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# Algebraic quantum gravity (AQG): II. Semiclassical analysis

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**Abstract**

In the previous paper (Giesel and Thiemann 2006 Conceptual setup *Preprint* gr-qc/0607099) a new combinatorial and thus purely algebraical approach to quantum gravity, called algebraic quantum gravity (AQG), was introduced. In the framework of AQG, existing semiclassical tools can be applied to operators that encode the dynamics of AQG such as the master constraint operator. In this paper, we will analyse the semiclassical limit of the (extended) algebraic master constraint operator and show that it reproduces the correct infinitesimal generators of general relativity. Therefore, the question of whether general relativity is included in the semiclassical sector of the theory, which is still an open problem in LQG, can be significantly improved in the framework of AQG. For the calculations, we will substitute $SU(2)$ with $U(1)^3$. That this substitution is justified will be demonstrated in the third paper (Giesel and Thiemann 2006 Semiclassical perturbation theory *Preprint* gr-qc/0607101) of this series.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

In the previous companion paper [1] of this series we introduced a new top down approach to quantum gravity, called algebraic quantum gravity (AQG). This combinatorial approach is very much inspired by the ideas and concepts of LQG [3, 4]. However, it departs in a crucial way from LQG by discarding the notion of embedded graphs and considering algebraic graphs instead. Since these graphs are algebraic, we lose information such as topology and the

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differential structure of the spatial manifold that are fundamental for LQG. Nevertheless, we showed that all physical (gauge invariant) operators such as the master constraint operator can be formulated in an algebraic (i.e. embedding independent) way and thus be lifted from LQG to AQG. In this sense, AQG offers a technically simpler approach since one just has to deal with one fundamental infinite algebraic graph, while within LQG one considers an infinite number of finite embedded graphs. The missing information in AQG about the topology and the differential structure of the spacetime manifold as well as the background metric to be approximated is encoded in the coherent states and thus only of interest in the semiclassical limit. As pointed out in [1] the analysis of the semiclassical limit of the dynamics of LQG could not be performed so far, because existing semiclassical tools fail to be applied to graph-changing operators such as the Hamiltonian or the graph-changing version of the master constraint operator [5–8]. The reason for the failure in the case of the Hamiltonian constraint operator is that in order to quantize these operators without anomalies, it has to be formulated in a graph-changing fashion. The action of a graph-changing operator on coherent states will necessarily add degrees of freedom to the coherent states under consideration. The fluctuation of these additional degrees of freedom are not well suppressed by the coherent states leading to an unacceptable semiclassical approximation of the Hamilton constraint operator. The graph-changing version of the master constraint operator is spatially diffeomorphism invariant. In [9], it was shown that such operators have to be defined directly on the spatially diffeomorphism invariant Hilbert space. Hence, we would need spatially diffeomorphism invariant coherent states, that so far have not been defined in LQG.

In contrast within the framework of AQG, we work with the (extended) master constraint operator, which is quantized in a graph-non-changing formulation. Therefore, the dynamics will not change the degrees of freedom and thus existing semiclassical tools can be used to analyse the semiclassical behaviour of the AQG-dynamics. Furthermore, since we have only one fundamental or maximal graph in AQG, we are able to remove the graph-dependence that is present in the semiclassical tools of LQG.

In this paper, we will display the semiclassical analysis of the (extended) algebraic master constraint operator associated with an algebraic graph of cubic symmetry and show that AQG reproduces the correct infinitesimal generators of general relativity in the semiclassical limit. We will use the semiclassical tools developed in [10–12]. Since we are working on the algebraic level, the restriction to an algebraic graph of cubic symmetry incorporates all graphs of valence 6 or lower[1]. We will substitute $SU(2)$ with $U(1)^3$, because this will simplify the calculation enormously. That this substitution is satisfied was already shown in [10], where it was proven that the electric fluxes and holonomies for $SU(2)$ are well approximated in the semiclassical limit. Additionally, we will prove in our companion paper [2] that the $U(1)^3$ substitution is also satisfied for operators such as the (extended) master constraint operator. Here, we will only consider the gravitational sector. However, the techniques used here carry over to all standard matter coupling. Since the Gauss constraint consists of a linear combination of flux operators, and for those the correct semiclassical limit has been already demonstrated in [10], we neglect the Gauss constraint in our analysis.

Due to the fact that the relation $\left\{H_E^{(1)}, V\right\} = \int_\sigma \mathrm{d}^3x\, K_a^j E_j^a$ for $SU(2)$, on which equation (2.18) in [1] relies, fails to hold, we cannot approximate the Lorentzian part of the Hamiltonian by $U(1)^3$ correctly. Hence, we will only consider the Euclidean part here. However, the discussion in [2] shows that the correct $SU(2)$ calculation reproduces the correct semiclassical limit.

[1] The graphs with valence $(6-n)$ with $3 \leqslant n < 6$ can be obtained by simply not exiting $n$ edges at each vertex of the algebraic graph.

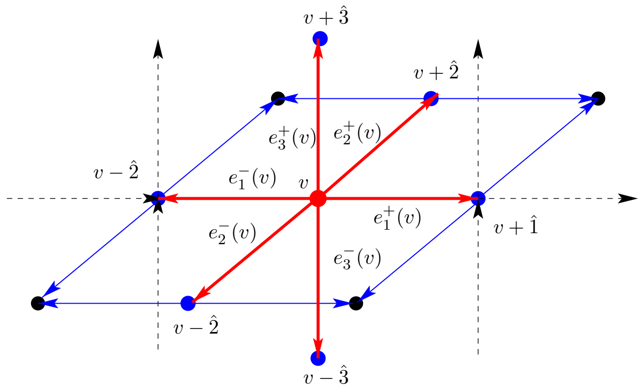


**Figure 1.** Sketch of a graph of cubic topology.

This paper is organized as follows. In section 2, we introduce the necessary technical tools in order to perform the semiclassical analysis. We discuss the notion of the infinite algebraic graph of cubic symmetry as well as explaining in detail how the $U(1)$ approximation is performed. Afterwards we introduce the (extended) algebraic master constraint operator in the $U(1)^3$ approximation. Here we follow the ideas of [13] and generalize them to our case. Note, that, in contrast to the master constraint operator, the operators considered in [13] contained no loop operators. We will display certain details of the calculation in section 2 in the appendix and just refer to them in the main text. We decided to present this calculation in a very detailed form, because as pointed out in our companion paper [2], this is the first time that semiclassical perturbation theory within AQG allows us to compute expectation values of dynamical operators. In section 3, we discuss the leading-order (LO) contribution of the expectation value of the master constraint operator. In section 4, we analyse in detail how this LO contribution is related to the classical master constraint. In section 5, we demonstrate the next-to-leading-order (NLO) term of the expectation value of the master constraint operator. In section 6, we discuss our results and finally conclude.

## 2. The master constraint operator for an algebraic infinite graph of cubic topology and within the $U(1)^3$ approximation

### *2.1. The infinite algebraic graph of cubic topology*

We will consider an algebraic graph with cubic topology, sketched in figure 1. Each vertex is six-valent with three edges going out and three edges going in. We will choose the embedding such that for a given vertex $v$ all six edges are outgoing, as shown in figure 1. Since the algebraic master constraint operator acts on vertices only and, moreover, consists of a sum of the contributions at each vertex, it is always possible to restrict attention to one vertex only. For a given vertex $v$, we will label the six edges with $e_J^\sigma(v)$, whereby $\sigma = \{+, -\}$ and $J = \{1, 2, 3\}$. We will label the outgoing edges by $e_J(v)$ and choose an ordering such that the triple $\{e_1, e_3, e_3\}$ is right-handed with respect to the given orientation of $\Sigma$. We use the notation $e_J^+(v) := e_J(v)$ and $e_J^-(v) := e_J(v - \hat{J})$, where $v - \hat{J}$ denotes the point translated one unit along the $\hat{J}$ axis while the other two directions do not change. The dual surfaces associated with $e_J^+(v)$ is $S_{e_J}$ while the one belonging to $e_J^-(v)$ is $S_{e_J(v-\hat{J})}$ with its orientation reversed. Beside the six edges directly connected to the vertex $v$ (the red or thick ones respectively in figure 1), the action of the algebraic master constraint involves additional edges of the next neighbouring vertices. In figure 1, these edges are the blue or thinner ones respectively that are not dashed. The next neighbouring vertices are also blue or thinner respectively. We will choose the orientation of these edges such that, when embedding the graph via coherent states, the orientation of the edge $e_J^\sigma(v+\sigma'\hat{I})$ agrees with the orientation of $e_J^\sigma$.

### *2.2. The $U(1)^3$-approximation*

In our calculations, we will use the approximation that $SU(2)$ is replaced by $U(1)^3$. Former work [10] showed that although replacing $SU(2)$ with $U(1)^3$ is incorrect the results for expectation values of powers of flux and holonomy operators are reproduced qualitatively. Moreover, the main advantage of this approximation is that the $U(1)^3$ volume operator counterpart diagonalizes the $U(1)^3$ counterparts of the SNF, often called charge network functions (CNFs). Thus calculations involving the volume operator, as is the case for the master constraint operator, become very much easier. Since we are mainly interested in the question of whether the zeroth order of the expectation value of the master constraint operator with respect to coherent states reproduces the correct classical expression, the approximation should be appropriate for our purpose. In [2], we will justify this approximation rigorously.

Let us denote the $U(1)^3$ holonomy by $h_e$ and the dimensionless electrical flux by $p^e$. Note that in order to emphasize the difference between $SU(2)$ and $U(1)^3$, we will not choose the letters $A(e)$ and $E(e)$ here. The $U(1)^3$ approximation includes then the following replacements:

$$A(e) \to h_e := \left(h_e^1, h_e^2, h_e^3\right) \qquad E(e) \to p^e := \left(p_1^e, p_2^e, p_3^e\right), \tag{2.1}$$

where

$$h_e^j(m) := \exp\left(\mathrm{i}\int_e A^j\right) \qquad \text{and} \qquad p_j^e(m) := \frac{1}{a_e^2}\int_{S_e}(*E)_j. \tag{2.2}$$

The Poisson algebra of $h_e^j$ and $p_j^e$ given by

$$\left\{p_j^e, h_{e'}^k\right\} = \mathrm{i}\frac{\kappa}{a_e^2}h_{e'}^k\delta_j^k\delta_{e'}^e \qquad \left\{p_j^e, p_k^e\right\} = \left\{h_e^k, h_e^j\right\} = 0 \tag{2.3}$$

leads to the following commutator relations:

$$\left[\hat{p}_j^e, \hat{h}_{e'}^k\right] = -\frac{\ell_p^2}{a_e^2}\hat{h}_e^j\delta_j^k\delta_{e'}^e \qquad \left[\hat{p}_j^e, \hat{p}_k^e\right] = \left[\hat{h}_e^j, \hat{h}_e^k\right] = 0. \tag{2.4}$$

Here, we introduced a parameter $a_e$ with a dimension of length, in order to work with dimensionless fluxes. Its relation with the classicality parameter $t_e$ of the coherent states is $t_e = \ell_P a_e^2$. Working with dimensionless fluxes will be convenient for the later discussion of the quantum fluctuations. For the holonomies and fluxes of our cubic graph, we use the following abbreviations in order to keep our notation as simple as possible:

$$\widehat{h}_{e_J^\sigma}^j := \hat{h}_{J\sigma j v}, \qquad \widehat{p}_j^{e_J^\sigma} := \hat{p}_{J\sigma j v}. \tag{2.5}$$

### *2.3. The algebraic (extended) master constraint operator for an algebraic graph of cubic topology*

The algebraic extended $SU(2)$ master constraint reads

$$\mathbf{M} = \sum_v \left[\sum_a \mathrm{Tr}\left(A\left(\beta_v^a\right)A\left(e_v^a\right)\left[A\left(e_v^a\right)^{-1}, \sqrt{V_v}\right]\right)\right]^2 + \sum_{\ell=1}^{3}\mathrm{Tr}\left(\tau_\ell A\left(\beta_v^a\right)A\left(e_v^a\right)\left[A\left(e_v^a\right)^{-1}, \sqrt{V_v}\right]\right)\right]^2, \tag{2.6}$$

where $\beta_v^a$ denotes the minimal plaquettes loop in the $x^a = \text{const}$ direction. The substitution of $U(1)^3$ for $SU(2)$ replaces

$$\mathrm{Tr}\big(A\big(\beta_v^a\big)A\big(e_v^a\big)\big[A\big(e_v^a\big)^{-1}, \sqrt{V_v}\big]\big)\big]^2 \to \sum_{I_0J_0K_0} \epsilon^{I_0J_0K_0} h^{n_0}_{\beta_{I_0J_0}} h^{n_0}_{e_{K_0}(v)} \big\{(h^{n_0}_{e_{K_0}})^{-1}(v), \sqrt{V_v}\big\}$$

$$\sum_{\ell_0=1}^{3} \mathrm{Tr}\big(\tau_{\ell_0} A\big(\beta_v^a\big)A\big(e_v^a\big)\big[A\big(e_v^a\big)^{-1}, \sqrt{V_v}\big]\big)\big]^2$$

$$\to \sum_{\ell_0=1}^{3} \sum_{I_0J_0K_0} \epsilon^{I_0J_0K_0} \epsilon_{\ell_0 m_0 n_0} h^{m_0}_{\beta_{I_0J_0}} (h^{n_0}_{e_{K_0}})^{-1}(v) \big\{h^{-1}_{e_{K_0}(v)}, \sqrt{V_v}\big\} \tag{2.7}$$

where $h^{n_0}_{\beta_{I_0J_0}}$ denotes a minimal loop of $U(1)^3$ holonomies along the edges $e_{I_0}, e_{J_0}$. Let us parametrize the minimal loops by the parameters $I_0, \sigma_0, J_0, \sigma_0'$, then any possible loop can be written as

$$\beta_{\{I_0,\sigma_0,J_0,\sigma_0',v\}} = e^{\sigma_0}_{I_0}(v) \circ e^{\sigma_0'}_{J_0}(v + \sigma_0 \hat{I}_0) \circ \big(e^{\sigma_0}_{I_0}\big)^{-1}(v + \sigma_0' \hat{J}_0) \circ \big(e^{\sigma_0'}_{J_0}\big)^{-1}(v), \tag{2.8}$$

and hence the $U(1)^3$ loop is given by

$$\widehat{h}_{\beta_{I_0\sigma_0J_0\sigma_0'm_0v}} = \widehat{h}_{I_0\sigma_0 m_0 v} \circ \widehat{h}_{J_0\sigma_0' m_0 v+\sigma_0\hat{I}_0} \circ \widehat{h}^{-1}_{I_0\sigma_0 m_0 v+\sigma_0'\hat{J}_0} \circ \widehat{h}^{-1}_{J_0\sigma_0' m_0 v}. \tag{2.9}$$

The summation over all possible minimal loops $\beta_{\{I_0,\sigma_0,J_0,\sigma_0',v\}}$ can be expressed in terms of $\epsilon_{ijk}$ tensors such that the algebraic master constraint operator associated with a graph having cubic topology for $U(1)^3$ is given by

$$\widehat{\mathbf{M}}_v = \sum_{\mu=0}^{3} \widehat{C}^{\dagger}_{\mu,v} \widehat{C}_{\mu,v}, \tag{2.10}$$

where

$$\widehat{C}_{0,v} = \sum_{I_0J_0K_0} \sum_{\sigma_0=+,-} \sum_{\sigma_0'=+,-} \sum_{\sigma_0''=+,-} \frac{4}{\kappa} \epsilon^{I_0J_0K_0} \widehat{h}_{\beta_{I_0\sigma_0'J_0\sigma_0''\ell_0 v}} \widehat{h}_{K_0\sigma_0\ell_0 v} \frac{1}{\mathrm{i}\hbar} \big[\widehat{h}^{-1}_{K_0\sigma_0\ell_0 v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\big] \tag{2.11}$$

$$\widehat{C}_{\ell_0,v} = \sum_{I_0J_0K_0} \sum_{\sigma_0=+,-} \sum_{\sigma_0'=+,-} \sum_{\sigma_0''=+,-} \frac{4}{\kappa} \epsilon^{I_0J_0K_0} \epsilon_{\ell_0 m_0 n_0} \widehat{h}_{\beta_{I_0\sigma_0'J_0\sigma_0''m_0 v}} \widehat{h}_{K_0\sigma_0 n_0 v} \frac{1}{\mathrm{i}\hbar} \big[\widehat{h}^{-1}_{K_0 n_0\sigma_0 v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\big]. \tag{2.12}$$

When considering the master constraint operator, we realize that for fixed values of $K_0, \sigma_0$, we have four possible minimal loops. These loops are shown in figure 2 for the case $K_0 = 3, \sigma_0 = +$. Throughout our calculation we want to use the simplification $\sigma_0 = \sigma_0' = \sigma_0''$. This assumption will not affect our final semiclassical result[2], but has the advantage that the four loops reduce to only one loop. For instance in figure 2 only the loop $\widehat{h}_{\beta_{\{1,+,2,+,m_0,v\}}}$ fulfils $\sigma_0 = \sigma_0' = \sigma_0''$. Hence, in total we will have less edges involved in the action of the master constraint, whose $\widehat{C}_{\mu,v}$ operators, inserting our assumption $\sigma_0 = \sigma_0' = \sigma_0''$, are

$$\begin{aligned} \widehat{C}_{0,v} &= \sum_{I_0J_0K_0} \sum_{\sigma_0=+,-} \frac{4}{\kappa} \epsilon^{I_0J_0K_0} \widehat{h}_{\beta_{I_0J_0\sigma_0\ell_0 v}} \widehat{h}_{K_0\sigma_0\ell_0 v} \frac{1}{\mathrm{i}\hbar} \big[\widehat{h}^{-1}_{K_0\sigma_0\ell_0 v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\big] \\ \widehat{C}_{\ell_0,v} &= \sum_{I_0J_0K_0} \sum_{\sigma_0=+,-} \frac{4}{\kappa} \epsilon^{I_0J_0K_0} \epsilon_{\ell_0 m_0 n_0} \widehat{h}_{\beta_{I_0J_0\sigma_0 m_0 v}} \widehat{h}_{K_0\sigma_0 n_0 v} \frac{1}{\mathrm{i}\hbar} \big[\widehat{h}^{-1}_{K_0 n_0\sigma_0 v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\big] \end{aligned} \tag{2.13}$$

[2] When considering four loops we have to divide by a factor of 4 and hence semiclassically this factor is cancelled.

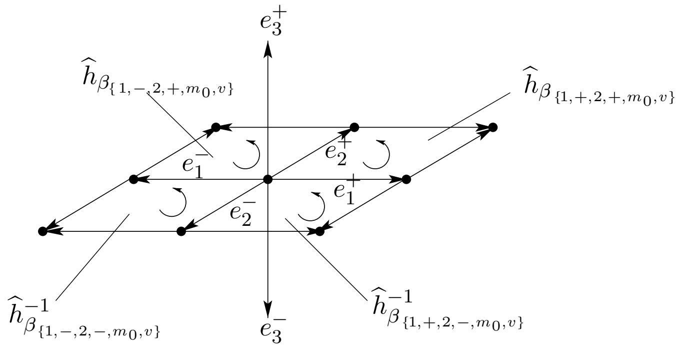


**Figure 2.** The four possible minimal loops $\widehat{h}_{\beta_{1+2,+m_0v}}$, $\widehat{h}_{\beta_{1-2+m_0v}}$, $\widehat{h}^{-1}_{\beta_{1-2-m_0v}}$ and $\widehat{h}^{-1}_{\beta_{1+2-m_0v}}$ for $K_0 = 3, \sigma_0 = +$.

where we introduced $\widehat{h}_{\beta_{I_0J_0\sigma_0m_0v}} := \widehat{h}_{\beta_{I_0J_0\sigma_0\sigma_0m_0v}}$, which we will use as the notation for the loops from now on because we always have $\sigma_0' = \sigma_0''$ and do not have to carry a separated $\sigma_0$ label for $I_0$ and $J_0$. By introducing the operators $\widehat{X}_j^{e_J^\sigma} := \widehat{X}_{J\sigma jv} = \mathrm{i}\hat{h}_{J\sigma jv}\partial/\partial\hat{h}_{J\sigma jv}$ and taking advantage of the cubic symmetry of $\alpha$, the square root of the volume operator denoted by $\widehat{V}^{\frac{1}{2}}_{\alpha,v}$ can be rewritten as

$$\widehat{V}^{\frac{1}{2}}_{\alpha,v} = \left(\ell_p^3\sqrt{\left|\epsilon^{jkl}\left[\frac{\widehat{X}_{1+jv} - \widehat{X}_{1-jv}}{2}\right]\left[\frac{\widehat{X}_{2+kv} - \widehat{X}_{2-kv}}{2}\right]\left[\frac{\widehat{X}_{3+lv} - \widehat{X}_{3-lv}}{2}\right]\right|}\right)^{\frac{1}{2}}. \tag{2.14}$$

The eigenvalue of $\widehat{V}^{\frac{1}{2}}_{\alpha,v}$ is given by

$$\lambda^{\frac{1}{2}}(\{n_{J\sigma j\tilde{v}}\}) = \left(\ell_p^3\sqrt{\left|\epsilon^{jkl}\left[\frac{n_{1+jv} - n_{1-jv}}{2}\right]\left[\frac{n_{2+kv} - n_{2-kv}}{2}\right]\left[\frac{n_{3+lv} - n_{3-lv}}{2}\right]\right|}\right)^{\frac{1}{2}}. \tag{2.15}$$

Note that we use the embedding-dependent operator introduced in [15], because the embedding-independent operator [16] has been ruled out by a recent analysis [17].

### 2.4. $U(1)^3$ coherent states associated with a graph of cubic symmetry

The $U(1)^3$ coherent states are given by

$$\Psi^t_{\alpha,m} = \prod_{e\in E(\alpha)}\prod_{j=1,2,3}\Psi^{t_e}_{g_e(m)}, \tag{2.16}$$

where

$$\Psi^{t_e}_{g(m)}(h) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{-t_e n^2/2}(g_e h^{-1})^n \tag{2.17}$$

and $g^j_{e_J^\sigma} := \mathrm{e}^{p_{J\sigma j\tilde{v}}} h_{J\sigma jv} = g_{J\sigma jv}$ and $t_e := \ell_p^2/a_e^2$ is the so-called classical parameter. Now we want to calculate expectation values of $\widehat{\mathbf{M}}_v$ for coherent $U(1)^3$ states

$$\begin{aligned}\frac{\langle\Psi^t_{\alpha,m}|\widehat{\mathbf{M}}_v|\Psi^t_{\alpha,m}\rangle}{\|\Psi^t_{\alpha,m}\|^2} &= \frac{\sum_{\mu=0}^3\langle\Psi^t_{\alpha,m}|\widehat{C}^\dagger_{\mu,v}\widehat{C}_{\mu,v}|\Psi^t_{\alpha,m}\rangle}{\|\Psi^t_{\alpha,m}\|^2}\\ &= \frac{\sum_{\mu=0}^3\langle\widehat{C}_{\mu,v}\Psi^t_{\alpha,m}|\widehat{C}_{\mu,v}\Psi^t_{\alpha,m}\rangle}{\|\Psi^t_{\alpha,m}\|^2}.\end{aligned} \tag{2.18}$$

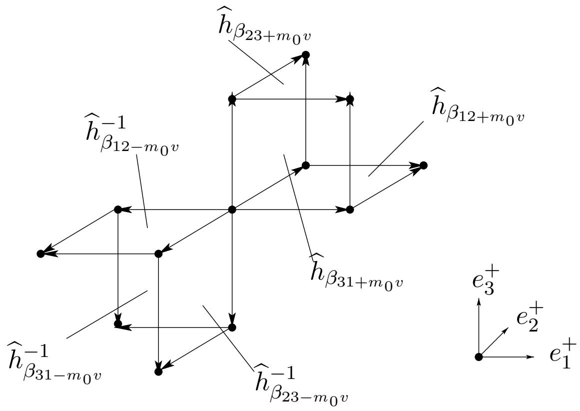


**Figure 3.** The eighteen edges involved in the action of the master constraint at a given vertex and the six corresponding minimal loops.

Let us discuss in a bit more detail how many and precisely which edges are involved in the action of $\widehat{\mathbf{M}}_v$ at a given vertex $v$. Since we chose $\sigma_0 = \sigma_0' = \sigma_0''$ for simplicity, for each chosen $(K_0, \sigma_0) \in (\{1, 2, 3\}, \{+.-\})$ there is only one possible minimal loop $\widehat{h}_{\beta_{I_0 J_0 \sigma_0 m_0 v}}$. Hence, when summing over all possible loops in total we have $6+3\cdot 4 = 18$ edges which are involved in the action of $\widehat{\mathbf{M}}_v$, whereby the additional 12 edges are not directly connected to the vertex $v$. The volume operator considers only the six edges $\{e_J^\sigma \mid \sigma = +, -;\ J = 1, 2, 3\}$ that are directly connected to $v$, hence we do not get any additional edges to consider from the commutator term. These 18 edges are shown in figure 3.

Equation (2.16) states that a coherent state associated with a graph $\alpha$ can be written in terms of the product of the coherent states associated with each edge $e \in E(\alpha)$ of the graph. Consequently, when considering expectation values of the form in equation (2.18), all edges that are not involved in the operator action will simply be cancelled by their corresponding norm in the denominator. Hence, the expectation value of $\widehat{\mathbf{M}}_v$ with respect to $\Psi^t_{\alpha,m}$ is equivalent to the expectation value of $\widehat{\mathbf{M}}_v$ with respect to $\Psi^t_{\alpha_{18},m}$, where $\Psi^t_{\alpha_{18},m}$ denotes the coherent state associated with the graph with 18 edges as shown in figure 3.

### *2.5. The basic building blocks of the expectation value of the master constraint operator*

Let us introduce the following shorthand:

$$\widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v} := \widehat{h}_{\beta_{I_0 J_0 \sigma_0 m_0 v}} \widehat{h}_{K_0 \sigma_0 n_0 v} \frac{1}{\mathrm{i}\hbar}\big[\widehat{h}^{-1}_{K_0 \sigma_0 n_0 v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\big], \tag{2.19}$$

then the basic building block of the master constraint operator is given by

$$\frac{\big\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0 \widetilde{\sigma}_0 v} \Psi^t_{\alpha,m} \mid \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v} \Psi^t_{\alpha,m}\big\rangle}{\big\|\Psi^t_{\alpha,m}\big\|^2}. \tag{2.20}$$

If we know the explicit value of the expectation value of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0 \widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$ in equation (2.20) for general $I_0, J_0, K_0, \sigma_0, m_0, n_0$ and $\widetilde{I}_0, \widetilde{J}_0, \widetilde{K}_0, \widetilde{\sigma}_0, \widetilde{m}_0, \widetilde{n}_0$ respectively, the expectation value of the master constraint can be expressed in terms of a sum of expectation values of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0 \widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$. Due to the fact that, in general, we have fixed $I_0, J_0, K_0, \sigma_0, m_0, n_0$ and $\widetilde{I}_0, \widetilde{J}_0, \widetilde{K}_0, \widetilde{\sigma}_0, \widetilde{m}_0, \widetilde{n}_0$ here, only ten out of the 18 different

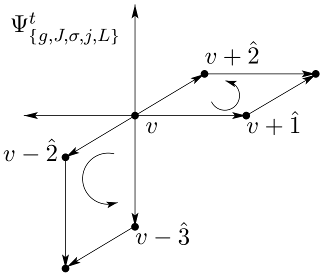


**Figure 4.** $\Psi^t_{\{g,J,\sigma,j,L\}}$ with $L = L(3,2,1,-,1,2,3,+,v)$ for the loops $\widehat{h}_{\beta_{32-m_0 v}} = \widehat{h}^{-1}_{\beta_{23-m_0 v}}$ and $\widehat{h}_{\beta_{12+m_0 v}}$.

edges of $\Psi^t_{\alpha_{18},m}$ are considered by the operator $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$. Let us introduce the set $L(\widetilde{I}_0, \widetilde{J}_0, \widetilde{K}_0, \widetilde{\sigma}_0, I_0, J_0, K_0, \sigma_0, v)$ that contains these ten edges and use the following notation:

$$
\begin{aligned}
L_v &:= \big\{ \mathrm{e}^\sigma_J(v) \,\big|\, \sigma = +.-\,,\ J = 1,2,3 \big\} \\
L &:= L(\widetilde{I}_0, \widetilde{J}_0, \widetilde{K}_0, \widetilde{\sigma}_0, I_0, J_0, K_0, \sigma_0, v) \\
&:= L_v \cup \big\{ e^{\sigma_0}_{J_0}\big(v+\sigma_0\hat{I}_0\big), \big(e^{\sigma_0}_{I_0}\big)^{-1}\big(v+\sigma_0\hat{J}_0\big), \mathrm{e}^{\widetilde{\sigma}_0}_{\widetilde{J}_0}\big(v+\widetilde{\sigma}_0\hat{\widetilde{I}}_0\big), \big(\mathrm{e}^{\widetilde{\sigma}_0}_{\widetilde{I}_0}\big)^{-1}\big(v+\widetilde{\sigma}_0\hat{\widetilde{J}}_0\big)\big\}.
\end{aligned}
\tag{2.21}
$$

Apart from the six edges that are directly connected to the vertex $v$, at most four different additional edges that are not connected to $v$ are modified by the action of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$. An example of such a graph is shown in figure 4. When defining the coherent state associated with this graph consisting of at most ten edges, we have to take the product of the coherent states associated with each edge; see equation (2.16). Introducing the set of vertices $V := \{v, v+\sigma_0\hat{I}_0, v+\sigma_0\hat{J}_0, v+\widetilde{\sigma}_0\hat{\widetilde{I}}_0, v+\widetilde{\sigma}_0\hat{\widetilde{J}}_0\}$, we can parametrize these ten edges by the labels $J, \sigma, j, \tilde{v}$, whereby $j, J \in \{1,2,3\}$, $\sigma \in \{+,-\}$ and $\tilde{v} \in V$. Denoting the coherent state associated with this graph by $\Psi^t_{\{g,J,\sigma,j,L\}}$, we can express it as

$$
\begin{aligned}
\Psi^t_{\{g,J,\sigma,j,L\}} &= \prod_{\tilde{v}\in V} \prod_{\substack{(J,\sigma,j)\\ \in L}} \Psi^t_{gJ\sigma j\tilde{v}} \\
&= \prod_{\tilde{v}\in V} \prod_{\substack{(J,\sigma,j)\\ \in L}} \sum_{\substack{n_{J\sigma j\tilde{v}}\\ \in\mathbb{Z}}} \mathrm{e}^{-\frac{t_{J\sigma j\tilde{v}} n^2_{J\sigma j\tilde{v}}}{2}}\, \mathrm{e}^{+p_{J\sigma j\tilde{v}} n_{J\sigma j\tilde{v}}}\, \mathrm{e}^{+\mathrm{i}\varphi_{J\sigma j\tilde{v}} n_{J\sigma j\tilde{v}}}\, \mathrm{e}^{-\mathrm{i}\vartheta_{J\sigma j\tilde{v}} n_{J\sigma j\tilde{v}}}.
\end{aligned}
\tag{2.22}
$$

Consequently, as explained already in the case of the 18-edges graph, when discussing the action of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$ on $\Psi_{\alpha_{18},m}$ it is enough to know the action on $\Psi^t_{\{g,J,\sigma,j,L\}}$. Thus we have

$$
\frac{\big\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\Psi_{\alpha_{18},m} \,\big|\, \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\Psi_{\alpha_{18},m}\big\rangle}{\big\|\Psi^t_{\alpha,m}\big\|^2} = \frac{\big\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\Psi^t_{\{g,J,\sigma,j,L\}} \,\big|\, \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\Psi^t_{\{g,J,\sigma,j,L\}}\big\rangle}{\big\|\Psi^t_{\{g,J,\sigma,j,L\}}\big\|^2}. \tag{2.23}
$$

Note that most generally, the classicality parameter $t_{J\sigma j\tilde{v}}$ can be different for each single edge. Hence, we would have to take ten different limits $t_{J\sigma j\tilde{v}} \to 0$ when actually calculating our expectation values. Since we already need a lot of notation throughout our calculation and the final result will not be affected in general when we choose $t := t_{J\sigma j\tilde{v}}$ for all $g, J, m, \sigma, \tilde{v}$, we will do this in the following discussion.

*2.6. The action of the operator* $\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}$

For the benefit of the reader we will discuss the explicit action of $\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}$ on coherent states in detail. First we will analyse the action of the loop operator contained in $\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}$, then the action of the remaining holonomy and commutator term and afterwards combining both into the total action of $\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}$.

*2.6.1. The action of the loop operator* $\widehat{h}_{\beta_{I_0J_0\sigma_0m_0v}}$. The loop $\widehat{h}_{\beta_{I_0J_0\sigma_0m_0v}}$ expressed in terms of four single holonomies reads

$$\widehat{h}_{\beta_{I_0J_0\sigma_0m_0v}} = \widehat{h}_{I_0\sigma_0m_0v} \circ \widehat{h}_{J_0\sigma_0m_0v+\sigma_0\hat{I}_0} \circ \widehat{h}^{-1}_{I_0m_0\sigma_0v+\sigma_0\hat{J}_0} \circ \widehat{h}^{-1}_{J_0m_0\sigma_0v}. \tag{2.24}$$

Hence, the action of $\widehat{h}_{\beta_{I_0J_0\sigma_0m_0v}}$ is given by

$$\begin{aligned}
&\widehat{h}_{\beta_{I_0J_0\sigma_0m_0v}} \frac{\Psi^t_{\{g,J,\sigma,j,L\}}}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|} = \widehat{h}_{I_0\sigma_0m_0v} \circ \widehat{h}_{J_0\sigma_0m_0v+\sigma_0\hat{I}_0} \circ \widehat{h}^{-1}_{I_0m_0\sigma_0v+\sigma_0\hat{J}_0} \circ \widehat{h}^{-1}_{J_0m_0\sigma_0v} \frac{\Psi^t_{\{g,J,\sigma,j,L\}}}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|} \\
&= \frac{1}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|} \prod_{\tilde{v}\in V} \prod_{\substack{(J,\sigma,j)\\ \in L}} \sum_{\substack{n_{J\sigma j\tilde{v}}\\ \in\mathbb{Z}}} \mathrm{e}^{-\frac{1}{2}(t(n_{J\sigma j\tilde{v}})^2)} \, \mathrm{e}^{+(p_{J\sigma j\tilde{v}}n_{J\sigma j\tilde{v}})} \, \mathrm{e}^{+\mathrm{i}(\varphi_{J\sigma j\tilde{v}}n_{J\sigma j\tilde{v}})} \\
&\times \mathrm{e}^{-\mathrm{i}\vartheta_{J\sigma j\tilde{v}}(n_{J\sigma j\tilde{v}}+\delta_{(J_0,\sigma_0,m_0,v),(J,\sigma,j,\tilde{v})}+\delta_{(I_0,\sigma_0,v+\sigma_0\hat{J}_0,m_0),(J,\sigma,j,\tilde{v})}-\delta_{(I_0,\sigma_0,v,m_0),(J,\sigma,j,\tilde{v})}-\delta_{(J_0,\sigma_0,v+\sigma_0\hat{I}_0,m_0),(J,\sigma,j,\tilde{v})})},
\end{aligned} \tag{2.25}$$

where

$$\delta_{(J_0,\sigma_0,m_0,v),(J,\sigma,j,\tilde{v})} = \delta_{J_0,J}\delta_{\sigma_0,\sigma}\delta_{m_0,j}\delta_{v,\tilde{v}}. \tag{2.26}$$

In order to get a succinct expression for the $\delta$ functions, we introduce the abbreviation

$$\begin{aligned}
\Delta(I_0, J_0, \sigma_0, m_0, v, J, \sigma, j, \tilde{v}) := (&+\delta_{(J_0,\sigma_0,m_0,v),(J,\sigma,j,\tilde{v})} + \delta_{(I_0,\sigma_0,m_0,v+\sigma_0\hat{J}_0),(J,\sigma,j,\tilde{v})} \\
&- \delta_{(I_0,\sigma_0,m_0,v),(J,\sigma,j,\tilde{v})} - \delta_{(J_0,\sigma_0,m_0,v+\sigma_0\hat{I}_0),(J,\sigma,j,\tilde{v})}).
\end{aligned} \tag{2.27}$$

*2.6.2. The action of* $\widehat{h}_{K_0\sigma_0n_0v}\frac{1}{\mathrm{i}\hbar}\big[\widehat{h}^{-1}_{n_0K_0\sigma_0v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\big]$. The action of $\widehat{h}_{K_0\sigma_0n_0v}\big[\widehat{h}^{-1}_{K_0\sigma_0n_0v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\big]$ involves, for a given vertex $v$, only the six edges that are directly connected to the vertex $v$.

$$\begin{aligned}
&\frac{\widehat{h}_{K_0\sigma_0n_0v}\frac{1}{\mathrm{i}\hbar}\big[\widehat{h}^{-1}_{K_0\sigma_0n_0v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\big]\Psi^t_{\{g,J,\sigma,j,L\}}}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|} = \frac{\frac{1}{\mathrm{i}\hbar}\big(\widehat{V}^{\frac{1}{2}} - \widehat{h}_{K_0\sigma_0n_0v}\widehat{V}^{\frac{1}{2}}_{\alpha,v}\widehat{h}^{-1}_{K_0\sigma_0n_0v}\big)\Psi^t_{\{g,J,\sigma,j,L\}}}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|} \\
&= \left(\prod_{\substack{\tilde{v}\neq v\\ \in V}} \prod_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} \sum_{\substack{n_{J\sigma j\tilde{v}}\\ \in\mathbb{Z}}} \mathrm{e}^{-\frac{1}{2}(t(n_{J\sigma j\tilde{v}})^2)} \, \mathrm{e}^{+(p_{J\sigma j\tilde{v}}n_{J\sigma j\tilde{v}})} \, \mathrm{e}^{+\mathrm{i}(\varphi_{J\sigma j\tilde{v}}n_{J\sigma j\tilde{v}})}\right) \\
&\times \left(\prod_{\substack{(J,\sigma,j)\\ \in L_v}} \sum_{\substack{n_{J\sigma jv}\\ \in\mathbb{Z}}} \big(\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}\}) - \lambda^{\frac{1}{2}}(\{n_{J\sigma jv} + \delta_{(J,\sigma,j,v),(K_0,\sigma_0,n_0,v)}\})\big)\right. \\
&\left.\times \mathrm{e}^{-\frac{1}{2}(t(n_{J\sigma jv})^2)} \, \mathrm{e}^{+(\hat{p}_{J\sigma jv}n_{J\sigma jv})} \, \mathrm{e}^{+\mathrm{i}(\varphi_{J\sigma jv}n_{J\sigma jv})}\right) \Big/ \left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|,
\end{aligned} \tag{2.28}$$

where

$$\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}\}) = a^{\frac{3}{2}} \frac{t^{\frac{3}{4}}}{\mathrm{i}\hbar} \left(\sqrt{\left|\epsilon^{jkl}\left[\frac{n_{1+jv}-n_{1-jv}}{2}\right]\left[\frac{n_{2+kv}-n_{2-kv}}{2}\right]\left[\frac{n_{3+lv}-n_{3-lv}}{2}\right]\right|}\right)^{\frac{1}{2}}. \tag{2.29}$$

Note that we have inserted a factor of 1 into the eigenvalue $\lambda^{\frac{1}{2}}$ by multiplying and, at the same time, by dividing the whole term by a factor of $a^{\frac{3}{2}}$ since $t = \ell_p/a^2$. This will be convenient for our later notation.

The volume operator acts on edges directly connected to the vertex $v$ only. Therefore, the parts of the coherent state associated with edges at $\tilde{v} \neq v$ commute with the volume operator and can therefore be moved to the left-hand side of the holonomy-commutator term. Recall from equation (2.21) that the set $L_v = \left\{\mathrm{e}_I^\sigma \,\middle|\, \sigma = +, -;\ I = 1, 2, 3\right\}$. Combining together the separate action of the loop and the commutator term, we end up with the following action of $\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}$:

$$\begin{aligned}
&\frac{1}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|}\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\\
&= \frac{1}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|}\widehat{h}_{\beta_{I_0J_0\sigma_0m_0v}}\widehat{h}_{K_0\sigma_0n_0v}\frac{1}{\mathrm{i}\hbar}\left[(\widehat{h}^{-1}_{K_0\sigma_0n_0v}, \widehat{V}^{\frac{1}{2}}_{\alpha,v}\right]\Psi^t_{\{g,J,\sigma,j,L\}}\\
&= \frac{1}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|}\left(\prod_{\substack{\tilde{v}\neq v\\ \in V}}\prod_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\sum_{\substack{n_{J\sigma j\tilde{v}}\\ \in\mathbb{Z}}}\mathrm{e}^{-\frac{1}{2}(t(n_{J\sigma j\tilde{v}})^2)}\,\mathrm{e}^{+(p_{J\sigma j\tilde{v}}n_{J\sigma j\tilde{v}})}\right.\\
&\left.\times\,\mathrm{e}^{+\mathrm{i}(\varphi_{J\sigma j\tilde{v}}n_{J\sigma j\tilde{v}})}\,\mathrm{e}^{+\mathrm{i}\vartheta_{J\sigma j\tilde{v}}(-n_{J\sigma j\tilde{v}}-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,\tilde{v}))}\right)\\
&\left(\prod_{\substack{(J,\sigma,j)\\ \in L_v}}\sum_{\substack{n_{J\sigma jv}\\ \in\mathbb{Z}}}\left(\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}\}) - \lambda^{\frac{1}{2}}(\{n_{J\sigma jv}+\delta_{(J,\sigma,j,v),(K_0,\sigma_0,n_0,v)}\})\right)\mathrm{e}^{-\frac{1}{2}(t(n_{J\sigma jv})^2)}\,\mathrm{e}^{+(\hat{p}_{J\sigma jv}n_{J\sigma jv})}\right.\\
&\left.\times\,\mathrm{e}^{+\mathrm{i}(\varphi_{J\sigma jv}n_{J\sigma jv})}\,\mathrm{e}^{+\mathrm{i}\vartheta_{J\sigma jv}(-n_{J\sigma jv}-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v))}\right).
\end{aligned} \tag{2.30}$$

Hence, we are able to give an expression for the expectation value of $\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}$:

$$\begin{aligned}
&\frac{\left\langle\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2} = \frac{1}{\left\|\left|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}\\
&\underbrace{\left(\prod_{\substack{\tilde{v}\neq v\\ \in V}}\prod_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\sum_{\substack{n_{J\sigma j\tilde{v}}\\ \in\mathbb{Z}}}\sum_{\substack{\widetilde{n}_{J\sigma j\tilde{v}}\\ \in\mathbb{Z}}}\mathrm{e}^{-\frac{1}{2}(t((n_{J\sigma j\tilde{v}})^2+(\widetilde{n}_{J\sigma j\tilde{v}})^2))}\,\mathrm{e}^{+(p_{J\sigma j\tilde{v}}(n_{J\sigma j\tilde{v}}+\widetilde{n}_{J\sigma j\tilde{v}}))}\,\mathrm{e}^{+\mathrm{i}(\varphi_{J\sigma j\tilde{v}}(n_{J\sigma j\tilde{v}}-\widetilde{n}_{J\sigma j\tilde{v}}))}\right.\\
&\left.\int\mathrm{d}\vartheta_{J\sigma j\tilde{v}}\,\mathrm{e}^{+\mathrm{i}\vartheta_{J\sigma j\tilde{v}}(-n_{J\sigma j\tilde{v}}-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,\tilde{v})+\widetilde{n}_{J\sigma j\tilde{v}}+\Delta(\widetilde{I}_0,\widetilde{J}_0,\widetilde{\sigma}_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}\right)}_{\delta(\widetilde{n}_{J\sigma j\tilde{v}}+\Delta(\widetilde{I}_0,\widetilde{J}_0,\widetilde{\sigma}_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})-n_{J\sigma j\tilde{v}}-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,\tilde{v}))}
\end{aligned}$$

$$\left(\prod_{\substack{(J,\sigma,j)\\ \in L_v}} \sum_{\substack{n_{J\sigma j\tilde{v}}\\ \in\mathbb{Z}}} \sum_{\widetilde{n}_{J\sigma jv}\in\mathbb{Z}} \mathrm{e}^{-\frac{1}{2}(t((n_{J\sigma jv})^2+(\widetilde{n}_{J\sigma jv})^2))}\,\mathrm{e}^{+(\hat{p}_{J\sigma jv}(n_{J\sigma jv}+\widetilde{n}_{J\sigma jv}))}\,\mathrm{e}^{+\mathrm{i}(\varphi_{J\sigma jv}(n_{J\sigma jv}-\widetilde{n}_{J\sigma jv}))}\right.$$
$$\times\left[\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}\})-\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}+\delta_{(J,\sigma,j,v),(K_0,\sigma_0,n_0,v)}\})\right]$$
$$\times\left[\lambda^{\frac{1}{2}}(\{\widetilde{n}_{J\sigma jv}\})-\lambda^{\frac{1}{2}}(\{\widetilde{n}_{J\sigma jv}+\delta_{(J,\sigma,j,v),(\widetilde{K}_0,\widetilde{\sigma}_0,\widetilde{n}_0,v)}\})\right]$$
$$\left.\underbrace{\int \mathrm{d}\vartheta_{J\sigma jv}\,\mathrm{e}^{+\mathrm{i}\vartheta_{J\sigma jv}(-n_{J\sigma jv}-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v)+\widetilde{n}_{J\sigma jv}+\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v))}\right)}_{\delta(\widetilde{n}_{J\sigma jv}+\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v)-n_{J\sigma jv}-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v))}. \tag{2.31}$$

The $\delta$ function forces the following condition on $n_{J\sigma j\tilde{v}}, n_{J\sigma jv}$ and $\widetilde{n}_{J\sigma j\tilde{v}}, \widetilde{n}_{J\sigma jv}$ respectively:

$$\begin{aligned}\widetilde{n}_{J\sigma j\tilde{v}} &= n_{J\sigma j\tilde{v}}+\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,\tilde{v})-\Delta(\widetilde{I}_0,\widetilde{J}_0,\widetilde{\sigma}_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\\ \widetilde{n}_{J\sigma jv} &= n_{J\sigma jv}+\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v)-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v).\end{aligned} \tag{2.32}$$

Introducing

$$\begin{aligned}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}) &:= \Delta(\widetilde{I}_0,\widetilde{J}_0,\widetilde{\sigma}_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\\ &\quad-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,\tilde{v})\\ \Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v) &:= \Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v)\\ &\quad-\Delta(I_0,J_0,\sigma_0,m_0,v,J,\sigma,j,v),\end{aligned} \tag{2.33}$$

the condition for $n_{J\sigma j\tilde{v}}, n_{J\sigma jv}$ can be rewritten as

$$\begin{aligned}\widetilde{n}_{J\sigma j\tilde{v}} &= n_{J\sigma j\tilde{v}}-\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\\ \widetilde{n}_{J\sigma jv} &= n_{J\sigma jv}-\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v).\end{aligned} \tag{2.34}$$

Reinserting this condition into the expectation value of $\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}$, we obtain

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}=\frac{1}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}$$
$$\left(\prod_{\substack{\tilde{v}\neq v\\ \in V}}\prod_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} \mathrm{e}^{-p_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\,\mathrm{e}^{+\mathrm{i}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\right.$$
$$\times\,\mathrm{e}^{-\frac{t}{2}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}$$
$$\left.\sum_{\substack{n_{J\sigma j\tilde{v}}\\ \in\mathbb{Z}}} \mathrm{e}^{-t((n_{J\sigma j\tilde{v}})^2-2n_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}\,\mathrm{e}^{+2p_{J\sigma j\tilde{v}}n_{J\sigma j\tilde{v}}}\right)$$
$$\left(\prod_{(J,\sigma,j)} \mathrm{e}^{-\hat{p}_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}\,\mathrm{e}^{+\mathrm{i}\varphi_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}\right.$$
$$\times\,\mathrm{e}^{-\frac{t}{2}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v))^2}$$
$$\sum_{\substack{n_{J\sigma jv}\\ \in\mathbb{Z}}} \mathrm{e}^{-t((n_{J\sigma jv})^2-2n_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v))}\,\mathrm{e}^{+2\hat{p}_{J\sigma jv}n_{J\sigma jv}}$$
$$\left[\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}\})-\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}+\delta_{(J,\sigma,j,v),(K_0,\sigma_0,n_0,v)}\})\right]$$

$$\Big[+\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}-\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\})$$
$$-\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}-\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)+\delta_{(J,\sigma,j,v),(K_0,\widetilde{\sigma}_0,\widetilde{n}_0,v)}\})\Big]\Bigg). \tag{2.35}$$

Note that $\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)$ is the special case where $\tilde{v}=v$, since only edges that are directly connected to $v$ are considered. Thus four out of the eight Kronecker deltas can be neglected, and we have

$$\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)=\big(\delta_{(\widetilde{J}_0,\sigma_0,\widetilde{m}_0,v),(J,\sigma,j,v)}-\delta_{(\widetilde{I}_0,\widetilde{\sigma}_0,\widetilde{m}_0,v),(J,\sigma,j,v)}$$
$$-\delta_{(J_0,\sigma_0,m_0,v),(J,\sigma,j,v)}+\delta_{(I_0,\sigma_0,m_0,v),(J,\sigma,j,v)}\big). \tag{2.36}$$

*2.7. Application of the Poisson resummation theorem*

The aim of this work is to discuss the semiclassical behaviour of the algebraic master constraint; thus we are mainly interested in the properties of the expectation value in equation (2.35) for tiny values of the classicality parameter $t$. Looking at equation (2.35), tiny values of $t$ will correspond to a slow convergence behaviour when considering the sum over $n_{J\sigma j\tilde{v}}$. Therefore, we will perform a Poisson resummation in which $t$ gets replaced by $1/t$. Then the series converges rapidly when considering small, tiny values of the classicality parameter. Let us introduce the following quantities:

$$T:=\sqrt{t} \qquad x_{J\sigma j\tilde{v}}:=Tn_{J\sigma j\tilde{v}} \qquad x_{J\sigma jv}:=Tn_{J\sigma jv}, \tag{2.37}$$

with the help of whose all quantities can be expressed in terms of $x_{J\sigma jv}$

$$\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}\})=T^{-\frac{3}{4}}\lambda^{\frac{1}{2}}(\{Tn_{J\sigma jv}\})=T^{-\frac{3}{4}}\lambda^{\frac{1}{2}}(\{x_{J\sigma jv}\}) \tag{2.38}$$

and the expectation value can be rewritten in terms of $x_{J\sigma j\tilde{v}}$ as

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0v}\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}=\frac{T^{-\frac{3}{2}}}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}$$
$$\Bigg(\mathrm{e}^{-\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}p_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\,\mathrm{e}^{+\mathrm{i}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{((J,\sigma,j)\\ \in L\setminus L_v}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}$$
$$\times\,\mathrm{e}^{-\frac{t}{2}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}$$
$$\sum_{x_{J\sigma j\tilde{v}}\in\mathbb{Z}T}\mathrm{e}^{-\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(x_{J\sigma j\tilde{v}})^2-x_{J\sigma j\tilde{v}}(\frac{2}{T}p_{J\sigma j\tilde{v}}+T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}$$
$$\Bigg(\mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}\hat{p}_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}\,\mathrm{e}^{+\mathrm{i}\sum\limits_{\substack{((J,\sigma,j)\\ \in L_v}}\varphi_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}$$
$$\times\,\mathrm{e}^{-\frac{t}{2}\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v))^2}$$
$$\sum_{x_{J\sigma jv}\in\mathbb{Z}T}\mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}(x_{J\sigma jv})^2-x_{J\sigma jv}(\frac{2}{T}\hat{p}_{J\sigma jv}+T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v))}$$

$$\Lambda^{\frac{1}{2}}\big(\{x_{J\sigma jv}\}, e^{\sigma_0}_{K_0}(v), n_0\big)$$
$$\times\,\Lambda^{\frac{1}{2}}\big(\{x_{J\sigma jv} - T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\}, e^{\widetilde{\sigma}_0}_{\widetilde{K}_0}(v),\widetilde{n}_0\big)\Bigg), \tag{2.39}$$

where we introduced

$$\Lambda^{\frac{1}{2}}\big(\{x_{J\sigma jv}\}, e^{\sigma_0}_{K_0}(v), n_0\big) := T^{-\frac{3}{4}}\big[\lambda^{\frac{1}{2}}(\{x_{J\sigma jv}\}) - \lambda^{\frac{1}{2}}(\{x_{J\sigma jv} + T\delta_{(J,\sigma,j,v),(K_0,\sigma_0,n_0,v)}\})\big] \tag{2.40}$$

in order to keep the expression as short as possible. Moreover, the denominator can be re-expressed as

$$\begin{aligned}\big\|\Psi^t_{\{g,J,\sigma,j,L\}}\big\|^2 &= \prod_{(J,\sigma,j)}\sqrt{\frac{\pi}{t}}\,\mathrm{e}^{-\frac{1}{t}(p_{J\sigma j\tilde v})^2}[1+K_t(p)]\\ &= \left(\sqrt{\frac{\pi}{t}}\right)^{30}\mathrm{e}^{+\frac{1}{t}\sum\limits_{\tilde v\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}}(p_{J\sigma j\tilde v})^2}[1+K_t(p)]^{30}.\end{aligned} \tag{2.41}$$

The application of the Poisson resummation formula leads therefore to the following expectation value:

$$\begin{aligned}&\frac{\big\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\Psi^t_{\{g,J,\sigma,j,L\}}\,\big|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\Psi^t_{\{g,J,\sigma,j,L\}}\big\rangle}{\big\|\Psi^t_{\{g,J,\sigma,j,L\}}\big\|^2} = \frac{1}{\big(\sqrt{\frac{\pi}{t}}\big)^{30}[1+K_t(p)]^{30}}\,\frac{(\sqrt{\pi})^{12}}{T^{30}}\\
&\Bigg(\mathrm{e}^{+\mathrm{i}\sum\limits_{\substack{\tilde v\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\varphi_{J\sigma j\tilde v}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde v)}\,\mathrm{e}^{-\frac{t}{4}\sum\limits_{\substack{\tilde v\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde v))^2}\\
&\sum_{n_{J\sigma j\tilde v}\in\mathbb{Z}}\mathrm{e}^{-\frac{\pi^2}{t}\sum\limits_{\substack{\tilde v\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(n_{J\sigma j\tilde v})^2}\,\mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{\tilde v\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}n_{J\sigma j\tilde v}(p_{J\sigma j\tilde v}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde v))}\Bigg)\\
&\Bigg(\mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}\hat p_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}\,\mathrm{e}^{+\mathrm{i}\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}\varphi_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}\\
&\times\mathrm{e}^{-\frac{t}{2}\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v))^2}\,\mathrm{e}^{-\frac{1}{t}\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}(\hat p_{J\sigma jv})^2}\\
&\sum_{n_{J\sigma jv}\in\mathbb{Z}}\mathrm{e}^{-\frac{\pi^2}{t}\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}(n_{J\sigma jv})^2}\,\mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}n_{J\sigma jv}(\hat p_{J\sigma jv}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde v))}\\
&\int_{\mathbb{R}^{18}}\mathrm{d}^{18}x_{J\sigma jv}\,\mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}}(x_{J\sigma jv})^2-\frac{2}{T}x_{J\sigma jv}(\hat p_{J\sigma jv}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)-\mathrm{i}\pi n_{J\sigma jv})}\\
&\Lambda^{\frac{1}{2}}\big(\{x_{J\sigma jv}\}, e^{\sigma_0}_{K_0}(v), n_0\big)\Lambda^{\frac{1}{2}}\big(\{x_{J\sigma jv}\\
&-T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\}, e^{\widetilde{\sigma}_0}_{\widetilde{K}_0}(v),\widetilde{n}_0\big)\Bigg).\end{aligned} \tag{2.42}$$

Similar to [13], we introduce new $x_{J\sigma jv}$ variables denoted by $(x_{Jjv})^+ := \frac{1}{2}(x_{J+jv} + x_{J-jv})$ and $(x_{Jjv})^- := \frac{1}{2}(x_{J+jv} - x_{J-jv})$. These variables have the advantage that the $\Lambda^{\frac{1}{2}}$ are functions on $(x_{Jjv})^-$ only. Hence, the nine-dimensional integral over $(x_{Jjv})^+$ no longer contains $\Lambda^{\frac{1}{2}}$ and can be easily computed, because it has become a usual complex Gaussian integral. Additionally, all the other quantities as $n_{J\sigma jv}$, $\hat{p}_{J\sigma jv}$ undergo analogous transformations. The transformation for the terms involving $\delta$ functions will depend on the sign of $\sigma_0$ and $\widetilde{\sigma}_0$ respectively and will be of the general form

$$\begin{aligned}\frac{1}{2}\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)} &:= \frac{\delta_{(J,+,j,v),(K_0,\sigma_0,n_0,v)} - \delta_{(J,-,j,v),(K_0,\sigma_0,n_0,v)}}{2}\\ \frac{1}{2}(\delta)^+_{(J,j,v),(K_0,n_0,v)} &:= \frac{\delta_{(J,+,j,v),(K,\sigma_0,n_0,v)} + \delta_{(J,-,j,v),(K_0,\sigma_0,n_0,v)}}{2}.\end{aligned} \tag{2.43}$$

Note that it was necessary to re-express $\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, v)$ in terms of the difference of $\Delta(I_0, J_0, \sigma_0, m_0, v, J, \sigma, j, v) - \Delta(I_0, J_0, \sigma_0, m_0, v, J, \sigma, j, v)$ (see equation (2.33)) since there is no global sign term to factor out in this case

$$\begin{aligned}&\frac{1}{2}\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0, J_0, v, J, j, v)\\ &\qquad := \frac{\Delta(I_0, J_0, \sigma_0, m_0, v, J, +, j, v) - \Delta(I_0, J_0, \sigma_0, m_0, v, J, -, j, v)}{2}\\ &\frac{1}{2}(\Delta)^+(I_0, J_0, v, J, j, v) := \frac{\Delta(I_0, J_0, \sigma_0, m_0, v, J, +, j, v) + \Delta(I_0, J_0, \sigma_0, m_0, v, J, -, j, v)}{2}.\end{aligned} \tag{2.44}$$

For the details of this transformation, see appendix B. The change of variables and the performance of the Gaussian integral simplify the expectation value to

$$\begin{aligned}&\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\Psi^t_{\{g,J,\sigma,j,L\}} \,\middle|\, \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}\\
&= \frac{\mathrm{e}^{+\mathrm{i}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\ \mathrm{e}^{-\frac{t}{4}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}}\\
&\quad\frac{(\sqrt{\pi})^{12}}{T^{30}}2^9\left(\sqrt{\frac{\pi}{2}}\right)^9\\
&\quad\left(\sum_{n_{J\sigma j\tilde{v}}\in\mathbb{Z}}\mathrm{e}^{-\frac{\pi^2}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(n_{J\sigma j\tilde{v}})^2}\ \mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}n_{J\sigma j\tilde{v}}(p_{J\sigma j\tilde{v}}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}\right)\\
&\quad\left(\mathrm{e}^{-\frac{t}{8}\sum_{\substack{(J,j)\\ \in L_v}}(((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v))^2)}\ \mathrm{e}^{+\frac{i}{2}\sum_{\substack{(J,j)\\ \in L_v}}((\varphi_{Jjv})^+((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right.\\
&\quad\left.\sum_{(n_{Jjv})^+\in\mathbb{Z}}\mathrm{e}^{-2\frac{\pi^2}{t}\sum_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^+)^2}\ \mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^+((p_{Jjv})^++\frac{T^2}{4}((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right)\\
&\quad\left(\mathrm{e}^{-\frac{1}{2}\sum_{\substack{(J,j)\\ \in L_v}}((p_{Jjv})^-(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}\right.\\
&\quad\times\mathrm{e}^{+\frac{i}{2}\sum_{\substack{(J,j)\\ \in L_v}}((\varphi_{Jjv})^-(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}\\
&\quad\times\mathrm{e}^{-\frac{2}{t}\sum_{\substack{(J,j)\\ \in L_v}}(((p_{Jjv})^-)^2)}\ \mathrm{e}^{-\frac{t}{4}\sum_{\substack{(J,j)\\ \in L_v}}((\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(I_0,J_0,v,J,j,v))^2)}\end{aligned}$$

$$
\sum_{(n_{Jjv})^- \in \mathbb{Z}} \int_{\mathbb{R}^9} \mathrm{d}^9 (x_{Jjv})^-
$$
$$
\times \mathrm{e}^{-2 \sum_{\substack{(J,j,v)\\ \in L_v}} (((x_{Jjv})^-)^2 - \frac{2}{T}((p_{Jjv})^- + \frac{T^2}{2}[\frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - \frac{1}{2}\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)] - \mathrm{i}\pi(n_{Jjv})^-))}
$$
$$
\Lambda^{\frac{1}{2}} \left( \left\{ (x_{Jjv})^- - \frac{T}{2}[\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0, \widetilde{J}_0, v, J, j, v) \right.\right.
$$
$$
\left.\left. - \mathrm{sgn}(\sigma_0)(\Delta)^-(I_0, J_0, v, J, j, v)] \right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0) e_{\widetilde{K}_0}(v), \widetilde{n}_0 \right)
$$
$$
\left. \Lambda^{\frac{1}{2}} \left( \{(x_{Jjv})^-\}, \frac{1}{2}\mathrm{sgn}(\sigma_0) e_{K_0}(v), n_0 \right) \right). \tag{2.45}
$$

## *2.8. Only the term with $n_{J\sigma j\tilde{v}} = 0$ matters*

The remaining integral in equation (2.45) cannot be calculated in a closed form for the reason that the $\Lambda^{\frac{1}{2}}$ functions prevent it from being a usual Gaussian integral. Moreover, we have an infinite summation over $(n_{Jjv})^-$ occurring in the argument of the exponential function. Therefore, we also have to discuss which terms in this $n_{J\sigma j\tilde{v}}$ summation have to be considered and which can be neglected. The problem with completing the square in the exponent is that we will have to continue the $\Lambda^{\frac{1}{2}}$ function into the complex plane. Since $\Lambda^{\frac{1}{2}}$ is not analytic in $\mathbb{C}^9$, we cannot use a simple contour argument in order to estimate the remaining integral. In order to get the integrand univalent, we express $\Lambda^{\frac{1}{2}}$ in terms of squares of determinants by simply squaring the usual expression of the determinant in $\Lambda^{\frac{1}{2}}$ in equation (2.40) and at the same time taking the square root out of the squared determinant

$$
\begin{aligned}
\Lambda^{\frac{1}{2}} & \left( \left\{ (\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^- - \mathrm{i}\pi(n_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- \right.\right. \\
& \qquad \left.\left. - \mathrm{sgn}(\sigma_0)(\Delta)^-] \right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0) e_{K_0}(v), n_0 \right) \\
& = t^{\frac{3}{8}} \left( \left[ \left( \det \left( \left\{ (\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^- - \mathrm{i}\pi(n_{Jjv})^-) \right.\right.\right.\right.\right. \\
& \qquad \left.\left.\left.\left. + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-] \right\} \right) \right)^2 \right]^{\frac{1}{8}} \\
& \qquad - \left[ \left( \det \left( \left\{ (\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^- - \mathrm{i}\pi(n_{Jjv})^-) \right.\right.\right.\right. \\
& \qquad \left.\left.\left.\left.\left. + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-] - \frac{T}{2}\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)} \right) \right\} \right)^2 \right]^{\frac{1}{8}} \right)
\end{aligned} \tag{2.46}
$$

and furthermore using the trivial identity

$$
\left(\left[\det\left(\left\{(\tilde{x}_{Jjv})^- + B^{K_0}_{Jjv}\right\}\right)\right]^2\right)^{\frac{1}{8}} = \exp\left(\frac{1}{8}\ln\left(\left[\det\left(\left\{(\tilde{x}_{Jjv})^- + B^{K_0}_{Jjv}\right\}\right)\right]^2\right)\right), \tag{2.47}
$$

whereby $B^{K_0}_{Jjv}$ denotes symbolically all the additional terms to $(\tilde{x}_{Jjv})^-$ that occur in the argument of the determinant and we have to use the branch of the $\ln(z) = \ln(|z|\,\mathrm{e}^{\mathrm{i}\phi})$ for any complex number $z = |z|\,\mathrm{e}^{\mathrm{i}\phi}$ with $\phi \in [0, 2\pi)$. With this branch in mind, the integrand becomes indeed univalent on the entire complex plane $\mathbb{C}^9$ except at the points where $\det\big(\big\{(\tilde{x}_{Jjv})^- + B^{K_0}_{Jjv}\big\}\big) = 0$. Now we have a univalent integrand, a contour argument can be found that allows us to move the integration path away from the real hyperplane $\mathbb{R}^9$ in $\mathbb{C}^9$ without changing the result. Consequently, we can now complete the square in the exponent and obtain by using

$$(\tilde{x}_{Jjv})^- := (x_{Jjv})^- - \frac{1}{T}(p_{Jjv})^- + \frac{T^2}{4}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-) - \frac{\mathrm{i}\pi}{T}(n_{Jjv})^-,$$

where the shorthands

$$(\Delta)^- := (\Delta)^-(I_0, J_0, v, J, j, v) \qquad (\widetilde{\Delta})^- := (\Delta)^-(\widetilde{I}_0, \widetilde{J}_0, v, J, j, v) \tag{2.48}$$

were introduced. This results in the following form of the expectation value:

$$\begin{aligned}
&\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\Psi^t_{\{g,J,\sigma,j,L\}} \,\middle|\, \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2} \\
&\frac{\mathrm{e}^{+\mathrm{i}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\;\mathrm{e}^{-\frac{t}{4}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}} \\
&\frac{(\sqrt{\pi})^{12}}{T^{30}}2^9\left(\sqrt{\frac{\pi}{2}}\right)^9 \\
&\left(\sum_{n_{J\sigma j\tilde{v}}\in\mathbb{Z}}\mathrm{e}^{-\frac{\pi^2}{t}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(n_{J\sigma j\tilde{v}})^2}\;\mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}n_{J\sigma j\tilde{v}}(p_{J\sigma j\tilde{v}}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}\right) \\
&\left(\sum_{(n_{Jjv})^+\in\mathbb{Z}}\mathrm{e}^{-2\frac{\pi^2}{t}\sum\limits_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^+)^2}\;\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^+((p_{Jjv})^++\frac{T^2}{4}((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right) \\
&\left(\sum_{(n_{Jjv})^-\in\mathbb{Z}}\mathrm{e}^{-2\frac{\pi^2}{t}\sum\limits_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^-)^2}\right. \\
&\times\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^-((p_{Jjv})^-+\frac{T^2}{4}(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))} \\
&\int_{\mathbb{R}^9}\mathrm{d}^9(\tilde{x}_{Jjv})^-\mathrm{e}^{-2\sum\limits_{(J,j)}((\tilde{x}_{Jjv})^-)^2} \\
&\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^- - \mathrm{i}\pi(n_{Jjv})^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-\right.\right. \\
&\qquad\qquad \left.-\mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\Big) \\
&\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^- - \mathrm{i}\pi(n_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-\right.\right. \\
&\qquad\qquad \left.\left.\left.-\mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)\right).
\end{aligned} \tag{2.49}$$

Here we have combined all the exponentials that do not depend on $n_{J\sigma j\tilde{v}}$ of the various edges to a compact form summing over $\tilde{v}$ and $J, j, \sigma$ again.

In appendix C, we show that the only term that contributes to the infinite sum over $(n_{Jjv})^-$ is the term with $(n_{Jjv})^- = 0$ all other terms are of the order $O(t^\infty)$. Hence, up to order $O(t^\infty)$

the expectation value is given by

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}$$
$$= \frac{\mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{(J,\sigma,j)\in L}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\;\mathrm{e}^{-\frac{t}{4}\sum_{\tilde{v}\in V}\sum_{(J,\sigma,j)\in L}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}}$$
$$\frac{(\sqrt{\pi})^{12}}{T^{30}}2^9\left(\sqrt{\frac{\pi}{2}}\right)^9\int_{\mathbb{R}^9}\mathrm{d}^9(\tilde{x}_{Jjv})^-\,\mathrm{e}^{-2\sum_{(J,j)\in L_v}((\tilde{x}_{Jjv})^-)^2}$$
$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)$$
$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right). \tag{2.50}$$

### 2.9. Expansion of the $\Lambda^{\frac{1}{2}}$ functions

Although the expectation value simplifies a lot when considering the $n_{J\sigma j\tilde{v}} = 0$ term only, the integral in equation (2.50) cannot be performed analytically due to the occurrence of $\Lambda^{\frac{1}{2}}$ functions. The way out of this problem is to expand these functions in terms of powers of $(\tilde{x}_{Jjv})^-$ which yields integrals of the form $\int_{\mathbb{R}^9}\mathrm{d}^9(\tilde{x}_{Jjv})^-\mathrm{e}^{-2((\tilde{x}_{Jjv})^-)^2}((\tilde{x}_{Jjv})^-)^k$ with $k\in\mathbb{N}$ that can be solved analytically. Here we will use the same technique that was introduced in [13]. Recalling again the definition of $\Lambda^{\frac{1}{2}}$ in terms of the determinants and furthermore introducing the dimensionless matrix

$$(q_{Jjv})^- := (p_{Jjv})^- t^{-\alpha}, \qquad \text{with} \qquad s = t^{\frac{1}{2}-\alpha}, \tag{2.51}$$

where $t^\alpha = \frac{\epsilon^2}{a^2}$. This relation takes its origin in the analysis of the estimation of the discretization as well as the quantization error. Roughly speaking, on one hand the discretization error will be proportional to $\left(\frac{\epsilon}{a}\right)^n$, where $n > 0$. On the other hand, we have the quantum fluctuations that are proportional to $\frac{t}{(\frac{\epsilon}{a})^m}$ with $m > 0$. Thus, the discretization error decreases when $\epsilon$ gets smaller, while the quantum fluctuation error increases and might even diverge in the limit $\epsilon \to 0$. Therefore, we can conclude that the total error will be minimized for $\frac{\epsilon}{a} \propto t^{\frac{\alpha}{2}}$ for some $\alpha > 0$. The concrete value of the optimal $\alpha$ will strongly depend on whether one is interested in a very small classical error and accepts larger, but still finite quantum fluctuation, or whether one wants to keep the quantum fluctuations as small as possible and deals with a larger discretization error, or whether one takes the point of view that both errors should be treated with equal weight. In [14], a value of $\alpha = \frac{1}{6}$ was proposed by having made a rough estimate. We do not want to fix the value of $\alpha$ here, but rather keep it general as long as possible.

In any case, with the help of $(q_{Jjv})^-$, we can rewrite $\Lambda^{\frac{1}{2}}$ as

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right) cd$$
$$= \frac{t^{\frac{3}{8}}}{T^{\frac{3}{4}}}\left(\frac{a^{\frac{3}{2}}}{\mathrm{i}\hbar}\right)\left(|\det((p)^-)|^{\frac{1}{4}}\right)$$

$$\times \left[ \left| \det\left(1 + s(q^{-1})^{-}(\tilde{x})^{-} + \frac{T}{4}s(q^{-1})^{-}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-} - \mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right) \right|^{\frac{1}{4}} \right.$$
$$- \left| \det\left(1 + s(q^{-1})^{-}(\tilde{x})^{-} + \frac{T}{4}s(q^{-1})^{-}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-} - \mathrm{sgn}(\sigma_0)(\Delta)^{-}] \right.\right.$$
$$\left.\left.\left. + \frac{T}{2}s(q^{-1})^{-}\mathrm{sgn}(\sigma_0)(\delta)^{-}_{(J,j,v),(K_0,n_0,v)}\right) \right|^{\frac{1}{4}} \right]. \tag{2.52}$$

The terms involving determinants can be further expanded by using the following identity:

$$\det(1 + A)^2 = 1 + \underbrace{2z'_A + (z'_A)^2}_{=:\, z_A} =: 1 + z_A, \qquad \text{with}$$
$$z'_A = \mathrm{tr}(A) + \tfrac{1}{2}([\mathrm{tr}(A)]^2 - \mathrm{tr}(A^2)) + \det(A). \tag{2.53}$$

In our case, we have to consider four different $A$ matrices. Let us denote them by $A_1 := sq^{-1}x + \Delta$, $A_2 := sq^{-1}x - \Delta$, $A_3 := sq^{-1}x + \Delta + \delta$ and $A_4 := sq^{-1}x - \Delta + \delta$. Thus, we need the explicit expressions for $z_{sq^{-1}x+\Delta}$, $z_{sq^{-1}x-\Delta}$, $z_{sq^{-1}x+\Delta+\delta}$ and $z_{sq^{-1}x-\Delta+\delta}$ in order to expand all four determinants contained in the two $\Lambda^{\frac{1}{2}}$ functions. These explicit expressions are derived in appendix D.1.

We then define

$$\begin{aligned} y &:= 1 + z_{sq^{-1}x+\Delta} & y_1 &:= 1 + z_{sq^{-1}x+\Delta+\delta} \\ \widetilde{y} &:= 1 + z_{sq^{-1}x-\Delta} & \widetilde{y}_1 &:= 1 + z_{sq^{-1}x-\Delta+\tilde{\delta}}, \end{aligned} \tag{2.54}$$

and due to equation (2.53), we can express the $\Lambda^{\frac{1}{2}}$ functions as

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^{-} + \frac{1}{T}((p_{Jjv})^{-}) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-} - \mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)$$
$$= \frac{a^{\frac{3}{2}}|\det((p)^{-})|^{\frac{1}{4}}}{\mathrm{i}\hbar}\left(y^{\frac{1}{8}} - y_1^{\frac{1}{8}}\right) \tag{2.55}$$

and

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^{-} + \frac{1}{T}((p_{Jjv})^{-}) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-} - \mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)$$
$$= \frac{a^{\frac{3}{2}}|\det((p)^{-})|^{\frac{1}{4}}}{(-\mathrm{i}\hbar)}\left(\widetilde{y}^{\frac{1}{8}} - \widetilde{y}_1^{\frac{1}{8}}\right). \tag{2.56}$$

In order to continue the calculation we will expand $y^{\frac{1}{8}}, y_1^{\frac{1}{8}}, \widetilde{y}^{\frac{1}{8}}, \widetilde{y}_1^{\frac{1}{8}}$ around $y = y_1 = \widetilde{y} = \widetilde{y}_1 = 1$. Here we follow the guideline given in [13]. The general expansion yields

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^{-} + \frac{1}{T}((p_{Jjv})^{-}) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-} - \mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)$$
$$= \frac{a^{\frac{3}{2}}|\det((p)^{-})|^{\frac{1}{4}}}{\mathrm{i}\hbar}\left\{(y - y_1)\left(\sum_{k=1}^{n} f^{(k)}_{\frac{1}{8}}(1)\sum_{l=0}^{k-1}(y-1)^l(y_1-1)^{k-1-l}\right)\right.$$
$$\left. + \underbrace{\left[f^{(n+1)}_{\frac{1}{8}}(y)(y-1)^{n+1} - f^{(n+1)}_{\frac{1}{8}}(y_1)(y_1-1)^{(n+1)}\right]}_{\text{remainder}}\right\} \tag{2.57}$$

and similar for the second $\Lambda^{\frac{1}{2}}$ functions where $y$, $y_1$ are replaced by $\widetilde{y}, \widetilde{y}_1$. Here, we will only be interested in the semiclassical limit (leading order) and the first quantum correction (next-to-leading order). Noting that $t = \ell_p^2/a^2$ and therefore $\hbar = t/\kappa a^2$, we obtain the following power counting in $s$ for each single $\Lambda^{\frac{1}{2}}$ function:

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\operatorname{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \operatorname{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\operatorname{sgn}(\sigma_0)e_{K_0}(v), n_0\right)$$
$$= \frac{sT}{t}(1 + s(x_{Jjv})^- + s^2((x_{Jjv})^-)^2 + O(sT)). \tag{2.58}$$

The fact of whether the $s^3$ or the $sT$ contribution is the next-to-next-to leading-order term depends on the value of $\alpha$. The quotient $sT/s^3 = t^{\frac{1}{2}-2\alpha}$ is small as long as $\alpha \leqslant \frac{1}{4}$. When $\alpha$ passes the value of $\frac{1}{4}$, the $s^3$ contribution becomes larger than the corresponding one coming from $sT$. Similar to [13], we consider the $sT$ contribution as the next-to-next-to leading-order (NNLO) term. If one wants to work with an $\alpha$ being greater than $\frac{1}{4}$, one should replace $O(sT)$ by $O(s^3)$ in the power counting above. Since the expectation value contains a product of two $\Lambda^{\frac{1}{2}}$ functions, we will expand the expectation value up to order $\left(\frac{sT}{t}\right)^2 s^2$. In appendix D.2 it is shown that as long as $n \leqslant n_0$, the integral over the remainder when the expansion is reinserted into the expectation value is smaller than the $s^n$ contribution. As mentioned in appendix D.2, the precise value of $n_0$ will depend on the chosen value of $\alpha$. In our case, we have to ensure that when expanding up to order $s^{n'}$ with $n' > 2$ that $s^{n'+1} \ll sTs^2$. This is equivalent to the condition $s^{n'-2} \ll T = t^{\frac{1}{2}}$ from which the minimal value of $n'$ can be computed. The result reads $n' > \frac{\frac{1}{2}+1-2\alpha}{\frac{1}{2}-\alpha}$. For instance for the suggested value of $\alpha = \frac{1}{6}$ in [14], the minimal value of $n'$ is $n' = 4$ which is well below the value of $n_0 \gg 1$. Thus the error of neglecting the remainder is indeed of higher order in $s$ than $\left(\frac{sT}{t}\right)^2 s^2$. Moreover, when $\alpha$ is coming closer to the value of $\frac{1}{2}$, the value of $n'$ increases strongly and a careful analysis can be done on whether $n' < n_0$ is necessary.

In appendix E, we derive the explicit forms of the $y, y_1, \widetilde{y}$ and $\widetilde{y}_1$ up to the necessary orders. It turns out that the lowest contribution in the terms $(y - y_1)$ and $(\widetilde{y} - \widetilde{y}_1)$ respectively is already of the order $sT$. The highest order we want to consider is the next-to-leading-order term of order $(s^3T/t)$. Since all other terms occurring in the $\Lambda^{\frac{1}{2}}$ expansion are multiplied by $(y - y_1)$ and $(\widetilde{y} - \widetilde{y}_1)$ respectively, we expand $(y - y_1)$ and $(\widetilde{y} - \widetilde{y}_1)$ respectively up to order $(s^3T/t)$ and all the other terms occur in the expansion up to order $O(s^3)$.[3]

The expansion of the $\Lambda^{\frac{1}{2}}$ functions up to $O((sT)^2/t)$ yields

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\operatorname{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \operatorname{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\operatorname{sgn}(\sigma_0)e_{K_0}(v), n_0\right)$$
$$= \frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\mathrm{i}\hbar}(y - y_1)\big\{\big(f^{(1)}_{\frac{1}{8}}(1) + f^{(2)}_{\frac{1}{8}}(1)(2(y-1) + (y_1 - y))$$
$$+ f^{(3)}_{\frac{1}{8}}(1)(3(y_1-1)^2 + 3(y-1)(y_1-1) + (y_1-y)^2)\big)\big\} + O((sT)^2/t) \tag{2.59}$$

and

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}[\operatorname{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \operatorname{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\operatorname{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)$$
$$= \frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{(-\mathrm{i}\hbar)}(\widetilde{y} - \widetilde{y}_1)\big\{\big(f^{(1)}_{\frac{1}{8}}(1) + f^{(2)}_{\frac{1}{8}}(1)(2(\widetilde{y}-1) + (\widetilde{y}_1 - \widetilde{y}))$$
$$+ f^{(3)}_{\frac{1}{8}}(1)(3(\widetilde{y}_1-1)^2 + 3(\widetilde{y}-1)(\widetilde{y}_1-1) + (\widetilde{y}_1-\widetilde{y})^2)\big)\big\} + O((sT)^2/t). \tag{2.60}$$

[3] Note, that the proposed value of $\alpha = \frac{1}{6} \leqslant \frac{1}{4}$ in [14] corresponds to a NNLO-term of order $sT$.

## 3. Leading order of the expectation value

Let us summarize the structure of our calculation. The aim is to calculate the expectation value of the algebraic master constraint operator $\widehat{\mathbf{M}}_v$ with respect to certain coherent states. We have introduced an operator $\widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$ which has the advantage that the expectation value of $\widehat{\mathbf{M}}_v$ can be expressed in terms of a sum over $I_0$, $J_0$ $K_0$, $\widetilde{I}_0$, $\widetilde{J}_0$, $\widetilde{K}_0$ of expectation values of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$. By actually analysing the expectation value of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$, we saw that the so-called $\Lambda^{\frac{1}{2}}$ function occurring in the expectation value cannot be integrated analytically. Therefore, we are forced to expand these functions in terms of powers as $sT/t(1 + s(\tilde{x})^- + s^2((\tilde{x})^-)^2 + O(sT))$.

In this section, we are interested in the leading order of the expectation value of $\widehat{\mathbf{M}}_v$. Consequently, we need the leading order of $\Lambda^{\frac{1}{2}}$ in order to calculate the leading order of the expectation value of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$. With the knowledge of the result of the expectation value of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$, finally we are able to give an expression for the leading order of the expectation value of $\widehat{\mathbf{M}}_v$.

The detailed analysis of the explicit expressions for $y$, $y_1$, $\widetilde{y}$ and $\widetilde{y}_1$ which are derived in appendix E shows that the leading of $\Lambda^{\frac{1}{2}}$ is of order $sT/t$. This is due to the fact that any term in the expansion is multiplied by a term of the form $(y - y_1)$ and $(\widetilde{y} - \widetilde{y}_1)$ respectively. The lowest order of these terms is $sT$ which together with the $1/\hbar \propto 1/t$ in equations (2.59) and (2.60) respectively combines into terms of the order $sT/t$. The explicit expressions are given by

$$y - y_1|_{sT} = -sT\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0 n_0} \qquad \widetilde{y} - \widetilde{y}_1|_{sT} = -sT\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}. \tag{3.1}$$

Hence, the leading-order expansion for the $\Lambda^{\frac{1}{2}}$ functions are

$$\begin{aligned}\Lambda^{\frac{1}{2}}&\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)\\ &= \frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\mathrm{i}\hbar} f^{(1)}_{\frac{1}{8}}(1)(-sT\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0 n_0}).\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\Lambda^{\frac{1}{2}}&\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)\\ &= \frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{(-\mathrm{i}\hbar)} f^{(1)}_{\frac{1}{8}}(1)(-sT\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}).\end{aligned} \tag{3.3}$$

We know that the expectation value of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$ contains a product of these two leading-order $\Lambda^{\frac{1}{2}}$ functions. Consequently, the leading order of the expectation value will be of the order $O((sT/t)^2$. Reinserting the leading order $\Lambda^{\frac{1}{2}}$ into the expression of the expectation value of the $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}$ leads to

$$\frac{\big\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\Psi^t_{\{g,J,\sigma,j,L\}} \,\big|\, \widehat{O}^{m_0,n_0}_{I_0 J_0 K_0 \sigma_0 v}\Psi^t_{\{g,J,\sigma,j,L\}}\big\rangle}{\big\|\Psi^t_{\{g,J,\sigma,j,L\}}\big\|^2}$$

$$= \frac{\mathrm{e}^{+\mathrm{i}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}} \varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\ \mathrm{e}^{-\frac{t}{4}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1 + K_t(p)]^{30}}$$

$$\frac{(\sqrt{\pi})^{12}}{T^{30}} 2^9 \left(\sqrt{\frac{\pi}{2}}\right)^9 \int_{\mathbb{R}^9} \mathrm{d}^9 (\tilde{x}_{Jjv})^- \, \mathrm{e}^{-2 \sum_{\substack{(J,j)\\ \in L_v}} ((x_{Jjv})^-)^2}$$
$$\left(\frac{a^{\frac{3}{2}} |\det((p)^-)|^{\frac{1}{4}}}{\hbar}\right)^2 \left(f^{(1)}_{\frac{1}{8}}(1)\right)^2 (-sT \operatorname{sgn}(\sigma_0)(q^{-1})^-_{K_0 n_0})(-sT \operatorname{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0 \widetilde{n}_0})$$
$$= \mathrm{e}^{+\mathrm{i} \sum_{\tilde{v}\in V} \sum_{\substack{(J,\sigma,j)\\ \in L}} \varphi_{J\sigma j \tilde{v}} \Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v})}$$
$$\left(\frac{a^{\frac{3}{2}} |\det((p)^-)|^{\frac{1}{2}}}{\hbar}\right)^2 (sT)^2 \left(f^{(1)}_{\frac{1}{8}}(1)\right)^2 (\operatorname{sgn}(\sigma_0)(q^{-1})^-_{K_0 n_0})(\operatorname{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0 \widetilde{n}_0}). \tag{3.4}$$

In the last step, we expanded the exp function

$$\mathrm{e}^{-\frac{t}{4} \sum_{\tilde{v}\in V} \sum_{\substack{(J,\sigma,j)\\ \in L}} (\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v}))^2}$$
$$= 1 - \frac{t}{4} \sum_{\tilde{v}\in V} \sum_{\substack{(J,\sigma,j)\\ \in L}} (\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v}))^2 + \cdots \tag{3.5}$$

and took only the leading term, that is the one in the expansion, since we are collecting terms of order $(sT/t)^2$ only. Finally, we use the above result in order to build the expectation value of $\widehat{\mathbf{M}}_v$ out of it. This yields

$$\frac{\left\langle \Psi^t_{\{g,J,\sigma,j,L\}} \middle| \widehat{\mathbf{M}}_v \middle| \Psi^t_{\{g,J,\sigma,j,L\}} \right\rangle}{\|\Psi^t_{\{g,J,\sigma,j,L\}}\|^2} = \sum_{I_0 J_0 K_0} \sum_{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0} \sum_{\sigma_0=+,-} \sum_{\widetilde{\sigma}_0=+,-} \epsilon^{I_0 J_0 K_0} \epsilon^{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0} \left(\frac{4}{\kappa}\right)^2$$
$$\times \left\{ \left(\delta_{m_0,n_0} \delta_{\widetilde{m}_0,\widetilde{n}_0} + \sum_{\ell_0=1}^{3} \epsilon_{\ell_0 m_0 n_0} \epsilon_{\ell_0 \widetilde{m}_0 \widetilde{n}_0}\right) \left(\frac{a^{\frac{3}{2}} |\det((p)^-)|^{\frac{1}{4}}}{\hbar}\right)^2 (sT)^2 \right.$$
$$\times \mathrm{e}^{+\mathrm{i} \sum_{(J,\sigma,j)} \varphi_{J\sigma j \tilde{v}} \Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v})}$$
$$\left. \times \left(f^{(1)}_{\frac{1}{8}}(1)\right)^2 (\operatorname{sgn}(\sigma_0)(q^{-1})^-_{K_0 n_0})(\operatorname{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0 \widetilde{n}_0}) \right\} + O((sT/t)^2). \tag{3.6}$$

## 4. (One) semiclassical limit of algebraic quantum gravity

Recall the philosophy of algebraic quantum gravity. On the algebraic level, we have an algebraic master constraint operator $\widehat{\mathbf{M}}$ which acts on algebraic graphs $\alpha$. Furthermore, we can define coherent states associated with an embedded image $\gamma = X(\alpha)$ of the algebraic graph. Hence, when the coherent states enter the picture, the missing information such as the topology, the differential structure and the background metric to be approximated are encoded in these coherent states. This has the consequence that algebraic quantum gravity does not have a single semiclassical limit; rather, for each set of coherent states that represent a different topology, the differential structure and background metric to approximate the semiclassical limit will be different in general.

### *4.1. Comparison of the leading-order expectation value with the classical (discretized) master constraint*

In this section, we want to show that the leading order of the expectation value of the algebraic master constraint with respect to the coherent states used in the calculations can indeed be interpreted as the classical master constraint of general relativity. Hence, we can demonstrate that there exists coherent states such that the semiclassical limit of algebraic quantum gravity reproduces the infinitesimal generators of general relativity. Thus, the problem of whether the semiclassical sector includes general relativity, that is still unsolved within the framework of loop quantum gravity, is significantly improved in the context of algebraic quantum gravity.

In order to show that the semiclassical limit of the expectation value of the algebraic master constraint with respect to these coherent states associated with a cubic graph $\alpha$ is the classical master constraint (associated with a cubic graph), we will use three steps as follows.

(1) Recall that the coherent states are labelled by the so-called classicality parameter $t$. Thus we will take the limit $\lim_{t\to 0}\langle\widehat{\mathbf{M}}\rangle_t$ in order to subtract the semiclassical limit out of the expectation value.
(2) We will show that $\lim_{t\to 0}\langle\widehat{\mathbf{M}}\rangle_t = \sum_{v\in E(\alpha)} \mathbf{M}^{\text{cubic}}_v =: \mathbf{M}^{\text{cubic}}$ whereby $\mathbf{M}^{\text{cubic}}$ can be interpreted as a discretized version of the classical master constraint on a lattice with cubic symmetry, i.e. a cubic graph with infinitely many edges.
(3) We will investigate the limit of $\mathbf{M}^{\text{cubic}}$ in which we shrink the lattice length $\epsilon$ to zero and show that this limit is exactly the continuum expression $M$, i.e. the classical master constraint. Note, that since $\mathbf{M}^{\text{cubic}}$ does not depend on the lattice length explicitly due to the fundamental background independence of the theory, this limit is rather easy to take.

*Step 1.* This was already done in the last section when we were actually calculating the leading-order contribution of the expectation value of the algebraic master constraint operator $\widehat{\mathbf{M}}$. We therefore take the result of the last sections as our starting point and proceed with step 2.

*Step 2.* The discretization of the classical master constraint on a cubic graph is given by

$$\mathbf{M}^{\text{cubic}} = \sum_{v\in V(\alpha)} \mathbf{M}^{\text{cubic}}_v \qquad \mathbf{M}^{\text{cubic}}_v = \sum_{\mu=0}^{3} \left[C^{\text{cubic}}_{\mu,v}\right]^2, \tag{4.1}$$

where

$$\begin{aligned} C^{\text{cubic}}_{0,v} &= \sum_{I_0J_0K_0}\sum_{\sigma_0=+,-} \frac{4}{\kappa} a^{\frac{3}{2}} \epsilon^{I_0J_0K_0} h_{\beta_{I_0J_0\sigma_0m_0v}} h_{K_0\sigma_0m_0v}\left\{h^{-1}_{K_0\sigma_0m_0v}, V^{\frac{1}{2}}_{\alpha,v}\right\} \\ C^{\text{cubic}}_{\ell_0,v} &= \sum_{I_0J_0K_0}\sum_{\sigma_0=+,-} \frac{4}{\kappa} a^{\frac{3}{2}} \epsilon_{\ell_0m_0n_0} \epsilon^{I_0J_0K_0} h_{\beta_{I_0J_0\sigma_0m_0v}} h_{K_0\sigma_0n_0v}\left\{h^{-1}_{K_0\sigma_0n_0v}, V^{\frac{1}{2}}_{\alpha,v}\right\}, \end{aligned} \tag{4.2}$$

where $V_{\alpha,v}$ denotes the dimensionless volume of a cube centred around the vertex $v$ with the edge parameter length $\epsilon$. This cube can be parametrized by an embedding $X^a_v : \left[-\frac{\epsilon}{2}, +\frac{\epsilon}{2}\right]^3 \to \sigma$ with $(t^1, t^2, t^3) \mapsto X^a_v(t^1, t^2, t^3)$ and $X^a_v(0) = v$. The dimensionless volume can be expressed in terms of the dimensionless fluxes $p_{J\sigma jv}$:

$$V_{\alpha,v} = \sqrt{\left|\epsilon^{jkl}\left(\frac{p_{1+jv} - p_{1-jv}}{2}\right)\left(\frac{p_{2+kv} - p_{2-kv}}{2}\right)\left(\frac{p_{3+lv} - p_{3-lv}}{2}\right)\right|}. \tag{4.3}$$

Recall that in our notation $p_{1+jv}$ denotes the dimensionless $j$ component of the flux through the surface $S_{e_1^+(v)}$ etc. Introducing $(p_{Jjv})^- := \frac{1}{2}(p_{J+jv} - p_{J-jv})$ the volume can be re-expressed as

$$V_{\alpha,v} = \sqrt{|\det((p_{Jjv})^-)|}. \tag{4.4}$$

We will show

$$h_{\beta_{I_0J_0\sigma_0m_0v}} h_{K_0\sigma_0n_0v}\left\{(h_{K_0\sigma_0n_0v})^{-1}, V^{\frac{1}{2}}_{\alpha,v}\right\} = \mathrm{i}\frac{\kappa}{8a^2} h_{\beta_{I_0J_0m_0\sigma_0v}} |\det((p_{J\sigma jv})^-)|^{\frac{1}{4}} [(p_{K_0\sigma_0n_0v})^-]^{-1}, \tag{4.5}$$

where

$$h_{\beta_{I_0 J_0 \sigma_0 m_0 v}} = \mathrm{e}^{+\mathrm{i}(\varphi_{I_0\sigma_0 m_0 v}+\varphi_{J_0\sigma_0 m_0 v+\sigma_0 \hat{I}_0}-\varphi_{J_0\sigma_0 m_0 v}-\varphi_{I_0\sigma_0 m_0 v+\sigma_0 \hat{J}_0})}. \tag{4.6}$$

The classical Poisson bracket reads

$$h_{K_0\sigma_0 n_0 v}\big\{(h_{K_0\sigma_0 n_0 v})^{-1}, V^{\frac{1}{2}}_{\alpha,v}\big\} = \kappa h_{K_0\sigma_0 n_0 v}\int \mathrm{d}^3 z\left(\frac{\delta(h_{K_0\sigma_0 n_0 v})^{-1}}{\delta A^k_b(z)}\right)\left(\frac{\delta V^{\frac{1}{2}}\alpha, v((p_{Jjv})^-)}{\delta E^b_k(z)}\right). \tag{4.7}$$

We have

$$\begin{aligned} h_{K_0\sigma_0 n_0 v}\left(\frac{\delta(h_{K_0\sigma_0 n_0 v})^{-1}}{\delta A^k_b(z)}\right) &= \mathrm{i}\int_0^1 \mathrm{d}t\,\dot{e}^{\sigma_0}_{K_0}(t)\delta^k_{n_0}\delta^b_a\delta(t,z)\\ \left(\frac{\delta V^{\frac{1}{2}}_{\alpha,v}((p_{J\sigma jv})^-)}{\delta E^b_k(z)}\right) &= \frac{1}{4}|\det((p_{Jjv})^-)|^{-\frac{3}{4}}\operatorname{sgn}(\det((p_{Jjv})^-))\frac{\delta\det((p_{Jjv})^-)}{\delta E^b_k(z)}. \end{aligned} \tag{4.8}$$

Recalling

$$(p_{Jjv})^- = \frac{1}{2a^2}\left(\int_{S_{e^+_J}} n_a^{S_{e^+_J}} E^a_j - \int_{S_{e^-_J}} n_a^{S_{e^-_J}} E^a_j\right), \tag{4.9}$$

we obtain

$$\begin{aligned} \frac{\delta\det((p_{Jjv})^-)}{\delta E^b_k(z)} &= \frac{\det((p_{Jjv})^-)}{2a^2}(p^{-1}_{Iiv})^-\\ &\times\left(\int_{S_{e^+_I}}\mathrm{d}^2u\, n^{S_{e^+_I}}(u_1,u_2)\delta^a_b\delta^k_i\delta(x(u),z) - \int_{S_{e^-_I}}\mathrm{d}^2u\, n^{S_{e^-_I}}(u_1,u_2)\delta^a_b\delta^k_i\delta(x(u),z)\right). \end{aligned} \tag{4.10}$$

Thus, we get

$$\begin{aligned} &h_{\beta_{I_0 J_0\sigma_0 m_0 v}} h_{K_0 n_0\sigma_0 v}\{h^{-1}_{K_0 n_0\sigma_0 v}, V^{\frac{1}{2}}_{\alpha,v}\}\\ &\quad= \frac{\mathrm{i}\kappa}{8a^2}h_{\beta_{I_0 J_0\sigma_0 m_0 v}}|\det((p_{Jjv})^-)|^{-\frac{3}{4}}\operatorname{sgn}(\det((p_{Jjv})^-)\det((p_{Jjv})^-)(p^{-1}_{Iiv})^-\\ &\qquad\times\Bigg(\underbrace{\int_0^1\mathrm{d}t\int_{S_{e^+_I}}\mathrm{d}^2u\,\dot{e}^{\sigma_0}_{K_0}(t)n_a^{S_{e^+_I}}(u_1,u_2)\delta(x(u),z)\delta^k_{n_0}\delta^i_k\delta^b_a\delta^a_b}_{\delta_{(K_0,\sigma_0,n_0,v)(I,+,i,v)}}\\ &\qquad-\underbrace{\int_0^1\mathrm{d}t\int_{S_{e^-_I}}\mathrm{d}^2u\,\dot{e}^{\sigma_0}_{K_0}(t)n_a^{S_{e^-_I}}(u_1,u_2)\delta(x(u),z)\delta^k_{n_0}\delta^i_k\delta^b_a\delta^a_b}_{\delta_{(K_0,\sigma_0,n_0,v)(I,-,i,v)}}\Bigg)\\ &\quad= \frac{\mathrm{i}\kappa}{8a^2}h_{\beta_{I_0 J_0\sigma_0 m_0 v}}|\det((p_{Jjv})^-)|^{\frac{1}{4}}\operatorname{sgn}(\sigma_0)(p^-_{K_0 n_0 v})^-. \end{aligned} \tag{4.11}$$

Consequently, we have

$$\begin{aligned} &\big(a^{\frac{3}{2}}h_{\beta_{I_0 J_0\tilde{\sigma}_0 m_0 v}}h_{K_0 n_0\sigma_0 v}\{h^{-1}_{K_0 n_0\sigma_0 v}, V^{\frac{1}{2}}_{\alpha,v}\}\big)^* a^{\frac{3}{2}}h_{\beta_{I_0 J_0\tilde{\sigma}_0 m_0 v}}h_{K_0 n_0\sigma_0 v}\big\{h^{-1}_{K_0 n_0\sigma_0 v}, V^{\frac{1}{2}}_{\alpha,v}\big\}\\ &\quad= \left(\frac{\kappa a^{\frac{3}{2}}}{8a^2}\right)^2 (h_{\beta_{\tilde{I}_0\tilde{J}_0\tilde{\sigma}_0\tilde{m}_0 v}})^{-1}\widehat{h}_{\beta_{I_0 J_0\sigma_0 m_0 v}}\big(|\det((p_{Jjv})^-)|^{\frac{1}{4}}\big)^2\\ &\qquad\times\operatorname{sgn}(\sigma_0)\big(p^{-1}_{K_0 n_0 v}\big)^-\operatorname{sgn}\big(\widetilde{\sigma}_0\big)(p^{-1}_{\widetilde{K}_0\widetilde{n}_0 v})^-. \end{aligned} \tag{4.12}$$

From our semiclassical calculation, we get

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0 J_0 K_0\sigma_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}$$
$$= \mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\left(\frac{a^{\frac{3}{2}}}{\hbar}|\det((p_{Jjv})^-)|^{\frac{1}{4}}\right)^2$$
$$\times(-sT)f^{(1)}_{\frac{1}{8}}(1)\mathrm{sgn}(\sigma_0)((q^{-1})^-)_{K_0 n_0}(-sT)f^{(1)}_{\frac{1}{8}}(1)\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-)_{\widetilde{K}_0\widetilde{n}_0}. \tag{4.13}$$

Therefore, we have to show that the result above agrees with the expression in equation (4.12) in order to show the correctness of the leading order.

Using

$$\left(q^{-1}_{K_0 n_0 v}\right)^- = \left(p^{-1}_{K_0 n_0 v}\right)^- t^{\alpha}, \tag{4.14}$$

we obtain

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0 J_0 K_0\sigma_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}$$
$$= \left(\frac{a^{\frac{3}{2}}}{\hbar}\right)^2 \mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}|\det((p_{Jjv})^-)|^{\frac{1}{2}}t^{2\alpha}$$
$$\times(-sT)f^{(1)}_{\frac{1}{8}}\mathrm{sgn}(\sigma_0)\left(p^{-1}_{K_0 n_0 v}\right)^-(-sT)f^{(1)}_{\frac{1}{8}}\mathrm{sgn}(\widetilde{\sigma}_0)\left(p^{-1}_{\widetilde{K}_0\widetilde{n}_0 v}\right)^-$$
$$= \left(\frac{t^{2\alpha}a^{\frac{3}{2}}}{\hbar}\right)^2 \mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}|\det((p_{Jjv})^-)|^{\frac{1}{2}}\left(t^{\frac{1}{2}-\alpha}t^{\frac{1}{2}}\right)^2$$
$$\times\frac{1}{8}\mathrm{sgn}(\sigma_0)((p_{K_0 n_0 v})^-)^{-1}\frac{1}{8}\mathrm{sgn}(\widetilde{\sigma}_0)((p_{\widetilde{K}_0\widetilde{n}_0 v})^-)^{-1}$$
$$= \left(\frac{\ell_p^2 a^{\frac{3}{2}}}{8\hbar a^2}\right)^2 \mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}|\det((p_{Jjv})^-)|^{\frac{1}{2}}$$
$$\times\mathrm{sgn}(\sigma_0)(p^{-1}_{K_0 n_0 v})^-\mathrm{sgn}(\sigma_0)(p^{-1}_{\widetilde{K}_0\widetilde{n}_0 v})^-$$
$$= \left(\frac{\kappa a^{\frac{3}{2}}}{8a^2}\right)^2 \mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}|\det((p_{Jjv})^-)|^{\frac{1}{2}}$$
$$\times\mathrm{sgn}(\sigma_0)\left(p^{-1}_{K_0 n_0 v}\right)^-\mathrm{sgn}(\widetilde{\sigma}_0)\left(p^{-1}_{\widetilde{K}_0\widetilde{n}_0 v}\right)^-, \tag{4.15}$$

whereby we used in the second line $s = t^{\frac{1}{2}-\alpha}$, $T = t^{\frac{1}{2}}$, in the third line $ta^2 = \ell_p^2$ and in the last step the fact that $\kappa = \ell_p^2/\hbar$.

Using the explicit definition of $\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v})$, we get

$$\mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})} = \mathrm{e}^{+\mathrm{i}(\varphi_{\widetilde{J}_0\widetilde{\sigma}_0\widetilde{m}_0 v}+\varphi_{\widetilde{I}_0\widetilde{\sigma}_0\widetilde{m}_0 v+\widetilde{\sigma}_0\hat{\widetilde{J}}_0}-\varphi_{\widetilde{I}_0\widetilde{\sigma}_0\widetilde{m}_0 v}+\varphi_{\widetilde{J}_0\widetilde{\sigma}_0\widetilde{m}_0 v+\widetilde{\sigma}_0\hat{\widetilde{I}}_0})}$$
$$\times\,\mathrm{e}^{-\mathrm{i}(\varphi_{J_0\sigma_0 m_0 v}+\varphi_{I_0\sigma_0 m_0 v+\sigma_0\hat{J}_0}-\varphi_{I_0\sigma_0 m_0 v}+\varphi_{J_0\sigma_0 m_0 v+\sigma_0\hat{I}_0})}h^{-1}_{\beta_{\widetilde{I}_0\widetilde{J}_0\widetilde{\sigma}_0\widetilde{m}_0 v}}h_{\beta_{I_0 J_0\sigma_0 m_0 v}}. \tag{4.16}$$

Finally,

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}$$
$$= \left(\frac{\kappa a^{\frac{3}{2}}}{8a^2}\right)^2 h^{-1}_{\beta_{\widetilde{I}_0\widetilde{J}_0\widetilde{\sigma}_0\widetilde{m}_0 v}} h_{\beta_{I_0J_0\sigma m_0 v}} |\det((p_{Jjv})^-)|^{\frac{1}{2}} \operatorname{sgn}(\sigma_0)\left(p^{-1}_{K_0n_0v}\right)^- \operatorname{sgn}(\widetilde{\sigma}_0)\left(p^{-1}_{\widetilde{K}_0\widetilde{n}_0 v}\right)^-$$
$$= \left(a^{\frac{3}{2}} h^{-1}_{\beta_{\widetilde{I}_0\widetilde{J}_0\widetilde{\sigma}_0\widetilde{m}_0 v}} h_{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0 v}\{h^{-1}_{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0 v}, V^{\frac{1}{2}}_{\alpha,v}\}\right)^* a^{\frac{3}{2}} h_{\beta_{I_0J_0\sigma_0 m_0 v}} h_{K_0\sigma_0 n_0 v}\left\{h^{-1}_{K_0\sigma_0 n_0 v}, V^{\frac{1}{2}}_{\alpha,v}\right\}. \tag{4.17}$$

Therefore, the basic building blocks of the leading order reproduce the correct classical building blocks of the classical discretized master constraint. Thus, the semiclassical limit of the algebraic master constraint can indeed be interpreted as the discretization of the classical master constraint on a cubic lattice, since the discrete master constraint differs from the expectation value of $\left(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\right)^\dagger \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$ only by an additional summation over $I_0, J_0, K_0, \widetilde{I}_0, \widetilde{J}_0, \widetilde{K}_0, \sigma_0, \widetilde{\sigma}_0$ and a multiplication with $(\mathrm{i}/2)^2$. The same summation and multiplication has to be performed on the classical side as well; thus we have shown that each summand in the sum has the correct semiclassical limit

$$\sum_{v\in V(\alpha)} \left.\frac{\left\langle \Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{\mathbf{M}}_v\,\middle|\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2}\right|_{LO} = \mathbf{M}^{\mathrm{cubic}} \tag{4.18}$$

and therefore, we are done with step two.

*Step 3*. The last step that remains to be shown is that the discretized version of the classical master constraint $\mathbf{M}^{\mathrm{cubic}}$ yields the continuum constraint when we shrink the parameter interval length $\epsilon$ to zero. For this purpose, we will expand the discretized master constraint $\mathbf{M}^{\mathrm{cubic}}$ in terms of powers of $\epsilon$. Recall the form of $\mathbf{M}^{\mathrm{cubic}}$

$$\mathbf{M}^{\mathrm{cubic}} = \left(\frac{4}{\kappa}\right)^2 a^{\frac{3}{2}} \sum_{v\in V(\alpha)}\sum_{I_0J_0K_0}\sum_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0}\sum_{\sigma_0=+,-}\sum_{\widetilde{\sigma}_0=+,-} \epsilon^{I_0J_0K_0}\epsilon^{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0}$$
$$\times\left\{\left(\delta_{m_0,n_0}\delta_{\widetilde{m}_0,\widetilde{n}_0} + \sum_{\ell_0=1}^3 \epsilon_{\ell_0 m_0 n_0}\epsilon_{\ell_0\widetilde{m}_0\widetilde{n}_0}\right)\right.$$
$$\times h^{-1}_{\beta_{\widetilde{I}_0\widetilde{J}_0\widetilde{\sigma}_0\widetilde{m}_0 v}} h_{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0 v}\left\{h^{-1}_{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0 v}, V^{\frac{1}{2}}_{\alpha,v}\right\} h_{\beta_{I_0J_0\sigma_0 m_0 v}} h_{K_0\sigma_0 n_0 v}\left\{h^{-1}_{K_0\sigma_0 n_0 v}, V^{\frac{1}{2}}_{\alpha,v}\right\}$$
$$= \sum_{v\in V(\alpha)}\sum_{I_0J_0K_0}\sum_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0}\sum_{\sigma_0=+,-}\sum_{\widetilde{\sigma}_0=+,-} \epsilon^{I_0J_0K_0}\epsilon^{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0}\left\{\left(\delta_{m_0,n_0}\delta_{\widetilde{m}_0,\widetilde{n}_0} + \sum_{\ell_0=1}^3 \epsilon_{\ell_0 m_0 n_0}\epsilon_{\ell_0\widetilde{m}_0\widetilde{n}_0}\right)\right.$$
$$\times\left(\frac{a^{\frac{3}{2}}}{2a^2}\right)^2 \operatorname{sgn}(\widetilde{\sigma}_0)\operatorname{sgn}(\sigma_0)\left(h_{\beta_{\widetilde{I}_0\widetilde{J}_0\widetilde{\sigma}_0\widetilde{m}_0 v}}\right)^{-1}\widehat{h}_{\beta_{I_0J_0\sigma_0 m_0 v}}|\det((p_{Jjv})^-)|^{\frac{1}{2}}$$
$$\times\left(p^{-1}_{K_0n_0v}\right)^-\left(p^{-1}_{\widetilde{K}_0\widetilde{n}_0 v}\right)^- \tag{4.19}$$

whereby we used the result derived in step 2 in the last line. The first thing we do is to re-express $(p_{K_0n_0v})^-$ in terms of $p_{K_0+n_0v}$ and $p_{K_0-n_0v}$. The relation is given in equation (B.2). Inverting this equation then yields

$$p_{K_0+n_0v} = (p_{K_0n_0v})^+ + (p_{K_0n_0v})^- \qquad p_{K_0-n_0v} = (p_{K_0n_0v})^+ - (p_{K_0n_0v})^-. \tag{4.20}$$

So, for a general $\sigma_0 \in \{+, -\}$, we have

$$p_{K_0\sigma_0 n_0 v} = (p_{K_0 n_0 v})^+ + \mathrm{sgn}(\sigma_0)(p_{K_0 n_0 v})^- \quad \Leftrightarrow \quad \mathrm{sgn}(\sigma_0)\big(p^{-1}_{K_0 n_0 v}\big)^- = p^{-1}_{K_0\sigma_0 n_0 v} - \big(p^{-1}_{K_0 n_0 v}\big)^+. \tag{4.21}$$

Now, we have to express $p^{-1}_{K_0\sigma_0 n_0 v}$ in terms of $p_{K_0\sigma_0 n_0 v}$. The relation is given by

$$\begin{aligned} p^{-1}_{K_0\sigma_0 n_0 v} &= \frac{1}{2}\epsilon_{K_0 MN}\epsilon_{n_0 mn}\frac{p_{M\sigma_0 mv}\,p_{N\sigma_0 nv}}{\det(p_{J\sigma_0 jv})} \\ &= \frac{a^2}{2}\epsilon_{K_0 MN}\epsilon_{n_0 mn}\frac{E^{M\sigma_0}_{mv}E^{N\sigma_0}_{nv}}{\det\big(E^{J\sigma_0}_{jv}\big)}, \end{aligned} \tag{4.22}$$

where we introduced $E^{J\sigma}_{jv} := \int_{S_{e^\sigma_J(v)}} n_a^{S_{e^\sigma_J(v)}} E^a_j$. Analogously, we get

$$\big(p^{-1}_{K_0 n_0 v}\big)^+ = \frac{a^2}{2}\epsilon_{K_0 MN}\epsilon_{n_0 mn}\left(\frac{E^{M+}_{mv}E^{N+}_{nv}}{\det\big(E^{J+}_{jv}\big)} + \frac{E^{M-}_{mv}E^{N-}_{nv}}{\det\big(E^{J-}_{jv}\big)}\right). \tag{4.23}$$

Now, we expand the loops and the fluxes in powers of $\epsilon$. With the orientation of the edges we have chosen and explained at the beginning, we have

$$\begin{aligned} h_{\beta_{I_0 J_0 + m_0 v}} &= h_{I_0 + m_0 v} \circ h_{J_0 + m_0 v + \hat{I}_0} \circ h^{-1}_{I_0 + m_0 v + \hat{J}_0} \circ h^{-1}_{J_0 + m_0 v} \\ h_{\beta_{I_0 J_0 - m_0 v}} &= h_{I_0 - m_0 v} \circ h_{J_0 - m_0 v - \hat{I}_0} \circ h^{-1}_{I_0 - m_0 v - \hat{J}_0} \circ h^{-1}_{J_0 - m_0 v}. \end{aligned} \tag{4.24}$$

Thus, the expansion yields

$$\begin{aligned} h_{\beta_{I_0 J_0 \sigma_0 m_0 v}} &\approx 1 + \epsilon^2 \mathrm{sgn}(\sigma_0) F^{m_0}_{I_0 J_0}(v) + O(\epsilon^3) \\ h^{-1}_{\beta_{\tilde{I}_0 \tilde{J}_0 \tilde{\sigma}_0 \tilde{m}_0 v}} &\approx 1 - \mathrm{sgn}(\widetilde{\sigma}_0)\epsilon^2 F^{\widetilde{m}_0}_{\widetilde{I}_0 \widetilde{J}_0}(v) + O(\epsilon^3) = 1 + \epsilon^2 \mathrm{sgn}(\widetilde{\sigma}_0) F^{\widetilde{m}_0}_{\widetilde{J}_0 \widetilde{I}_0}(v) + O(\epsilon^3) \end{aligned} \tag{4.25}$$

and for the flux

$$E^{J\sigma}_{jv} = \int_{S_{e^{\sigma_0}_{K_0}}} n_a^{K_0\sigma_0} E^a_{n_0} \approx E^a_{n_0}(v)\epsilon^2 n_a^{K_0\sigma_0}(v) + O(\epsilon^3), \tag{4.26}$$

whereby we introduced the shorthand $n_a^{S_{e^{\sigma_0}_{K_0}}}(v) = n^{K_0\sigma_0}(v)$. The determinant of the fluxes is therefore approximated by

$$\det\big(E^{J\sigma_0}_{jv}\big) \approx \det\big(E^a_j(v)\epsilon^2 n_a^{J\sigma_0}(v) + O(\epsilon^3)\big) = \epsilon^6 \det\big(E^a_j(v)\big)\det\big(n_a^{J\sigma_0}(v)\big) + O(\epsilon^8). \tag{4.27}$$

Due to the fact that $\det\big(n_a^{J+}(v)\big) = -\det\big(n_a^{J-}(v)\big)$, we conclude that $\big(p^{-1}_{K_0 n_0 v}\big)^+$ vanishes in the leading order, because the two terms cancel each other exactly. Therefore, we get

$$\begin{aligned} \mathrm{sgn}(\sigma_0)\big(p^{-1}_{K_0 n_0 v}\big)^- &\approx \frac{a^2}{2}\epsilon_{K_0 MN}\epsilon^{n_0 mn}\frac{E^a_m(v)n_a^{M\sigma_0}(v)E^b_n(v)n_b^{N\sigma_0}(v)}{\det\big(E^a_j(v)\big)\det\big(n_a^{J\sigma_0}(v)\big)}\left(\frac{1 + O(\epsilon^2)}{\epsilon^2 + O(\epsilon^4)}\right) \\ &= \frac{a^2}{2}\epsilon_{K_0 MN}\epsilon^{n_0 mn}\mathrm{sgn}(\sigma_0)\frac{E^a_m(v)n_a^{M\sigma_0}(v)E^b_n(v)n_b^{N\sigma_0}(v)}{\det\big(E^a_j(v)\big)\big|\det\big(n_a^{J\sigma_0}(v)\big)\big|}\left(\frac{1 + O(\epsilon^2)}{\epsilon^2 + O(\epsilon^4)}\right). \end{aligned} \tag{4.28}$$

Reinserting the expanded terms into $\mathbf{M}^{\text{cubic}}$, we end up with

$$\mathbf{M}^{\text{cubic}} \approx \sum_{v\in V(\alpha)} \sum_{I_0 J_0 K_0} \sum_{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0} \sum_{\sigma_0=+,-} \sum_{\widetilde{\sigma}_0=+,-} \epsilon^{I_0 J_0 K_0} \epsilon^{\widetilde{I}_0 \widetilde{J}_0 \widetilde{K}_0} \left\{ \left( \delta_{m_0,n_0}\delta_{\widetilde{m}_0,\widetilde{n}_0} + \sum_{\ell_0=1}^{3} \epsilon_{\ell_0 m_0 n_0}\epsilon_{\ell_0 \widetilde{m}_0 \widetilde{n}_0} \right)\right.$$
$$\times \left(\frac{a^{\frac{3}{2}}}{2a^2}\right)^2 \left(1+\epsilon^2 \operatorname{sgn}(\widetilde{\sigma}_0) F^{\widetilde{m}_0}_{\widetilde{J}_0 \widetilde{I}_0}(v) + O(\epsilon^3)\right)\left(1+\operatorname{sgn}(\sigma_0)\epsilon^2 F^{m_0}_{I_0 J_0}(v) + O(\epsilon^3)\right)$$
$$\times \left(\frac{\epsilon^{\frac{3}{2}}}{a^{\frac{3}{2}}} \left|\det\left(E^a_j\right) \det\left(n^{J\sigma_0}_a\right)\right|^{\frac{1}{4}} + O(\epsilon^4)\right) \left(\frac{\epsilon^{\frac{3}{2}}}{a^{\frac{3}{2}}} \left|\det\left(E^a_j\right) \det\left(n^{J\widetilde{\sigma}_0}_a\right)\right|^{\frac{1}{4}} + O(\epsilon^2)\right)$$
$$\times \left(\frac{a^2}{2}\epsilon_{K_0 MN}\epsilon^{n_0 mn}\operatorname{sgn}(\sigma_0)\frac{E^a_m(v) n^{M\sigma_0}_a(v) E^b_n(v) n^{N\sigma_0}_b(v)}{\det\left(E^a_j(v)\right)\left|\det\left(n^{J\sigma_0}_a(v)\right)\right|}\left(\frac{1+O(\epsilon^2)}{\epsilon^2+O(\epsilon^4)}\right)\right)$$
$$\left.\times \left(\frac{a^2}{2}\epsilon_{K_0 \widetilde{M}\widetilde{N}}\epsilon^{\widetilde{n}_0 \widetilde{m}\widetilde{n}}\operatorname{sgn}(\widetilde{\sigma}_0)\frac{E^a_{\widetilde{m}}(v) n^{\widetilde{M}\widetilde{\sigma}_0}_a(v) E^b_{\widetilde{n}}(v) n^{\widetilde{N}\widetilde{\sigma}_0}_b(v)}{\det\left(E^a_j(v)\right)\left|\det\left(n^{J\widetilde{\sigma}_0}_a(v)\right)\right|}\left(\frac{1+O(\epsilon^2)}{\epsilon^2+O(\epsilon^4)}\right)\right)\right\}. \tag{4.29}$$

Let us consider the summation over $I_0$, $J_0$, $K_0$ separately

$$\sum_{I_0 J_0 K_0} \epsilon^{I_0 J_0 K_0}\left(1+\operatorname{sgn}(\sigma_0)\epsilon^2 F^{m_0}_{I_0 J_0}(v)\right)\epsilon_{K_0 MN}\epsilon^{n_0 mn}\operatorname{sgn}(\sigma_0)\frac{E^a_m(v) n^{M\sigma_0}_a(v) E^b_n(v) n^{N\sigma_0}_b(v)}{\det(E^a_j(v))|\det(n^{J\sigma_0}_a(v))|}$$
$$= 2\operatorname{sgn}(\sigma_0)\epsilon^2 F^{m_0}_{I_0 J_0}(v)\epsilon^{n_0 mn}\operatorname{sgn}(\sigma_0)\frac{E^a_m(v) n^{I_0\sigma_0}_a(v) E^b_n(v) n^{J_0\sigma_0}_b(v)}{\det(E^a_j(v))|\det(n^{J\sigma_0}_a(v))|}$$
$$= 2\epsilon^2 F^{m_0}_{I_0 J_0}(v)\epsilon^{n_0 mn}\frac{E^a_m(v) n^{I_0\sigma_0}_a(v) E^b_n(v) n^{J_0\sigma_0}_b(v)}{\det(E^a_j(v))|\det(n^{J\sigma_0}_a(v))|}. \tag{4.30}$$

In the last step, $\operatorname{sgn}^2(\sigma_0) = 1$ was used. Using $F^{m_0}_{I_0 J_0}(v) = F^{m_0}_{ab}(v)\frac{\partial X^a(v)}{\partial t^{I_0}}\frac{\partial X^b(v)}{\partial t^{J_0}}$, we obtain

$$\sum_{I_0 J_0 K_0} \epsilon^{I_0 J_0 K_0}\left(1+\operatorname{sgn}(\sigma_0)\epsilon^2 F^{m_0}_{I_0 J_0}(v)\right)\epsilon_{K_0 MN}\epsilon^{n_0 mn}\operatorname{sgn}(\sigma_0)\frac{E^a_m(v) n^{M\sigma_0}_a(v) E^b_n(v) n^{N\sigma_0}_b(v)}{\det(E^a_j(v))|\det(n^{J\sigma_0}_a(v))|}$$
$$= 2\epsilon^2 F^{m_0}_{ab}(v)\epsilon^{n_0 mn}\frac{E^c_m(v) E^d_n(v)}{\det(E^f_j(v))|\det(n^{J\sigma_0}_f(v))|}\left(n^{I_0\sigma_0}_c(v)\frac{\partial X^a(v)}{\partial t^{I_0}}\right)\left(n^{J_0\sigma_0}_d(v)\frac{\partial X^b(v)}{\partial t^{J_0}}\right). \tag{4.31}$$

Taking advantage of the identity

$$n^{I\sigma_0}_a(v)\frac{\partial X^b(v)}{\partial t^I} = \delta^a_b \operatorname{sgn}(\sigma_0)\left|\det\left(\frac{\partial X^a(v)}{\partial t^I}\right)\right|, \tag{4.32}$$

we immediately see

$$\det(n^{I\sigma_0}_a(v)) = \left|\det\left(\frac{\partial X^a(v)}{\partial t^I}\right)\right|^2. \tag{4.33}$$

Consequently, the two last terms cancel the $|\det(n^{J\sigma_0}_f(v))|$ in the denominator, and we have

$$\sum_{I_0 J_0 K_0} \epsilon^{I_0 J_0 K_0}\left(1+\operatorname{sgn}(\sigma_0)\epsilon^2 F^{m_0}_{I_0 J_0}(v)\right)\epsilon_{K_0 MN}\epsilon^{n_0 mn}\operatorname{sgn}(\sigma_0)\frac{E^a_m(v) n^{M\sigma_0}_a(v) E^b_n(v) n^{N\sigma_0}_b(v)}{\det(E^a_j(v))|\det(n^{J\sigma_0}_a(v))|}$$
$$= \frac{2\epsilon^2 F^{m_0}_{ab}(v)\epsilon^{n_0 mn} E^a_m(v) E^b_n(v)}{\det(E^c_j(v))}. \tag{4.34}$$

Reinserting the result of the summation over $I_0, J_0, K_0$ and $\widetilde{I}_0, \widetilde{J}_0, \widetilde{K}_0$ respectively into the expectation value of $\mathbf{M}^{\text{cubic}}$ results

$$
\begin{aligned}
\mathbf{M}^{\text{cubic}} &\approx \sum_{v\in V(\alpha)} \sum_{\sigma_0=+,-} \sum_{\widetilde{\sigma}_0=+,-} \left\{ \left( \delta_{m_0,n_0}\delta_{\widetilde{m}_0,\widetilde{n}_0} + \sum_{\ell_0=1}^{3} \epsilon_{\ell_0 m_0 n_0}\epsilon_{\ell_0 \widetilde{m}_0 \widetilde{n}_0} \right) \left( \frac{a^{\frac{3}{2}}a^2}{a^{\frac{3}{2}}a^2} \right)^2 \right. \\
&\times \left( \frac{\epsilon^3}{4} \right) \left( \frac{F^{m_0}_{ab}(v)\epsilon^{n_0 mn} E^a_m(v) E^b_n(v) F^{\widetilde{m}_0}_{\widetilde{a}\widetilde{b}}(v)\epsilon^{\widetilde{n}_0 \widetilde{m}\widetilde{n}} E^{\widetilde{a}}_{\widetilde{m}}(v) E^{\widetilde{b}}_{\widetilde{n}}(v)}{|\det(E^c_j(v))|^{\frac{3}{2}}} \right) \left| \det\left( \frac{\partial X^a}{\partial t^I} \right) \right| \left( \frac{1+O(\epsilon)}{1+O(\epsilon^4)} \right) \\
&= \lim_{\epsilon\to 0} \int_\sigma d^3x \left\{ \left( \frac{\left[\epsilon^{m_0 mn} F^{m_0}_{ab}(x) E^a_m(x) E^b_n(x)\right]^2}{(\sqrt{\det(q(x))})^3} \right) + \sum_{\ell_0=1}^{3} \left( \frac{\left[\epsilon_{\ell_0 m_0 n_0}\epsilon^{n_0 mn} F^{m_0}_{ab}(x) E^a_m(x) E^b_n(x)\right]^2}{(\sqrt{\det(q(x))})^3} \right) \right. \\
&= \left\{ \int_\sigma d^3x \frac{C^2}{(\sqrt{\det(q)})^3} + \frac{q^{ab}C_aC_b(x)}{\sqrt{\det(q)}} \right\} \\
&= \mathbf{M},
\end{aligned} \tag{4.35}
$$

with

$$
C := \epsilon^{n_0 mn} F^{n_0}_{ab}(x) E^a_m(x) E^b_n(x) \qquad \text{and} \qquad C_a := F^j_{ab} E^b_j. \tag{4.36}
$$

Here, as a first step we performed the sum over $\sigma_0, \widetilde{\sigma}_0$ which leads to a factor of 4 and cancels the $\frac{1}{4}$. Secondly, we used $|\det(E^c_j(v))| = |\det(q(v))|$ and finally in the limit $\epsilon \to 0$, we replaced

$$
\sum_{v\in V(\alpha)} \epsilon^3 \left| \det\left( \frac{\partial X^a}{\partial t^I} \right) \right| \longrightarrow \int_\sigma d^3x \tag{4.37}
$$

and realized that all terms above the leading order in $\epsilon$ vanish in the limit $\epsilon \to 0$. Note that we have a denominator of $(\sqrt{\det(q(x))})^3$ for the Hamiltonian constraint here, because we used the version of the constraint which is a scalar density of weight two.

Finally, summarizing steps 1, 2 and 3 we have proved the following identity:

$$
\boxed{\frac{\langle \Psi^t_{\alpha,m} | \widehat{\mathbf{M}} | \Psi^t_{\alpha,m} \rangle}{\| \Psi^t_{\alpha,m} \|^2} = \sum_{v\in V(\alpha)} \frac{\langle \Psi^t_{\{g,J,\sigma,j,L\}} | \widehat{\mathbf{M}}_v | \Psi^t_{\{g,J,\sigma,j,L\}} \rangle}{\| \Psi^t_{\{g,J,\sigma,j,L\}} \|^2} \overset{\lim_{t\to 0}}{=} \mathbf{M}^{\text{cubic}}[m] \overset{\lim_{\epsilon\to 0}}{=} \mathbf{M}[m].} \tag{4.38}
$$

This equation has to be understood in the following way. We have calculated the expectation value of the algebraic master constraint with respect to coherent states. These states carry a classicality label $t \propto \hbar$. Taking $\lim_{t\to 0}$ that corresponds to $\lim_{\hbar\to 0}$, we obtain an expression that can be identified with a discretization $\mathbf{M}^{\text{cubic}}$ of the classical master constraint $\mathbf{M}$ on a cubic lattice. The parameter of this discretization is $\epsilon$, the so-called parameter interval length. Considering $\lim_{\epsilon\to 0}$, we showed in step 3 that $\mathbf{M}^{\text{cubic}}$ coincides with the classical master constraint $\mathbf{M}$.

## 5. The next-to-leading order contribution to the expectation value of the algebraic master constraint

In this section, we will discuss the next-to-leading-order term of the expectation value of the algebraic master constraint operator $\widehat{\mathbf{M}}_v$. As before, our first task is to derive the next-to-leading-order terms of the $\Lambda^{\frac{1}{2}}$ functions. These are the terms denoted by $(sT/t)s(\tilde{x})^-$ and $(sT/t)s^2((\tilde{x})^-)^2$ in the power counting equation (2.58). The derivation for the expanded

$\Lambda^{\frac{1}{2}}$ functions up to $O(s^2(sT/t)^2)$ can be found in appendix F. The product of these two $\Lambda^{\frac{1}{2}}$ functions which is entering the expectation value of $\left(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\right)^{\dagger}\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$ is then given by

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)$$
$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)$$
$$= \left(\frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\hbar}\right)^2 (sT)^2$$
$$\Big(C^{K_0\sigma_0 n_0} + s(\tilde{x})^-_{Mm}C^{Mm,K_0\sigma_0 n_0} + s^2(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\left[\frac{1}{2}C^{Mm,K_0\sigma_0 n_0}C^{Nn}\right.$$
$$\left. + \frac{1}{3!}\epsilon_{ijk}\left[\epsilon_{n_0mn}C^{K_0\sigma_0 i}C^{Mj}C^{Nk} + \epsilon_{\ell_0 0 n}C^{K_0\sigma_0 j}C^{Li}C^{Nk} + \epsilon_{\ell m n_0}C^{K_0\sigma_0 k}C^{Li}C^{Mj}\right]\Big)\right]$$
$$\Big(C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0} + s(\tilde{x})^-_{Mm}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0} + s^2(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\left[\frac{1}{2}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Nn}\right.$$
$$\left. + \frac{1}{3!}\epsilon_{ijk}\left[\epsilon_{\widetilde{n}_0mn}C^{\widetilde{K}_0\widetilde{\sigma}_0 i}C^{Mj}C^{Nk} + \epsilon_{\ell\widetilde{n}_0 n}C^{\widetilde{K}_0\widetilde{\sigma}_0 j}C^{Li}C^{Nk} + \epsilon_{\ell m\widetilde{n}_0}C^{\widetilde{K}_0\widetilde{\sigma}_0 k}C^{Li}C^{Mj}\right]\Big)\right]$$
$$\left[\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2 + s(\tilde{x})^-_{Mm}C^{Mm}\left(8f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\right) + s^2(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\left[C^{Mm,Nn}\left(4f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\right)\right.\right.$$
$$\left.\left.\left. + C^{Mm}C^{Nn}\left(40f^{(1)}_{\frac{1}{8}}(1)f^{(3)}_{\frac{1}{8}}(1) + 32\left(f^{(2)}_{\frac{1}{8}}(1)\right)^2\right)\right]\right]\right|_{s^2(sT/t)^2)}$$
$$+ O(s^2(sT/t)^2), \tag{5.1}$$

where we introduced the shorthands

$$\begin{aligned} C^{Mm} &:= (q^{-1})^-_{Mm} \\ C^{Mm,Nn} &:= 2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm} \\ C^{K_0\sigma_0 n_0} &:= \mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0n_0} \\ C^{Mm,K_0\sigma_0 n_0} &:= \mathrm{sgn}(\sigma_0)\big(2(q^{-1})^-_{Mm}(q^{-1})^-_{K_0n_0} - (q^{-1})^-_{K_0n}(q^{-1})^-_{Nn_0}\big) \end{aligned} \tag{5.2}$$

and $|_{s^2(sT/t)^2)}$ denotes that only terms up to order $s^2(sT/t)^2$ are considered. The expansion has the following structure:

$$\begin{aligned} \left(\frac{sT}{t}\right)^2 &(\alpha_0 + \alpha_1 s(\tilde{x})^- + \alpha_2 s^2((\tilde{x})^-)^2)(\beta_0 + \beta_1 s(\tilde{x})^- + \beta_2 s^2((\tilde{x})^-)^2)(\gamma_0 + \gamma_1 s(\tilde{x})^- \\ &+ \gamma_2 s^2((\tilde{x})^-)^2) = \alpha_0\beta_0\gamma_0 + s^2((\tilde{x})^-)^2\,[\alpha_2(\beta_0+\gamma_0) + \beta_2(\gamma_2+\alpha_2) \\ &+ \gamma_2(\alpha_0+\beta_0) + \alpha_1\beta_1\gamma_0 + \alpha_1\beta_0\gamma_1 + \alpha_0\beta_1\gamma_1] \\ &+ \mathrm{lin}((\tilde{x})^-) + O(s^2(sT/t)^2), \end{aligned} \tag{5.3}$$

whereby $\mathrm{lin}((\tilde{x})^-)$ denotes all the terms linear in $(\tilde{x})^-$ which we do not show in detail as they will not contribute to the final result, because they vanish when integrated against the even function $\exp(-2((\tilde{x})^-)^2)$. The integration of $\Lambda^{\frac{1}{2}}$ multiplied with the Gaussian $\exp(-2((\tilde{x})^-_{Mm})^2)$, which is contained in the expression of the expectation value of $\left(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\right)^{\dagger}\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$ yields $\sqrt{\pi/2}^9$ for the zeroth power and $(9/4)\sqrt{\pi/2}^9$ for the second power in $(\tilde{x})^-_{Mm}$ respectively. Note that we have a factor

$\mathrm{e}^{-\frac{t}{4}\sum_{\tilde{v}\in V}\sum_{(J,\sigma,j)\in L}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}$ in the expression for the expectation value. Therefore, we have to expand this function in powers of $t$. The linear term in $t$ leads to a term having a minimal order of $(sT)^2/t$. This order is already smaller than the terms of the order $s^2(sT/t)^2$, because $[s^2(sT/t)^2]\left[t/(sT)^2\right] = s^2/t = 1/t^{2\alpha} \gg 1$. Fortunately, we can neglect the linear term in $t$ in the expansion of the exp function. We refrain from listing the explicit form of the expectation value of $\left(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\right)^{\dagger}\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$ here which can be found in appendix F in equation (F.13) and discuss directly the final expression of the expectation value of $\widehat{\mathbf{M}}$ given by

$$
\begin{aligned}
&\frac{\left\langle \Psi^t_{\{g,J,\sigma,j,L\}}\middle|\widehat{\mathbf{M}}\middle|\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2} = \mathbf{M} + \frac{9}{4}s^2\sum_{v\in V(\alpha)}\left[\sum_{I_0J_0K_0}\sum_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0}\sum_{\sigma_0=+,-}\sum_{\widetilde{\sigma}_0=+,-}\right.\\
&\times\left\{\epsilon^{I_0J_0K_0}\epsilon^{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0}\left(\delta_{m_0,n_0}\delta_{\widetilde{m}_0,\widetilde{n}_0}+\sum_{\ell_0=1}^{3}\epsilon_{\ell_0 m_0 n_0}\epsilon_{\ell_0\widetilde{m}_0\widetilde{n}_0}\right)\left(\frac{4a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\kappa\hbar}\right)^2\right.\\
&\times(sT)^2\,\mathrm{e}^{+\mathrm{i}\sum\limits_{(J,\sigma,j)}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\\
&\times\Big\{\Big(C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}+\big(f^{(1)}_{\frac{1}{8}}(1)\big)^2\Big)\left[\frac{1}{2}C^{Mm,K_0\sigma_0n_0}C^{Nn}+\frac{1}{3!}\epsilon_{ijk}\big[\epsilon_{n_0mn}C^{K_0\sigma_0i}C^{Mj}C^{Nk}\right.\\
&\qquad\left.+\epsilon_{\ell_{0}0n}C^{K_0\sigma_0j}C^{Li}C^{Nk}+\epsilon_{\ell mn_0}C^{K_0\sigma_0k}C^{Li}C^{Mj}\big]\Big)\right]+\Big(C^{K_0\sigma_0n_0}+\big(f^{(1)}_{\frac{1}{8}}(1)\big)^2\Big)\\
&\left[\frac{1}{2}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Nn}+\frac{1}{3!}\epsilon_{ijk}\big[\epsilon_{n_0mn}C^{\widetilde{K}_0\widetilde{\sigma}_0i}C^{Mj}C^{Nk}+\epsilon_{\ell\widetilde{n}_0n}C^{\widetilde{K}_0\widetilde{\sigma}_0j}C^{Li}C^{Nk}\right.\\
&\qquad\left.+\epsilon_{\ell m\widetilde{n}_0}C^{\widetilde{K}_0\widetilde{\sigma}_0k}C^{Li}C^{Mj}\big]\Big)\right]+(C^{K_0\sigma_0n_0}+C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0})\\
&\left[C^{Mm,Nn}\left(4f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\right)+C^{Mm}C^{Nn}\Big(40f^{(1)}_{\frac{1}{8}}(1)f^{(3)}_{\frac{1}{8}}(1)+32\big(f^{(2)}_{\frac{1}{8}}(1)\big)^2\Big)\right]\\
&\qquad+(C^{K_0\sigma_0n_0}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}+C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Mm,K_0\sigma_0n_0})C^{Nn}\Big(8f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\Big)\\
&\qquad\left.\left.+C^{Mm,K_0\sigma_0n_0}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}\big(f^{(1)}_{\frac{1}{8}}(1)\big)^2\right\}\right]+O(s^2(sT/t)^2).
\end{aligned}
\tag{5.4}
$$

From the above result, one can conclude that the magnitude of the quantum fluctuations (NLO) compared to the leading order (LO) adopts an additional $s^2$ factor, because NLO/LO $\propto s^2$. Recalling $s = t^{\frac{1}{2}-\alpha}$, we can conclude that as long as $0 < \alpha < \frac{1}{2}$ and $t$ is a tiny number as assumed throughout all the calculations, the quantum fluctuations are finite and small compared to the LO term.

With this in mind, we could proceed similarly to the discussion of the LO term and rewrite the quantum fluctuations as $s^2/2$ times a discretized integral over certain powers of the fluxes and the field strengths. In this work, we are not interested in the precise value of this Riemann sum, rather in the question whether it is finite. Hence, as long as we choose $\alpha < \frac{1}{2}$, this is indeed the case, because the $C^{K_0\sigma_0}$, $C^{Mm}$, $C^{Mm,Nn}$ and $C^{Mm,K_0\sigma_0}$ are all of order unity since $(q^{-1})^-$ is of order unity by construction.

## 6. Conclusion

In this paper, we investigated the semiclassical limit of the (extended) algebraic master constraint operator $\widehat{\mathbf{M}}$ associated with an algebraic graph of cubic symmetry. We showed in detail that the leading order of the expectation value of $\widehat{\mathbf{M}}$ with respect to coherent states can be interpreted as the discretized version of the (extended) master constraint operator on a cubic lattice, denoted by $\mathbf{M}^{\text{cubic}}$. In a further analysis, we proved that $\mathbf{M}^{\text{cubic}}$ agrees with the classical (extended) master constraint $\mathbf{M}$ in the limit where the lattice parameter interval length is sent to zero. Hence, we have the following identity:

$$\boxed{\frac{\left\langle\Psi^t_{\alpha,m}\middle|\widehat{\mathbf{M}}\middle|\Psi^t_{\alpha,m}\right\rangle}{\left\|\Psi^t_{\alpha,m}\right\|^2} = \sum_{v\in V(\alpha)} \frac{\left\langle\Psi^t_{\{g,J,\sigma,j,L\}}\middle|\widehat{\mathbf{M}}_v\middle|\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2} \overset{\lim_{t\to 0}}{=} \mathbf{M}^{\text{cubic}}[m] \overset{\lim_{\epsilon\to 0}}{=} \mathbf{M}[m],} \tag{6.1}$$

whereby $t$ is the so-called classicality parameter and the limit $t \to 0$ corresponds to extracting the leading order out of the semiclassical expectation value. The second limit $\epsilon \to 0$ denotes the transition from a discretized into a continuum theory. Consequently, we have shown that the dynamics of AQG which are encoded in $\widehat{\mathbf{M}}$ reproduce the correct infinitesimal generators of general relativity. Furthermore, we discussed the next-to-leading-order contribution of the expectation value of $\widehat{\mathbf{M}}$ and could show that these quantum fluctuations are finite. A more detailed analysis of the quantum fluctuations will be postponed for future research.

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## Appendix A. Application of the Poisson resummation formula

The aim of this work is to discuss the semiclassical behaviour of the algebraic master constraint, thus we are mainly interested in the properties of the expectation value in equation (2.35) for tiny values of the classicality parameter $t$. Looking at equation (2.35), tiny values of $t$ will correspond to a slow convergence behaviour when considering the sum over $n_{J\sigma j\tilde{v}}$. Therefore, we will perform a Poisson resummation in which $t$ gets replaced by $1/t$. Then the series converges rapidly when considering small, tiny values of the classicality parameter. Let us introduce the following quantities:

$$T := \sqrt{t} \qquad x_{J\sigma j\tilde{v}} := T n_{J\sigma j\tilde{v}} \qquad x_{J\sigma jv} := T n_{J\sigma jv}, \tag{A.1}$$

with the help of which all quantities can be expressed in terms of $x_{J\sigma jv}$:

$$\lambda^{\frac{1}{2}}(\{n_{J\sigma jv}\}) = T^{-\frac{3}{4}}\lambda^{\frac{1}{2}}(\{T n_{J\sigma jv}\}) = T^{-\frac{3}{4}}\lambda^{\frac{1}{2}}(\{x_{J\sigma jv}\}), \tag{A.2}$$

and the expectation value can be rewritten in terms of $x_{J\sigma j\tilde{v}}$ as

$$\begin{aligned}
&\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\Psi^t_{\{g,J,\sigma,j,L\}} \,\middle|\, \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{||\Psi^t_{\{g,J,\sigma,j,L\}}||^2} \\
&= \frac{T^{-\frac{3}{2}}}{||\Psi^t_{\{g,J,\sigma,j,L\}}||^2}\left(\mathrm{e}^{-\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} p_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\right. \\
&\times \mathrm{e}^{+\mathrm{i}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{((J,\sigma,j)\\ \in L\setminus L_v}} \varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\ \mathrm{e}^{-\frac{t}{2}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2} \\
&\times \sum_{x_{J\sigma j\tilde{v}}\in\mathbb{Z}T}\mathrm{e}^{-\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\left(x_{J\sigma j\tilde{v}}\right)^2 - x_{J\sigma j\tilde{v}}\left(\frac{2}{T}p_{J\sigma j\tilde{v}}+T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)}
\end{aligned}$$

$$
\left(\mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}} \hat{p}_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}\;\mathrm{e}^{+\mathrm{i}\sum\limits_{\substack{((J,\sigma,j)\\ \in L_v}} \varphi_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}\right.
$$

$$
\times\, \mathrm{e}^{-\frac{t}{2}\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}} \left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right)^2}
$$

$$
\times \sum_{x_{J\sigma jv}\in\mathbb{Z}T} \mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}} \left(x_{J\sigma jv}\right)^2 - x_{J\sigma jv}\left(\frac{2}{T}\hat{p}_{J\sigma jv} + T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right)}
$$

$$
\left[\lambda^{\frac{1}{2}}(\{x_{J\sigma jv}\}) - \lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv} + T\delta_{(J,\sigma,j,v),(K_0,\sigma_0,n_0,v)}\}\right)\right]
$$

$$
\left[+\lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv} - T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\}\right)\right.
$$

$$
\left.\left. -\lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv} - T(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v) + \delta_{(J,\sigma,j,v),(K_0,\sigma_0,\widetilde{n}_0,v)})\}\right)\right]\right). \tag{A.3}
$$

Since the first three exp functions are not involved in the summations, we can rearrange the terms considering $\tilde{v}\neq v$ and $\tilde{v}=v$ together again

$$
\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{||\Psi^t_{\{g,J,\sigma,j,L\}}||^2}
$$

$$
= \frac{T^{-\frac{3}{2}}}{||\Psi^t_{\{g,J,\sigma,j,L\}}||^2}
$$

$$
\times\,\mathrm{e}^{-\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}} p_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\;\mathrm{e}^{+\mathrm{i}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{((J,\sigma,j)\\ \in L}} \varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}
$$

$$
\times\,\mathrm{e}^{-\frac{t}{2}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}} \left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2}
$$

$$
\left(\sum_{x_{J\sigma j\tilde{v}}\in\mathbb{Z}T} \mathrm{e}^{-\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} \left(x_{J\sigma j\tilde{v}}\right)^2 - x_{J\sigma j\tilde{v}}\left(\frac{2}{T}p_{J\sigma j\tilde{v}} + T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)}\right)
$$

$$
\left(\sum_{x_{J\sigma jv}\in\mathbb{Z}T} \mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}} \left(x_{J\sigma jv}\right)^2 - x_{J\sigma jv}\left(\frac{2}{T}\hat{p}_{J\sigma jv} + T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right)}\right.
$$

$$
\Lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv}\}, e^{\sigma_0}_{K_0}(v), n_0\right)\Lambda^{\frac{1}{2}}(\{x_{J\sigma jv} - T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\},
$$

$$
\left. e^{\widetilde{\sigma}_0}_{\widetilde{K}_0}(v), \widetilde{n}_0)\right) \tag{A.4}
$$

where we introduced

$$
\Lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv}\}, e^{\sigma_0}_{K_0}(v), n_0\right) := T^{-\frac{3}{4}}\left[\lambda^{\frac{1}{2}}(\{x_{J\sigma jv}\}) - \lambda^{\frac{1}{2}}(\{x_{J\sigma jv} + T\delta_{(J,\sigma,j,v),(K_0,\sigma_0,n_0,v)}\})\right] \tag{A.5}
$$

in order to keep the expression as short as possible. Moreover, the denominator can be reexpressed as

$$
\begin{aligned}
\left\|\Psi^t_{\{g,J,\sigma,j,L\}}\right\|^2 &= \prod_{(J,\sigma,j)}\sqrt{\frac{\pi}{t}}\mathrm{e}^{-\frac{1}{t}(p_{J\sigma j\tilde{v}})^2}[1+K_t(p)] \\
&= \left(\sqrt{\frac{\pi}{t}}\right)^{30}\mathrm{e}^{+\frac{1}{t}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}}(p_{J\sigma j\tilde{v}})^2}[1+K_t(p)]^{30}.
\end{aligned} \tag{A.6}
$$

Here we get a power of 30 since we have ten edges involved and each edge has three labels. Inserting the expression for the norm above, we get

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\|\Psi^t_{\{g,J,\sigma,j,L\}}\|^2}$$
$$= \frac{1}{(\sqrt{\frac{\pi}{t}})^{30}[1+K_t(p)]^{30}}\,\mathrm{e}^{-\frac{1}{t}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}(p_{J\sigma j\tilde{v}})^2}$$
$$\times\,\mathrm{e}^{-\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}} p_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\,\mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{((J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}$$
$$\times\,\mathrm{e}^{-\frac{t}{2}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2}$$
$$\left(\sum_{x_{J\sigma j\tilde{v}}\in\mathbb{Z}T}\mathrm{e}^{-\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\left(x_{J\sigma j\tilde{v}}\right)^2-x_{J\sigma j\tilde{v}}\left(\frac{2}{T}p_{J\sigma j\tilde{v}}+T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)}\right)$$
$$\left(\sum_{x_{J\sigma jv}\in\mathbb{Z}T}\mathrm{e}^{-\sum_{\substack{(J,\sigma,j)\\ \in L_v}}\left(x_{J\sigma jv}\right)^2-x_{J\sigma jv}\left(\frac{2}{T}\hat{p}_{J\sigma jv}+T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right)}\right.$$
$$\Lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv}\},e^{\sigma_0}_{K_0}(v),n_0\right)\Lambda^{\frac{1}{2}}(\{x_{J\sigma jv}-T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\},$$
$$\left.e^{\widetilde{\sigma}_0}_{\widetilde{K}_0}(v),\widetilde{n}_0)\right). \tag{A.7}$$

The Poisson resummation formula reads

$$\sum_{n_{J\sigma j\tilde{v}}} f(Tn_{J\sigma j\tilde{v}}) = \frac{1}{T}\sum_{n_{J\sigma j\tilde{v}}}\tilde{f}\left(\frac{2\pi n_{J\sigma j\tilde{v}}}{T}\right). \tag{A.8}$$

In order to apply these formulae, we need the Fourier transformation of the following functions:

$$f(x_{J\sigma jv}) := \mathrm{e}^{-\sum_{\substack{(J,\sigma,j)\\ \in L_v}}(x_{J\sigma jv})^2-x_{J\sigma jv}\left(\frac{2}{T}\hat{p}_{J\sigma jv}+T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right)}$$
$$\Lambda^{\frac{1}{2}}(\{x_{J\sigma jv}\},e^{\sigma_0}_{K_0}(v),n_0)\Lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv}-T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\},\right.$$
$$\left.e^{\widetilde{\sigma}_0}_{\widetilde{K}_0}(v),\widetilde{n}_0\right)g(x_{J\sigma j\tilde{v}}) := \mathrm{e}^{-\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\left(x_{J\sigma j\tilde{v}}\right)^2-x_{J\sigma j\tilde{v}}\left(\frac{2}{T}p_{J\sigma j\tilde{v}}+T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)}. \tag{A.9}$$

It is simply given by

$$\tilde{f}(k_{J\sigma jv}) := \frac{1}{(2\pi)^{18}}\int_{\mathbb{R}^{18}}\mathrm{d}^{18}x_{J\sigma jv}\,\mathrm{e}^{-\mathrm{i}\sum_{\substack{(J,\sigma,j)\\ \in L_v}}k_{J\sigma jv}x_{J\sigma jv}}$$
$$\times\,\mathrm{e}^{-\sum_{\substack{(J,\sigma,j)\\ \in L_v}}\left(x_{J\sigma jv}\right)^2-x_{J\sigma jv}\left(\frac{2}{T}\hat{p}_{J\sigma jv}+T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right)}$$
$$\Lambda^{\frac{1}{2}}(\{x_{J\sigma jv}\},e^{\sigma_0}_{K_0}(v),n_0)\Lambda^{\frac{1}{2}}\left(\left\{x_{J\sigma jv}-T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right\},\right.$$
$$\left.e^{\widetilde{\sigma}_0}_{\widetilde{K}_0}(v),\widetilde{n}_0\right) \tag{A.10}$$

and

$$\tilde{g}(k_{J\sigma j\tilde{v}}) := \frac{1}{(2\pi)^{12}} \int\limits_{\mathbb{R}^{12}} \mathrm{d}^{12}x_{J\sigma j\tilde{v}}\, \mathrm{e}^{-\mathrm{i}\sum\limits_{\tilde{v}\neq v\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} k_{J\sigma j\tilde{v}}x_{J\sigma j\tilde{v}}}$$
$$\times\, \mathrm{e}^{-\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} (x_{J\sigma j\tilde{v}})^2 - x_{J\sigma j\tilde{v}}\left(\frac{2}{T}p_{J\sigma j\tilde{v}} + T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)}$$
$$= \frac{(\sqrt{\pi})^{12}}{(2\pi)^{12}}\, \mathrm{e}^{+\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} p_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\, \mathrm{e}^{+\frac{1}{t}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} (p_{J\sigma j\tilde{v}})^2}\, \mathrm{e}^{+\sum\limits_{\tilde{v}\neq v\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} (k_{J\sigma j\tilde{v}})^2}$$
$$\times\, \mathrm{e}^{+\frac{t}{4}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} (\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}$$
$$\times\, \mathrm{e}^{-\sum\limits_{\tilde{v}\neq v\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} (\hat{p}_{J\sigma jv} + \frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))k_{J\sigma j\tilde{v}}}. \tag{A.11}$$

The last line follows from the fact that the integral is a usual (complex) Gaussian integral that can easily be performed. Thus

$$\tilde{f}\left(\frac{2\pi n_{J\sigma jv}}{T}\right) = \frac{1}{(2\pi)^{18}} \int\limits_{\mathbb{R}^{18}} \mathrm{d}^{18}x_{J\sigma jv}\, \mathrm{e}^{-\mathrm{i}\frac{2\pi}{T}\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}} x_{J\sigma jv}n_{J\sigma jv}}$$
$$\times\, \mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}} (x_{J\sigma jv})^2 - x_{J\sigma jv}\left(\frac{2}{T}\hat{p}_{J\sigma jv} + T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right)}$$
$$\Lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv}\}, e^{\sigma_0}_{K_0}(v), n_0\right) \Lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv} - T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\}, e^{\sigma_0}_{K_0}(v), \widetilde{n}_0\right)$$
$$= \frac{1}{(2\pi)^{18}} \int\limits_{\mathbb{R}^{18}} \mathrm{d}^{18}x_{J\sigma jv}\mathrm{e}^{-\sum\limits_{\substack{(J,\sigma,j)\\ \in L_v}} (x_{J\sigma jv})^2 - \frac{2}{T}x_{J\sigma jv}(\hat{p}_{J\sigma jv} + \frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v) - \mathrm{i}\pi n_{J\sigma jv})}$$
$$\Lambda^{\frac{1}{2}}(\{x_{J\sigma jv}\}, e^{\sigma_0}_{K_0}(v), n_0)\Lambda^{\frac{1}{2}}(\{x_{J\sigma jv} - T\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\}, e^{\widetilde{\sigma}_0}_{\widetilde{K}_0}(v), \widetilde{n}_0) \tag{A.12}$$

and

$$\tilde{g}\left(\frac{2\pi n_{J\sigma j\tilde{v}}}{T}\right) = \frac{(\sqrt{\pi})^{12}}{(2\pi)^{12}}\mathrm{e}^{+\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} p_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\, \mathrm{e}^{+\frac{1}{t}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} (p_{J\sigma j\tilde{v}})^2}$$
$$\times\, \mathrm{e}^{+\frac{t}{4}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} (\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))^2}$$
$$\times\, \mathrm{e}^{-\frac{\pi^2}{t}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} \left(n_{J\sigma j\tilde{v}}\right)^2}\, \mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} n_{J\sigma j\tilde{v}}\left(p_{J\sigma j\tilde{v}} + \frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)}. \tag{A.13}$$

Additionally, we have to Fourier-transform the expression in the denominator as well, thus the application of the Poisson resummation formula leads therefore to the following expectation value:

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0v}\,\Psi^t_{\{g,J,\sigma,j,L\}} \,\middle|\, \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{||\Psi^t_{\{g,J,\sigma,j,L\}}||^2} = \frac{1}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}}\frac{(\sqrt{\pi})^{12}}{T^{30}}$$
$$\left(\mathrm{e}^{+\mathrm{i}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} \varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\, \mathrm{e}^{-\frac{t}{4}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} \left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2}\right.$$

$$
\sum_{n_{J\sigma j\tilde{v}}\in\mathbb{Z}} \mathrm{e}^{-\frac{\pi^2}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\left(n_{J\sigma j\tilde{v}}\right)^2} \mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} n_{J\sigma j\tilde{v}}(p_{J\sigma j\tilde{v}}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}\Bigg)
$$

$$
\Bigg(\mathrm{e}^{-\sum_{\substack{(J,\sigma,j)\\ \in L_v}}\hat{p}_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)} \mathrm{e}^{+\mathrm{i}\sum_{\substack{(J,\sigma,j)\\ \in L_v}}\varphi_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}
$$

$$
\times\,\mathrm{e}^{-\frac{t}{2}\sum_{\substack{(J,\sigma,j)\\ \in L_v}}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\right)^2} \mathrm{e}^{-\frac{1}{t}\sum_{\substack{(J,\sigma,j)\\ \in L_v}}(\hat{p}_{J\sigma jv})^2}
$$

$$
\sum_{n_{J\sigma jv}\in\mathbb{Z}} \mathrm{e}^{-\frac{\pi^2}{t}\sum_{\substack{(J,\sigma,j)\\ \in L_v}}(n_{J\sigma jv})^2} \mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum_{\substack{(J,\sigma,j)\\ \in L_v}} n_{J\sigma jv}(\hat{p}_{J\sigma jv}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}
$$

$$
\int_{\mathbb{R}^{18}} \mathrm{d}^{18}x_{J\sigma jv}\mathrm{e}^{-\sum_{\substack{(J,\sigma,j)\\ \in L_v}}\left(x_{J\sigma jv}\right)^2-\frac{2}{T}x_{J\sigma jv}(\hat{p}_{J\sigma jv}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)-\mathrm{i}\pi n_{J\sigma jv})}
$$

$$
\Lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv}\},e^{\sigma_0}_{K_0}(v),n_0\right)\Lambda^{\frac{1}{2}}\left(\{x_{J\sigma jv}-T\,\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)\},e^{\widetilde{\sigma}_0}_{\widetilde{K}_0}(v),\widetilde{n}_0\right)\Bigg). \tag{A.14}
$$

## Appendix B. $(x_{Jjv})^-$ transformation

Furthermore, similar to [13] we introduce another transformation of variables that reduces the 18-dimensional integral down to a 9-dimensional integral times another 9-dimensional integral which contains no $\Lambda^{\frac{1}{2}}$ functions.

This transformation leads to 18 new variables called $(x_{Jjv})^-$ and $(x_{Jjv})^+$ which are defined by

$$
(x_{Jjv})^- := \frac{x_{J+jv}-x_{J-jv}}{2}, \qquad (x_{Jjv})^+ := \frac{x_{J+jv}+x_{J-jv}}{2}. \tag{B.1}
$$

While the other quantities as $n_{J\sigma j\tilde{v}}$ and $p_{J\sigma j\tilde{v}}$ undergo an analogous transformation given by

$$
\begin{aligned}
(n_{Jjv})^- &:= \frac{n_{J+jv}-n_{J-jv}}{2}, \qquad & (n_{Jjv})^+ &:= \frac{n_{J+jv}+n_{J-jv}}{2}\\
(p_{Jjv})^- &:= \frac{p_{J+jv}-p_{J-jv}}{2}, \qquad & (p_{Jjv})^+ &:= \frac{p_{J+jv}+p_{J-jv}}{2},
\end{aligned} \tag{B.2}
$$

the corresponding transformation for the Kronecker-$\delta$ function reads

$$
\begin{aligned}
\frac{1}{2}\,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)} &:= \frac{\delta_{(J,+,j,v),(K_0,\sigma_0,n_0,v)}-\delta_{(J,-,j,v),(K_0,\sigma_0,n_0,v)}}{2}\\
\frac{1}{2}(\delta)^+_{(J,j,v),(K_0,n_0,v)} &:= \frac{\delta_{(J,+,j,v),(K,\sigma_0,n_0,v)}+\delta_{(J,-,j,v),(K_0,\sigma_0,n_0,v)}}{2}\\
\frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)(\delta)^-_{(J,j,v),(\widetilde{K}_0,\widetilde{n}_0,v)} &:= \frac{\delta_{(J,+,j,v),(\widetilde{K}_0,\widetilde{\sigma}_0,\widetilde{n}_0,v)}-\delta_{(J,-,j,v),(\widetilde{K}_0,\widetilde{\sigma}_0,\widetilde{n}_0,v)}}{2}\\
\frac{1}{2}(\delta)^+_{(J,j,v),(\widetilde{K}_0,\widetilde{n}_0,v)} &:= \frac{\delta_{(J,+,j,v),(\widetilde{K}_0,\widetilde{\sigma}_0,\widetilde{n}_0,v)}+\delta_{(J,-,j,v),(\widetilde{K}_0,\widetilde{\sigma}_0,\widetilde{n}_0,v)}}{2}.
\end{aligned} \tag{B.3}
$$

The Jacobean of this transformation is simply

$$
\left|\det\left(\frac{\partial x_{J\sigma jv}}{\partial((x_{Jjv})^-,(x_{Jjv})^+)}\right)\right| = 2^{\frac{1}{2}\dim(x_{J\sigma jv})} = 2^9. \tag{B.4}
$$

The corresponding transformation for the $\Delta$ functions can only be given when decomposing $\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, v)$ again into $\Delta(I_0, J_0, \sigma_0, m_0, v, J, \sigma, j, v)$ and $\Delta(I_0, J_0, \sigma_0, m_0, v, J, \sigma, j, v)$ since we cannot factor out a global $\mathrm{sgn}(\sigma)$ factor due to the fact that $\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, v)$ contains $\sigma_0$ and $\widetilde{\sigma}_0$.

$$
\begin{aligned}
&\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, v) \\
&\qquad := \Delta(I_0, J_0, \sigma_0, m_0, v, J, \sigma, j, v) - \Delta(I_0, J_0, \sigma_0, m_0, v, J, \sigma, j, v).
\end{aligned}
\tag{B.5}
$$

We then define

$$
\begin{aligned}
&\frac{1}{2}\,\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0, J_0, v, J, j, v) \\
&\qquad := \frac{\Delta(I_0, J_0, \sigma_0, m_0, v, J, +, j, v) - \Delta(I_0, J_0, \sigma_0, m_0, v, J, -, j, v)}{2} \\
&\frac{1}{2}(\Delta)^+(I_0, J_0, v, J, j, v) \\
&\qquad := \frac{\Delta(I_0, J_0, \sigma_0, m_0, v, J, +, j, v) + \Delta(I_0, J_0, \sigma_0, m_0, v, J, -, j, v)}{2} \\
&\frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0, \widetilde{J}_0, v, J, j, v) \\
&\qquad := \frac{\Delta(\widetilde{I}_0, \widetilde{J}_0, \widetilde{\sigma}_0, \widetilde{m}_0, v, J, +, j, v) - \Delta(\widetilde{I}_0, \widetilde{J}_0, \widetilde{\sigma}_0, \widetilde{m}_0, v, J, -, j, v)}{2} \\
&\frac{1}{2}(\Delta)^+(\widetilde{I}_0, \widetilde{J}_0, v, J, j, v) \\
&\qquad := \frac{\Delta(\widetilde{I}_0, \widetilde{J}_0, \widetilde{\sigma}_0, \widetilde{m}_0, v, J, +, j, v) + \Delta(\widetilde{I}_0, \widetilde{J}_0, \widetilde{\sigma}_0, \widetilde{m}_0, v, J, -, j, v)}{2}.
\end{aligned}
\tag{B.6}
$$

When now expressing $x_{J+jv}$ by the sum of $(x_{Jjv})^+$ and $(x_{Jjv})^-$ and $x_{J-jv}$ by $(x_{Jjv})^+ - (x_{Jjv})^-$ (the same for all the other occurring terms) and performing the square in the exponential, we realize that all terms that involve mixed $()^+$ and $()^-$ terms as for instance $(x_{Jjv})^+(x_{Jjv})^-$ will drop out, while the terms involving only $()^+$ or $()^-$ respectively occur twice. Furthermore, the $\Lambda^{\frac{1}{2}}$ functions do only depend on the variable $(x_{Jjv})^-$, consequently, we can rewrite the integral as

$$
\begin{aligned}
&\int_{\mathbb{R}^{18}} \mathrm{d}^{18} x_{J\sigma jv} F(x_{J\sigma jv}, n_{J\sigma jv}) \\
&\qquad = \int_{\mathbb{R}^9} \mathrm{d}^9 (x_{Jjv})^+ G_1((x_{Jjv})^+, (n_{Jjv})^+) \int_{\mathbb{R}^9} \mathrm{d}^9 (x_{Jjv})^- G_2((x_{Jjv})^-, (n_{Jjv})^-),
\end{aligned}
\tag{B.7}
$$

where

$$
\begin{aligned}
G_1((x_{Jjv})^+) &:= \mathrm{e}^{-2\sum_{\substack{(J,j,v)\\ \in L_v}} (((x_{Jjv})^+)^2 - \frac{2}{T}(x_{Jjv})^+((p_{Jjv})^+ + \frac{T^2}{4}[(\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - (\Delta)^+(I_0,J_0,v,J,j,v)] - \mathrm{i}\pi(n_{Jjv})^+))} \\
G_2((x_{Jjv})^-) &:= \mathrm{e}^{-2\sum_{\substack{(J,j,v)\\ \in L_v}} (((x_{Jjv})^-)^2 - \frac{2}{T}(x_{Jjv})^-((p_{Jjv})^- + \frac{T^2}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - \mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)] - \mathrm{i}\pi(n_{Jjv})^-))} \\
&\Lambda^{\frac{1}{2}}\left(\left\{(x_{Jjv})^- - \frac{T}{2}\left[\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0, \widetilde{J}_0, v, J, j, v) - \mathrm{sgn}(\sigma_0)(\Delta)^-(I_0, J_0, v, J, j, v)\right]\right\},\right. \\
&\qquad \left.\frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0) e^-_{\widetilde{K}_0}(v), \widetilde{n}_0\right) \\
&\Lambda^{\frac{1}{2}}\left(\{(x_{Jjv})^-, \frac{1}{2}\mathrm{sgn}(\sigma_0) e^-_{K_0}(v), n_0\right).
\end{aligned}
\tag{B.8}
$$

Thus, we get

$$
\begin{aligned}
&2^9 \int_{\mathbb{R}^9} \mathrm{d}^9 (x_{Jjv})^+ \, \mathrm{e}^{-2 \sum_{(J,j) \in L_v} (((x_{Jjv})^+)^2 - \frac{2}{T}(x_{Jjv})^+((p_{Jjv})^+ + \frac{T^2}{4}\left[(\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - (\Delta)^+(I_0,J_0,v,J,j,v)\right] - \mathrm{i}\pi(n_{Jjv})^+))} \\
&2^9 \int_{\mathbb{R}^9} \mathrm{d}^9 (x_{Jjv})^- \\
&\times \mathrm{e}^{-2 \sum_{(J,j) \in L_v} (((x_{Jjv})^-)^2 - \frac{2}{T}(x_{Jjv})^-((p_{Jjv})^- + \frac{T^2}{4}\left[\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - \mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)\right] - \mathrm{i}\pi(n_{Jjv})^-))} \\
&\Lambda^{\frac{1}{2}}\left((x_{Jjv})^- - \frac{T}{2}[\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0, \widetilde{J}_0, v, J, j, v) - \mathrm{sgn}(\sigma_0)(\Delta)^-(I_0, J_0, v, J, j, v)]\},\right. \\
&\qquad\qquad \left.\frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0) e_{\widetilde{K}_0}(v), \widetilde{n}_0\right) \\
&\Lambda^{\frac{1}{2}}\left(\{(x_{Jjv})^-\}, \frac{1}{2}\mathrm{sgn}(\sigma_0) e_{K_0}(v), n_0\right) \\
&= 2^9 \left(\sqrt{\frac{\pi}{2}}\right)^9 \mathrm{e}^{-\frac{1}{2}\sum_{(J,j) \in L_v} \left((p_{Jjv})^+((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - (\Delta)^+(I_0,J_0,v,J,j,v))s\right)} \mathrm{e}^{+\frac{2}{t}\sum_{(J,j,v) \in L_v} ((p_{Jjv})^+)^2} \\
&\times \mathrm{e}^{+\frac{t}{8}\sum_{(J,j,v) \in L_v} ((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - (\Delta)^+(I_0,J_0,v,J,j,v))^2} \\
&\times \mathrm{e}^{-2\frac{\pi^2}{t}\sum_{(J,j) \in L_v} \left((n_{Jjv})^+\right)^2} \mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{(J,j) \in L_v} (n_{Jjv})^+((p_{Jjv})^+ + \frac{T^2}{4}((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - (\Delta)^+(I_0,J_0,v,J,j,v)))} \\
&2^9 \int_{\mathbb{R}^9} \mathrm{d}^9 (x_{Jjv})^- \\
&\times \mathrm{e}^{-2 \sum_{(J,j) \in L_v} (((x_{Jjv})^-)^2 - \frac{2}{T}(x_{Jjv})^-((p_{Jjv})^- + \frac{T^2}{4}\left[\frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) - \frac{1}{2}\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)\right] - \mathrm{i}\pi(n_{Jjv})^-))} \\
&\Lambda^{\frac{1}{2}}\left((x_{Jjv})^- - T[(\Delta)^-(\widetilde{I}_0, \widetilde{J}_0, v, J, j, v) - (\Delta)^-(I_0, J_0, v, J, j, v)]\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0) e_{\widetilde{K}_0}(v), \widetilde{n}_0\right) \\
&\Lambda^{\frac{1}{2}}\left(\{(x_{Jjv})^-\}, \frac{1}{2}\mathrm{sgn}(\sigma_0) e_{K_0}(v), n_0\right)
\end{aligned}
\tag{B.9}
$$

with

$$
\begin{aligned}
&\Lambda^{\frac{1}{2}}\left(\{(x_{Jjv})^-\}, \frac{1}{2}\,\mathrm{sgn}(\sigma_0) e_{K_0}(v), n_0\right) := T^{-\frac{3}{4}}\left[\lambda^{\frac{1}{2}}(\{(x_{Jjv})^-\}) - \lambda^{\frac{1}{2}}\left(\left\{(x_{Jjv})^-\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \frac{T}{2}\,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\right\}\right)\right]
\end{aligned}
\tag{B.10}
$$

$$
\lambda^{\frac{1}{2}}(\{(x_{Jjv})^-\}) = t^{\frac{3}{4}}\left(\sqrt{|\det((x_{Jjv})^-)|}\right)^{\frac{1}{2}}.
$$

The integral over $(x_{Jjv})^+$ is a usual (complex) Gaussian integral that can be easily performed and yields a factor $(\sqrt{\pi/2})^9$. Thus, the expectation simplifies to

$$
\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\|\Psi^t_{\{g,J,\sigma,j,L\}}\|^2}
$$

$$
= \frac{\mathrm{e}^{+\mathrm{i}\sum_{\tilde v\neq v \in V}\sum_{(J,\sigma,j)\in L\setminus L_v}\varphi_{J\sigma j\tilde v}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde v)}\;\mathrm{e}^{-\frac{t}{4}\sum_{\tilde v\neq v \in V}\sum_{(J,\sigma,j)\in L\setminus L_v}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde v)\right)^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}}
$$

$$
\frac{(\sqrt{\pi})^{12}}{T^{30}}2^9\left(\sqrt{\frac{\pi}{2}}\right)^9
$$

$$
\left(\sum_{n_{J\sigma j\tilde v}\in\mathbb{Z}}\mathrm{e}^{-\frac{\pi^2}{t}\sum_{\tilde v\neq v\in V}\sum_{(J,\sigma,j)\in L\setminus L_v}\left(n_{J\sigma j\tilde v}\right)^2}\;\mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum_{\tilde v\neq v\in V}\sum_{(J,\sigma,j)\in L\setminus L_v}n_{J\sigma j\tilde v}(p_{J\sigma j\tilde v}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde v))}\right)
$$

$$
\left(\mathrm{e}^{-\frac{t}{8}\sum_{(J,j)\in L_v}(((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v))^2)}\;\mathrm{e}^{+\frac{i}{2}\sum_{(J,j)\in L_v}((\varphi_{Jjv})^+((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right.
$$

$$
\left.\sum_{(n_{Jjv})^+\in\mathbb{Z}}\mathrm{e}^{-2\frac{\pi^2}{t}\sum_{(J,j)\in L_v}\left((n_{Jjv})^+\right)^2}\;\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{(J,j)\in L_v}(n_{Jjv})^+((p_{Jjv})^++\frac{T^2}{4}((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right)
$$

$$
\left(\mathrm{e}^{-\frac{1}{2}\sum_{(J,j)\in L_v}((p_{Jjv})^-(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}\right.
$$

$$
\times\,\mathrm{e}^{+\frac{i}{2}\sum_{(J,j)\in L_v}\left((\varphi_{Jjv})^-(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v))\right)}
$$

$$
\times\,\mathrm{e}^{-\frac{2}{t}\sum_{(J,j)\in L_v}(((p_{Jjv})^-)^2)}\;\mathrm{e}^{-\frac{t}{4}\sum_{(J,j)\in L_v}((\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(I_0,J_0,v,J,j,v))^2)}
$$

$$
\sum_{(n_{Jjv})^-\in\mathbb{Z}}\int_{\mathbb{R}^9}\mathrm{d}^9(x_{Jjv})^-
$$

$$
\times\,\mathrm{e}^{-2\sum_{(J,j,v)\in L_v}(((x_{Jjv})^-)^2-\frac{2}{T}((p_{Jjv})^-+\frac{T^2}{2}\left[\frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\frac{1}{2}\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)\right]-\mathrm{i}\pi(n_{Jjv})^-))}
$$

$$
\Lambda^{\frac{1}{2}}\left(\left\{(x_{Jjv})^- - \frac{T}{2}[\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)\right.\right.
$$

$$
\left.\left.-\,\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)]\right\},\frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v),\widetilde{n}_0\right)
$$

$$
\Lambda^{\frac{1}{2}}\left(\{(x_{Jjv})^-\},\frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v),n_0\right)\Bigg), \tag{B.11}
$$

where we used

$$
\mathrm{e}^{-\sum_{(J,\sigma,j)\in L_v}\hat p_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}
$$

$$
= \mathrm{e}^{-\frac{1}{2}\sum_{(J,j)\in L_v}((p_{Jjv})^+((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v))+(p_{Jjv})^-(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}
$$

$$
\mathrm{e}^{+\mathrm{i}\sum_{(J,\sigma,j)\in L_v}\varphi_{J\sigma jv}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v)}
$$

$$= \mathrm{e}^{+\frac{i}{2}\sum_{\substack{(J,j)\\ \in L_v}}((\varphi_{Jjv})^+((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v))+(\varphi_{Jjv})^-(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v) -\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}$$

$$\mathrm{e}^{-\frac{1}{t}\sum_{\substack{(J,\sigma,j)\\ \in L_v}}(\hat{p}_{J\sigma jv})^2} = \mathrm{e}^{-\frac{2}{t}\sum_{\substack{(J,j)\\ \in L_v}}(((p_{Jjv})^+)^2+((p_{Jjv})^-)^2)}$$

$$\mathrm{e}^{-\frac{t}{2}\sum_{\substack{(J,\sigma,j)\\ \in L_v}}(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,v))^2}$$

$$= \mathrm{e}^{-\frac{t}{4}\sum_{\substack{(J,j)\\ \in L_v}}(((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v))^2+(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(I_0,J_0,v,J,j,v))^2)} \tag{B.12}$$

## Appendix C. Only the term with $n_{J\sigma jv} = 0$ matters

In the following, we will show that only the term with $n_{J\sigma jv} = 0$ contributes and all other terms are of order $O(t^\infty)$. Thus we consider the following estimation:

$$\left|\frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2} - \frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2}\right|_{n_{J\sigma j\tilde{v}}=0}\Big| = \left|\frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2}\right. \tag{C.1}$$

$$-\frac{\mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\ \mathrm{e}^{-\frac{t}{4}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}}$$

$$\frac{(\sqrt{\pi})^{12}}{T^{30}}2^9\left(\sqrt{\frac{\pi}{2}}\right)^9\int_{\mathbb{R}^9}\mathrm{d}^9(\tilde{x}_{Jjv})^-\mathrm{e}^{-2\sum_{\substack{(J,j)\\ \in L_v}}\left((x_{Jjv})^-\right)^2}$$

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}\left((p_{Jjv})^-\right) - \frac{T}{4}\left[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-\right]\right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)$$

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)\Bigg|. \tag{C.2}$$

Let us neglect the explicit $J, j, v$ and introduce the following abbreviations in order to make the expressions more convenient:

$$(x_{Jjv})^- := (\tilde{x})^-, \qquad (p_{Jjv})^- := (p)^-, \qquad n_{J\sigma j\tilde{v}} := (n)^- \tag{C.3}$$

and use the expression of the $\Lambda^{\frac{1}{2}}$ functions in terms of the determinants in equation (2.46) yields

$$\left|\frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2} - \frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2}\right|_{n_{J\sigma j\tilde{v}}=0}\Big| \leqslant \left|\frac{(\sqrt{\pi})^{12}}{T^{30}}2^9\left(\sqrt{\frac{\pi}{2}}\right)^9\right. \tag{C.4}$$

$$\frac{\mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\ \mathrm{e}^{-\frac{t}{4}\sum_{\tilde{v}\in V}\sum_{(J,\sigma,j)}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}}$$

$$\left(\sum_{\substack{n_{J\sigma j\tilde{v}}\neq 0\\ \in\mathbb{Z}}} \mathrm{e}^{-\frac{\pi^2}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\left(n_{J\sigma j\tilde{v}}\right)^2}\,\mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} n_{J\sigma j\tilde{v}}\left(p_{J\sigma j\tilde{v}}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)}\right)$$

$$\left(\sum_{\substack{(n_{Jjv})^+\neq 0\\ \in\mathbb{Z}}} \mathrm{e}^{-2\frac{\pi^2}{t}\sum_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^+)^2}\,\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^+((p_{Jjv})^+ +\frac{T^2}{4}((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right)$$

$$\left(\sum_{\substack{(n_{Jjv})^-\neq 0\\ \in\mathbb{Z}}} \mathrm{e}^{-2\frac{\pi^2}{t}\sum_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^-)^2}\,\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^-((p_{Jjv})^- +\frac{T^2}{4}(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}\right.$$

$$\int_{\mathbb{R}^9} \mathrm{d}^9(\tilde{x}_{Jjv})^- \mathrm{e}^{-2\sum_{(J,j)}((\tilde{x}_{Jjv})^-)^2}$$

$$\left[\exp\left(\ln\left(\frac{1}{4}\det\left(\left\{(\tilde{x})^- + \frac{1}{T}((p)^- - \mathrm{i}\pi(n)^-) + \frac{T}{4}\left[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-\right]\right\}\right)\right)\right)\right.$$

$$+\exp\left(\ln\left(\frac{1}{4}\det\left(\left\{(\tilde{x})^- + \frac{1}{T}((p)^- - \mathrm{i}\pi(n)^-) + \frac{T}{4}\left[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-\right]\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left. - \frac{T}{2}\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\right\}\right)\right)\right)\right]$$

$$\left[\exp\left(\ln\left(\frac{1}{4}\det\left(\left\{(\tilde{x})^- + \frac{1}{T}((p)^- - \mathrm{i}\pi(n)^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}\right)\right)\right)\right.$$

$$+\exp\left(\ln\left(\frac{1}{4}\det\left(\left\{(\tilde{x})^- + \frac{1}{T}((p)^- - \mathrm{i}\pi(n)^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.\left.\left. - \frac{T}{2}\mathrm{sgn}(\widetilde{\sigma}_0)(\delta)^-_{(J,j,v),(\widetilde{K}_0,\widetilde{n}_0,v)}\right\}\right)\right)\right)\right]\right)\right|, \tag{C.5}$$

where we have used $|a-b| \leqslant |a+b|$ for the exp functions. Let $(\omega_{\{J,j\}})$ be a matrix of complex numbers and define the norm to be $\|\omega\|^2 = \sum_{J,j}|\omega_{\{J,j\}}|^2$. Then, certainly, $\|\omega_1+\omega_2\| \leqslant \|\omega_1\|+\|\omega_2\|$ and $\|\omega\| \geqslant |\omega_{\{J,j\}}|$ for all $J, j$. In particular, $\det(\{\omega_{\{J,j\}}\}) \leqslant 6\|\omega\|^3$. Consequently, we get

$$\begin{aligned}
&\det\left(\left\{(\tilde{x})^- + \frac{1}{T}((p)^- - \mathrm{i}\pi(n)^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}\right)\\
&\qquad \leqslant 6\left(T\|(\tilde{x})^-\| + \|(p)^-\| + \pi\|(n)^-\| + \frac{T^2}{4}\|\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-\|\right)^3\\
&\qquad \leqslant 6\left(T\|(\tilde{x})^-\| + \|(p)^-\| + \pi\|(n)^-\| + \frac{T^2}{4} + \frac{T^2}{4}\right)^3\\
&\qquad \leqslant 6(T\|(\tilde{x})^-\| + \|(p)^-\| + \pi\|(n)^-\| + T^2)^3.
\end{aligned} \tag{C.6}$$

Analogously,

$$
\begin{aligned}
&\det\left(\left\{(\tilde{x})^- + \frac{1}{T}((p)^- - \mathrm{i}\pi(n)^-) + \frac{T}{4}[\operatorname{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \operatorname{sgn}(\sigma_0)(\Delta)^-]\right.\right.\\
&\qquad\left.\left. - \frac{T}{2}\operatorname{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\right\}\right)\\
&\qquad \leqslant 6\left(T\|(\tilde{x})^-\| + \|(p)^-\| + \pi\|(n)^-\| + \frac{T^2}{4}\|\operatorname{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \operatorname{sgn}(\sigma_0)(\Delta)^-\|\right.\\
&\qquad\left. - \frac{T^2}{2}\|\operatorname{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\|\right)^3\\
&\qquad \leqslant 6\left(T\|(\tilde{x})^-\| + \|(p)^-\| + \pi\|(n)^-\| + \frac{T^2}{4}\|\operatorname{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-\| + \frac{T}{4}\|\operatorname{sgn}(\sigma_0)(\Delta)^-\|\right.\\
&\qquad\left. + \frac{T^2}{2}\|\operatorname{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\|\right)^3\\
&\qquad \leqslant 6\left(T\|(\tilde{x})^-\| + \|(p)^-\| + \pi\|(n)^-\| + \frac{T^2}{4} + \frac{T^2}{4} + \frac{T^2}{2}\right)^3\\
&\qquad = 6(T\|(\tilde{x})^-\| + \|(p)^-\| + \pi\|(n)^-\| + T^2)^3. 
\end{aligned}
\tag{C.7}
$$

The same is true for the second term involving the determinants, thus we obtain

$$
\left|\frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2} - \frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2}\right|_{n_{J\sigma j\tilde{v}}=0}\Big| \leqslant \left|\frac{(\sqrt{\pi})^{12}}{T^{30}} 2^9\left(\sqrt{\frac{\pi}{2}}\right)^9\right. \tag{C.8}
$$

$$
\frac{\mathrm{e}^{+\mathrm{i}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\,\mathrm{e}^{-\frac{t}{4}\sum\limits_{\tilde{v}\in V}\sum\limits_{(J,\sigma,j)}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2}}{(\sqrt{\frac{\pi}{t}})^{30}[1+K_t(p)]^{30}}
$$

$$
\left(\sum_{\substack{n_{J\sigma j\tilde{v}}\neq 0\\ \in\mathbb{Z}}}\mathrm{e}^{-\frac{\pi^2}{t}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(n_{J\sigma j\tilde{v}})^2}\,\mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{\tilde{v}\neq v\\ \in V}}\sum\limits_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}n_{J\sigma j\tilde{v}}(p_{J\sigma j\tilde{v}}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}\right)
$$

$$
\left(\sum_{\substack{(n_{Jjv})^+\neq 0\\ \in\mathbb{Z}}}\mathrm{e}^{-2\frac{\pi^2}{t}\sum\limits_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^+)^2}\,\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^+((p_{Jjv})^++\frac{T^2}{4}((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right)
$$

$$
\left(\sum_{\substack{(n_{Jjv})^-\neq 0\\ \in\mathbb{Z}}}\mathrm{e}^{-2\frac{\pi^2}{t}\sum\limits_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^-)^2}\,\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum\limits_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^-((p_{Jjv})^-+\frac{T^2}{4}(\operatorname{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\operatorname{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}\right.
$$

$$
\int_{\mathbb{R}^9}\mathrm{d}^9(\tilde{x}_{Jjv})^-\,\mathrm{e}^{-2\sum\limits_{(J,j)}((\tilde{x}_{Jjv})^-)^2}
$$

$$
\begin{aligned}
&\left[\exp\left(\ln\left(\frac{1}{4}6\left(T||(\tilde{x})^-|| + ||(p)^-|| + \pi||(n)^-|| + T^2\right)^3\right)\right)\right.\\
&\qquad\left. + \exp\left(\ln\left(\frac{1}{4}6\left(T||(\tilde{x})^-|| + ||(p)^-|| + \pi||(n)^-|| + T^2\right)^3\right)\right)\right]
\end{aligned}
$$

$$\left[\exp\left(\ln\left(\frac{1}{4}6\left(T||(\tilde{x})^-|| + ||(p)^-|| + \pi||(n)^-|| + T^2\right)^3\right)\right)\right.$$

$$\left.\left.\left. + \exp\left(\ln\left(\frac{1}{4}6\left(T||(\tilde{x})^-|| + ||(p)^-|| + \pi||(n)^-|| + T^2\right)^3\right)\right)\right]\right)\right| \qquad (C.9)$$

and

$$\left|\frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2} - \frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2}\right|_{n_{J\sigma j\tilde{v}}=0} | \leqslant \left|\frac{(\sqrt{\pi})^{12}}{T^{30}} 2^9 \left(\sqrt{\frac{\pi}{2}}\right)^9\right. \qquad (C.10)$$

$$\left(\exp\left(+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}} \varphi_{J\sigma j\tilde{v}}\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v})\right)\right.$$

$$\left.\times \exp\left(-\frac{t}{4}\sum_{\tilde{v}\in V}\sum_{(J,\sigma,j)} (\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v}))^2\right)\right) \Big/$$

$$\left(\left(\sqrt{\frac{\pi}{t}}\right)^{30} [1 + K_t(p)]^{30}\right)$$

$$\left(\sum_{\substack{n_{J\sigma j\tilde{v}}\neq 0\\ \in\mathbb{Z}}} \mathrm{e}^{-\frac{\pi^2}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}(n_{J\sigma j\tilde{v}})^2} \mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}} n_{J\sigma j\tilde{v}}(p_{J\sigma j\tilde{v}}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}\right)$$

$$\left(\sum_{\substack{(n_{Jjv})^+\neq 0\\ \in\mathbb{Z}}} \mathrm{e}^{-2\frac{\pi^2}{t}\sum_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^+)^2} \mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^+((p_{Jjv})^+ + \frac{T^2}{4}((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right)$$

$$\left(\sum_{\substack{(n_{Jjv})^-\neq 0\\ \in\mathbb{Z}}} \mathrm{e}^{-2\frac{\pi^2}{t}\sum_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^-)^2} \mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^-((p_{Jjv})^- + \frac{T^2}{4}(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}\right.$$

$$\left.\int_{\mathbb{R}^9} \mathrm{d}^9(\tilde{x})^- \mathrm{e}^{-2\|(\tilde{x})^-\|^2}\left[\frac{1}{4} + T^2\|(\tilde{x})^-\|^2 + \|(p)^-\| + \pi\|(n)^-\| + T^2\right]^{\left[\frac{3}{2}\right]+1}\right|, \qquad (C.11)$$

where we used $x \leqslant x^2 + \frac{1}{4}$ in the last step and $\left[\frac{3}{2}\right]$ is the Gauß bracket, i.e. the smallest integer equal or lower than $\frac{3}{2}$, hence $\left[\frac{3}{2}\right] = 1$.

Estimation of the integral is

$$I_k := \left(\sqrt{\frac{2}{\pi}}\right)^9 \int_{\mathbb{R}^9} \mathrm{d}^9x \mathrm{e}^{-2\|x\|^2}\|x\|^{2k}, \qquad I_k = \frac{9 + 2(k-1)}{4} I_{k-1}, \qquad I_0 = 1. \qquad (C.12)$$

In our case, $k = 2$; therefore, we get $I_1 = \frac{9}{4}$, $I_2 = \frac{99}{16}$. Using

$$\left(\frac{1}{4} + T^2\|(\tilde{x})^-\|^2 + \|(p)^-\| + \pi\|(n)^-\| + T^2\right)^2$$
$$= T^4\|(\tilde{x})^-\|^4 + 2T^2\|(\tilde{x})^-\|^2\left(\frac{1}{4} + T^2\|(\tilde{x})^-\|^2 + \|(p)^-\| + \pi\|(n)^-\| + T^2\right)$$
$$+ \left(\frac{1}{4} + T^2\|(\tilde{x})^-\|^2 + \|(p)^-\| + \pi\|(n)^-\| + T^2\right)^2, \tag{C.13}$$

we get

$$\int_{\mathbb{R}^9} \mathrm{d}^9(\tilde{x})^- \mathrm{e}^{-2\|(\tilde{x})^-\|^2}\left[\frac{1}{4} + T^2\|(\tilde{x})^-\|^2 + \|(p)^-\| + \pi\|(n)^-\| + T^2\right]^2$$
$$= \left(\sqrt{\frac{\pi}{2}}\right)^9 \left(T^4 I_2 + 2T^2\left(\frac{1}{4} + T^2\|(\tilde{x})^-\|^2 + \|(p)^-\| + \pi\|(n)^-\| + T^2\right) I_1\right.$$
$$\left. + \left(\frac{1}{4} + T^2\|(\tilde{x})^-\|^2 + \|(p)^-\| + \pi\|(n)^-\| + T^2\right)^2 I_0\right)$$
$$= \left(\sqrt{\frac{\pi}{2}}\right)^9 \left(\left(\frac{9}{4}T^2 + \frac{1}{4} + T^2\|(\tilde{x})^-\|^2 + \|(p)^-\| + \pi\|(n)^-\| + T^2)\right)^2 + \frac{9}{4}T^4\right). \tag{C.14}$$

Hence, we have

$$\left|\frac{\langle .\,,\,.\rangle}{\|\,.\,\|^2} - \frac{\langle .\,,\,.\rangle}{\|\,.\,||^2}\right|_{n_{J\sigma j\tilde{v}}=0} \tag{C.15}$$

$$\frac{\mathrm{e}^{+\mathrm{i}\sum_{\tilde{v}\in V}\sum_{(J,\sigma,j)\in L}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\;\mathrm{e}^{-\frac{t}{4}\sum_{\tilde{v}\in V}\sum_{(J,\sigma,j)}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}}$$
$$\left(\sum_{\substack{n_{J\sigma j\tilde{v}}\neq 0\\ \in\mathbb{Z}}}\mathrm{e}^{-\frac{\pi^2}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}\left(n_{J\sigma j\tilde{v}}\right)^2}\;\mathrm{e}^{-2\mathrm{i}\frac{\pi}{t}\sum_{\substack{\tilde{v}\neq v\\ \in V}}\sum_{\substack{(J,\sigma,j)\\ \in L\setminus L_v}}n_{J\sigma j\tilde{v}}(p_{J\sigma j\tilde{v}}+\frac{T^2}{2}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v}))}\right)$$
$$\left(\sum_{\substack{(n_{Jjv})^+\neq 0\\ \in\mathbb{Z}}}\mathrm{e}^{-2\frac{\pi^2}{t}\sum_{\substack{(J,j)\\ \in L_v}}\left((n_{Jjv})^+\right)^2}\;\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^+((p_{Jjv})^+ + \frac{T^2}{4}((\Delta)^+(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-(\Delta)^+(I_0,J_0,v,J,j,v)))}\right)$$
$$\left(\sum_{\substack{(n_{Jjv})^-\neq 0\\ \in\mathbb{Z}}}\mathrm{e}^{-2\frac{\pi^2}{t}\sum_{\substack{(J,j)\\ \in L_v}}((n_{Jjv})^-)^2}\;\mathrm{e}^{-4\mathrm{i}\frac{\pi}{t}\sum_{\substack{(J,j)\\ \in L_v}}(n_{Jjv})^-((p_{Jjv})^- + \frac{T^2}{4}(\mathrm{sgn}(\widetilde{\sigma}_0)(\Delta)^-(\widetilde{I}_0,\widetilde{J}_0,v,J,j,v)-\mathrm{sgn}(\sigma_0)(\Delta)^-(I_0,J_0,v,J,j,v)))}\right.$$
$$\left(\sqrt{\frac{\pi}{2}}\right)^9\left(\left(\frac{9}{4}T^2 + \frac{1}{4} + T^2||(\tilde{x})^-||^2 + ||(p)^-|| + \pi||(n)^-|| + T^2)\right)^2 + \frac{9}{4}T^4\right)\Bigg|. \tag{C.16}$$

This is obviously of the order $O(t^\infty)$ for $n_{J\sigma j\tilde{v}} \neq 0$. Consequently, up to order $O(t^\infty)$ we have

$$\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}} \,\middle|\, \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{||\Psi^t_{\{g,J,\sigma,j,L\}}||^2}$$

$$= \frac{\mathrm{e}^{+\mathrm{i}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}} \varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})}\;\mathrm{e}^{-\frac{t}{4}\sum\limits_{\tilde{v}\in V}\sum\limits_{\substack{(J,\sigma,j)\\ \in L}}\left(\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})\right)^2}}{\left(\sqrt{\frac{\pi}{t}}\right)^{30}[1+K_t(p)]^{30}}$$

$$\frac{(\sqrt{\pi})^{12}}{T^{30}}2^9\left(\sqrt{\frac{\pi}{2}}\right)^9\int\limits_{\mathbb{R}^9}\mathrm{d}^9(\tilde{x}_{Jjv})^-\,\mathrm{e}^{-2\sum\limits_{\substack{(J,j)\\ \in L_v}}((\tilde{x}_{Jjv})^-)^2}$$

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}\left[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-\right]\right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)$$

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right). \tag{C.17}$$

## Appendix D. The expansion of the $\Lambda^{\frac{1}{2}}$ functions

### *D.1. Calculation of the $z_A$ terms*

As mentioned in the main text, we have to calculate the necessary $z'_A$ up to order $O(sT)$ and $O(s^3)$ respectively. For this purpose, we use the following index notation $A_{ik} = s\sum_M (q^{-1})^-_{Mi}(\tilde{x})^-_{Mk}$:

$$\begin{aligned}\mathrm{tr}&(s(q^{-1})^-(\tilde{x})^- \pm \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-])\\ &= s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm} \pm \frac{sT}{4}(q^{-1})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Mm} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm}).\end{aligned} \tag{D.1}$$

Thus

$$\begin{aligned}&\left[\mathrm{tr}(s(q^{-1})^-(\tilde{x})^- \pm \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-])\right]^2\\ &\quad= s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} \pm \frac{s^2T}{2}(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}\\ &\qquad\times(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}) + \frac{(sT)^2}{16}(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Mm}\\ &\qquad - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm})(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}).\end{aligned} \tag{D.2}$$

Moreover,

$$\mathrm{tr}\left(\left[s(q^{-1})^-(\tilde{x})^- \pm \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right]^2\right) = s^2(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}$$

$$\times(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} \pm \frac{s^2T}{2}(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})$$

$$+\frac{(sT)^2}{16}(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Mm}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm})(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}). \tag{D.3}$$

$$\det\left(s(q^{-1})^-(\tilde{x})^-\pm\frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-]\right) \tag{D.4}$$

$$\begin{aligned}
&=\frac{1}{3!}\epsilon_{ijk}\epsilon_{\ell mn}\left((s(q^{-1})^-(\tilde{x})^-)_{i\ell}\pm\frac{T}{4}s((q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-])_{i\ell}\right)\\
&\left((s(q^{-1})^-(\tilde{x})^-)_{jm}\pm\frac{T}{4}s((q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-])_{jm}\right)\\
&\left((s(q^{-1})^-(\tilde{x})^-)_{kn}\pm\frac{T}{4}s((q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-])_{kn}\right)\\
&=s^3\det((q^{-1})^-)\det((\tilde{x})^-)\pm\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}\\
&\left[+\left(\frac{T}{4}(q^{-1})^-_{Li}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell}]\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\right.\\
&+\left(\frac{T}{4}(q^{-1})^-_{Mj}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Mm}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&\left.+\left(\frac{T}{4}(q^{-1})^-_{Nk}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\right]\\
&+O((sT)^2).
\end{aligned} \tag{D.5}$$

Hence, we obtain

$$\begin{aligned}
z'_{sq^{-1}x\pm\Delta}&=s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}\pm\frac{sT}{4}(\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{Mm}(\widetilde{\Delta})^-_{Mm}-\mathrm{sgn}(\sigma_0)(q^{-1})^-_{Mm}(\Delta)^-_{Mm})\\
&+\frac{s^2}{2}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\\
&\pm\frac{s^2T}{4}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})\\
&(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})\\
&+s^3\det((q^{-1})^-)\det((\tilde{x})^-)\pm\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}\\
&\left[+\left(\frac{T}{4}(q^{-1})^-_{Li}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell}]\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\right.\\
&+\left(\frac{T}{4}(q^{-1})^-_{Mj}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Mm}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&\left.+\left(\frac{T}{4}(q^{-1})^-_{Nk}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\right]\\
&+O((sT)^2)
\end{aligned} \tag{D.6}$$

and

$$\begin{aligned}
(z'_{sq^{-1}x\pm\Delta})^2&=s^2(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\\
&\pm\frac{s^2T}{4}(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})
\end{aligned}$$

$$+\frac{s^3}{2}(q^{-1})^-_{L\ell}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}(\tilde{x})^-_{L\ell}$$
$$\pm\frac{s^3T}{4}(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{L\ell}-(q^{-1})^-_{M\ell}(q^{-1})^-_{Lm})(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell})$$
$$\times(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}+s^4(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}\det((q^{-1})^-)\det((\tilde{x})^-)+o((sT)^2). \tag{D.7}$$

Therefore, we get

$$\left[\det\left(1+s(q^{-1})^-(\tilde{x})^-\pm\frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-]\right)\right]^2$$
$$=1+2z'_{sq^{-1}x\pm\Delta}+(z'_{q^{-1}xs\pm\Delta})^2$$
$$=1+2s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}\pm\frac{sT}{2}(\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{Mm}(\widetilde{\Delta})^-_{Mm}-\mathrm{sgn}(\sigma_0)(q^{-1})^-_{Mm}(\Delta)^-_{Mm})$$
$$+s^2(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}+2s^3\det((q^{-1})^-)\det((\tilde{x})^-)$$
$$\pm\frac{s^2T}{2}(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})$$
$$+\frac{s^3}{2}(q^{-1})^-_{L\ell}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}(\tilde{x})^-_{L\ell}$$
$$\pm2\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}$$
$$\left[+\left(\frac{T}{4}(q^{-1})^-_{Li}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell}]\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\right.$$
$$+\left(\frac{T}{4}(q^{-1})^-_{Mj}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Mm}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})$$
$$\left.+\left(\frac{T}{4}(q^{-1})^-_{Nk}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\right]$$
$$+s^4(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}\det((q^{-1})^-)\det((\tilde{x})^-)$$
$$\pm\frac{s^3T}{4}(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{L\ell}-(q^{-1})^-_{M\ell}(q^{-1})^-_{Lm})(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell})$$
$$\times(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}+O((sT)^2). \tag{D.8}$$

Now, we have to repeat this calculation for the second term involving the Kronecker deltas

$$\mathrm{tr}(s(q^{-1})^-(\tilde{x})^-\pm\frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-])+\frac{T}{2}s(q^{-1})^-$$
$$\times\,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}=s(q^{-1})^-_{mM}(\tilde{x})^-_{Mm}\pm\frac{sT}{4}(\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{Mm}(\widetilde{\Delta})^-_{Mm}$$
$$-\mathrm{sgn}(\sigma_0)(q^{-1})^-_{Mm}(\Delta)^-_{Mm})+\frac{sT}{2}(q^{-1})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm}. \tag{D.9}$$

Thus

$$\left[\mathrm{tr}\left(s(q^{-1})^-(\tilde{x})^-\pm\frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-]\right.\right.$$
$$\left.\left.+\frac{T}{2}s(q^{-1})^-\,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\right)\right]^2=s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}$$
$$\pm\frac{s^2T}{2}(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})$$
$$\pm s^2T(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}+O((sT)^2). \tag{D.10}$$

$$\mathrm{tr}\left(\left[s(q^{-1})^-(\tilde{x})^- \pm \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma_0})(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right.\right.$$
$$\left.\left. + \frac{T}{2}s(q^{-1})^- \,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\right]^2\right) = s^2(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}$$
$$\pm\frac{s^2T}{2}(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma_0})(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}) \pm s^2T(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}$$
$$\times(\tilde{x})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn} + O((sT)^2). \tag{D.11}$$

$$\det\left(s(q^{-1})^-(\tilde{x})^- \pm \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma_0})(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right.$$
$$\left. + \frac{T}{2}s(q^{-1})^-\,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\right) = s^3\det((q^{-1})^-)\det((\tilde{x})^-)$$
$$\pm\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}$$
$$\left[+\left(\frac{T}{4}(q^{-1})^-_{Li}[\mathrm{sgn}(\widetilde{\sigma_0})(\widetilde{\Delta})^-_{L\ell} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell}]\right)((q^{-1})^-_{Jm}(\tilde{x})^-_{Mm})((q^{-1})^-_{Kn}(\tilde{x})^-_{Nn})\right.$$
$$+\left(\frac{T}{4}(q^{-1})^-_{Mj}[\mathrm{sgn}(\widetilde{\sigma_0})(\widetilde{\Delta})^-_{Mm} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})$$
$$\left.+\left(\frac{T}{4}(q^{-1})^-_{Nk}[\mathrm{sgn}(\widetilde{\sigma_0})(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\right]$$
$$+\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}$$
$$\left[+\left(\frac{T}{2}(q^{-1})^-_{Li}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{L\ell}\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\right.$$
$$+\left(\frac{T}{2}(q^{-1})^-_{Mj}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm}\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})$$
$$\left.+\left(\frac{T}{2}(q^{-1})^-_{Nk}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\right]$$
$$+O((sT)^2). \tag{D.12}$$

Hence, we obtain

$$z'_{sq^{-1}x\pm\Delta+\delta} = s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm} \pm \frac{sT}{4}(\mathrm{sgn}(\widetilde{\sigma_0})(q^{-1})^-_{Mm}(\widetilde{\Delta})^-_{Mm} - \mathrm{sgn}(\sigma_0)(q^{-1})^-_{Mm}(\Delta)^-_{Mm})$$
$$+\frac{sT}{2}(q^{-1})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm}$$
$$+\frac{s^2}{2}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}$$
$$\pm\frac{s^2T}{4}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mm}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma_0})(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})$$
$$+\frac{s^2T}{2}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}$$
$$+s^3\det((q^{-1})^-)\det((\tilde{x})^-)$$

$$\pm \frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}$$
$$\left[+\left(\frac{T}{4}(q^{-1})^-_{Li}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell}]\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\right.$$
$$+\left(\frac{T}{4}(q^{-1})^-_{Mj}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Mm} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})$$
$$\left.+\left(\frac{T}{4}(q^{-1})^-_{Nk}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\right]$$
$$+\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}$$
$$\left[+\left(\frac{T}{2}(q^{-1})^-_{Li}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{L\ell}\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\right.$$
$$+\left(\frac{T}{2}(q^{-1})^-_{Mj}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm}\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})$$
$$\left.+\left(\frac{T}{2}(q^{-1})^-_{KN}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}\right)((q^{-1})^-_{IL}(\tilde{x})^-_{L\ell})((q^{-1})^-_{JM}(\tilde{x})^-_{Mm})\right]$$
$$+O((sT)^2). \tag{D.13}$$
$$(z'_{sq^{-1}x\pm\Delta+\delta})^2 = s^2(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}$$
$$\pm\frac{s^2T}{4}(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})$$
$$+\frac{s^2T}{2}(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}$$
$$+\frac{s^3}{2}(q^{-1})^-_{L\ell}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}(\tilde{x})^-_{L\ell}$$
$$\pm\frac{s^3T}{4}(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{L\ell} - (q^{-1})^-_{M\ell}(q^{-1})^-_{Lm})(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell}$$
$$-\mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + \frac{s^3T}{2}(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{L\ell} - (q^{-1})^-_{M\ell}(q^{-1})^-_{Lm})(\tilde{x})^-_{Mm}$$
$$\times\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{L\ell}(\tilde{x})^-_{Nn}$$
$$+s^4(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}\det((q^{-1})^-)\det((\tilde{x})^-) + O((sT)^2). \tag{D.14}$$

Therefore, we obtain

$$\left[\det\left(1 + s(q^{-1})^-(\tilde{x})^- \pm \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right.\right.$$
$$\left.\left.+\frac{T}{2}s(q^{-1})^-\,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\right)\right]^2 = 1 + 2z'_{sq^{-1}x\pm\Delta+\delta} + (z'_{q^{-1}xs\pm\Delta+\delta})^2$$
$$= 1 + 2s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm} \pm \frac{sT}{2}(\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{Mm}(\widetilde{\Delta})^-_{Mm} - \mathrm{sgn}(\sigma_0)(q^{-1})^-_{Mm}(\Delta)^-_{Mm})$$
$$+sT(q^{-1})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm}$$
$$+s^2(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}$$
$$\pm\frac{s^2T}{2}(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn} - \mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn})$$
$$+s^2T(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}$$

$$
\begin{aligned}
&+2s^3\det((q^{-1})^-)\det((\tilde{x})^-)\\
&\pm 2\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}\\
&\Bigg[+\left(\frac{T}{4}(q^{-1})^-_{Li}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell}]\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\left(\frac{T}{4}(q^{-1})^-_{Mj}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Mm}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Mm}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\left(\frac{T}{4}(q^{-1})^-_{Nk}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{Nn}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{Nn}]\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Bigg]\\
&+2\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}\\
&\Bigg[+\left(\frac{T}{2}(q^{-1})^-_{Li}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{L\ell}\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\left(\frac{T}{2}(q^{-1})^-_{Mj}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm}\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\left(\frac{T}{2}(q^{-1})^-_{Nk}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Bigg]\\
&+\frac{s^3}{2}(q^{-1})^-_{L\ell}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}(\tilde{x})^-_{L\ell}\pm\frac{s^3T}{4}(q^{-1})^-_{Nn}\\
&\times((q^{-1})^-_{Mm}(q^{-1})^-_{L\ell}-(q^{-1})^-_{M\ell}(q^{-1})^-_{Lm})(\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^-_{L\ell}-\mathrm{sgn}(\sigma_0)(\Delta)^-_{L\ell})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\\
&+\frac{s^3T}{2}(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{L\ell}-(q^{-1})^-_{M\ell}(q^{-1})^-_{Lm})(\tilde{x})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{L\ell}(\tilde{x})^-_{Nn}\\
&+s^4(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}\det((q^{-1})^-)\det((\tilde{x})^-)+O((sT)^2).
\end{aligned}
\tag{D.15}
$$

We easily see that we have the following equality:

$$
\begin{aligned}
&\Bigg[\det\Bigg(1+s(q^{-1})^-(\tilde{x})^-\pm\frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-]\\
&+\frac{T}{2}s(q^{-1})^-\,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\Bigg)\Bigg]^2\\
&=\left[\det\left(1+s(q^{-1})^-(\tilde{x})^-\pm\frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-]\right)\right]^2\\
&+sT(q^{-1})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm}\\
&+s^2T(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}\,\mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}\\
&+2\frac{s^3}{3!}\epsilon_{ijk}\epsilon_{\ell mn}\\
&\Bigg[+\left(\frac{T}{2}(q^{-1})^-_{Li}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{L\ell}\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\left(\frac{T}{2}(q^{-1})^-_{Mj}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm}\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\left(\frac{T}{2}(q^{-1})^-_{Nk}(\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Nn}\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Bigg]\\
&+\frac{s^3T}{2}(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{L\ell}-(q^{-1})^-_{M\ell}(q^{-1})^-_{Lm})(\tilde{x})^-_{Mm}
\end{aligned}
$$

$$\times\, \mathrm{sgn}(\sigma_0)((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{L\ell}(\tilde{x})^-_{Nn} + O((sT)^2). \tag{D.16}$$

Considering the terms that contain Kronecker deltas, we get

$$\begin{aligned}
&(q^{-1})^-_{Mm}((\delta)^-_{(J,j,v),(K_0,n_0,v)})_{Mm} = (q^{-1})^-_{Mm}\delta^-_{(M,m,v),(K_0,\sigma_0,v)} = (q^{-1})^-_{K_0,n_0}\\
&(q^{-1})^-_{Mm}((\widetilde{\Delta})^-)_{Mm} = (q^{-1})^-_{\widetilde{J}_0\widetilde{m}_0} - (q^{-1})^-_{\widetilde{I}_0\widetilde{m}_0}\\
&(q^{-1})^-_{Mm}((\Delta)^-)_{Mm} = (q^{-1})^-_{J_0m_0} - (q^{-1})^-_{I_0m_0}.
\end{aligned}\tag{D.17}$$

The only difference that occurs when considering the term $z'_{sq^{-1}x-\Delta+\delta}$ is the $K_0, \sigma_0, n_0$ get replaced by $\widetilde{K}_0, \widetilde{\sigma}_0, \widetilde{n}_0$.

Consequently, by reinserting the above results into the expression for $[\det]^2$, we obtain the following four expressions:

$$\begin{aligned}
&\left[\det\left(1 + s(q^{-1})^-(\tilde{x})^- + \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right)\right]^2\\
&\qquad\qquad = 1 + 2s(q^{-1})^-_{Mm}\\
&(\tilde{x})^-_{Mm} + \frac{sT}{2}\big(\mathrm{sgn}(\widetilde{\sigma}_0)\big((q^{-1})^-_{\widetilde{J}_0\widetilde{m}_0} - (q^{-1})^-_{\widetilde{I}_0\widetilde{m}_0}\big) - \mathrm{sgn}(\sigma_0)\big((q^{-1})^-_{J_0m_0} - (q^{-1})^-_{I_0m_0}\big)\big)\\
&+ s^2(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\\
&+ \frac{s^2T}{2}(\tilde{x})^-_{Mm}\Big[2(\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-_{\widetilde{J}_0\widetilde{m}_0} - (q^{-1})^-\widetilde{I}_0\widetilde{m}_0) - \mathrm{sgn}(\sigma_0)((q^{-1})^-_{J_0m_0} - (q^{-1})^-_{I_0m_0}))(q^{-1})^-_{Mm}\\
&- ((q^{-1})^-_{M\widetilde{m}_0}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-_{\widetilde{J}_0m} - (q^{-1})^-_{\widetilde{I}_0m}) - (q^{-1})^-_{Mm_0}\,\mathrm{sgn}(\sigma_0)((q^{-1})^-_{J_0m} - (q^{-1})^-_{I_0m}))\Big]\\
&+ 2s^3\det((q^{-1})^-)\det((\tilde{x})^-)\\
&+ 2\frac{s^3}{3!}\epsilon_{ijk}\\
&\Bigg[+\left(\frac{T}{4}\big[\epsilon_{\widetilde{m}_0mn}\,\mathrm{sgn}(\widetilde{\sigma}_0)\big((q^{-1})^-_{\widetilde{J}_0i} - (q^{-1})^-_{\widetilde{I}_0i}\big) - \epsilon_{m_0mn}\,\mathrm{sgn}(\sigma_0)\big((q^{-1})^-_{J_0i} - (q^{-1})^-_{I_0i}\big)\big]\right)\\
&((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\left(\frac{T}{4}\big[\epsilon_{\ell\widetilde{m}_0n}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-_{\widetilde{J}_0j} - (q^{-1})^-_{\widetilde{I}_0j}) - \epsilon_{\ell m_0n}\,\mathrm{sgn}(\sigma_0)\big((q^{-1})^-_{J_0j} - (q^{-1})^-_{I_0j}\big)\big]\right)\\
&((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\left(\frac{T}{4}\big[\epsilon_{\ell m\widetilde{m}_0}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-_{\widetilde{J}_0k} - (q^{-1})^-_{\widetilde{I}_0k}) - \epsilon_{\ell mm_0}\,\mathrm{sgn}(\sigma_0)\big((q^{-1})^-_{J_0k} - (q^{-1})^-_{I_0k}\big)\big]\right)\\
&((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Bigg]\\
&+\frac{s^3}{2}(q^{-1})^-_{L\ell}((q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}(\tilde{x})^-_{L\ell} + \frac{s^3T}{4}\\
&\big[(q^{-1})^-_{Nn}\Big[(q^{-1})^-_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-_{\widetilde{J}_0\widetilde{m}_0} - (q^{-1})^-_{\widetilde{I}_0\widetilde{m}_0}) - \mathrm{sgn}(\sigma_0)((q^{-1})^-_{J_0m_0} - (q^{-1})^-_{I_0m_0}))\big)\\
&-((q^{-1})^-_{M\widetilde{m}_0}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-_{\widetilde{J}_0m} - (q^{-1})^-_{\widetilde{I}_0m}) - (q^{-1})^-_{Mm_0}\,\mathrm{sgn}(\sigma_0)((q^{-1})^-_{J_0m} - (q^{-1})^-_{I_0m}))\big]\\
&\times(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\Big] + s^4(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}\det((q^{-1})^-)\det((\tilde{x})^-) + O((sT)^2).
\end{aligned}\tag{D.18}$$

$$\begin{aligned}
&\Bigg[\det\Bigg(1 + s(q^{-1})^-(\tilde{x})^- + \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\\
&+\frac{T}{2}s(q^{-1})^-\,\mathrm{sgn}(\sigma_0)(\delta)^-_{(J,j,v),(K_0,n_0,v)}\Bigg)\Bigg]^2
\end{aligned}$$

$$
\begin{aligned}
&= \left[\det\left(1 + s(q^{-1})^{-}(\tilde{x})^{-} + \frac{T}{4}s(q^{-1})^{-}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-} - \mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right)\right]^2 \\
&+ sT\,\mathrm{sgn}(\sigma_0)(q^{-1})^{-}_{K_0n_0} + s^2T\,\mathrm{sgn}(\sigma_0)(2(q^{-1})^{-}_{K_0n_0}(q^{-1})^{-}_{Mm} - (q^{-1})^{-}_{K_0m}(q^{-1})^{-}_{Mn_0})(\tilde{x})^{-}_{Mm} \\
&+ 2\frac{s^3}{3!}\epsilon_{ijk} \\
&\left[+\left(\frac{T}{2}\epsilon_{n_0mn}(q^{-1})^{-}_{K_0i}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^{-}_{Mj}(\tilde{x})^{-}_{Mm})((q^{-1})^{-}_{Nk}(\tilde{x})^{-}_{Nn})\right. \\
&+\left(\frac{T}{2}\epsilon_{\ell_{00}n}(q^{-1})^{-}_{K_0j}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^{-}_{Li}(\tilde{x})^{-}_{L\ell})((q^{-1})^{-}_{Nk}(\tilde{x})^{-}_{Nn}) \\
&\left.+\left(\frac{T}{2}\epsilon_{\ell mn_0}(q^{-1})^{-}_{K_0k}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^{-}_{Li}(\tilde{x})^{-}_{L\ell})((q^{-1})^{-}_{Mj}(\tilde{x})^{-}_{Mm})\right] + \frac{s^3T}{2} \\
&\times\mathrm{sgn}(\sigma_0)(q^{-1})^{-}_{Nn}\big((q^{-1})^{-}_{Mm}(q^{-1})^{-}_{K_0n_0} - (q^{-1})^{-}_{Mn_0}(q^{-1})^{-}_{K_0m}\big)(\tilde{x})^{-}_{Mm}(\tilde{x})^{-}_{Nn} + O((sT)^2). \qquad \text{(D.19)}
\end{aligned}
$$

$$
\begin{aligned}
&\left[\det\left(1 + s(q^{-1})^{-}(\tilde{x})^{-} - \frac{T}{4}s(q^{-1})^{-}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-} - \mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right)\right]^2 \\
&= 1 + 2s(q^{-1})^{-}_{Mm}(\tilde{x})^{-}_{Mm} - \frac{sT}{2}(\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^{-}_{\widetilde{J}_0\widetilde{m}_0} - (q^{-1})^{-}_{\widetilde{I}_0\widetilde{m}_0}) - \mathrm{sgn}(\sigma_0)((q^{-1})^{-}_{J_0m_0} \\
&- (q^{-1})^{-}_{I_0m_0})) + s^2(2(q^{-1})^{-}_{Mm}(q^{-1})^{-}_{Nn} - (q^{-1})^{-}_{Mn}(q^{-1})^{-}_{Nm})(\tilde{x})^{-}_{Mm}(\tilde{x})^{-}_{Nn} \\
&- \frac{s^2T}{2}(\tilde{x})^{-}_{Mm}\Big[2(\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^{-}_{\widetilde{J}_0\widetilde{m}_0} - (q^{-1})^{-}_{\widetilde{I}_0\widetilde{m}_0}) - \mathrm{sgn}(\sigma_0)((q^{-1})^{-}_{J_0m_0} - (q^{-1})^{-}_{I_0m_0}))(q^{-1})^{-}_{Mm} \\
&- ((q^{-1})^{-}_{M\widetilde{m}_0}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^{-}_{\widetilde{J}_0m} - (q^{-1})^{-}_{\widetilde{I}_0m}) - (q^{-1})^{-}_{Mm_0}\,\mathrm{sgn}(\sigma_0)((q^{-1})^{-}_{J_0m} - (q^{-1})^{-}_{I_0m}))\Big] \\
&- 2\frac{s^3}{3!}\epsilon_{ijk} \\
&\left[+\left(\frac{T}{4}\big[\epsilon_{\widetilde{m}_0mn}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^{-}_{\widetilde{J}_0i} - (q^{-1})^{-}_{\widetilde{I}_0i}) - \epsilon_{m_0mn}\,\mathrm{sgn}(\sigma_0)\big((q^{-1})^{-}_{J_0i} - (q^{-1})^{-}_{I_0i}\big)\big]\right)\right. \\
&((q^{-1})^{-}_{Mj}(\tilde{x})^{-}_{Mm})((q^{-1})^{-}_{Nk}(\tilde{x})^{-}_{Nn}) \\
&+\left(\frac{T}{4}\big[\epsilon_{\ell\widetilde{m}_0n}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^{-}_{\widetilde{J}_0j} - (q^{-1})^{-}_{\widetilde{I}_0j}) - \epsilon_{\ell m_0n}\,\mathrm{sgn}(\sigma_0)\big((q^{-1})^{-}_{J_0j} - (q^{-1})^{-}_{I_0j}\big)\big]\right) \\
&((q^{-1})^{-}_{Li}(\tilde{x})^{-}_{L\ell})((q^{-1})^{-}_{Nk}(\tilde{x})^{-}_{Nn}) \\
&+\left(\frac{T}{4}\big[\epsilon_{\ell m\widetilde{m}_0}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^{-}_{\widetilde{J}_0k} - (q^{-1})^{-}_{\widetilde{I}_0k}) - \epsilon_{\ell mm_0}\,\mathrm{sgn}(\sigma_0)\big((q^{-1})^{-}_{J_0k} - (q^{-1})^{-}_{I_0k}\big)\big]\right) \\
&\left.((q^{-1})^{-}_{Li}(\tilde{x})^{-}_{L\ell})((q^{-1})^{-}_{Mj}(\tilde{x})^{-}_{Mm})\right] \\
&+ \frac{s^3}{2}(q^{-1})^{-}_{L\ell}((q^{-1})^{-}_{Mm}(q^{-1})^{-}_{Nn} - (q^{-1})^{-}_{Mn}(q^{-1})^{-}_{Nm})(\tilde{x})^{-}_{Mm}(\tilde{x})^{-}_{Nn}(\tilde{x})^{-}_{L\ell} \\
&- \frac{s^3T}{4}\big[(q^{-1})^{-}_{Nn}\big[(q^{-1})^{-}_{Mm}(\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^{-}_{\widetilde{J}_0\widetilde{m}_0} - (q^{-1})^{-}_{\widetilde{I}_0\widetilde{m}_0}) \\
&- \mathrm{sgn}(\sigma_0)((q^{-1})^{-}_{J_0m_0} - (q^{-1})^{-}_{I_0m_0}))\big) - ((q^{-1})^{-}_{M\widetilde{m}_0}\,\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^{-}_{\widetilde{J}_0m} - (q^{-1})^{-}_{\widetilde{I}_0m}) \\
&- (q^{-1})^{-}_{Mm_0}\,\mathrm{sgn}(\sigma_0)((q^{-1})^{-}_{J_0m} - (q^{-1})^{-}_{I_0m}))\big](\tilde{x})^{-}_{Mm}(\tilde{x})^{-}_{Nn}\big] \\
&+ s^4(q^{-1})^{-}_{Mm}(\tilde{x})^{-}_{Mm}\det((q^{-1})^{-})\det((\tilde{x})^{-}) + O((sT)^2). \qquad \text{(D.20)}
\end{aligned}
$$

$$\Big[\det\Big(1+s(q^{-1})^-(\tilde{x})^- - \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]$$
$$+\frac{T}{2}s(q^{-1})^-\,\mathrm{sgn}(\widetilde{\sigma}_0)(\delta)^-_{(J,j,v),(\widetilde{K}_0,\widetilde{n}_0,v)}\Big)\Big]^2$$
$$=\Big[\det\Big(1+s(q^{-1})^-(\tilde{x})^- - \frac{T}{4}s(q^{-1})^-[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\Big)\Big]^2$$
$$+sT\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}+s^2T\,\mathrm{sgn}(\widetilde{\sigma}_0)(2(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}(q^{-1})^-_{Mm}-(q^{-1})^-_{\widetilde{K}_0m}(q^{-1})^-_{M\widetilde{n}_0})(\tilde{x})^-_{Mm}$$
$$+2\frac{s^3}{3!}\epsilon_{ijk}\Big[+\Big(\frac{T}{2}\epsilon_{\widetilde{n}_0mn}(q^{-1})^-_{\widetilde{K}_0i}\,\mathrm{sgn}(\widetilde{\sigma}_0)\Big)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})$$
$$+\Big(\frac{T}{2}\epsilon_{\ell\widetilde{n}_0n}(q^{-1})^-_{\widetilde{K}_0j}\,\mathrm{sgn}(\widetilde{\sigma}_0)\Big)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})$$
$$+\Big(\frac{T}{2}\epsilon_{\ell m\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0k}\,\mathrm{sgn}(\widetilde{\sigma}_0)\Big)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Big]$$
$$+\frac{s^3T}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}-(q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}+O((sT)^2). \tag{D.21}$$

*D.2. Expansion of the y and $y_1$ terms: estimation of the remainder in the expansion*

In this section, we will estimate the remainder in the expansion of $y^{\frac{1}{8}}, y_1^{\frac{1}{8}}, \widetilde{y}^{\frac{1}{8}}, \widetilde{y}_1^{\frac{1}{8}}$ around $y = y_1 = \widetilde{y} = \widetilde{y}_1 = 1$. For this, we use the tools developed in [13]. Here we have the case where $L = 1$ and $M = 8$. The expansion of $\Lambda^{\frac{1}{2}}$ [13] reads in this case

$$\Lambda^{\frac{1}{2}}\Big(\Big\{(\tilde{x}_{Jjv})^-+\frac{1}{T}((p_{Jjv})^-)+\frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\Big\},\frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v),n_0\Big)$$
$$=\frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\mathrm{i}\hbar}$$
$$\Big\{(y-y_1)\Big(\sum_{k=1}^{n}f^{(k)}_{\frac{1}{8}}(1)\sum_{l=0}^{k-1}(y-1)^l(y_1-1)^{k-1-l}\Big)$$
$$+\underbrace{\Big[f^{(n+1)}_{\frac{1}{8}}(y)(y-1)^{n+1}-f^{(n+1)}_{\frac{1}{8}}(y_1)(y_1-1)^{(n+1)}\Big]}_{\text{remainder}}\Big\}. \tag{D.22}$$

By using the explicit expression for $\Lambda^{\frac{1}{2}}$ in terms of $y^{\frac{1}{8}}, y_1^{\frac{1}{8}}$ and $\widetilde{y}^{\frac{1}{8}}, \widetilde{y}_1^{\frac{1}{8}}$ respectively, the remainder in lemma 4.1 of [13] will lead to a Gaussian integral of the form

$$\int_{\mathbb{R}^9}\mathrm{d}^9(x_{Jjv})^-\mathrm{e}^{-2\|(\tilde{x})^-\|^2}\Big(f^{(n+1)}_{\frac{1}{8}}(y)(y-1)^{(n+1)}-f^{(n+1)}_{\frac{1}{8}}(y_1)(y_1-1)^{(n+1)}\Big) \tag{D.23}$$

for the one $\Lambda^{\frac{1}{2}}$ function, and

$$\int_{\mathbb{R}^9}\mathrm{d}^9(x_{Jjv})^-\mathrm{e}^{-2\|(\tilde{x})^-\|^2}\Big(f^{(n+1)}_{\frac{1}{8}}(\widetilde{y})(\widetilde{y}-1)^{(n+1)}-f^{(n+1)}_{\frac{1}{8}}(\widetilde{y}_1)(\widetilde{y}_1-1)^{(n+1)}\Big) \tag{D.24}$$

for the other one respectively.

We come to the estimation of the two remainders now.

First of all we need an estimation for $z_{sq^{-1}x+\Delta}$, $z_{sq^{-1}x-\Delta}$, $z_{sq^{-1}x+\Delta+\delta}$, $z_{sq^{-1}x-\Delta+\tilde{\delta}}$:

$$\begin{aligned} z_{sq^{-1}x\pm\Delta} = {} & \mathrm{tr}(s(q^{-1})^-(\tilde{x})^- \pm \Delta) + \frac{1}{2}[(\mathrm{tr}(s(q^{-1})^-(\tilde{x})^- \pm \Delta))^2 \\ & - \mathrm{tr}([s(q^{-1})^-(\tilde{x})^- \pm \Delta)^2)] + \det(s(q^{-1})^-(\tilde{x})^- \pm \Delta). \end{aligned} \tag{D.25}$$

We obtain

$$\begin{aligned} |\mathrm{tr}(s(q^{-1})^-(\tilde{x})^- \pm \Delta)| \leqslant {} & s\|(q^{-1})^-\|\|(\tilde{x})^-\| + \frac{sT}{4}\big(|(q^{-1})^-_{\tilde{J}_0\tilde{m}_0}| + |(q^{-1})^-_{\tilde{I}_0\tilde{m}_0}| + |(q^{-1})^-_{J_0m_0}| \\ & + |(q^{-1})^-_{I_0m_0}|\big) \leqslant s\|(q^{-1})^-\|\|(\tilde{x})^-\| + sT\|(q^{-1})^-\| \end{aligned} \tag{D.26}$$

$$\begin{aligned} |[\mathrm{tr}(s(q^{-1})^-(\tilde{x})^- \pm \Delta)]^2| \leqslant {} & |\,\mathrm{tr}(s(q^{-1})^-(\tilde{x})^- \pm \Delta)|^2 \leqslant s^2\|(q^{-1})^-\|^2\|(\tilde{x})^-\|^2 \\ & + \frac{s^2T^2}{16}\big(|(q^{-1})^-_{\tilde{J}_0\tilde{m}_0}| + |(q^{-1})^-_{\tilde{I}_0\tilde{m}_0}| + |(q^{-1})^-_{J_0m_0}| + |(q^{-1})^-_{I_0m_0}|\big)^2 \\ & + \frac{sT}{2}\|(q^{-1})^-\|\|(\tilde{x})^-\|\big(|(q^{-1})^-_{\tilde{J}_0\tilde{m}_0}| + |(q^{-1})^-_{\tilde{I}_0\tilde{m}_0}| + |(q^{-1})^-_{J_0m_0}| + |(q^{-1})^-_{I_0m_0}|\big) \\ \leqslant {} & s^2\|(q^{-1})^-\|^2\|(\tilde{x})^-\|^2 + 2sT\|(q^{-1})^-\|^2\|(\tilde{x})^-\| + s^2T^2\|(q^{-1})^-\|^2 \end{aligned} \tag{D.27}$$

$$\begin{aligned} |\,\mathrm{tr}([s(q^{-1})^-(\tilde{x})^- \pm \Delta]^2)| \leqslant {} & s^2\|(q^{-1})^-\|^2\|(\tilde{x})^-\|^2 + \frac{s^2T^2}{16}\big(|(q^{-1})^-_{\tilde{J}_0\tilde{m}_0}| + |(q^{-1})^-_{\tilde{I}_0\tilde{m}_0}| \\ & + |(q^{-1})^-_{J_0m_0}| + |(q^{-1})^-_{I_0m_0}|\big)^2 + \frac{sT}{2}(|((q^{-1})^-(\tilde{x})^-(q^{-1})^-)_{\tilde{J}_0\tilde{m}_0}| \\ & + |((q^{-1})^-(\tilde{x})^-(q^{-1})^-)_{\tilde{I}_0\tilde{m}_0}| + |((q^{-1})^-(\tilde{x})^-(q^{-1})^-)_{J_0m_0}| \\ & + |((q^{-1})^-(\tilde{x})^-(q^{-1})^-)_{I_0m_0}|) \leqslant s^2\|(q^{-1})^-\|^2\|(\tilde{x})^-\|^2 \\ & + 2sT\|(q^{-1})^-\|^2\|(\tilde{x})^-\| + s^2T^2\|(q^{-1})^-\|^2 \end{aligned} \tag{D.28}$$

$$\begin{aligned} |\det(s(q^{-1})^-(\tilde{x})^- \pm \Delta)| \leqslant {} & 6s^3\left(\|(q^{-1})^-\|\|(\tilde{x})^-\| + \frac{T}{4}\big(|(q^{-1})^-_{\tilde{J}_0\tilde{m}_0}| + |(q^{-1})^-_{\tilde{I}_0\tilde{m}_0}| \right. \\ & \left. + |(q^{-1})^-_{J_0m_0}| + |(q^{-1})^-_{I_0m_0}|\big)\right)^3 \leqslant 6s^3(\|(q^{-1})^-\|^3\|(\tilde{x})^-\|^3 \\ & + 3T\|(q^{-1})^-\|^3\|(\tilde{x})^-\|^2 + 3T\|(\tilde{x})^-\|\|(q^{-1})^-\|^3 + T^3\|(q^{-1})^-\|^3). \end{aligned} \tag{D.29}$$

With these intermediate estimations, we can estimate $z_{sq^{-1}x+\Delta}$ and $z_{sq^{-1}x-\Delta}$ now:

$$\begin{aligned} |z_{sq^{-1}x\pm\Delta}| \leqslant {} & s\|(q^{-1})^-\|\|(\tilde{x})^-\| + sT\|(q^{-1})^-\| \\ & + \frac{1}{2}\big[+s^2\|(q^{-1})^-\|^2\|(\tilde{x})^-\|^2 + 2sT\|(q^{-1})^-\|^2\|(\tilde{x})^-\| + s^2T^2\|(q^{-1})^-\|^2 \\ & - s^2\|(q^{-1})^-\|^2\|(\tilde{x})^-\|^2 + 2sT\|(q^{-1})^-\|^2\|(\tilde{x})^-\| + s^2T^2\|(q^{-1})^-\|^2\big] \\ & + 6s^3(\|(q^{-1})^-\|^3\|(\tilde{x})^-\|^3 + 3T\|(q^{-1})^-\|^3\|(\tilde{x})^-\|^2 + 3T\|(\tilde{x})^-\|\|(q^{-1})^-\|^3 \\ & + T^3\|(q^{-1})^-\|^3) \\ = {} & s\|(q^{-1})^-\|\|(\tilde{x})^-\| + 18s^3T\|(q^{-1})^-\|^3\|(\tilde{x})^-\| \\ & + 18s^3T\|(q^{-1})^-\|^3\|(\tilde{x})^-\|^2 + 6s^3\|(q^{-1})^-\|^3\|(\tilde{x})^-\|^3 + sT\|(q^{-1})^-\| \\ & + 6s^3T^3\|(q^{-1})^-\|^3 \\ =: {} & u(\|(\tilde{x})^-\|), \end{aligned} \tag{D.30}$$

then

$$|y-1| \leqslant 2u + u^2 =: P(\|x\|) \quad \text{and} \quad |\widetilde{y}-1| \leqslant 2u + u^2 =: P(\|x\|). \tag{D.31}$$

In an analogous way, we obtain

$$|z_{sq^{-1}x+\Delta+\delta}| \leqslant |z_{sq^{-1}x\pm\Delta}| + sT\|(q^{-1})^-\| + s^2T\|(q^{-1})^-\|^2\|(\tilde{x})^-\| =: u_1(\|x\|) \tag{D.32}$$

and

$$|z_{sq^{-1}x-\Delta+\tilde{\delta}}| \leqslant |z_{sq^{-1}x\pm\Delta}| + sT\|(q^{-1})^-\| + s^2T\|(q^{-1})^-\|^2\|(\tilde{x})^-\| =: u_1(\|x\|). \tag{D.33}$$

Thus, we have

$$|y_1-1| \leqslant 2u_1 + u_1^2 =: P_1(\|x\|) \quad \text{and} \quad |\widetilde{y}_1-1| \leqslant 2u_1 + u_1^2 =: P_1(\|x\|). \tag{D.34}$$

Since $y$ and $\widetilde{y}$ and $y_1$ and $\widetilde{y}_1$ respectively are estimated by exactly the same polynomials $P(\|(\tilde{x})^-\|)$ and $P_1(\|(\tilde{x})^-\|)$ respectively, we can estimate the two remainders by the same term. Thus

$$\begin{aligned}
&\left|\int_{\mathbb{R}^9} \mathrm{d}^9(x_{Jjv})^- \mathrm{e}^{-2\|(\tilde{x})^-\|^2}\left(f_{\frac{1}{8}}^{(n+1)}(y)(y-1)^{(n+1)} - f_{\frac{1}{8}}^{(n+1)}(y_1)(y_1-1)^{(n+1)}\right)\right| \\
&\qquad \leqslant \int_{\mathbb{R}^9} \mathrm{d}^9(x_{Jjv})^- \mathrm{e}^{-2\|(\tilde{x})^-\|^2}(3\cdot 8)^{(n+1)}\left[P(\|(\tilde{x})^-\|)^{(n+2)} + 2P(\|(\tilde{x})^-\|)^{(n+1)}\right. \\
&\qquad\quad \left. + P_1(\|(\tilde{x})^-\|)^{(n+2)} + 2P_1(\|(\tilde{x})^-\|)^{(n+1)}\right]
\end{aligned} \tag{D.35}$$

and

$$\begin{aligned}
&\left|\int_{\mathbb{R}^9} \mathrm{d}^9(x_{Jjv})^- \mathrm{e}^{-2\|(\tilde{x})^-\|^2}\left(f_{\frac{1}{8}}^{(n+1)}(\widetilde{y})(\widetilde{y}-1)^{(n+1)} - f_{\frac{1}{8}}^{(n+1)}(\widetilde{y}_1)(\widetilde{y}_1-1)^{(n+1)}\right)\right| \\
&\qquad \leqslant \int_{\mathbb{R}^9} \mathrm{d}^9(x_{Jjv})^- \mathrm{e}^{-2\|(\tilde{x})^-\|^2}(3\cdot 8)^{(n+1)}\left[P(\|(\tilde{x})^-\|)^{(n+2)} + 2P(\|(\tilde{x})^-\|)^{(n+1)}\right. \\
&\qquad\quad \left. + P_1(\|(\tilde{x})^-\|)^{(n+2)} + 2P_1(\|(\tilde{x})^-\|)^{(n+1)}\right];
\end{aligned} \tag{D.36}$$

thus we can restrict our further estimation to one term only.

Express $P(\|(\tilde{x})^-\|)$ as an arbitrary polynomial of the order sixth

$$P(\|(\tilde{x})^-\|) = \sum_{k=0}^{6} a_k\|(\tilde{x})^-\|^k. \tag{D.37}$$

By the multinomial theorem, we obtain

$$[P(\|(\tilde{x})^-\|)]^n = \sum_{n_0+..+n_6=n} \frac{n!}{(n_0!)\ldots(n_6)!}\left[\prod_{k=0}^{6} a_k^n\right]\|(\tilde{x})^-\|^{\sum_{k=0}^{6} kn_k}. \tag{D.38}$$

Consider the Gaussian integral of the form

$$\sqrt{\frac{2}{\pi}}^9 \int_{\mathbb{R}^9} \mathrm{d}^9(\tilde{x})^-\, \mathrm{e}^{-2\|(\tilde{x})^-\|^2}\|(\tilde{x})^-\|^n = V_8\sqrt{\frac{2}{\pi}}^9 \int_0^\infty \mathrm{d}r\, \mathrm{e}^{-2r^2} r^{n+8} =: V_8\sqrt{\frac{2}{\pi}}^9 J_{n+8}, \tag{D.39}$$

where $V_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}$ is the volume of $S^m$. Now

$$\begin{aligned}
J_n &= \frac{\sqrt{2\pi}}{4} 2^{-\frac{3n}{2}} \frac{n!}{(\frac{n}{2})!} \quad \text{for n even} \\
J_n &= \frac{1}{2} 2^{-\frac{(n-1)}{2}} \left(\frac{n-1}{2}\right)! \quad \text{for } n \text{ odd.}
\end{aligned} \tag{D.40}$$

Introducing again the Gauß bracket [.], we can immediately check that

$$J_n \leqslant \frac{\sqrt{2\pi}\left[\frac{n}{2}\right]!}{4\cdot 2^{\left[\frac{n}{2}\right]}}. \tag{D.41}$$

Using $n! \leqslant e\left(\frac{(n+1)}{e}\right)^{(n+1)}$, we may further estimate

$$J_n \leqslant \frac{e\sqrt{2\pi}}{4}\frac{\left(\frac{(n+1)}{2e}\right)^{\frac{(n+1)}{2}}}{2^{\frac{(n-1)}{2}}} = \frac{e\sqrt{2\pi}}{4}2^{-n}\left(\frac{n+1}{e}\right)^{\frac{(n+1)}{2}}, \tag{D.42}$$

where $\frac{(n-1)}{2} \leqslant \left[\frac{n}{2}\right] \leqslant \frac{n}{2}$.

Finally, if $n \leqslant n_9$, then

$$\sqrt{\frac{2}{\pi}}^9 \int_{\mathbb{R}^9} \mathrm{d}^9(\tilde{x})^- \,\mathrm{e}^{-2\|(\tilde{x})^-\|^2} P(\|(\tilde{x})^-\|)^n \leqslant V_8\sqrt{\frac{2}{\pi}}^9\left(\frac{9+6n}{4e}\right)^{\frac{9}{2}}\left[\sum_{k=0}^{6} a_k\sqrt{\frac{9+6n}{4e}}^k\right]^n$$
$$=: K_{9,6}\left(\frac{9+6n}{4e}\right)^{\frac{9}{2}} P\left(\sqrt{\frac{9+6n}{4e}}\right). \tag{D.43}$$

Using the above estimate, we can bound the remainder from above

$$\begin{aligned}&\left|\int_{\mathbb{R}^9}\mathrm{d}^9(x_{Jjv})^-\,\mathrm{e}^{-2\|(\tilde{x})^-\|^2}\left(f_{\frac{1}{8}}^{(n+1)}(\tilde{y})(\tilde{y}-1)^{(n+1)} - f_{\frac{1}{8}}^{(n+1)}(\tilde{y}_1)(\tilde{y}_1-1)^{(n+1)}\right)\right|\\
&\quad\leqslant \int_{\mathbb{R}^9}\mathrm{d}^9(x_{Jjv})^-\,\mathrm{e}^{-2\|(\tilde{x})^-\|^2}(3\cdot 8)^{(n+1)}\left[P(\|(\tilde{x})^-\|)^{(n+2)} + 2P(\|(\tilde{x})^-\|)^{(n+1)}\right.\\
&\qquad\left. + P_1(\|(\tilde{x})^-\|)^{(n+2)} + 2P_1(\|(\tilde{x})^-\|)^{(n+1)}\right]\\
&\quad\leqslant K_{9,6}(3\cdot 8)^{(n+1)}\left\{\left(\frac{9+6(n+2)}{4e}\right)^{\frac{9}{2}}\left[\left(P\left(\sqrt{\frac{9+6(n+2)}{4e}}\right)\right)^{(n+2)}\right.\right.\\
&\qquad\left. + \left(P_1\left(\sqrt{\frac{9+6(n+2)}{4e}}\right)\right)^{(n+2)}\right] + 2\left(\frac{9+6(n+1)}{4e}\right)^{\frac{9}{2}}\\
&\qquad\left.\times\left[\left(P\left(\sqrt{\frac{9+6(n+1)}{4e}}\right)\right)^{(n+1)} + \left(P_1\left(\sqrt{\frac{9+6(n+1)}{4e}}\right)\right)^{(n+1)}\right]\right\}. \end{aligned}\tag{D.44}$$

As pointed out in [13], for small values of $n$ the error connected with the remainder is proportional to $s^{n+1}$. However, for larger values of $n$ the size of the error becomes comparable to the order of accuracy (in powers of $s$) up to which we have performed the expansion. Thus, we are interested in the value $n_0$ from where onwards the error becomes so large that it does not make sense to compute corrections. An estimate for $n_0$ can be derived from the condition

$$\frac{\left|\int_{\mathbb{R}^9}\mathrm{d}^9(x_{Jjv})^-\,\mathrm{e}^{-2\|(\tilde{x})^-\|^2}\left(f_{\frac{1}{8}}^{(n+2)}(\tilde{y})(\tilde{y}-1)^{(n+2)} - f_{\frac{1}{8}}^{(n+2)}(\tilde{y}_1)(\tilde{y}_1-1)^{(n+2)}\right)\right|}{\left|\int_{\mathbb{R}^9}\mathrm{d}^9(x_{Jjv})^-\,\mathrm{e}^{-2\|(\tilde{x})^-\|^2}\left(f_{\frac{1}{8}}^{(n+1)}(\tilde{y})(\tilde{y}-1)^{(n+1)} - f_{\frac{1}{8}}^{(n+1)}(\tilde{y}_1)(\tilde{y}_1-1)^{(n+1)}\right)\right|} \geqslant 1. \tag{D.45}$$

Since the upper bound in equation (D.44) looks rather complicated $n_0$ cannot be computed analytically. Nevertheless, the order of magnitude of $n_0$ can be obtained under the assumption that the value of $n_0$ is supposed to be quite large and therefore that the change of $P\left(\frac{9+6(n_0+2)}{4e}\right)$ as we replace $n_0$ by $n_0+1$ is much smaller than the value of $n_0$ itself. Under this assumption, the value of $n_0$ [13] is given by

$$n_0 = \frac{4e\left(\frac{\tau_0(8)}{s\|(q_{Jjv})^-\|}\right)^2 - 9}{6} - 3, \tag{D.46}$$

whereby $\tau_0(8)$ is of order unity[13]. Consequently, since $(q_{Jjv})^-$ is of order unity, we conclude as long as $s = t^{\frac{1}{2}-\alpha}$ is small, the value of $n_0 \gg 1$. Hence, the precise value of $n_0$ depends on the chosen value for $\alpha$.

## Appendix E. Explicit expressions for $y, y_1, \widetilde{y}$ and $\widetilde{y}_1$

In this section, we will derive the explicit terms for $y$, $y_1$, $\widetilde{y}$ and $\widetilde{y}_1$ that occur in the expansion of the $\Lambda^{\frac{1}{2}}$ functions up to order $O((sT)^2)$:

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)$$
$$= \frac{a^{\frac{3}{2}}}{\mathrm{i}\hbar}|\det((p)^-)|^{\frac{1}{4}}\big\{(y-y_1)\big(f^{(1)}_{\frac{1}{8}}(1) + f^{(2)}_{\frac{1}{8}}(1)(y-y_1-2) + f^{(3)}_{\frac{1}{8}}(1)((y_1-1)^2$$
$$+ (y-1)(y_1-1) + (y-1)^2)\big)\big\} \tag{E.1}$$

and

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)$$
$$= \frac{a^{\frac{3}{2}}}{(-\mathrm{i}\hbar)}|\det((p)^-)|^{\frac{1}{4}}\big\{(\widetilde{y}-\widetilde{y}_1)\big(f^{(1)}_{\frac{1}{8}}(1) + f^{(2)}_{\frac{1}{8}}(1)(\widetilde{y}-\widetilde{y}_1-2)$$
$$+ f^{(3)}_{\frac{1}{8}}(1)((\widetilde{y}_1-1)^2 + (\widetilde{y}-1)(\widetilde{y}_1-1) + (\widetilde{y}-1)^2)\big)\big\}. \tag{E.2}$$

The lowest order in the term $(y - y_1)$ and $(\widetilde{y} - \widetilde{y}_1)$ respectively is $sT$. Since we have a global term $(y - y_1)$ and $(\widetilde{y} - \widetilde{y}_1)$ respectively and the highest order we want to consider is $s^3T$, we will expand $(y - y_1)$ and $(\widetilde{y} - \widetilde{y}_1)$ respectively up to order $O((sT)^2)$ and all the others terms that are multiplied with these terms up to order $O(s^3)$. Using the definition of $y$, $y_1$, $\widetilde{y}$ and $\widetilde{y}_1$ in terms of the corresponding $z_{sq^{-1}x+\Delta}$, $z_{sq^{-1}x+\Delta+\delta}$, $z_{sq^{-1}x-\Delta}$ and $z_{sq^{-1}x-\Delta+\delta}$, we obtain

$$y - y_1 = z_{sq^{-1}x+\Delta} - z_{sq^{-1}x+\Delta+\delta}$$
$$= -sT\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0n_0} - s^2T\,\mathrm{sgn}(\sigma_0)(2(q^{-1})^-_{K_0n_0}(q^{-1})^-_{Mm}$$
$$-(q^{-1})^-_{K_0m}(q^{-1})^-_{Mn_0})(\tilde{x})^-_{Mm} - 2\frac{s^3T}{3!}\epsilon_{ijk}$$
$$\left[+\left(\frac{1}{2}\epsilon_{n_0mn}(q^{-1})^-_{K_0i}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\right.$$
$$+\left(\frac{1}{2}\epsilon_{\ell_0 0n}(q^{-1})^-_{K_0j}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})$$
$$\left.-\left(\frac{1}{2}\epsilon_{\ell m\widetilde{n}_0}(q^{-1})^-_{K_0k}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\right]$$
$$-\frac{s^3T}{2}\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{K_0n_0}$$
$$-(q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{K_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O((sT)^2). \tag{E.3}$$

$$y - 1 = z_{sq^{-1}x+\Delta}$$
$$= 2s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm} + \frac{sT}{2}(\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-_{\widetilde{J}_0\widetilde{m}_0} - (q^{-1})^-_{\widetilde{I}_0\widetilde{m}_0}) - \mathrm{sgn}(\sigma_0)((q^{-1})^-_{J_0m_0}$$
$$-(q^{-1})^-_{I_0m_0})) + s^2(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn} - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}$$

$$
\begin{aligned}
&+\frac{s^2T}{2}(\tilde{x})^-_{Mm}\big[\big(2\,\mathrm{sgn}(\widetilde{\sigma}_0)\big(\big((q^{-1})^-_{\widetilde{J}_0\widetilde{m}_0}-(q^{-1})^-_{\widetilde{I}_0\widetilde{m}_0}\big)-\mathrm{sgn}(\sigma_0)((q^{-1})^-_{J_0m_0}\\
&-(q^{-1})^-_{I_0m_0})\big)(q^{-1})^-_{Mm}-\big((q^{-1})^-_{M\widetilde{m}_0}\,\mathrm{sgn}(\widetilde{\sigma}_0)\big(((q^{-1})^-_{\widetilde{J}_0m}-(q^{-1})^-_{\widetilde{I}_0m}\big)\\
&-(q^{-1})^-_{Mm_0}\,\mathrm{sgn}(\sigma_0)((q^{-1})^-_{J_0m}-(q^{-1})^-_{I_0m}))\big]+O(s^3).
\end{aligned}
\tag{E.4}
$$

$$
\begin{aligned}
y_1-1=(y-1)&+sT\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0n_0}+s^2T\big(2(q^{-1})^-_{K_0n_0}(q^{-1})^-_{Mm}\\
&-(q^{-1})^-_{K_0m}(q^{-1})^-_{Mn_0}\big)(\tilde{x})^-_{Mm}+2\frac{s^3T}{3!}\epsilon_{ijk}\\
\Bigg[+\bigg(\frac{1}{2}&\epsilon_{n_0mn}(q^{-1})^-_{K_0i}\,\mathrm{sgn}(\sigma_0)\bigg)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\bigg(\frac{1}{2}\epsilon_{\ell_{00}n}(q^{-1})^-_{K_0j}\,\mathrm{sgn}(\sigma_0)\bigg)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&+\bigg(\frac{1}{2}\epsilon_{\ell mn_0}(q^{-1})^-_{K_0k}\,\mathrm{sgn}(\sigma_0)\bigg)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Bigg]\\
&+\frac{s^3T}{2}\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{K_0n_0}\\
&-(q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{K_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}+O((sT)^2)
\end{aligned}
\tag{E.5}
$$

and

$$
\begin{aligned}
\widetilde{y}-\widetilde{y}_1&=z_{sq^{-1}x-\Delta}-z_{sq^{-1}x-\Delta+\tilde{\delta}}\\
&=-sT(q^{-1})^-_{Mm}\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}-s^2T\,\mathrm{sgn}(\widetilde{\sigma}_0)\big(2(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}(q^{-1})^-_{Mm}\\
&\quad-(q^{-1})^-_{\widetilde{K}_0m}(q^{-1})^-_{M\widetilde{n}_0}\big)(\tilde{x})^-_{Mm}-2\frac{s^3T}{3!}\epsilon_{ijk}\\
\Bigg[+\bigg(\frac{1}{2}&\epsilon_{\widetilde{n}_0mn}(q^{-1})^-_{\widetilde{K}_0i}\,\mathrm{sgn}(\widetilde{\sigma}_0)\bigg)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&\quad+\bigg(\frac{1}{2}\epsilon_{\ell\widetilde{n}_0n}(q^{-1})^-_{\widetilde{K}_0j}\,\mathrm{sgn}(\widetilde{\sigma}_0)\bigg)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&\quad+\bigg(\frac{1}{2}\epsilon_{\ell m\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0k}\,\mathrm{sgn}(\widetilde{\sigma}_0)\bigg)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Bigg]\\
&\quad-\frac{s^3T}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{K_0\widetilde{n}_0}\\
&\quad-(q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}+O((sT)^2)
\end{aligned}
\tag{E.6}
$$

$$
\begin{aligned}
\widetilde{y}-1&=z_{sq^{-1}x-\Delta}\\
&=2s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}-\frac{sT}{2}\big(\mathrm{sgn}(\widetilde{\sigma}_0)((q^{-1})^-_{\widetilde{J}_0\widetilde{m}_0}-(q^{-1})^-_{\widetilde{I}_0\widetilde{m}_0}\big)-\mathrm{sgn}(\sigma_0)\big((q^{-1})^-_{J_0m_0}\\
&\quad-(q^{-1})^-_{I_0m_0})\big)+s^2(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\\
&\quad-\frac{s^2T}{2}(\tilde{x})^-_{Mm}\big[2(\mathrm{sgn}(\widetilde{\sigma}_0)\big((q^{-1})^-_{\widetilde{J}_0\widetilde{m}_0}-(q^{-1})^-_{\widetilde{I}_0\widetilde{m}_0}\big)-\mathrm{sgn}(\sigma_0)\big((q^{-1})^-_{J_0m_0}\\
&\quad-(q^{-1})^-_{I_0m_0}\big)\big)(q^{-1})^-_{Mm}-\big((q^{-1})^-_{M\widetilde{m}_0}\,\mathrm{sgn}(\widetilde{\sigma}_0)\big((q^{-1})^-_{\widetilde{J}_0m}-(q^{-1})^-_{\widetilde{I}_0m}\big)\\
&\quad-(q^{-1})^-_{Mm_0}\,\mathrm{sgn}(\sigma_0)\big((q^{-1})^-_{J_0m}-(q^{-1})^-_{I_0m}\big)\big)\big]+O(s^3)
\end{aligned}
\tag{E.7}
$$

$$\begin{aligned}
\widetilde{y}_1 - 1 = (\widetilde{y} - 1) &+ sT\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0} + s^2T(2(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}(q^{-1})^-_{Mm} \\
&- (q^{-1})^-_{\widetilde{K}_0m}(q^{-1})^-_{Mn_0})(\tilde{x})^-_{Mm} + 2\frac{s^3T}{3!}\epsilon_{ijk} \\
\Big[ + \Big(\frac{1}{2}\epsilon_{\widetilde{n}_0mn}&(q^{-1})^-_{\widetilde{K}_0i}\,\mathrm{sgn}(\widetilde{\sigma}_0)\Big)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn}) \\
&+ \Big(\frac{1}{2}\epsilon_{\ell\widetilde{n}_0n}(q^{-1})^-_{\widetilde{K}_0j}\,\mathrm{sgn}(\widetilde{\sigma}_0)\Big)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn}) \\
&+ \Big(\frac{1}{2}\epsilon_{\ell m\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0k}\,\mathrm{sgn}(\widetilde{\sigma}_0)\Big)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Big] \\
&+ \frac{s^3T}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0} \\
&- (q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O((sT)^2).
\end{aligned} \tag{E.8}$$

Since

$$y_1 - 1 = y - 1 + (y_1 - y) \qquad \widetilde{y}_1 - 1 = \widetilde{y} - 1 + (\widetilde{y}_1 - \widetilde{y}), \tag{E.9}$$

we get

$$\begin{aligned}
(y_1 - 1)^2 &= (y - 1)^2 + 2(y - 1)(y_1 - y) + (y_1 - y)^2 \\
(\widetilde{y}_1 - 1)^2 &= (\widetilde{y} - 1)^2 + 2(\widetilde{y} - 1)(\widetilde{y}_1 - \widetilde{y}) + (\widetilde{y}_1 - \widetilde{y})^2 \\
(y - 1)(y_1 - 1) &= (y - 1)^2 + (y_1 - y)(y - 1) \\
(\widetilde{y} - 1)(\widetilde{y}_1 - 1) &= (\widetilde{y} - 1)^2 + (\widetilde{y}_1 - \widetilde{y})(\widetilde{y} - 1).
\end{aligned} \tag{E.10}$$

Reinserting this into the expansion of the $\Lambda^{\frac{1}{2}}$ functions, we obtain

$$\begin{aligned}
\Lambda^{\frac{1}{2}}&\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right) \\
&= \frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\mathrm{i}\hbar}(y - y_1)\big\{\big(f^{(1)}_{\frac{1}{8}}(1) + f^{(2)}_{\frac{1}{8}}(1)(2(y - 1) + (y_1 - y)) \\
&\quad + f^{(3)}_{\frac{1}{8}}(1)(2(y_1 - 1)^2 + 3(y - 1)(y_1 - 1) + (y_1 - y)^2)\big)\big\}
\end{aligned} \tag{E.11}$$

and

$$\begin{aligned}
\Lambda^{\frac{1}{2}}&\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right) \\
&= \frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{(-\mathrm{i}\hbar)}(\widetilde{y} - \widetilde{y}_1)\big\{\big(f^{(1)}_{\frac{1}{8}}(1) + f^{(2)}_{\frac{1}{8}}(1)(2(\widetilde{y} - 1) + (\widetilde{y}_1 - \widetilde{y})) \\
&\quad + f^{(3)}_{\frac{1}{8}}(1)(2(\widetilde{y}_1 - 1)^2 + 3(\widetilde{y} - 1)(\widetilde{y}_1 - 1) + (\widetilde{y}_1 - \widetilde{y})^2))\big\}.
\end{aligned} \tag{E.12}$$

### *E.1. The leading-order term of $\Lambda^{\frac{1}{2}}$ functions*

The leading-order term of $\Lambda^{\frac{1}{2}}$ is of the order $sT/t$ due to the $\hbar$ in the denominator in equations (E.11) and (E.12). Hence, the leading-order contribution of $\Lambda^{\frac{1}{2}}$ is given by

$$\begin{aligned}
\Lambda^{\frac{1}{2}}&\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right) \\
&= \frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\mathrm{i}\hbar}f^{(1)}_{\frac{1}{8}}(1)(y - y_1)\Big|_{sT}
\end{aligned} \tag{E.13}$$

and

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^{-}+\frac{1}{T}((p_{Jjv})^{-})-\frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-}-\mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right\},\frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v),\widetilde{n}_0\right)$$
$$=\frac{a^{\frac{3}{2}}|\det((p)^{-})|^{\frac{1}{4}}}{\mathrm{i}\hbar}f^{(1)}_{\frac{1}{8}}(1)(\widetilde{y}-\widetilde{y}_1)\Big|_{sT}, \tag{E.14}$$

whereby in $(y-y_1)$ and $(\widetilde{y}-\widetilde{y}_1)$ only terms of order $sT$ are considered. Recalling equations (E.3)) and (E.6), we obtain for the $\Lambda^{\frac{1}{2}}$ functions in leading order the following result:

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^{-}+\frac{1}{T}((p_{Jjv})^{-})+\frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-}-\mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right\},\frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v),n_0\right)$$
$$=\frac{a^{\frac{3}{2}}|\det((p)^{-})|^{\frac{1}{4}}}{\mathrm{i}\hbar}f^{(1)}_{\frac{1}{8}}(1)\big(-sT\,\mathrm{sgn}(\sigma_0)(q^{-1})^{-}_{K_0n_0}\big) \tag{E.15}$$

and

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^{-}+\frac{1}{T}((p_{Jjv})^{-})-\frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-}-\mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right\},\frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v),\widetilde{n}_0\right)$$
$$=\frac{a^{\frac{3}{2}}|\det((p)^{-})|^{\frac{1}{4}}}{(-\mathrm{i}\hbar)}f^{(1)}_{\frac{1}{8}}(1)\big(-sT\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^{-}_{\widetilde{K}_0\widetilde{n}_0}\big). \tag{E.16}$$

## Appendix F. The next-to-leading-order contribution to the algebraic master constraint expectation value

Here we expand each $\Lambda^{\frac{1}{2}}$ function up to order $O(s^3T/t)$. Afterwards we take the product of these two functions and consider all terms up to the order $O((sT/t)^2s^2)$.

The precise expression for the expansion reads

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^{-}+\frac{1}{T}((p_{Jjv})^{-})+\frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-}-\mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right\},\frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v),n_0\right)$$
$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^{-}+\frac{1}{T}((p_{Jjv})^{-})-\frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^{-}-\mathrm{sgn}(\sigma_0)(\Delta)^{-}]\right\},\frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v),\widetilde{n}_0\right)$$
$$=\left(\frac{a^{\frac{3}{2}}|\det((p)^{-})|^{\frac{1}{4}}}{\hbar}\right)^2(y-y_1)(\widetilde{y}-\widetilde{y}_1)$$
$$\begin{aligned}&\Big[\big(f^{(1)}_{\frac{1}{8}}(1)\big)^2+f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)[2(y-1)+(y_1-1)+2(\widetilde{y}-1)+(\widetilde{y}_1-1)]\\
&\quad+f^{(1)}_{\frac{1}{8}}(1)f^{(3)}_{\frac{1}{8}}(1)[2(y-1)^2+3(y-1)(y_1-1)+(y_1-y)^2+2(\widetilde{y}-1)^2\\
&\quad+3(\widetilde{y}-1)(\widetilde{y}_1-1)+(\widetilde{y}_1-\widetilde{y})^2]+\big(f^{(2)}_{\frac{1}{8}}(1)\big)^2[2(y-1)+(y_1-1)]\\
&\quad\times[2(\widetilde{y}-1)+(\widetilde{y}_1-1)]+f^{(2)}_{\frac{1}{8}}(1)f^{(3)}_{\frac{1}{8}}(1)([2(\widetilde{y}-1)+(\widetilde{y}_1-1)][2(y-1)^2\\
&\quad+3(y-1)(y_1-1)+(y_1-y)^2]+[2(y-1)+(y_1-1)][2(\widetilde{y}-1)^2+3(\widetilde{y}-1)\\
&\quad\times(\widetilde{y}_1-1)+(\widetilde{y}_1-1)^2])+\big(f^{(3)}_{\frac{1}{8}}(1)\big)^2[2(\widetilde{y}-1)^2+3(\widetilde{y}-1)(\widetilde{y}_1-1)+(\widetilde{y}_1-1)^2]\\
&\quad\times[2(y_1)^2+3(y-1)(y_1-1)+(y_1-1)^2]\Big]\Big|_{s^2(sT/t)^2}+O(s^2(sT/t)^2),\end{aligned} \tag{F.1}$$

whereby $\big|_{s^2(sT/t)^2}$ means that only terms of maximal this order are considered, although apparently higher order terms will occur due to, for instance, the squares and products of

$y$, $y_1$, $\widetilde{y}$ and $\widetilde{y}_1$ respectively. As before, we expand $(y-y_1)$ and $(\widetilde{y}-\widetilde{y}_1)$ up to order $O((sT)^2)$ while the other occurring terms have to be expanded up to $O(s^3)$ only. The separated terms are given by

$$\begin{aligned}
y - y_1 = {}& -sT\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0n_0} - s^2T\,\mathrm{sgn}(\sigma_0)\big(2(q^{-1})^-_{K_0n_0}(q^{-1})^-_{Mm} \\
& - (q^{-1})^-_{K_0m}(q^{-1})^-_{Mn_0}\big)(\tilde{x})^-_{Mm} - 2\frac{s^3T}{3!}\epsilon_{ijk} \\
\Bigg[& + \left(\frac{1}{2}\epsilon_{n_0mn}(q^{-1})^-_{K_0i}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn}) \\
& + \left(\frac{1}{2}\epsilon_{\ell 0_0 n}(q^{-1})^-_{K_0j}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn}) \\
& + \left(\frac{1}{2}\epsilon_{\ell mn_0}(q^{-1})^-_{K_0k}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Bigg] \\
& - \frac{s^3T}{2}\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{K_0n_0} \\
& - (q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{K_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O((sT)^2)
\end{aligned} \tag{F.2}$$

$$\begin{aligned}
\widetilde{y} - \widetilde{y}_1 = {}& -sT(q^{-1})^-_{Mm}\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0} - s^2T\,\mathrm{sgn}(\widetilde{\sigma}_0)(2(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}(q^{-1})^-_{Mm} \\
& - (q^{-1})^-_{\widetilde{K}_0m}(q^{-1})^-_{M\widetilde{n}_0})(\tilde{x})^-_{Mm} - 2\frac{s^3T}{3!}\epsilon_{ijk} \\
\Bigg[& + \left(\frac{1}{2}\epsilon_{\widetilde{n}_0mn}(q^{-1})^-_{\widetilde{K}_0i}\,\mathrm{sgn}(\widetilde{\sigma}_0)\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn}) \\
& + \left(\frac{1}{2}\epsilon_{\ell\widetilde{n}_0n}(q^{-1})^-_{\widetilde{K}_0j}\,\mathrm{sgn}(\widetilde{\sigma}_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn}) \\
& + \left(\frac{1}{2}\epsilon_{\ell m\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0k}\,\mathrm{sgn}(\widetilde{\sigma}_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\Bigg] \\
& - \frac{s^3T}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{K_0\widetilde{n}_0} \\
& - (q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O((sT)^2)
\end{aligned} \tag{F.3}$$

$$\begin{aligned}
2(y-1) + (y_1 - y) &= 4s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm} + 2s^2(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn} \\
&\quad - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3) \\
2(\widetilde{y}-1) + (\widetilde{y}_1 - \widetilde{y}) &= 4s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm} + 2s^2(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn} \\
&\quad - (q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3) \\
(y-1)^2 &= 4s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3) \\
(y-1)(y_1-1) &= 4s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3) \\
(\widetilde{y}-1)(\widetilde{y}_1-1) &= 4s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3).
\end{aligned} \tag{F.4}$$

The terms $(y_1 - y)^2$ and $(\widetilde{y}_1 - y)$ are already of order $O(s^3)$ and hence do not have to be considered:

$$(\widetilde{y} - 1)^2 = 4s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3). \tag{F.5}$$

The terms $(\widetilde{y}_1 - \widetilde{y})^2$ and $(\widetilde{y} - 1)(\widetilde{y}_1 - y)$ are of order $O(s^3)$ and will therefore be neglected in the further discussion:

$$\begin{aligned}
(y-1)(\widetilde{y}-1) &= 4s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3)\\
(y-1)(\widetilde{y}_1-1) &= 4s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3)\\
(y_1-1)(\widetilde{y}-1) &= 4s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn} + O(s^3).
\end{aligned} \tag{F.6}$$

Thus, when neglecting the terms of order $O(s^2(sT/t)^2)$, the expansion reduces to

$$\begin{aligned}
&\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)\\
&\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)\\
&\quad = \left(\frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\hbar}\right)^2 (y-y_1)(\widetilde{y}-\widetilde{y}_1)\big[\big(f^{(1)}_{\frac{1}{8}}(1)\big)^2 + f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)[2(y-1)\\
&\qquad + (y_1-1) + 2(\widetilde{y}-1) + (\widetilde{y}_1-1)] + f^{(1)}_{\frac{1}{8}}(1)f^{(3)}_{\frac{1}{8}}(1)[2(y-1)^2 + 3(y-1)\\
&\qquad \times (y_1-1) + 2(\widetilde{y}-1)^2 + 3(\widetilde{y}-1)(\widetilde{y}_1-1)] + \big(f^{(2)}_{\frac{1}{8}}(1)\big)^2[4(y-1)(\widetilde{y}-1)\\
&\qquad + 2(y-1)(\widetilde{y}_1-1) + 2(y_1-1)(\widetilde{y}-1)]\big] + O(s^2(sT/t)^2).
\end{aligned} \tag{F.7}$$

Reinserting these various terms into the expansion of the $\Lambda^{\frac{1}{2}}$ functions, we obtain

$$\begin{aligned}
&\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right)\\
&\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right)\\
&\quad = \left(\frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\hbar}\right)^2 (sT)^2
\end{aligned}$$

$$\begin{aligned}
&\Big(\mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0n_0} + s\,\mathrm{sgn}(\sigma_0)(2(q^{-1})^-_{K_0n_0}(q^{-1})^-_{Nn} - (q^{-1})^-_{K_0n}(q^{-1})^-_{Nn_0})(\tilde{x})^-_{Mm} + 2\frac{s^2}{3!}\epsilon_{ijk}\\
&\bigg[+\left(\frac{1}{2}\epsilon_{n_0mn}(q^{-1})^-_{K_0i}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&\qquad + \left(\frac{1}{2}\epsilon_{\ell 0 0 n}(q^{-1})^-_{K_0j}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&\qquad + \left(\frac{1}{2}\epsilon_{\ell mn_0}(q^{-1})^-_{K_0k}\,\mathrm{sgn}(\sigma_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\bigg]\\
&\qquad + \frac{s^2}{2}\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{K_0n_0} - (q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{K_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\Big)
\end{aligned}$$

$$\begin{aligned}
&\Big(\mathrm{sgn}(\widetilde{\sigma}_0)(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0} + s\,\mathrm{sgn}(\widetilde{\sigma}_0)(2(q^{-1})^-_{\widetilde{K}_0\widetilde{n}_0}(q^{-1})^-_{Nn} - (q^{-1})^-_{\widetilde{K}_0n}(q^{-1})^-_{N\widetilde{n}_0})(\tilde{x})^-_{Mm} + 2\frac{s^2}{3!}\epsilon_{ijk}\\
&\bigg[+\left(\frac{1}{2}\epsilon_{n_0mn}(q^{-1})^-_{\widetilde{K}_0i}\,\mathrm{sgn}(\widetilde{\sigma}_0)\right)((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&\qquad + \left(\frac{1}{2}\epsilon_{\ell\widetilde{n}_0n}(q^{-1})^-_{\widetilde{K}_0j}\,\mathrm{sgn}(\widetilde{\sigma}_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Nk}(\tilde{x})^-_{Nn})\\
&\qquad + \left(\frac{1}{2}\epsilon_{\ell m\widetilde{n}_0}(q^{-1})^-_{\widetilde{K}_0k}\,\mathrm{sgn}(\widetilde{\sigma}_0)\right)((q^{-1})^-_{Li}(\tilde{x})^-_{L\ell})((q^{-1})^-_{Mj}(\tilde{x})^-_{Mm})\bigg]
\end{aligned}$$

$$+\frac{s^2}{2}\,\mathrm{sgn}(\sigma_0)(q^{-1})^-_{Nn}((q^{-1})^-_{Mm}(q^{-1})^-_{K_0n_0}-(q^{-1})^-_{M\widetilde{n}_0}(q^{-1})^-_{K_0m})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\Big)$$

$$\Big[\big(f^{(1)}_{\frac{1}{8}}(1)\big)^2+s(q^{-1})^-_{Mm}(\tilde{x})^-_{Mm}8f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)$$
$$+s^2(2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm})(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\big(4f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\big)$$
$$+s^2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\big(40f^{(1)}_{\frac{1}{8}}(1)f^{(3)}_{\frac{1}{8}}(1)+32\big(f^{(2)}_{\frac{1}{8}}(1)\big)^2\big)\Big]$$
$$+O(s^2(sT/t)^2). \tag{F.8}$$

We will order the expansion in powers of $(\tilde{x})^-$ since this is quite useful for the later integration over $(\tilde{x})^-$. Moreover, we will neglect the linear powers of $(\tilde{x})^-$, because they will cancel in the integration when integrating against the even function $\exp(-2((\tilde{x})^-)^2)$. Let us introduce the following shorthands:

$$\begin{aligned}
C^{Mm} &:= (q^{-1})^-_{Mm}\\
C^{Mm,Nn} &:= 2(q^{-1})^-_{Mm}(q^{-1})^-_{Nn}-(q^{-1})^-_{Mn}(q^{-1})^-_{Nm}\\
C^{K_0\sigma_0n_0} &:= \mathrm{sgn}(\sigma_0)(q^{-1})^-_{K_0n_0}\\
C^{Mm,K_0\sigma_0n_0} &:= \mathrm{sgn}(\sigma_0)\big(2(q^{-1})^-_{Mm}(q^{-1})^-_{K_0n_0}-(q^{-1})^-_{K_0n}(q^{-1})^-_{Nn_0}\big).
\end{aligned} \tag{F.9}$$

Then we can rewrite the $\Lambda^{\frac{1}{2}}$ expansion as

$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^-+\frac{1}{T}((p_{Jjv})^-)+\frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-]\right\},\frac{1}{2}\,\mathrm{sgn}(\sigma_0)e_{K_0}(v),n_0\right)$$
$$\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^-+\frac{1}{T}((p_{Jjv})^-)-\frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^--\mathrm{sgn}(\sigma_0)(\Delta)^-]\right\},\frac{1}{2}\,\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v),\widetilde{n}_0\right)$$
$$=\left(\frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\hbar}\right)^2(sT)^2$$
$$\Bigg(C^{K_0\sigma_0n_0}+s(\tilde{x})^-_{Mm}C^{Mm,K_0\sigma_0n_0}+s^2(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}$$
$$\Bigg[\frac{1}{2}C^{Mm,K_0\sigma_0n_0}C^{Nn}+\frac{1}{3!}\epsilon_{ijk}\big[\epsilon_{n_0mn}C^{K_0\sigma_0i}C^{Mj}C^{Nk}$$
$$+\epsilon_{\ell_{00}n}C^{K_0\sigma_0j}C^{Li}C^{Nk}+\epsilon_{\ell mn_0}C^{K_0\sigma_0k}C^{Li}C^{Mj}\big]\Bigg)\Bigg]$$
$$\Bigg(C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}+s(\tilde{x})^-_{Mm}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}+s^2(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}$$
$$\Bigg[\frac{1}{2}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Nn}+\frac{1}{3!}\epsilon_{ijk}\big[\epsilon_{\widetilde{n}_0mn}C^{\widetilde{K}_0\widetilde{\sigma}_0i}C^{Mj}C^{Nk}$$
$$+\epsilon_{\ell\widetilde{n}_0n}C^{\widetilde{K}_0\widetilde{\sigma}_0j}C^{Li}C^{Nk}+\epsilon_{\ell m\widetilde{n}_0}C^{\widetilde{K}_0\widetilde{\sigma}_0k}C^{Li}C^{Mj}\big]\Bigg)\Bigg]$$
$$\Bigg[\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2+s(\tilde{x})^-_{Mm}C^{Mm}\left(8f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\right)$$
$$+s^2(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\big[C^{Mm,Nn}\big(4f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\big)$$

$$+ C^{Mm}C^{Nn}\big(40 f^{(1)}_{\frac{1}{8}}(1) f^{(3)}_{\frac{1}{8}}(1) + 32\big(f^{(2)}_{\frac{1}{8}}(1)\big)^2\big)\big]\Bigg] + O(s^2(sT/t)^2). \qquad \text{(F.10)}$$

The expansion has the following structure:

$$\begin{aligned}
&\left(\frac{sT}{t}\right)^2 (\alpha_0 + \alpha_1 s(\tilde{x})^- + \alpha_2 s^2((\tilde{x})^-)^2)(\beta_0 + \beta_1 s(\tilde{x})^- \\
&\qquad + \beta_2 s^2((\tilde{x})^-)^2)(\gamma_0 + \gamma_1 s(\tilde{x})^- + \gamma_2 s^2((\tilde{x})^-)^2) \\
&\quad = \alpha_0\beta_0\gamma_0 + s^2((\tilde{x})^-)^2[\alpha_2(\beta_0+\gamma_0) + \beta_2(\gamma_2+\alpha_2) + \gamma_2(\alpha_0+\beta_0) \\
&\qquad + \alpha_1\beta_1\gamma_0 + \alpha_1\beta_0\gamma_1 + \alpha_0\beta_1\gamma_1] + \mathrm{lin}((\tilde{x})^-) + O(s^2(sT/t)^2),
\end{aligned} \qquad \text{(F.11)}$$

whereby $\mathrm{lin}((\tilde{x})^-)$ denotes all terms linear in $(\tilde{x})^-$ which we do not show in detail as they will not contribute to the final result, because they vanish when integrated against the even function $\exp(-2((\tilde{x})^-)^2)$. Precisely, when considering what the $\alpha$, $\beta$ and $\gamma$ coefficients actually are, we obtain

$$\begin{aligned}
&\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) + \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\sigma_0)e_{K_0}(v), n_0\right) \\
&\Lambda^{\frac{1}{2}}\left(\left\{(\tilde{x}_{Jjv})^- + \frac{1}{T}((p_{Jjv})^-) - \frac{T}{4}[\mathrm{sgn}(\widetilde{\sigma}_0)(\widetilde{\Delta})^- - \mathrm{sgn}(\sigma_0)(\Delta)^-]\right\}, \frac{1}{2}\mathrm{sgn}(\widetilde{\sigma}_0)e_{\widetilde{K}_0}(v), \widetilde{n}_0\right) \\
&\qquad = \left(\frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\hbar}\right)^2 (sT)^2 \\
&\left[C^{K_0\sigma_0 n_0}C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2 + s^2(\tilde{x})^-_{Mm}(\tilde{x})^-_{Nn}\left\{+\left(C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0} + \left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\right)\right.\right. \\
&\left[\frac{1}{2}C^{Mm,K_0\sigma_0 n_0}C^{Nn} + \frac{1}{3!}\epsilon_{ijk}\big[\epsilon_{n_0 mn}C^{K_0\sigma_0 i}C^{Mj}C^{Nk} + \epsilon_{\ell_{0}0n}C^{K_0\sigma_0 j}C^{Li}C^{Nk}\right. \\
&\qquad \left.+ \epsilon_{\ell m n_0}C^{K_0\sigma_0 k}C^{Li}C^{Mj}\big]\big)\right] + \left(C^{K_0\sigma_0 n_0} + \left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\right) \\
&\left[\frac{1}{2}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Nn} + \frac{1}{3!}\epsilon_{ijk}\big[\epsilon_{n_0 mn}C^{\widetilde{K}_0\widetilde{\sigma}_0 i}C^{Mj}C^{Nk} + \epsilon_{\ell\widetilde{n}_0 n}C^{\widetilde{K}_0\widetilde{\sigma}_0 j}C^{Li}C^{Nk}\right. \\
&\qquad \left.+ \epsilon_{\ell m\widetilde{n}_0}C^{\widetilde{K}_0\widetilde{\sigma}_0 k}C^{Li}C^{Mj}\big]\big)\right] \\
&\qquad + (C^{K_0\sigma_0 n_0} + C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}) \\
&\left[C^{Mm,Nn}\left(4 f^{(1)}_{\frac{1}{8}}(1) f^{(2)}_{\frac{1}{8}}(1)\right) + C^{Mm}C^{Nn}\left(40 f^{(1)}_{\frac{1}{8}}(1) f^{(3)}_{\frac{1}{8}}(1) + 32\left(f^{(2)}_{\frac{1}{8}}(1)\right)^2\right)\right] \\
&\qquad + C^{K_0\sigma_0 n_0}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Nn}\left(8 f^{(1)}_{\frac{1}{8}}(1) f^{(2)}_{\frac{1}{8}}(1)\right) \\
&\qquad + C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Mm,K_0\sigma_0 n_0}C^{Nn}\left(8 f^{(1)}_{\frac{1}{8}}(1) f^{(2)}_{\frac{1}{8}}(1)\right) \\
&\qquad \left.\left.+ C^{Mm,K_0\sigma_0 n_0}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\right\}\right] + O(s^2(sT/t)^2).
\end{aligned} \qquad \text{(F.12)}$$

When integrating the $\Lambda^{\frac{1}{2}}$ functions multiplied with the Gaussian $\exp(-2((\tilde{x})^-_{Mm})^2)$, the integral is $\sqrt{\pi/2}^9$ and $(9/4)\sqrt{\pi/2}^9$ for the zeroth power and the second power in $(\tilde{x})^-_{Mm}$ respectively. Note that we have a factor $\exp\big(-\frac{t}{4}\sum_{\tilde{v}\in V}\sum_{\substack{(J,\sigma,j)\\ \in L}}\big(\Delta(I_0, \widetilde{I}_0, J_0, \widetilde{J}_0, \sigma_0, \widetilde{\sigma}_0, m_0, \widetilde{m}_0, v, J, \sigma, j, \tilde{v})\big)^2\big)$ in the expression for the expectation value. Therefore, we have to expand this function in powers of $t$. The linear term in $t$ leads to the term having a minimal order of $(sT)^2/t$. This order is already smaller than terms of the order $s^2(sT/t)^2$, because $[s^2(sT/t)^2][t/(sT)^2] = s^2/t = 1/t^{2\alpha} \gg 1$. Fortunately, we can neglect the linear term in $t$ in the expansion of the exp function. Consequently, the final expectation value of $\big(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\big)^\dagger \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$ is given by

$$\begin{aligned}
&\frac{\left\langle \widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v}\Psi^t_{\{g,J,\sigma,j,L\}} \,\middle|\, \widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\|\Psi^t_{\{g,J,\sigma,j,L\}}\|^2} \\
&= \mathrm{e}^{+\mathrm{i}\sum_{(J,\sigma,j)}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})} \\
&\left(\frac{a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\hbar}\right)^2 (sT)^2\Bigg[C^{K_0\sigma_0n_0}C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2+\frac{9}{4}s^2\Bigg\{\left(C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}+\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\right) \\
&\Bigg[\frac{1}{2}C^{Mm,K_0\sigma_0n_0}C^{Nn}+\frac{1}{3!}\epsilon_{ijk}\Big[\epsilon_{n_0mn}C^{K_0\sigma_0 i}C^{Mj}C^{Nk}+\epsilon_{\ell_{00}n}C^{K_0\sigma_0 j}C^{Li}C^{Nk} \\
&\qquad +\epsilon_{\ell mn_0}C^{K_0\sigma_0 k}C^{Li}C^{Mj}\Big]\Bigg)\Bigg]+\left(C^{K_0\sigma_0n_0}+\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\right) \\
&\Bigg[\frac{1}{2}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Nn}+\frac{1}{3!}\epsilon_{ijk}\Big[\epsilon_{n_0mn}C^{\widetilde{K}_0\widetilde{\sigma}_0 i}C^{Mj}C^{Nk}+\epsilon_{\ell\widetilde{n}_0n}C^{\widetilde{K}_0\widetilde{\sigma}_0 j}C^{Li}C^{Nk} \\
&\qquad +\epsilon_{\ell m\widetilde{n}_0}C^{\widetilde{K}_0\widetilde{\sigma}_0 k}C^{Li}C^{Mj}\Big]\Bigg)\Bigg]+(C^{K_0\sigma_0n_0}+C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}) \\
&\left[C^{Mm,Nn}\left(4f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\right)+C^{Mm}C^{Nn}\left(40f^{(1)}_{\frac{1}{8}}(1)f^{(3)}_{\frac{1}{8}}(1)+32\left(f^{(2)}_{\frac{1}{8}}(1)\right)^2\right)\right] \\
&+(C^{K_0\sigma_0n_0}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}+C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Mm,K_0\sigma_0n_0})C^{Nn}\left(8f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\right) \\
&\qquad +C^{Mm,K_0\sigma_0n_0}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\Bigg\}\Bigg]+O(s^2(sT/t)^2).
\end{aligned} \tag{F.13}$$

Recalling the definition of the master constraint, we have

$$\begin{aligned}
\widehat{C}_{0,\Box} &= \sum_{I_0J_0K_0}\sum_{\sigma_0=+,-}\frac{4}{\kappa}\epsilon^{I_0J_0K_0}\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}\delta_{m_0,n_0} \\
\widehat{C}_{\ell_0,\Box} &= \sum_{I_0J_0K_0}\sum_{\sigma_0=+,-}\frac{4}{\kappa}\epsilon^{I_0J_0K_0}\epsilon_{\ell_0m_0n_0}\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}.
\end{aligned} \tag{F.14}$$

Using the fact that the expectation value of $\widehat{\mathbf{M}}$ can be expressed in terms of the expectation value of $(\widehat{O}^{\widetilde{m}_0,\widetilde{n}_0}_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0\widetilde{\sigma}_0 v})^\dagger\widehat{O}^{m_0,n_0}_{I_0J_0K_0\sigma_0 v}$ and that we have shown that the leading order agrees with the classical master constraint, we have our final result given by

$$\begin{aligned}
&\frac{\left\langle\Psi^t_{\{g,J,\sigma,j,L\}}\,\middle|\,\widehat{\mathbf{M}}\,\middle|\,\Psi^t_{\{g,J,\sigma,j,L\}}\right\rangle}{\|\Psi^t_{\{g,J,\sigma,j,L\}}\|^2} = \mathbf{M}+\frac{9}{4}s^2\sum_{v\in V(\alpha)}\Bigg[\sum_{I_0J_0K_0}\sum_{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0}\sum_{\sigma_0=+,-}\sum_{\widetilde{\sigma}_0=+,-} \\
&\Bigg\{\epsilon^{I_0J_0K_0}\epsilon^{\widetilde{I}_0\widetilde{J}_0\widetilde{K}_0}\left(\delta_{m_0,n_0}\delta_{\widetilde{m}_0,\widetilde{n}_0}+\sum_{\ell_0=1}^{3}\epsilon_{\ell_0m_0n_0}\epsilon_{\ell_0\widetilde{m}_0\widetilde{n}_0}\right) \\
&\left(\frac{4a^{\frac{3}{2}}|\det((p)^-)|^{\frac{1}{4}}}{\kappa\hbar}\right)^2(sT)^2\,\mathrm{e}^{+\mathrm{i}\sum_{(J,\sigma,j)}\varphi_{J\sigma j\tilde{v}}\Delta(I_0,\widetilde{I}_0,J_0,\widetilde{J}_0,\sigma_0,\widetilde{\sigma}_0,m_0,\widetilde{m}_0,v,J,\sigma,j,\tilde{v})} \\
&\Bigg\{\left(C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}+\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\right)
\end{aligned}$$

$$\Bigg[\frac{1}{2}C^{Mm,K_0\sigma_0n_0}C^{Nn}+\frac{1}{3!}\epsilon_{ijk}\big[\epsilon_{n_0mn}C^{K_0\sigma_0i}C^{Mj}C^{Nk}+\epsilon_{\ell_00n}C^{K_0\sigma_0j}C^{Li}C^{Nk}+\epsilon_{\ell mn_0}C^{K_0\sigma_0k}C^{Li}C^{Mj}\big]\Big)\Bigg]$$

$$+\left(C^{K_0\sigma_0n_0}+\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\right)$$

$$\Bigg[\frac{1}{2}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Nn}+\frac{1}{3!}\epsilon_{ijk}\big[\epsilon_{n_0mn}C^{\widetilde{K}_0\widetilde{\sigma}_0i}C^{Mj}C^{Nk}+\epsilon_{\ell\widetilde{n}_0n}C^{\widetilde{K}_0\widetilde{\sigma}_0j}C^{Li}C^{Nk}+\epsilon_{\ell m\widetilde{n}_0}C^{\widetilde{K}_0\widetilde{\sigma}_0k}C^{Li}C^{Mj}\big]\Big)\Bigg]$$

$$+\left(C^{K_0\sigma_0n_0}+C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}\right)$$

$$\left[C^{Mm,Nn}\left(4f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\right)+C^{Mm}C^{Nn}\left(40f^{(1)}_{\frac{1}{8}}(1)f^{(3)}_{\frac{1}{8}}(1)+32\left(f^{(2)}_{\frac{1}{8}}(1)\right)^2\right)\right]$$

$$+\left(C^{K_0\sigma_0n_0}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}+C^{\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}C^{Mm,K_0\sigma_0n_0}\right)C^{Nn}\left(8f^{(1)}_{\frac{1}{8}}(1)f^{(2)}_{\frac{1}{8}}(1)\right)$$

$$+C^{Mm,K_0\sigma_0n_0}C^{Mm,\widetilde{K}_0\widetilde{\sigma}_0\widetilde{n}_0}\left(f^{(1)}_{\frac{1}{8}}(1)\right)^2\Bigg\}\Bigg]+O(s^2(sT/t)^2). \tag{F.15}$$

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