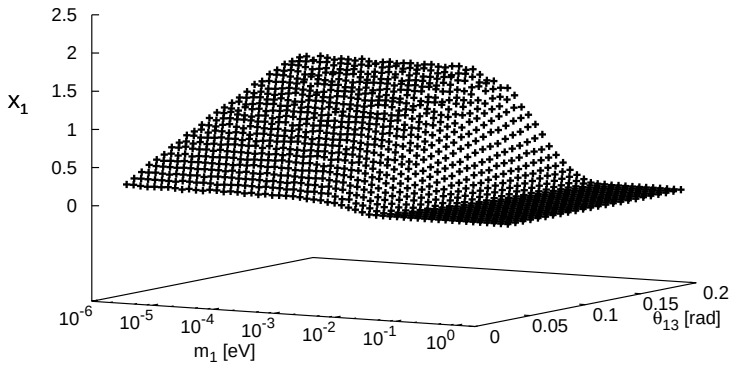

Figure 5.3: θ_{13}-dependence of the minimal X_1 for a normal spectrum

This reproduces the numerical result quite accurately. The prefactor (~ 10.8) of θ_{13} describes the linear dependence in Fig. 5.3 well. The largest possible X_1 is approximately given by large θ_{13}, small ϵ_r and phases that produce opposite signs for the two exponentials

$$X_1^{\max} \approx 2.8 + \mathcal{O}(1) \tag{5.11}$$

Now there are some extra terms coming from the non-TBM structure of θ_{12} and θ_{23} which are of the order of one. Hence, the analytic expression is compatible with the numerically calculated value $X_1^{\max} \approx 4$ in Figure 5.1. The maximal asymmetry X_1 is equally depending linearly on θ_{13} as shown in Fig. 5.2. The stability of $X_1^{\max}$ in the θ_{13}-m_1 space for small m_1 can be further analysed. A maximal asymmetry of X_1 would correspond to a vanishing denominator of X_1 in Eq. (5.4).

$$m_{12} + m_{13} = 0 \tag{5.12}$$

$$\sum_i U_{1i} \left(U_{2i} + U_{3i}\right) m_i = 0 \tag{5.13}$$

setting m_1 to zero results in

$$\frac{m_2}{m_3} = -\frac{U_{23} + U_{33}}{U_{12}\left(U_{22} + U_{32}\right)} U_{13} \tag{5.14}$$

with $|U_{13}|^2 = \sin^2\theta_{13}$ this can be translated into a relation for θ_{13}.

$$\theta_{13} = c\left(U_{12}, U_{22}, U_{23}, U_{32}, U_{33}\right) \frac{m_2}{m_3} \tag{5.15}$$

$$c \simeq \frac{e^{i(2\alpha-\beta)} c_{12} s_{12} (c_{23} - s_{23})}{c_{23} + s_{23}} \tag{5.16}$$

$$|c| \simeq \frac{c_{12} s_{12} (c_{23} - s_{23})}{c_{23} + s_{23}} \tag{5.17}$$

Parameter c is required to be small enough in order to be compatible with the allowed values for θ_{13} and r. The coefficient $|c|$ gets small for maximal ϵ_{23} while $\epsilon_{23} \to 0$ produces an infinite c. This leads to the requirement $|c| \lesssim 1.3$ but numerically one finds $|c| \gtrsim 3.1$ in the 3σ-range of the angles θ_{12} and θ_{23}. Hence, $X_1^{\max}$ has no singularities in a hierarchical spectrum.

The numerical maximal X_2 in Fig. 5.4 shows a stable range for small mass m_1 but no significant θ_{13} dependence. Similarly as for X_1, an expression for X_2 is derived via Taylor expansions. X_2 is mostly depending on the deviations of θ_{23}. Thus, the maximal

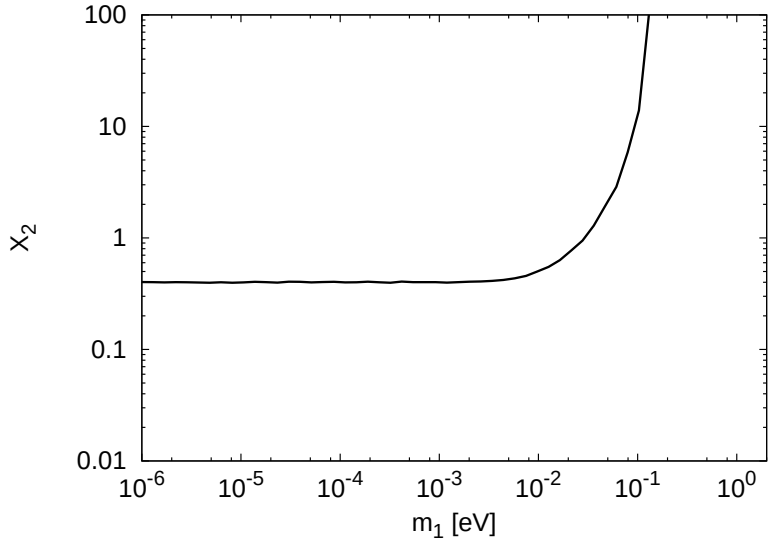

Figure 5.4: Maximal value of X_2 for a normal spectrum and fixed mass m_1.

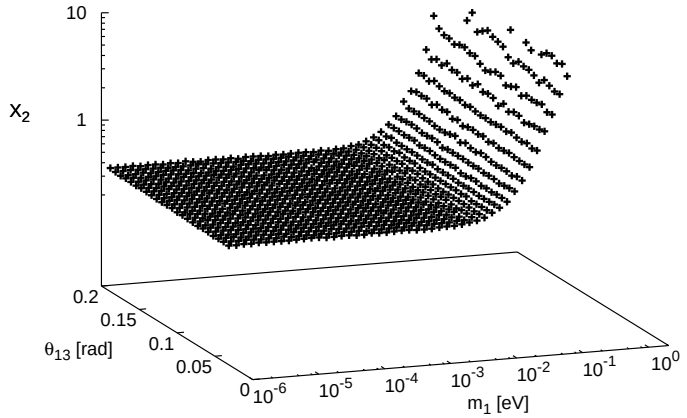

Figure 5.5: θ_{13}-dependence of the maximal X_2 for a normal spectrum

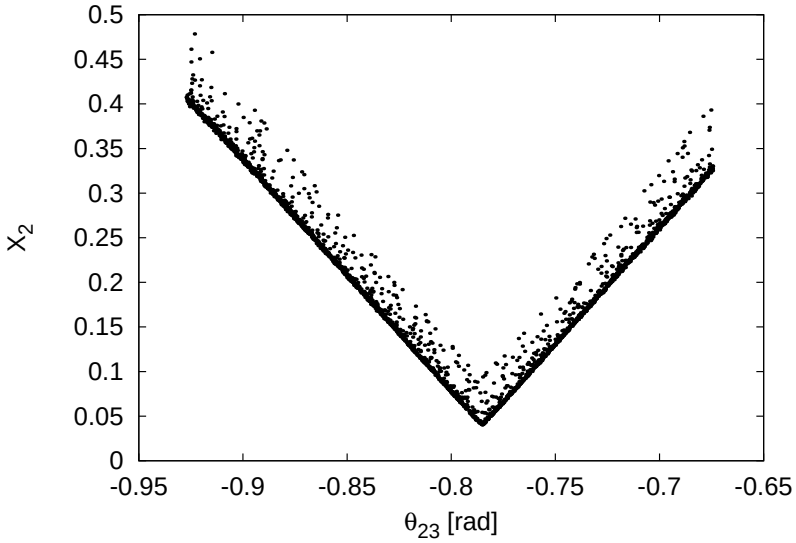

Figure 5.6: θ_{23}-dependence of the maximal X_2 for a normal spectrum plotted in the mass range $10^{-6}\,\mathrm{eV} \leq m_1 \leq 10^{-2}\,\mathrm{eV}$

value for X_2 is approximately given by the maximal ϵ_{23}.

$$X_2^{\max} \approx \left|-2\epsilon_{23} + \mathcal{O}(\epsilon^2)\right| \approx 0.3 + \mathcal{O}(\epsilon^2) \tag{5.18}$$

This explains why X_2 shows no remarkable θ_{13} dependence in Fig. 5.5. For θ_{23} being at its best-fit value this corresponds to $X_2^{\max} < 0.1$. That means that $X_2^{\max}$ could become rather small compared to the maxima of the other asymmetry parameters. In order to obtain a minima of X_2 the angle θ_{23} can be set to its TBM value. The lower bound is given by products of the non vanishing values ϵ_{13} and ϵ_r. The analytically derived linear dependence of θ_{23} with a slope of approximately 2 is reflected in Fig. 5.6. The numerically stable value of X_2 in a hierarchical spectrum in Fig.5.4 suggests that there are no singularities. This can be further investigated by looking at the explicit form of X_2.

$$X_2 = \frac{\cdots}{e^{2i\beta}c_{13}^2 m_3 + c_{12}^2(e^{2i\alpha}m_2 + e^{2i\delta}s_{13}^2 m_1) + s_{12}^2(m_1 + e^{2i(\alpha+\delta)}m_2 s_{13}^2)} \tag{5.19}$$

The denominator will be denoted by D_{X_2}. With the assumption of a nearly degenerated spectrum this simplifies to

$$D_{X_2} = e^{2i\beta}c_{13}^2 + c_{12}^2(e^{2i\alpha} + e^{2i\delta}s_{13}^2) + s_{12}^2(1 + e^{2i(\alpha+\delta)}s_{13}^2) \tag{5.20}$$

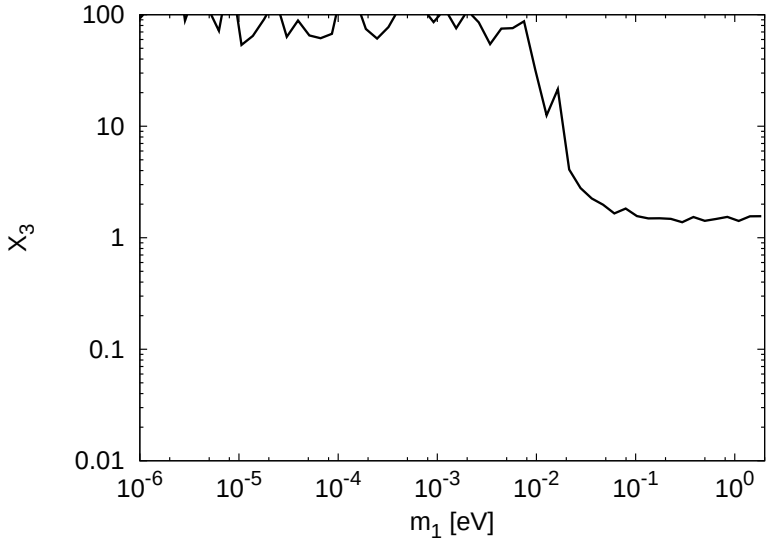

Figure 5.7: Maximal value of X_3 for a normal spectrum and fixed mass m_1.

This turns zero for $\alpha = 0$, $\beta = \delta = \frac{\pi}{2}$ leading to a maximal asymmetry X_2. Clearly this could only happen if the masses become totally equal which is impossible. A suitable choice of the phases where α, β, δ are functions of the three masses would probably minimize the denominator. In a normal spectrum the denominator D_{X_2} takes the following form under the assumptions $m_1 \to 0$ and $m_2 = rm_3$

$$D_{X_2} \approx m_3 \underbrace{\left(e^{2i\alpha}c_{12}^2 + e^{2i(\alpha+\delta)}s_{12}^2 s_{13}^2\right) r}_{\ll \cos^2\theta_{13}} + m_3 e^{2i\beta} c_{13}^2 \neq 0 \tag{5.21}$$

D_{X_2} cannot become zero for all possible complex phase configurations because the second term has a much larger absolute value than the first.

The mass dependence of X_3 in Fig. 5.7 differs from the other two asymmetry parameters. It has large values for small mass m_1 and a small stable range for $m_1 > 0.2\,\text{eV}$ with $X_3^{\max} \approx 1.7$ and a light θ_{13} dependence. After a Taylor expansion the parameter X_3 is roughly given by

$$X_3 \approx \left| \frac{3e^{2i\beta - i\delta} m_3}{2\sqrt{2} m_1} \theta_{13} + \epsilon_{23} \left(\frac{3e^{2i\beta} m_3}{2m_1} \right) + \mathcal{O}(\epsilon^2) \right| \approx \mathcal{O}(1) \tag{5.22}$$

This relation certainly only holds for a nearly degenerate spectrum because the $\mathcal{O}(\epsilon^2)$ contains terms with $\frac{m_3}{m_1}$ which becomes much larger than epsilon for small m_1. The numerical plot of X_3 in the m_1-θ_{13} space given in Fig. 5.8 suggests that for values at the lower bound of θ_{13} the maximal asymmetry stays finite.

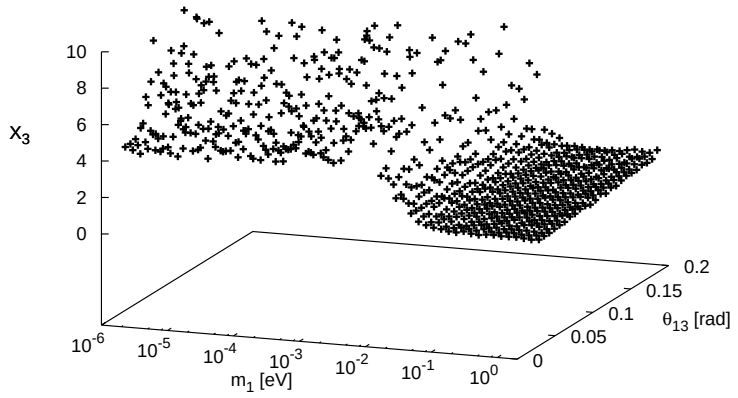

Figure 5.8: θ_{13}-dependence of the maximal X_3 for a normal spectrum

5.2. Inverted Spectrum

For the inverted spectrum all three asymmetry parameters can become minimal ($X_i < 10^{-3}$) for the whole mass range. This is in contrast to the normal spectrum where $X_1^{\min} > 0.2$ for small masses in Fig. 5.1. The maximal asymmetry X_1 is large for all masses and free parameter θ_{13}. Just for the lowest possible values of θ_{13} and small m_3 the asymmetry X_1 becomes roughly of order 10 as indicated in Fig. 5.9.

The asymmetry parameter X_2 given in Fig. 5.10 shows a stable range again. An approximate expression is once more derived by a Taylor expansion under the assumptions $m_3 \to 0$ and $m_1 \approx m_2$ for an inverted spectrum.

$$X_2 \approx 2 \left| \epsilon_{23} + \frac{\sqrt{2} e^{i\delta} \left(-1 + e^{2i\alpha}\right)}{1 + 2e^{2i\alpha}} \theta_{13} \right| + \mathcal{O}(\epsilon^2) \tag{5.23}$$

this becomes maximal for $\alpha = \frac{\pi}{2}$

$$X_2^{\max} \approx 0.3 + 5.7 \times \theta_{13} \tag{5.24}$$

This describes quite well the linear dependence of the maximal X_2 in Fig. 5.11. In both spectra, normal and inverted, X_2 becomes large if the value of the lightest neutrino mass corresponds to a nearly degenerated spectrum $m_{1,3} > 0.1$.

Like the normal spectrum D_{X_2} in Eq. (5.19) cannot become zero if the smallest neutrino mass, now m_3, is tending to zero. With $m_1 = m_2 + \epsilon_m$ and $s_{12}^2 = 1/3$ the scale of each

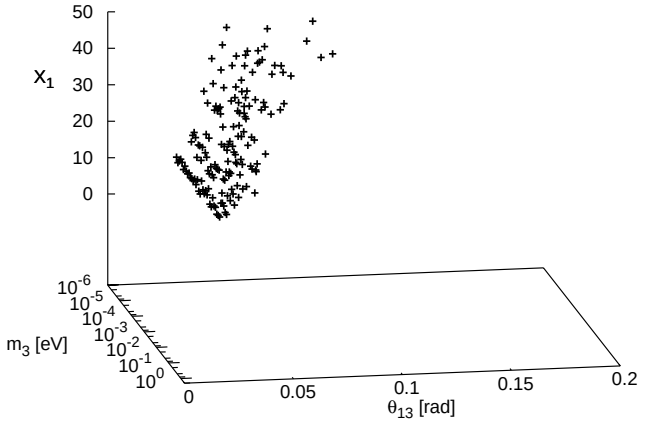

Figure 5.9: θ_{13}-dependence of the maximal X_1 for an inverted spectrum

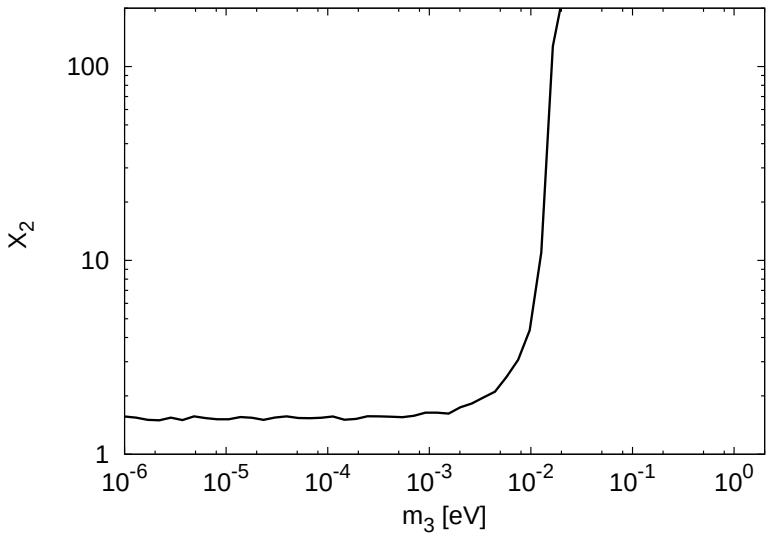

Figure 5.10: Maximal value of X_2 for an inverted spectrum and fixed mass m_3.

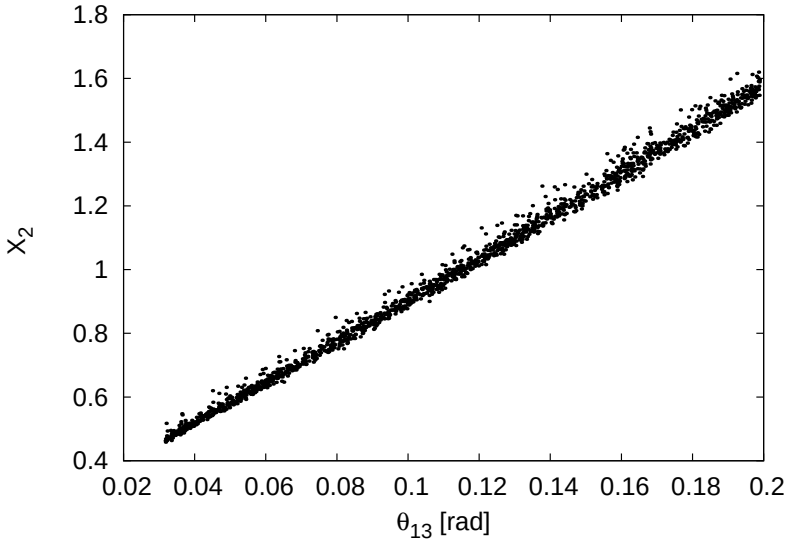

Figure 5.11: θ_{13}-dependence of the maximal X_2 for an inverted spectrum plotted in the mass range $10^{-6}\,\mathrm{eV} \leq m_3 \leq 10^{-3}\,\mathrm{eV}$

term in D_{X_2} can be evaluated by only considering its absolute value and using that s_{13}^2 is of order 10^{-2}.

$$D_{X_2} = \frac{m_2}{3}\left(2\underbrace{(e^{2i\alpha}+e^{2i\delta}s_{13}^2)}_{\approx e^{2i\alpha}} + \underbrace{(1+e^{2i(\alpha+\delta)}s_{13}^2)}_{\approx 1}\right) + \underbrace{\epsilon_m(\frac{2}{3}e^{2i\delta}s_{13}^2+\frac{1}{3})}_{\frac{\epsilon_m}{3}\sqrt{4s_{13}^4+1+4s_{13}^2\cos(2\delta)}\approx\frac{\epsilon_m}{3}} \tag{5.25}$$

$$\approx \frac{m_2}{3}\left(2e^{2i\alpha}+1\right) + \frac{\epsilon_m}{3} \approx \frac{m_2}{3}\underbrace{\sqrt{5+4\cos(2\alpha)}}_{>1} + \frac{\epsilon_m}{3} \tag{5.26}$$

Even if the two terms involving m and ϵ_m are antiparallel in the complex plane ($\arg(m_2\epsilon_m^*) = \pi$) they cannot cancel each other out because $m_2 > \sqrt{\Delta m^2_{\mathrm{atm}}} \gg \sqrt{\Delta m^2_{\odot}} = |\epsilon_m|$. Thus in the strong hierarchical case there are no singularities in X_2.

A Taylor expansion of the last asymmetry parameter X_3 yields a linear dependence of X_3 on the deviations ϵ_{12}, ϵ_{23} and on θ_{13} for both the hierarchical and the nearly degenerate spectrum. In the hierarchical case the assumptions $m_3 \to 0$ and $m_1 \approx m_2$ are used once again.

$$X_3 = \Bigg| -\underbrace{\frac{3e^{i(2\alpha+\delta)}}{2\sqrt{2}(1+2e^{2i\alpha})}}_{\sim 1.0\times\theta_{13}<0.2}\theta_{13} - \underbrace{\frac{3e^{2i\alpha}}{2+4e^{2i\alpha}}}_{\sim 1.5\times\epsilon_{23}<0.21}\epsilon_{23} - \underbrace{\frac{9(1-e^{2i\alpha})}{2\sqrt{2}(1+e^{2i\alpha})}}_{\sim 6.4\times\epsilon_{12}<0.45}\epsilon_{12} + \mathcal{O}(\epsilon^2)\Bigg| \tag{5.27}$$

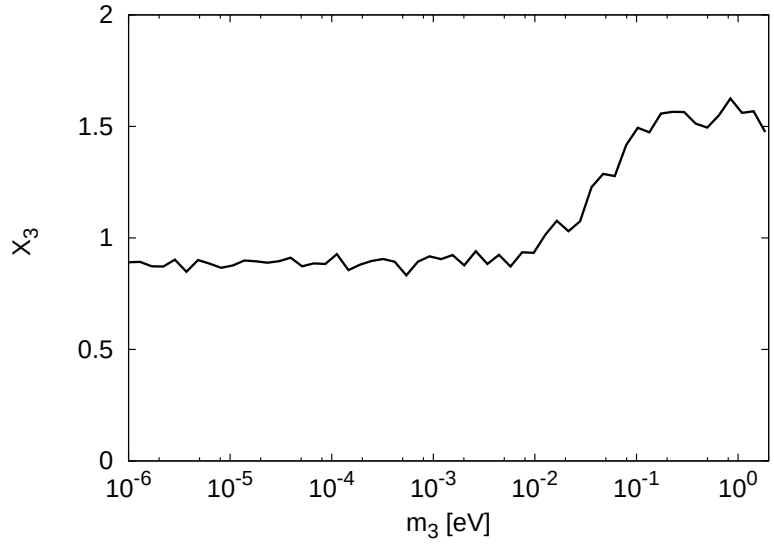

Figure 5.12: Maximal value of X_3 for an inverted spectrum and fixed mass m_3.

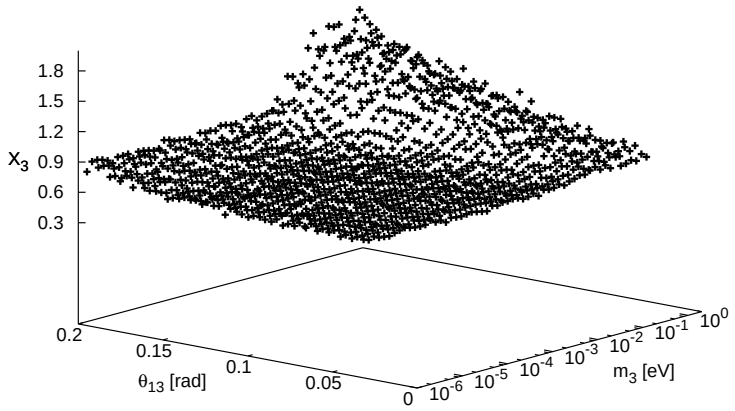

Figure 5.13: θ_{13}-dependence of the maximal X_3 for an inverted spectrum

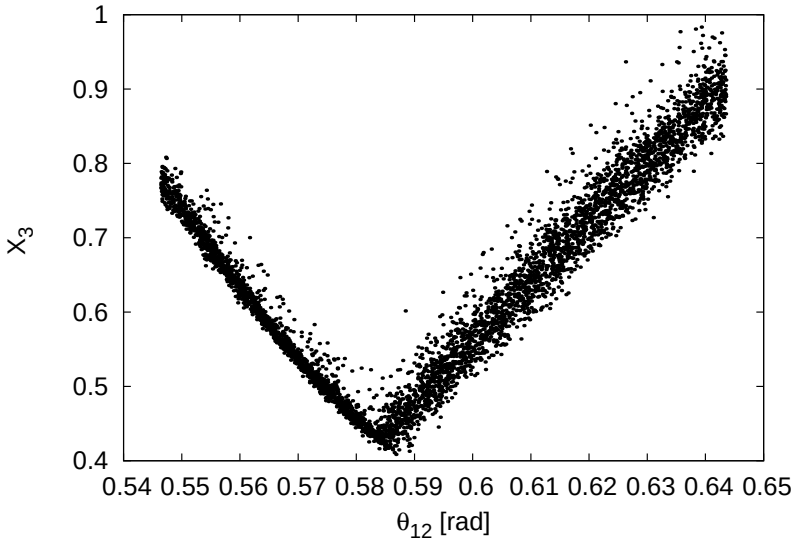

Figure 5.14: θ_{12}-dependence of the maximal X_3 for an inverted spectrum plotted in the mass range $1 \times 10^{-6} \leq m_3 \leq 1 \times 10^{-2}$

This expression is dominated by the ϵ_{12} dependence and coincides with the numerical results in Fig. 5.14. The numerical analysis in Fig. 5.12 diplays that X_3 has no singularities over the whole mass range. The exact value of X_3 has an upper limit of $X_3 \lesssim 1.8$. A summarizing discussion of the asymmetry parameters in the context of TBM-mixing and flavour symmetry models is given in the next section.

6. Discussion

All three asymmetry parameters acquire sizeable values for both mass spectra and the whole mass range of the lightest neutrino mass. Even though TBM-symmetries are not maximally violated for some mass ranges with $X_i \approx \mathcal{O}(1)$, the deviations still indicate a large discrepancy. Only for a hierarchical normal spectrum the maximal asymmetry X_2 becomes smaller than one ($X_2^{\max} \approx 0.4$) as displayed in Fig. 5.4. For the minimal possible asymmetry the normal hierarchical X_1 in Fig 5.1 differs from the other asymmetries. With $X_1^{\min} \approx 0.2$ it is considerably above the other minimal asymmetries ($< 10^{-3}$).

As a consequence, both normal and inverted ordering allow considerable deviations from the TBM-symmetry assumptions. In some stable mass ranges a linear dependence on the angles can be observed. These dependencies are further investigated analytically. For normal ordering two asymmetry parameters X_1 and X_2 have a stable range for a hierarchical spectrum and maximal asymmetries for a nearly degenerate spectrum. The third parameter X_3 shows opposite behaviour having large asymmetry for a hierarchi-

cal spectrum and a stable range for a nearly degenerate spectrum. For an inverted spectrum the second asymmetry parameter X_2 has a similar mass dependence to the normal case. The first parameter X_1 in an inverted ordering could become both maximal and minimal over the whole mass range. Whereas the third parameter X_3 is stable and large but non-maximal over the whole mass range. The maximal asymmetries are mostly caused by the phases which can vary in the total $[0, 2\pi]$ range as they are not experimentally restricted. Asymmetries rely moderately on θ_{12}. Only maximal X_3 in an inverted spectrum (see Fig. 5.14) shows significant θ_{12}-dependence. Since the other two mixing angle deviations contribute with the same order to X_3 the θ_{12}-dependence is softened. For X_1 in an inverted hierarchical ordering a maximal asymmetry is only possible for θ_{13}-values considerably above the lower three sigma bound (see Fig. 5.9). It is not surprising that the "strength" of the asymmetries is mostly driven by the angles θ_{13} and θ_{23}. The μ-τ-symmetry can already be explained by a maximal θ_{23} and zero θ_{13} without a fixed angle θ_{12}. This information is completely contained in X_1 and X_2 which therefore depend merely on θ_{23} and θ_{13}. The third asymmetry parameter X_3 which is more related to the experimentally quite well known θ_{12} cannot become maximal at all.

One needs to know if the deviations are explainable in an A_4 model. First of all there are certain mass ranges which are not compatible with the predictions of a particular A_4-model. In the effective A_4-model in Sec. 2.2 for example only the normal ordering would be relevant. If a symmetry is maximally violated this could generally be accommodated within an A_4 flavour symmetry as long as it allows deviations of the mixing parameters. Nevertheless, it poses the question whether the symmetry was a good starting point. Other forms of the neutrino mass matrix possibly allow for different flavour symmetries with less additional assumptions. The A_4-model presented in Sec. 2.2 had to introduce three A_4 scalar fields together with 14 new parameters in the scalar potential. In return it predicts the three mixing angles of U_{PMNS} but leaves the absolute mass range and the complex phases more or less undetermined. Most of the introduced parameters were restricted by the vacuum alignment which is not easy to explain using only the A_4 symmetry.

In addition to non-vanishing θ_{13}, asymmetries are induced by many unknowns of neutrinos. Especially it is still unclear whether neutrinos are Majorana or Dirac particles. The existence of the Majorana nature of neutrinos allows to describe mass models in which the smallness of neutrino masses is explained but requires to introduce two additional phases in U_{PMNS}. These mass models still leave the absolute mass scale and the ordering undetermined. However, this information would be essential to distinguish flavour symmetry models once the mixing angles are fixed.

A. Diagonalization of the Fermion Mass Matrix

In this section the problem of diagonalization of the matrices, arising in the Mass-Lagrangian, shall be illustrated. For this purpose the following theorem will be used.

Theorem 3. *Any complex $n \times n$ matrix $M \in GL(n, \mathbb{C})$ can be diagonalized by a biunitary transformation*

$$V^\dagger M U = \hat{M} = diag(m_1, m_2, \ldots, m_n) \tag{A.1}$$

with V and U unitary

The fermion masses are usually generated through the coupling of left- and right-handed fields.

$$\mathcal{L}_{mass} = \overline{\psi_{i,L}} M_{ij} \psi_{j,R} \tag{A.2}$$

If the fermion masses result from the coupling of the fermions to the Higgs-field the mass matrix is given by the Yukawa-Coupling matrix Y times the Higgs vacuum expectation value. The basis changes for the left and right-handed fields are done by two independent unitary transformations U and V. Because the elements of the diagonal matrix are interpreted as the physical masses, they should be nonnegative real numbers. This can always be achieved by a redefinition of the unitary matrices U and V by a multiplication with a diagonal phase matrix. The above theorem can be motivated by looking at the hermitian matrix $M^\dagger M$. Suppose this matrix is diagonalized by the unitary matrix U.

$$U^\dagger M^\dagger M U = \hat{M}^2 \tag{A.3}$$

$$\underbrace{\hat{M}^{-1} U^\dagger M^\dagger}_{\equiv V^\dagger} M U = \hat{M} \tag{A.4}$$

now one just has to show that $V^\dagger$ is unitary

$$V^\dagger V = \hat{M}^{-1} U^\dagger M^\dagger M U \underbrace{\left(\hat{M}^{-1}\right)^\dagger}_{=\hat{M}^{-1}} \tag{A.5}$$

$$= \hat{M}^{-1} \hat{M}^2 \hat{M}^{-1} \tag{A.6}$$

$$= \mathbf{1} \tag{A.7}$$

For a symmetric matrix M one can choose $V = U^*$ which results in

$$U^T M U = \hat{M} \tag{A.8}$$

This choice can be seen immediately by taking the transpose of Eq. (A.1). That means that a complex symmetric matrix M can be brought to diagonal form with nonnegative

entries by one unitary matrix. Obviously Eq. (A.8) is no eigenvalue equation. In consequence an eigenvalue of M is not necessarily a physical mass. But for the matrix $M^\dagger M$ one can calculate the masses via the solution of the eigenvalue problem.

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