On initial conditions and global existence for accelerating cosmologies from string theory

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Abstract

We construct a solution satisfying initial conditions for accelerating cosmologies from string/M-theory. Gowdy symmetric spacetimes with a positive potential are considered. Also, a global existence theorem for the spacetimes is shown.

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1 Introduction

It is expected that the inflation paradigm would be explained within superstring/M-theory. The theory predicts that spacetime dimension is greater than four. Since observable spacetime dimension is four, it is thought that the extra dimensions would be compactified within Planck scale. Recently, it has been pointed out that it is possible to find cosmological solutions which exhibit a transient phase of accelerated expansion of the universe (like inflation) if the size of the compactified internal hyperbolic space depends on time and/or if they are S(pacelike)-brane solutions [EG, TP, WMNR]. In these models, exponential potential terms like $V_0 e^{a\psi}$ appear, where ψ denotes the compactification volume or effective dilaton field, a is a coupling constant and V_0 is positive number. Explicitly, a typical action for the case is of the form

$$S = \int d^4x\sqrt{-g}\left[-\,{}^4R + \frac{1}{2}(\nabla\psi)^2 + V_0 e^{a\psi}\right]. \tag{1}$$

Then, it is explained that if it would be supposed that, in the case of $a > 0$, the field ψ starts at a large negative value (i.e. the potential term can be neglected) with high kinetic energy ($\partial_t\psi$ is positive and large enough) [2] near cosmological

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[2]In the case of $a < 0$, ψ and $\partial_t\psi$ start at large positive and negative values, respectively.

initial singularities, then, the scalar field runs up the exponential potential, turn around and falls back. At the turning point, the potential term becomes dominant, i.e. the universe makes accelerated expansion. Thus, the universe starts out in a decelerated expansion phase (asymptotic past) and enters an accelerating phase (intermediate era), and after these, the expansion becomes deceleration again (asymptotic future). We call this scenario paradigm-A.

We would like to investigate this interesting paradigm from viewpoint of mathematical relativity and cosmology. It is important to study rigorously whether or not the paradigm-A is acceptable. In particular, it should be shown that the assumption of the initial conditions for ψ is generic because, as indicated previously [EG], the accelerated expansion of the universe is all the result of the initial conditions. That is, (Q1): Are there singular solutions satisfying initial conditions in paradigm-A to the Einstein-matter equations in generic?

Furthermore, to be complete the scenario of paradigm-A, we should show global existence theorems, i.e., (Q2): Are there global solutions to the Einstein-matter equations with such exponential potentials? Unlike BKL [BKL] or cosmic no-hair conjectures [WR], which are problems in only asymptotic (local) regions of spacetimes, the paradigm-A is a *global* (in time) problem as mentioned already. In addition, it is also important as the first step to prove the strong cosmic censorship.

For (Q1), to construct solutions satisfying the initial condition of paradigm-A, we will use the Fuchsian algorithm developed by Kichenassamy and Rendall [KR]. It is interpreted that the class of solutions we are looking for here is a subclass of asymptotically velocity-terms dominated (AVTD) singular solutions since potential terms are neglected near the singularities and, in addition, signature of the time derivative of the scalar field is restricted. By using the method, it has been shown that there are AVTD singularities in (non-)vacuum Gowdy, polarized T^2-, polarized $U(1)$-symmetric spacetimes and the Einstein-scalar-p-from system without symmetry assumptions [AR, DHRW, IK, IM, NTM]. Also, systems with an exponential potential as given in (1) have been discussed formally in [DHRW, RA00]. Thus, our result is not only an answer for (Q1), but also it complements previous results.

For (Q2), we want to analyze Gowdy symmetric spacetimes. Future global existence theorems for spatially compact, locally homogeneous spacetimes [LH03, LH04, RA04] and hyperbolic symmetric spacetimes [TR] with a positive potential (or a positive cosmological constant) have been proved. These spacetimes do not include gravitational waves. Also, although global existence theorems for Gowdy (more generally, T^2-) symmetric spacetimes with or without matter have been shown [AH, ARW, BCIM, IW, MV, NM02, NM03, WM], it has not been prove the theorems for the spacetimes with a positive potential. Therefore, spacetimes with dynamical degrees of freedom of gravity and with the positive potential should be considered as the next step.

As a model, we choose the bosonic action arising in low energy effective superstring (supergravity) theory since we have a similar action with (1) after the toroidal compactification of the extra dimensions. There are anti-symmetric two-form, $B_{\mu\nu}$, and three-form, $C_{\mu\nu\rho}$ fields in the action. It is known that, in general, p-form fields in n-dimensional spacetimes may violate the strong energy condition for $p \geq n-1$ and then, accelerated expansion of the universe would be expected [GG]. Here, we do not consider hyperbolic compactification of the extra higher dimensions, but the only fluxes of four-form field strengths are investigated because these have essentially the same effects to obtain the exponential potential terms as (1) [EG, TP, WMNR].

Then, our purposes are to construct singular solutions satisfying conditions of paradigm-A and to show a global existence theorem for Gowdy symmetric spacetimes with stringy matter fields.

1.1 Action

The dimensionally reduced effective action in the Einstein frame is given by

$$S_{\mathrm{IIA}} = \int d^4x\sqrt{-g}\left[-\,{}^4R + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2\cdot 3!}e^{-2\lambda\phi}H^2 + \frac{1}{2\cdot 4!}e^{-2\lambda\phi}F^2\right], \tag{2}$$

where g is the determinant of the metric $g_{\mu\nu}$ on a four-dimensional spacetime manifold M, ${}^4R$ is the Ricci scalar of $g_{\mu\nu}$, ϕ is the dilaton field, $H = dB$ is the three-form field strength, $F = dC$ is the four-form field strength and λ is a coupling constant. If $\lambda = 1$, we have the action for the type IIA supergravity in the absence of vector fields and the Chern-Simons term [LWC].

In four dimensions, there is a duality between the three-form field strength and a one-form, which is interpreted as the gradient of a scalar field. Then, we may define the pseudo-scalar axion field σ as follows:

$$H^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\kappa}e^{2\lambda\phi}\nabla_\kappa\sigma. \tag{3}$$

Also, the field equation

$$\nabla_\mu\left(e^{-2\lambda\phi}F^{\mu\nu\rho\kappa}\right) = 0, \tag{4}$$

and the Bianchi identity

$$\partial_{[\alpha}F_{\mu\nu\rho\kappa]} = 0, \tag{5}$$

for the four-form field strength can be solved by

$$F^{\mu\nu\rho\kappa} = Q\epsilon^{\mu\nu\rho\kappa}e^{2\lambda\phi}, \tag{6}$$

where Q is an arbitrary constant. Thus, after taking the dual transformation and solving the field equations for F, we have a reduced effective action for the IIA system of the form

$$S_{\mathrm{IIA}*} = \int d^4x\sqrt{-g}\left[-\,{}^4R + \frac{1}{2}\left\{(\nabla\phi)^2 + e^{2\lambda\phi}(\nabla\sigma)^2 + Q^2e^{2\lambda\phi}\right\}\right]. \tag{7}$$

Hereafter, we assume $Q \neq 0$. Thus, we have the action which is the same from with (1).

1.2 Field equations for Gowdy symmetric spacetimes

The Gowdy symmetric spacetimes admit a T^2 isometry group with spacelike orbits and the twists associated to the group vanish [GR]. The topology of spatial section can be accepted S^3, $S^2 \times S^1$, T^3 or the lens space [CP]. In this paper, we assume T^3 spacelike topology.

Now, we will choose a coordinate, which is the *areal time* one. This means that time t is proportional to the geometric area of the orbits of the isometry group. Explicitly,

$$ds = -e^{2(\eta-U)}\alpha dt^2 + e^{2(\eta-U)}d\theta^2 + e^{2U}(dx + Ady)^2 + e^{-2U}t^2dy^2, \qquad (8)$$

where $\partial/\partial x$ and $\partial/\partial y$ are Killing vector fields generating the T^2 group action, and η, α, U and A are functions of $t \in (0, \infty)$ and $\theta \in S^1$. It is also assumed that functions describing behavior of matter fields are ones of t and θ.

Let us show the field equations obtained by varying the action (7) in the areal coordinate (8).

Constraint equations

$$\begin{aligned}\frac{\dot\eta}{t} = \dot U^2 &+ \alpha U'^2 + \frac{e^{4U}}{4t^2}(\dot A^2 + \alpha A'^2)\\ &+ \frac{1}{4}\left[\dot\phi^2 + \alpha\phi'^2 + e^{2\lambda\phi}(\dot\sigma^2 + \alpha\sigma'^2) + \alpha Q^2 e^{2\lambda\phi+2(\eta-U)}\right], \end{aligned}\qquad (9)$$

$$\frac{\eta'}{t} = 2\dot U U' + \frac{e^{4U}}{2t^2}\dot A A' - \frac{\alpha'}{2t\alpha} + \frac{1}{2}(\dot\phi\phi' + e^{2\lambda\phi}\dot\sigma\sigma'), \qquad (10)$$

$$\dot\alpha = -t\alpha^2 Q^2 e^{2\lambda\phi+2(\eta-U)}. \qquad (11)$$

Evolution equations

$$\begin{aligned}\ddot\eta - \alpha\eta'' = &\quad \frac{\eta'\alpha'}{2} + \frac{\dot\eta\dot\alpha}{2\alpha} - \frac{\alpha'^2}{4\alpha} + \frac{\alpha''}{2} - \dot U^2 + \alpha U'^2 + \frac{e^{4U}}{4t^2}(\dot A^2 - \alpha A'^2)\\ &+ \frac{1}{4}\left[-\dot\phi^2 + \alpha\phi'^2 + e^{2\lambda\phi}(-\dot\sigma^2 + \alpha\sigma'^2) + \alpha Q^2 e^{2\lambda\phi+2(\eta-U)}\right], \end{aligned}\qquad (12)$$

$$\ddot U - \alpha U'' = -\frac{\dot U}{t} + \frac{\dot\alpha\dot U}{2\alpha} + \frac{\alpha' U'}{2} + \frac{e^{4U}}{2t^2}(\dot A^2 - \alpha A'^2) + \frac{1}{4}\alpha Q^2 e^{2\lambda\phi+2(\eta-U)}, \qquad (13)$$

$$\ddot A - \alpha A'' = \frac{\dot A}{t} + \frac{\dot\alpha\dot A}{2\alpha} + \frac{\alpha' A'}{2} - 4(\dot A\dot U - \alpha A'U'), \qquad (14)$$

$$\ddot\phi - \alpha\phi'' = -\frac{\dot\phi}{t} + \frac{\dot\alpha\dot\phi}{2\alpha} + \frac{\alpha'\phi'}{2} + \lambda e^{2\lambda\phi}(\dot\sigma^2 - \alpha\sigma'^2) - \lambda\alpha Q^2 e^{2\lambda\phi+2(\eta-U)}, \qquad (15)$$

$$\ddot{\sigma} - \alpha\sigma'' = -\frac{\dot{\sigma}}{t} + \frac{\dot{\alpha}\dot{\sigma}}{2\alpha} + \frac{\alpha'\sigma'}{2} - 2\lambda(\dot{\phi}\dot{\sigma} - \alpha\phi'\sigma'). \tag{16}$$

Hereafter, dot and prime denote derivative with respect to t and θ, respectively. We will call this system of partial differential equations (PDEs) *Gowdy symmetric IIA system*. Note that these equations are not independent because the wave equation (12) for η can be derived from other equations. Indeed, there are only two dynamical degree of freedom (i.e. U and A) in the Gowdy symmetric spacetimes.

2 Initial singularities

Consider the problem (Q1). To begin with a brief review of the Fuchsian algorithm, which is a method to construct exact singular solutions to a PDE system near a singularity ($t = 0$). The algorithm is based on the following idea: near the singularity, decompose the singular formal solutions into a singular part, which depends on a number of arbitrary functions, and a regular part u. If the system can be written as a Fuchsian system of the form

$$[D + \mathcal{N}(x)]\, u = t f(t, x, u, \partial_x u), \tag{17}$$

where $D := t\partial_t$ and f is a vector-valued regular function, then the following theorem can be apply:

Theorem 1 [KR] *Assume that $\mathcal{N}$ is an analytic matrix near $x = x_0$ such that there is a constant C with $\|\Lambda^{\mathcal{N}}\| \leq C$ for $0 < \Lambda < 1$. In addition, suppose that f is a locally Lipschitz function of u and $\partial_x u$ which preserves analyticity in x and continuity in t. Then, the Fuchsian system (17) has a unique solution in a neighborhood of $x = x_0$ and $t = 0$ which is analytic in x and continuous in t and tend to zero as $t \to 0$.*

Thus, the regular part goes to zero and the singular part of the formal solution becomes an exact solution to the original PDE system near the singularity.

Unlike the vacuum Gowdy case, the evolution equations (13)-(16) do not decouple from the constraint equations (9)-(11), since they contain the function α. Therefore, according to [IK], we take equations (9), (11), (13)-(16) as effective evolution ones and (10) as the only effective constraint equation. This is not a standard setup for the initial-value problem for the Einstein-matter equations (see example [TM]). Therefore, it is not clear whether the initial-value problem for our case away from the singularity at $t = 0$ has a unique solution or not, unless it is shown that the constraint (10) propagates.

Let us show the local existence and uniqueness of our initial-value problem. We can obtain the following first-order system for $\vec{z}$ from the PDE system (9), (11), (13)-(16):

$$\partial_t \vec{z} = f(t, \theta, \vec{z}, \partial_\theta \vec{z}), \tag{18}$$

where $\vec{z} := (U, \dot{U}, U', A, \dot{A}, A', \phi, \dot{\phi}, \phi', \sigma, \dot{\sigma}, \sigma', \alpha, \eta)$. This means that the PDE system is of Cauchy-Kowalewskaya type. Thus, ignoring the constraint equation (10), we have a unique solution to the effective evolution equations by prescribing the analytic initial data for $t = t_0 > 0$ if all functions are analytic.

Now, to assure the local existence and uniqueness of the initial-value problem, we must show that the constraint (10) propagates. Let us set

$$N := \eta' - 2DUU' - \frac{e^{4U}}{2t^2}DAA' - \frac{1}{2}D\phi\phi' - \frac{e^{2\lambda\phi}}{2}D\sigma\sigma' + \frac{\alpha'}{2\alpha}, \tag{19}$$

Computing

$$\begin{aligned} 0 &= D\eta' - (D\eta)' \\ &= DN + D\left(2DUU' + \frac{e^{4U}}{2t^2}DAA' + \frac{1}{2}D\phi\phi' \frac{e^{2\lambda\phi}}{2}D\sigma\sigma' - \frac{\alpha'}{2\alpha}\right) - (D\eta)', \end{aligned} \tag{20}$$

we have a linear, homogeneous ordinary differential equation (ODE) for N of the form

$$DN - \frac{D\alpha}{2\alpha}N = 0. \tag{21}$$

Thus, the uniqueness theorem for ODEs guarantees that N is identically zero for any time t if we set initial data for $t = t_0$ such that $N(t_0) = 0$. Thus, the local existence and uniqueness of the initial-value problem for our case has been shown in the analytic case. In appendix, we shall consider the smooth version of the initial-value problem for our non-standard setup of the Gowdy symmetric IIA system.

2.1 Application of the Fuchsian algorithm

Let us construct AVTD singular solutions to the Gowdy symmetric IIA system. First, we will consider the case that a solution has a maximum number of free functions. In this sense, the solution (given in theorem 2) is generic.

Neglecting spatial derivative and potential terms in the effective evolution equations, we have velocity-terms dominated (VTD) equations as follow:

$$D\eta = (DU)^2 + \frac{e^{4U}}{4t^2}(DA)^2 + \frac{1}{4}(D\phi)^2 + \frac{e^{2\lambda\phi}}{4}(D\sigma)^2, \tag{22}$$

$$D\alpha = 0, \tag{23}$$

$$D^2U = \frac{1}{2\alpha}DUD\alpha + \frac{e^{4U}}{4t^2}(DA)^2, \tag{24}$$

$$D^2 A = 2DA + \frac{1}{2\alpha} DAD\alpha - 4DUDA, \tag{25}$$

$$D^2 \phi = \frac{1}{2\alpha} D\phi D\alpha + \lambda e^{2\lambda\phi} (D\sigma)^2, \tag{26}$$

$$D^2 \sigma = \frac{1}{2\alpha} D\sigma D\alpha - 2\lambda D\phi D\sigma. \tag{27}$$

Solving this system of VTD equations, we have a VTD solution. Then, the following formal solution is obtained:

$$\eta = \left(k(\theta)^2 + \frac{\kappa(\theta)^2}{4} \right) \ln t + \eta_0(\theta) + t^\epsilon \mu(t,\theta), \tag{28}$$

$$\alpha = \alpha_0(\theta) + t^\epsilon \beta(t,\theta), \tag{29}$$

$$U = k(\theta) \ln t + U_0(\theta) + t^\epsilon V(t,\theta), \tag{30}$$

$$A = h(\theta) + t^{2-4k} \left(A_0(\theta) + B(t,\theta) \right), \tag{31}$$

$$\phi = \kappa(\theta) \ln t + \phi_0(\theta) + t^\epsilon \Phi(t,\theta), \tag{32}$$

$$\sigma = \omega(\theta) + t^{-2\lambda\kappa} \left(\sigma_0(\theta) + \Sigma(t,\theta) \right), \tag{33}$$

where

$$\epsilon > 0, \qquad 0 < k(\theta) < \frac{1}{2}, \qquad \alpha_0 > 0 \tag{34}$$

and

$$-1 < \lambda\kappa(\theta) < 0. \tag{35}$$

Note that μ, β, V, B, Φ and Σ are regular parts and others are singular parts (=VTD solutions).

Inserting this formal solution into the Einstein-matter equations, we obtain the following Fuchsian system:

$$(D + \mathcal{N})\vec{u} = t^\delta f(t, \theta, \vec{u}, \partial_\theta \vec{u}), \tag{36}$$

where $\vec{u} := u_i = (V, DV, t^\epsilon V', B, DB, t^\epsilon B', \Phi, D\Phi, t^\epsilon \Phi', \Sigma, D\Sigma, t^\epsilon \Sigma', \beta, \mu)$, $i = 1, \cdots, 14$, f is a vector-valued regular function and

$$\mathcal{N} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon^2 & 2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2-4k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon^2 & 2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 \\ -2k\epsilon & -2k & 0 & 0 & 0 & 0 & -\frac{\kappa\epsilon}{2} & -\frac{\kappa}{2} & 0 & 0 & 0 & 0 & 0 & \epsilon \end{pmatrix}. \quad (37)$$

Note that $\delta > 0$ if the condition (34), (35) and

$$K := (k - \frac{1}{2})^2 + \frac{\kappa^2}{4} + \lambda\kappa + \frac{3}{4} > 0 \quad (38)$$

holds.

To apply theorem 1 to our Fuchsian system (36), we must to verify that the boundedness condition for the matrix $\mathcal{N}$ holds. To do this, we have $P^{-1}\mathcal{N}P = \mathcal{N}_0$, where

$$\mathcal{N}_0 = \begin{pmatrix} \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2-4k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \epsilon & 0 \\ 0 & 2k & 0 & 0 & 0 & 0 & 0 & \frac{\kappa}{2} & 0 & 0 & 0 & 0 & 0 & \epsilon \end{pmatrix}, \quad (39)$$

and

$$P=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\epsilon & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\epsilon & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \qquad (40)$$

Then,

$$\Lambda^{\mathcal{N}_0}=\begin{pmatrix}
\Lambda^\epsilon & \Lambda^\epsilon\ln\Lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \Lambda^\epsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \Lambda^{2-4k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \Lambda^\epsilon & \Lambda^\epsilon\ln\Lambda & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^\epsilon & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^{-2\lambda\kappa} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda^\epsilon & 0\\
0 & 2k\Lambda^\epsilon\ln\Lambda & 0 & 0 & 0 & 0 & 0 & \frac{\kappa}{2}\Lambda^\epsilon\ln\Lambda & 0 & 0 & 0 & 0 & 0 & \Lambda^\epsilon
\end{pmatrix}, \quad (41)$$

hence $P\Lambda^{\mathcal{N}_0}P^{-1}=\Lambda^{\mathcal{N}}$ is uniformly bounded for $0<\Lambda<1$ if the condition (34) and (35) hold.

Thus, there is a unique solution of the Fuchsian system (36) which goes to zero as $t\to 0$, and which is analytic in θ and continuous in t. Note that $(U,A,\phi,\sigma,\alpha,\eta)$ is a solution of the effective evolution equations of the Einstein-matter equations (9), (11), (13)-(16) if we construct $(U,A,\phi,\sigma,\alpha,\eta)$ from (28)-(33) with $V=u_1$, $B=u_4$, $\Phi=u_7$, $\Sigma=u_{10}$, $\beta=u_{13}$ and $\mu=u_{14}$. This fact follows from equations $D(u_{I+2}-t^\epsilon u_I')=0$, where $I=1,4,7,10$.

Now, we want to get a constraint condition to ensure that the solution obtained above is a genuine one to the full Einstein-matter equations. Since $D\alpha/\alpha=$

$\mathcal{O}(t^\epsilon)$,

$$\frac{\dot{N}}{N} = \frac{\dot{\alpha}}{2\alpha} = \mathcal{O}(t^{\epsilon-1}), \tag{42}$$

then, the right-hand-side of the above equation is integrable. From this result, we can put a function $P(t,\theta)$ such that

$$N \propto \exp P(t,\theta). \tag{43}$$

This means that N is identically zero if we would choose the singular data such that $N \to 0$ as $t \to 0$, and then, the constraint equation (10) is satisfied.

Inserting the formal solutions (28)-(33) into the constraint equation (19), we have

$$N = \eta_0' - 2kU_0' - e^{4U_0}(1-2k)h'A_0 - \frac{\kappa\phi_0'}{2} + e^{2\lambda\phi_0}\kappa\omega'\sigma_0 + \frac{\alpha_0'}{2\alpha_0} + \mathcal{O}(1), \tag{44}$$

where $\mathcal{O}(1)$ is some expression which tends to zero as $t \to 0$. Thus, the constraint holds iff the singular data satisfy

$$\eta_0' - 2kU_0' - e^{4U_0}(1-2k)h'A_0 - \frac{\kappa\phi_0'}{2} + e^{2\lambda\phi_0}\kappa\omega'\sigma_0 + \frac{\alpha_0'}{2\alpha_0} = 0. \tag{45}$$

To summarize, we have the following theorem:

Theorem 2 *Choose data such that conditions (34), (35) and (45) are satisfied. Suppose that ϵ is a positive constant less than* $\min\{4k, 2-4k, -2\lambda\kappa, 2+2\lambda\kappa, 2K\}$. *For any choice of the analytic singular data $\eta_0(\theta)$, $\alpha_0(\theta)$, $k(\theta)$, $U_0(\theta)$, $h(\theta)$, $A_0(\theta)$, $\kappa(\theta)$, $\phi_0(\theta)$, $\omega(\theta)$ and $\sigma_0(\theta)$, the Gowdy symmetric IIA system has a solution of the form (28)-(33), where μ, β, V, B, Φ and Σ tend to zero as $t \to 0$.* □

Although the solution given in theorem 2 is generic in the sense that the solution has a maximum number of free functions, conditions for paradigm-A does not hold since $\lambda\kappa < 0$, i.e. the universe starts with large potential and wrong sign of the time derivative of ϕ. To verify the validity of the paradigm-A we need to construct a solution allowing a condition $\lambda\kappa > 0$. Indeed, this problem can be overcame as follows.

If an AVTD solution with $\lambda\kappa > 0$ are needed, we replace expansion (33) with

$$\sigma = \omega(\theta) + t^\epsilon \Sigma(t,\theta). \tag{46}$$

In this case, $-2\lambda\kappa$ and $\Lambda^{-2\lambda\kappa}$ sitting the 11th line and the 11th row in the matrices $\mathcal{N}$ and $\Lambda^{\mathcal{N}_0}$ are replaced by ϵ and Λ^ϵ, respectively. Also, the constraint condition for the singular data becomes

$$\eta_0' - 2kU_0' - e^{4U_0}(1-2k)h'A_0 - \frac{\kappa\phi_0'}{2} + \frac{\alpha_0'}{2\alpha_0} = 0. \tag{47}$$

Thus, we have the following theorem which is consistent with conditions of paradigm-A.

Theorem 3 *Choose data such that conditions (34), (47) and $\lambda\kappa > -1/2$ are satisfied. Suppose that ϵ is a positive constant such that* $\max\{0, -2\lambda\kappa\} < \epsilon < \min\{4k, 2-4k\}$. *For any choice of the analytic singular data $\eta_0(\theta)$, $\alpha_0(\theta)$, $k(\theta)$, $U_0(\theta)$, $h(\theta)$, $A_0(\theta)$, $\kappa(\theta)$, $\phi_0(\theta)$ and $\omega(\theta)$, the Gowdy symmetric IIA system has a solution of the form (28)-(32) and (46), where μ, β, V, B, Φ and Σ tend to zero as $t \to 0$.* □

The positivity of K is automatically satisfied when $0 < k < 1/2$ and $\lambda\kappa > -1/2$ hold. Then, a solution to the Gowdy symmetric IIA system allowing the initial conditions for paradigm-A has been constructed. Note that we do not have the maximum number of free functions in this case. Thus, the solution given in theorem 3 is restricted than generic one given in theorem 2. The reason why we do not have the maximum number is the existence of dilaton coupling with kinetic terms of other fields (the axion field in our case). Generically, all fields arising in superstring/M-theory couple with the dilaton field. Therefore, we may not avoid such restriction for solutions to our problem unless the dilaton coupling is ignored.

3 Global existence

Now, consider the problem (Q2). We will show the following theorem:

Theorem 4 *Let (M, g, ϕ, σ) be the maximal Cauchy development of C^∞ initial data for the Gowdy symmetric IIA system. Suppose that the timelike convergence condition (TCC), which is $R_{\mu\nu}W^\mu W^\nu \geq 0$ for any timelike vector W^μ, holds and there is a positive constant $\bar{\lambda}$ such that $|\lambda| \leq \bar{\lambda} < 1/2$. Then, M can be covered by compact Cauchy surfaces of constant areal time t with each value in the range $(0, \infty)$.*

In the first place, we need a local existence theorem for the Gowdy symmetric IIA system, which is the Einstein-(minimally coupled) scalar system with a positive potential. Fortunately, there is no coupling caused by existence of such matter fields in the principal part of the PDE system. For this reason, since the local existence theorems for vacuum Gowdy (more generically, T^2-symmetric) spacetimes have been shown [MV, CP], the same theorem for the Gowdy symmetric IIA system can be shown as vacuum case [FR]. Thus, it is enough to verify uniform bounds of functions $(\eta, \alpha, U, A, \phi, \sigma)$ and their first and second derivatives to prove global existence [MA]. The strategy is similar with the case of T^2-symmetric Einstein(-Vlasov) system [AH, ARW, BCIM, IW, WM].

Let us define

$$\gamma := \eta + \frac{1}{2}\ln\alpha. \qquad (48)$$

By using γ we can rewrite the constraint equations as follows:

$$\frac{\dot{\gamma}}{t} = \mathcal{E} - \frac{Q^2}{4}e^{2(\gamma+\lambda\phi-U)}, \qquad (49)$$

$$\frac{\gamma'}{t} = \frac{\mathcal{F}}{\sqrt{\alpha}}, \tag{50}$$

$$\dot{\alpha} = -t\alpha Q^2 e^{2(\gamma+\lambda\phi-U)}, \tag{51}$$

where

$$\mathcal{E} := \dot{U}^2 + \alpha U'^2 + \frac{e^{4U}}{4t^2}\left(\dot{A}^2 + \alpha A'^2\right) + \frac{1}{4}\left[\dot{\phi}^2 + \alpha\phi'^2 + e^{2\lambda\phi}\left(\dot{\sigma}^2 + \alpha\sigma'^2\right)\right], \tag{52}$$

and

$$\mathcal{F} := \sqrt{\alpha}\left[2\dot{U}U' + \frac{e^{4U}}{2t^2}\dot{A}A' + \frac{1}{2}\left(\dot{\phi}\phi' + e^{2\lambda\phi}\dot{\sigma}\sigma'\right)\right]. \tag{53}$$

Define energies for the Gowdy symmetric IIA system

$$E(t) := \int_{S^1} \frac{1}{\sqrt{\alpha}}\left[\mathcal{E} + \frac{1}{4}\alpha Q^2 e^{2(\eta+\lambda\phi-U)}\right] d\theta, \tag{54}$$

and

$$\tilde{E}(t) := \int_{S^1} \frac{\mathcal{E}}{\sqrt{\alpha}} d\theta, \tag{55}$$

In our case, the TCC is as follows:

$$\dot{\phi}^2 + e^{2\lambda\phi}\dot{\sigma}^2 \geq \frac{1}{2}\alpha Q^2 e^{2(\eta+\lambda\phi-U)} \tag{56}$$

First, we will show energy decay and energy inequalities (see lemmas 1 and 3 in [IW]).

Lemma 1 *Suppose the TCC and the condition $|\lambda| \leq \bar{\lambda} < 1/2$. Then, E and $\tilde{E}$ decrease monotonically along time t, that is,*

$$\frac{dE(t)}{dt} < 0 \quad \text{and} \quad \frac{d\tilde{E}(t)}{dt} < 0, \tag{57}$$

and E and $\tilde{E}$ are bounded on (T_-, T_+), where $0 < T_- < t_i < T_+ < \infty$. Furthermore, there exists numbers, E_- and $\tilde{E}_-$, satisfying

$$E_- = \lim_{t\to T_-} E(t) \quad \text{and} \quad \tilde{E}_- = \lim_{t\to T_-} \tilde{E}(t). \tag{58}$$

Proof. One can calculate directly as follows:

$$\frac{dE(t)}{dt} = -\int_{S^1} \frac{1}{\sqrt{\alpha}t}\left(2\dot{U}^2 + \frac{e^{4U}}{2t^2}\alpha A'^2 + \frac{\dot{\phi}^2}{2} + \frac{e^{2\lambda\phi}\dot{\sigma}^2}{2}\right) d\theta \leq 0. \tag{59}$$

Thus, $E(t)$ is controlled by $E(t_i)$ for any $t \in [t_i, T_+)$.

The right-hand-side of equation (59) can be controlled by E:

$$\frac{dE}{dt} \geq -\frac{4}{t}E. \tag{60}$$

For any $t \in (T_-, t_i]$, we have

$$E(t) \leq E(t_i)\left(\frac{t_i}{t}\right)^4. \tag{61}$$

Then, $E(t) \leq E(t_i)\left(\frac{t_i}{T_-}\right)^4$ on (T_-, t_i). This boundedness and the monotonicity of $E(t)$ assert that $E(t)$ continuously extends to T_- and then E_- exists.

Next, we show the same results for $\tilde{E(t)}$. By direct calculation, we have

$$\begin{aligned} \frac{d\tilde{E}(t)}{dt} = \int_{S^1} & - \frac{2}{\sqrt{\alpha}t}\left(\dot{U}^2 + \frac{e^{4U}}{4t^2}\alpha A'^2 + \frac{\dot{\phi}^2}{4} + \frac{e^{2\lambda\phi}\dot{\sigma}^2}{4}\right) \\ & + \frac{\dot{\alpha}}{2\alpha}\left(\frac{\mathcal{E}}{\sqrt{\alpha}} + \frac{1}{t\sqrt{\alpha}}\left[\dot{U} - \lambda\dot{\phi}\right]\right) d\theta. \end{aligned} \tag{62}$$

We cannot conclude the monotonicity for $\tilde{E(t)}$ from the above form. Now,

$$-\frac{2\dot{U}^2}{t\sqrt{\alpha}} + \frac{\dot{\alpha}\dot{U}}{2t\alpha\sqrt{\alpha}} = -\frac{2}{t\sqrt{\alpha}}\left(\dot{U} - \frac{\dot{\alpha}}{8\alpha}\right)^2 + \frac{1}{32t\sqrt{\alpha}}\left(\frac{\dot{\alpha}}{\alpha}\right)^2, \tag{63}$$

and

$$-\frac{2\dot{\phi}^2}{4t\sqrt{\alpha}} - \frac{\lambda\dot{\alpha}\dot{\phi}}{2t\alpha\sqrt{\alpha}} = -\frac{1}{2t\sqrt{\alpha}}\left(\dot{\phi} + \frac{\lambda\dot{\alpha}}{2\alpha}\right)^2 + \frac{\lambda^2}{8t\sqrt{\alpha}}\left(\frac{\dot{\alpha}}{\alpha}\right)^2. \tag{64}$$

Therefore,

$$\begin{aligned} \frac{d\tilde{E}(t)}{dt} = \int_{S^1} & - \frac{2}{\sqrt{\alpha}t}\left(\left(\dot{U} - \frac{\dot{\alpha}}{8\alpha}\right)^2 + \frac{e^{4U}}{4t^2}\alpha A'^2 + \frac{1}{4}\left(\dot{\phi} + \frac{\lambda\dot{\alpha}}{2\alpha}\right)^2 + \frac{e^{2\lambda\phi}\dot{\sigma}^2}{4}\right) \\ & + \frac{\dot{\alpha}}{2\alpha}\left(\frac{\mathcal{E}}{\sqrt{\alpha}} - \left[\frac{1}{16} + \frac{\lambda^2}{4}\right]\sqrt{\alpha}Q^2 e^{2(\eta+\lambda\phi-U)}\right) d\theta, \end{aligned} \tag{65}$$

where equation (11) has been used. By using the TCC and the inequality $|\lambda| \leq \bar{\lambda} < 1/2$, we have the conclusion of the monotonic nonincreasing property for $\tilde{E(t)}$,

$$\frac{d\tilde{E}}{dt} \leq \int_{S^1} \frac{C_\lambda}{2}\frac{\dot{\alpha}}{\alpha}\frac{\mathcal{E}}{\sqrt{\alpha}} d\theta \leq 0, \tag{66}$$

where $C_\lambda < 1$ is a positive constant depending only λ.

Now, it follows that $\tilde{E(t)} \leq E(t)$ for any time t. Therefore, one can see that $\tilde{E}(t)$ also extend continuously to T_- by the monotonicity of it. □

Next two lemmas will be used to control dynamical parts (i.e. U, A, ϕ and σ) of the system. The method of the proof is based on the light cone estimate [MV, BCIM].

Lemma 2 *If $\dot{\alpha}\alpha^{-1}$ is bounded, $\mathcal{E}$ is bounded on $(T_-, T_+) \times S^1$.*

Proof. Differentiating quantities, $\mathcal{E}$ and $\mathcal{F}$, along null directions $\partial_\zeta := \partial_t - \sqrt{\alpha}\partial_\theta$ and $\partial_\xi := \partial_t + \sqrt{\alpha}\partial_\theta$, we have

$$\begin{aligned}\partial_\zeta(\mathcal{E}+\mathcal{F}) = \frac{\dot{\alpha}}{\alpha}(\mathcal{E}+\mathcal{F}) \quad &- \quad \frac{1}{t}\left[2\dot{U}^2 + \frac{e^{4U}}{2t^2}\alpha A'^2 + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}e^{2\lambda\phi}\dot{\sigma}^2 + \mathcal{F}\right] \\ &- \quad \frac{\dot{\alpha}}{2t\alpha}\left[\dot{U} + \sqrt{\alpha}U' - \lambda\left(\dot{\phi} + \sqrt{\alpha}\phi'\right)\right] =: L_+, \quad (67)\end{aligned}$$

and

$$\begin{aligned}\partial_\xi(\mathcal{E}-\mathcal{F}) = \frac{\dot{\alpha}}{\alpha}(\mathcal{E}-\mathcal{F}) \quad &- \quad \frac{1}{t}\left[2\dot{U}^2 + \frac{e^{4U}}{2t^2}\alpha A'^2 + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}e^{2\lambda\phi}\dot{\sigma}^2 - \mathcal{F}\right] \\ &\quad\ \frac{\dot{\alpha}}{2t\alpha}\left[\dot{U} - \sqrt{\alpha}U' - \lambda\left(\dot{\phi} - \sqrt{\alpha}\phi'\right)\right] =: L_-. \quad (68)\end{aligned}$$

Note that

$$\begin{aligned}\left(\dot{U} \pm \sqrt{\alpha}U'\right) - \lambda\left(\dot{\phi} \pm \sqrt{\alpha}\phi'\right) \quad &\leq \quad \left(\dot{U} \pm \sqrt{\alpha}U'\right)^2 + \lambda^2\left(\dot{\phi} \pm \sqrt{\alpha}\phi'\right)^2 + \frac{1}{2} \\ &\leq \quad (1+\lambda^2)(\mathcal{E}+\mathcal{F}) + \frac{1}{2} \\ &\leq \quad 2(1+\lambda^2)\mathcal{E} + \frac{1}{2}, \qquad (69)\end{aligned}$$

where $|\mathcal{F}| \leq \mathcal{E}$ has been used. Thus,

$$|L_\pm| \leq \left|\frac{\dot{\alpha}}{\alpha}\right|\left\{2\mathcal{E} + \frac{C\mathcal{E}}{t} + \frac{1}{4t}\right\} + \frac{3\mathcal{E}}{t}, \qquad (70)$$

where C is a positive constant.

Consider a point $(t,\theta) \in [t_i, T_+) \times S^1$. Integrating the both sides of equations (67) and (68) along null passes, ∂_ζ and ∂_ξ, from points (t_i, θ_+) and (t_i, θ_-) to the point (t,θ), respectively, we have

$$\int \partial_\zeta(\mathcal{E}+\mathcal{F})d\zeta = \mathcal{E}(t,\theta) + \mathcal{F}(t,\theta) - \mathcal{E}(t_i,\theta_+) - \mathcal{F}(t_i,\theta_+) = \int L_+ d\zeta, \qquad (71)$$

and

$$\int \partial_\xi(\mathcal{E}-\mathcal{F})d\xi = \mathcal{E}(t,\theta) - \mathcal{F}(t,\theta) - \mathcal{E}(t_i,\theta_-) + \mathcal{F}(t_i,\theta_-) = \int L_- d\xi. \qquad (72)$$

Adding these equations and using the inequality $|\mathcal{F}| \leq \mathcal{E}$,

$$\mathcal{E}(t,\theta) \leq \mathcal{E}(t_i,\theta_+) + \mathcal{E}(t_i,\theta_-) + \frac{1}{2}\left[\int |L_+|\, d\zeta + \int |L_-|\, d\xi\right]. \tag{73}$$

Taking supremums over all values of the space coordinate θ on the both sides of the inequality (73), we have

$$\begin{aligned} \sup_{S^1}\mathcal{E}(t,\theta) &\leq 2\sup_{S^1}\mathcal{E}(t_i,\theta) + \int_{t_i}^{t}\left[\left|\frac{\dot\alpha}{\alpha}\right|\left\{2\sup_{S^1}\mathcal{E}\left(1+\frac{C}{s}\right)+\frac{1}{4s}\right\}+\frac{3}{s}\sup_{S^1}\mathcal{E}\right]ds \\ &= C_1(t) + \int_{t_i}^{t} C_2(s)\sup_{S^1}\mathcal{E}(s,\theta)ds, \end{aligned} \tag{74}$$

where $C_i(t)$ are bounded and positive functions of t. We now apply Gronwall's lemma to this inequality (74), we have boundedness for $\mathcal{E}$ on $[t_i, T_+) \times S^1$. We can apply the same argument for $t \in (T_-, t_i] \times S^1$, and then we have the conclusion of this lemma. □

Lemma 3 *Let us define*

$$\tilde{\mathcal{E}} := \ddot{U}^2 + \alpha\dot{U}'^2 + \frac{e^{4U}}{4t^2}\left(\ddot{A}^2 + \alpha\dot{A}'^2\right) + \frac{1}{4}\left[\ddot\phi^2 + \alpha\dot\phi'^2 + e^{2\lambda\phi}\left(\ddot\sigma^2 + \alpha\dot\sigma'^2\right)\right], \tag{75}$$

and

$$\tilde{\mathcal{F}} := \sqrt{\alpha}\left[2\ddot{U}\dot{U}' + \frac{e^{4U}}{2t^2}\ddot{A}\dot{A}' + \frac{1}{2}\left(\ddot\phi\dot\phi' + e^{2\lambda\phi}\ddot\sigma\dot\sigma'\right)\right]. \tag{76}$$

If all functions and their first derivative, $\dot\alpha'$ and $\ddot\alpha$ are bounded, $\tilde{\mathcal{E}}$ is bounded on $(T_-, T_+) \times S^1$.

Proof. Taking time derivative of the wave equations (13)-(16) for U, A, ϕ and σ, we have wave equations for $\dot U$, $\dot A$, $\dot\phi$ and $\dot\sigma$. Now, $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ satisfy equations of the form

$$\partial_\zeta(\tilde{\mathcal{E}} + \tilde{\mathcal{F}}) = \tilde{L}_+ \quad \text{and} \quad \partial_\xi(\tilde{\mathcal{E}} - \tilde{\mathcal{F}}) = \tilde{L}_-, \tag{77}$$

where $\tilde{L}_\pm$ involve nothing but controlled quantities, together with terms quadratic in $\ddot U$, $\dot U'$, $\ddot A$, $\dot A'$, $\ddot\phi$, $\dot\phi'$, $\ddot\sigma$ and $\dot\sigma'$. Now, we can repeat the light cone argument and then, we have boundedness for $\tilde{\mathcal{E}}$ on $(T_-, T_+) \times S^1$. □

3.1 Past direction

Further estimates are given in the each case of past and future directions, separately. First, consider the past direction.

Lemma 4 *For any t, the function γ satisfies the following condition,*

$$\max_{S^1} \gamma(t,\theta) - \min_{S^1} \gamma(t,\theta) \leq tE(t). \tag{78}$$

Furthermore, for any $t \in (T_-, t_i]$, the functions U and ϕ satisfy the following conditions,

$$\max_{S^1} U(t,\theta) - \min_{S^1} U(t,\theta) \leq CE^{1/2}(t), \tag{79}$$

and

$$\max_{S^1} \phi(t,\theta) - \min_{S^1} \phi(t,\theta) \leq CE^{1/2}(t). \tag{80}$$

Proof. (cf. Step 1 of Section 5 in [AH]). For any $\theta_1, \theta_2 \in S^1$, we have

$$|\gamma(t,\theta_2) - \gamma(t,\theta_1)| = \left| \int_{\theta_1}^{\theta_2} \gamma' d\theta \right| \leq \int_{\theta_1}^{\theta_2} |\gamma'| \, d\theta \leq \int_{\theta_1}^{\theta_2} \frac{t\mathcal{E}}{\sqrt{\alpha}} d\theta \leq t\tilde{E}(t) \leq tE(t), \tag{81}$$

where equation (50) and the fact $|\mathcal{F}| \leq \mathcal{E}$ have been used. Since θ_1 and θ_2 are arbitrary, the first conclusion follows.

Similarly, for any $\theta_1, \theta_2 \in S^1$ and any $t \in (T_-, t_i]$, we have

$$\begin{aligned} |U(t,\theta_2) - U(t,\theta_1)| &= \left| \int_{\theta_1}^{\theta_2} U' d\theta \right| \\ &\leq \left(\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{\alpha}} \right)^{1/2} \left(\int_{\theta_1}^{\theta_2} \sqrt{\alpha} U'^2 d\theta \right)^{1/2} \\ &\leq \left(\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sqrt{\alpha(t_i)}} \right)^{1/2} \tilde{E}(t)^{1/2} \\ &\leq CE(t)^{1/2}, \end{aligned} \tag{82}$$

where the Hölder inequality and the monotonicity of α have been used.

The proof for ϕ is used the same argument. □

Lemma 5 *The function γ is bounded from above on $(T_-, t_i] \times S^1$.*

Proof. (cf. lemma 4 in [IW]). Note that

$$\dot{\phi}(t_i)^2 + e^{2\lambda\phi(t_i)} \dot{\sigma}(t_i)^2 \geq \frac{1}{2} \alpha(t_i) Q^2 e^{2[\eta(t_i) + \lambda\phi(t_i) - U(t_i)]} > 0, \tag{83}$$

since regular initial data at $t = t_i$ are supposed. This means $\tilde{E}(t_i) > 0$. From equation (66), we have

$$\frac{d\tilde{E}}{dt} \leq \int_{S^1} \frac{C_\lambda}{2} \frac{\dot{\alpha}}{\alpha} \frac{\mathcal{E}}{\sqrt{\alpha}} d\theta = -\frac{C_\lambda Q^2}{2} \int_{S^1} t e^{2(\gamma + \lambda\phi - U)} \frac{\mathcal{E}}{\sqrt{\alpha}} d\theta, \tag{84}$$

where $C_\lambda < 1$ is a positive constant depending on only the coupling constant λ. Suppose $\lambda \geq 0$. Integrating this inequality from t_i to t $(0 < t < t_i)$,

$$\begin{aligned}
\tilde{E}(t) &\geq \tilde{E}(t_i) + \frac{C_\lambda Q^2}{2}\int_t^{t_i}\left(\int_{S^1} s e^{2(\gamma+\lambda\phi-U)}\frac{\mathcal{E}}{\sqrt{\alpha}}d\theta\right)ds \\
&\geq \tilde{E}(t_i) + \frac{C_\lambda Q^2}{2}\int_t^{t_i} s\exp\left[2\left(\min_{S^1}\gamma + \lambda\min_{S^1}\phi - \max_{S^1}U\right)\right]\tilde{E}(s)ds \\
&\geq \tilde{E}(t_i)\left(1 + \frac{C_\lambda Q^2}{2}\int_t^{t_i} s\exp\left[2\left(\min_{S^1}\gamma + \lambda\min_{S^1}\phi - \max_{S^1}U\right)\right]ds\right), \quad (85)
\end{aligned}$$

where the monotonicity of $\tilde{E}$ has been used. From lemma 4,

$$\begin{aligned}
& \min_{S^1}\gamma + \lambda\min_{S^1}\phi - \max_{S^1}U \\
\geq\ & \max_{S^1}\gamma + \lambda\max_{S^1}\phi - \min_{S^1}U - \left(tE(t) + (C_1\lambda - C_2)E(t)^{1/2}\right) \\
\geq\ & \max_{S^1}\gamma + \lambda\max_{S^1}\phi - \min_{S^1}U - \left(t_iE(T_-) + (C_1\lambda - C_2)E(\tau)^{1/2}\right), \quad (86)
\end{aligned}$$

where C_1 and C_2 are positive constants, $\tau = t_i$ if $C_1\lambda - C_2 < 0$ and $\tau = T_-$ if $C_1\lambda - C_2 \geq 0$. Thus, we have

$$\tilde{E}(t) \geq \tilde{E}(t_i)\left(1 + \frac{C_\lambda Q^2}{2}e^{-2\left(t_iE(T_-)+(C_1\lambda-C_2)E(\tau)^{1/2}\right)}\int_t^{t_i} se^{2(\gamma+\lambda\phi-U)}ds\right), \quad (87)$$

and then,

$$\int_t^{t_i} se^{2(\gamma+\lambda\phi-U)}ds \leq \frac{2}{C_\lambda Q^2}e^{2\left(t_iE(T_-)+(C_1\lambda-C_2)E(\tau)^{1/2}\right)}\left(\frac{\tilde{E}(T_-)}{\tilde{E}(t_i)} - 1\right), \quad (88)$$

where the condition (83) has been used. When one consider the case of $\lambda < 0$, we have the same results by exchanging $\max_{S^1}\phi$ and $\min_{S^1}\phi$ in inequalities (85) and (86).

Now, integrating equation (49), we have

$$\begin{aligned}
\gamma(t,\theta) &= \gamma(t_i,\theta) - \int_t^{t_i}\left[s\mathcal{E} - \frac{sQ^2}{4}e^{2(\gamma+\lambda\phi-U)}\right]ds \\
&\leq \gamma(t_i,\theta) + \frac{Q^2}{4}\int_t^{t_i} se^{2(\gamma+\lambda\phi-U)}ds \\
&\leq \max_{S^1}\gamma(t_i,\theta) + \frac{1}{C_\lambda 2}e^{2\left(t_iE(T_-)+(C_1\lambda-C_2)E(\tau)^{1/2}\right)}\left(\frac{\tilde{E}(T_-)}{\tilde{E}(t_i)} - 1\right). \quad (89)
\end{aligned}$$

Thus, the boundedness of γ from above has been shown. □

Lemma 6 *For any numbers a and b, and for $n \leq \frac{1}{2}$, $\alpha^n e^{2\eta+a\phi-bU}$ is bounded on $(T_-, t_i] \times S^1$.*

Proof. (cf. lemma 5 in [WM]).

$$\begin{aligned}
& \partial_t \left(t^k \alpha^n e^{2\eta+a\phi-bU}\right) \\
= \; & \left(\frac{k}{t} + \frac{n\dot{\alpha}}{\alpha} + 2\dot{\eta} + a\dot{\phi} - b\dot{U}\right) t^k \alpha^n e^{2\eta+a\phi-bU} \\
= \; & \left[2t\left(\dot{U} - \frac{b}{4t}\right)^2 + \frac{t}{2}\left(\dot{\phi} + \frac{a}{t}\right)^2 + 2\alpha U'^2 + \frac{e^{4U}}{2t^2}\left(\dot{A}^2 + \alpha A'^2\right)\right. \\
& \left. + \frac{1}{2}\left(\alpha\phi'^2 + e^{2\lambda\phi}(\dot{\sigma}^2 + \alpha\sigma'^2)\right) + \left(\frac{1}{2} - n\right) t\alpha Q^2 e^{2(\eta+\lambda\phi-U)}\right] t^k \alpha^n e^{2\eta+a\phi-bU} \\
\geq \; & 0,
\end{aligned} \tag{90}$$

where we have chosen $8k = 4a^2 + b^2$. Then, we have

$$\alpha(t,\theta)^n e^{2\eta(t,\theta)+a\phi(t,\theta)-bU(t,\theta)} \leq \left(\frac{t_i}{T_-}\right)^k \alpha(t_i,\theta)^n e^{2\eta(t_i,\theta)+a\phi(t_i,\theta)-bU(t_i,\theta)}, \tag{91}$$

on $(T_-, t_i] \times S^1$. $\square$

Lemma 7 *α is bounded on $(T_-, t_i] \times S^1$.*

Proof. Integrating the constraint equation (11), we have

$$-\int_t^{t_i} \frac{\dot{\alpha}}{\alpha} ds = \ln\alpha(t) - \ln\alpha(t_i) = Q^2 \int_t^{t_i} s e^{2(\gamma+\lambda\phi-U)} ds, \tag{92}$$

for $t \in (T_-, t_i]$. By using inequality (88), we have boundedness of $\ln\alpha$ from above. As a result, $0 < \alpha$ is also bounded. $\square$

Lemma 8 *For any numbers a and b, $e^{\gamma+a\phi-bU} (= \sqrt{\alpha} e^{\eta+a\phi-bU})$ is bounded on $(T_-, t_i] \times S^1$.*

Proof. We have already a result that $e^{2\eta+a\phi-bU}$ is bounded on $(T_-, t_i] \times S^1$ (lemma 6). Combining this and lemma 7, the boundedness of $e^{\gamma+a\phi-bU}$ on $(T_-, t_i] \times S^1$ follows directly. $\square$

Corollary 1 *$\dot{\alpha}\alpha^{-1} = \partial_t(\ln\alpha)$ is bounded on $(T_-, t_i] \times S^1$. Thus, $\ln\alpha$ and $\dot{\alpha}$ are as well.*

Proof. Boundedness of $\alpha e^{2(\eta+a\phi-bU)}$ is obtained by lemma 8. From the constraint equation (51), we have $\dot{\alpha}\alpha^{-1} = -t\alpha Q^2 e^{2(\lambda\phi+\eta-U)}$. If we set $a = \lambda$ and $b = 1$, the boundedness of the right-hand-side of that equation is obtained. Thus, the conclusion of this lemma is shown. $\square$

Lemma 9 *The functions U, A, ϕ, σ and their first derivatives are bounded on $(T_-, t_i] \times S^1$.*

Proof. From lemma 2 and corollary 1, we have the boundedness for $\mathcal{E}$ on $(T_-, t_i] \times S^1$. Then, $\left|\dot{U}\right|$, $|U'|$, $\left|\dot{\phi}\right|$, $|\phi'|$, $\left|\left(e^{2U}/2t\right)\dot{A}\right|$, $\left|\left(e^{2U}/2t\right)A'\right|$, $\left|e^{\lambda\phi}\dot{\sigma}\right|$ and $\left|e^{\lambda\phi}\sigma'\right|$ are bounded for all $t \in (T_-, T_+)$. Once the boundedness on the first derivative of U and ϕ is obtained, it follows that U and ϕ are bounded for all $t \in (T_-, T_+)$. Then, we have bounds on $\dot{A}$, A', $\dot{\sigma}$ and σ', and consequently on A and σ. □

Lemma 10 *The functions α', $\dot{\alpha}'$ and $\ddot{\alpha}$ are bounded on $(T_-, t_i] \times S^1$. Also, η, $\dot{\eta}$ and η' are as well.*

Proof. (cf. Step 3 of Section 6 in [BCIM]). From the constraint equations (49) and (50), we have boundedness for $\dot{\gamma}$ and γ' directly. Then, γ is controlled. Differentiating both side of equation (51) with respect to θ, we have

$$\dot{\alpha}' = \alpha' \left(-tQ^2 e^{2(\gamma+\lambda\phi-U)}\right) - 2tQ^2 e^{2(\gamma+\lambda\phi-U)} \alpha \left(\gamma' + \lambda\phi' - U'\right). \tag{93}$$

Then, we have boundedness for α' by integrating this differential equation for α' in time since the coefficient of α' and the second term in the right-hand-side of the equation (93) are controlled. Thus, we have that η, $\dot{\eta}$ and η' is bounded.

The boundedness of $\dot{\alpha}'$ is obtained immediately from (93). Now, differentiating both side of equation (51) with respect to t, we have

$$\ddot{\alpha} = -Q^2 \alpha e^{2(\eta+\lambda\phi-U)} \left[\alpha + 2t\dot{\alpha} + 2t\alpha\left(\dot{\eta} + \lambda\dot{\phi} - \dot{U}\right)\right], \tag{94}$$

which implies that $\ddot{\alpha}$ is bounded. □

Lemma 11 *The second derivatives of U, A, ϕ and σ are bounded on $(T_-, t_i] \times S^1$.*

Proof. By lemma 3 we have the boundedness for $\tilde{\mathcal{E}}$ on $(T_-, t_i] \times S^1$. Then, we have uniform bounds on $\ddot{U}$, $\dot{U}'$, $\ddot{A}$, $\dot{A}'$, $\ddot{\phi}$, $\dot{\phi}'$, $\ddot{\sigma}$ and $\dot{\sigma}'$. Bounds on U'', A'', ϕ'' and σ'' follows from the wave equations (13)-(16) directly. □

Lemma 12 *α'', $\ddot{\eta}$, $\dot{\eta}'$ and η'' are bounded on $(T_-, t_i] \times S^1$.*

Proof. By taking the time derivative of (49) and (50), we have bounds on $\ddot{\gamma}$ and $\dot{\gamma}'$. Then, bounds on $\ddot{\eta}$ and $\dot{\eta}'$ are obtained by the definition of γ. Also, by taking the θ derivative of (50), we have bounds on γ''. The boundedness for α'' follows from the same argument in the proof of lemma 10. That is, differentiating both side of equation (93) with respect to θ, we have

$$\begin{aligned} \dot{\alpha}'' &= \alpha'' \left(-tQ^2 e^{2(\gamma+\lambda\phi-U)}\right) - 4tQ^2 \alpha' (\gamma' + \lambda\phi' - U') e^{2(\gamma+\lambda\phi-U)} \\ &- 2tQ^2 e^{2(\gamma+\lambda\phi-U)} \alpha \left[\gamma'' + \lambda\phi'' - U'' + 2\left(\gamma' + \lambda\phi' - U'\right)^2\right]. \end{aligned} \tag{95}$$

Therefore, we have boundedness for α'' by integrating this differential equation for α'' in time since the coefficient of α'' and the second and third terms in the right-hand-side of the equation (95) are bounded as shown already. Then, η'' is bounded by using the wave equation (12). $\square$

3.2 Future direction

Now, consider the future direction. We have already a monotonic decreasing property of $E(t)$ along increasing t, $dE/dt < 0$ (lemma 1). Therefore, for any $t \in [t_i, T_+)$,

$$E(t) \leq E(t_i). \tag{96}$$

Proofs of the following two lemmas are similar with the argument in Step 1 of Section 5 in [AH].

Lemma 13 *$\int_{\theta_1}^{\theta_2} \alpha^{-1/2} d\theta$ is bounded on $[t_i, T_+)$.*

Proof. The constraint equation (11) can be written as

$$\partial_t(\alpha^{-1/2}) = \frac{t}{2}\sqrt{\alpha} Q^2 e^{2(\eta+\lambda\phi-U)}. \tag{97}$$

Then,

$$\alpha^{-1/2}(t,\theta) - \alpha^{-1/2}(t_i,\theta) = \frac{Q^2}{2}\int_{t_i}^{t} s\sqrt{\alpha} e^{2(\eta+\lambda\phi-U)} ds, \tag{98}$$

for ant $t \in [t_i, T_+)$. Integrating this equation from θ_1 to θ_2 in S^1, we have

$$\begin{aligned}
\int_{\theta_1}^{\theta_2} \alpha^{-1/2}(t,\theta) d\theta &= \frac{Q^2}{2}\int_{t_i}^{t} s \int_{\theta_1}^{\theta_2} \sqrt{\alpha} e^{2(\eta+\lambda\phi-U)} d\theta ds + \int_{\theta_1}^{\theta_2} \alpha^{-1/2}(t_i,\theta) d\theta \\
&\leq \frac{Q^2}{2} E(t_i) \int_{t_i}^{t} s ds + 2\pi \sup_{S^1} \alpha^{-1/2}(t_i,\theta) \\
&\leq \frac{Q^2}{4} E(t_i)(t^2 - t_i^2) + C, \qquad (99)
\end{aligned}$$

where (96) has been used. $\square$

Lemma 14 *The functions U and ϕ are bounded on $[t_i, T_+) \times S^1$.*

Proof. For any θ_1, θ_2 and for each $t \in [t_i, T_+)$,

$$\begin{aligned}
|U(t,\theta_2) - U(t,\theta_1)| &= \left| \int_{\theta_1}^{\theta_2} U' d\theta \right| \\
&\leq \left(\int_{\theta_1}^{\theta_2} \alpha^{-1/2} d\theta \right)^{1/2} \left(\int_{\theta_1}^{\theta_2} \alpha^{1/2} U'^2 d\theta \right)^{1/2} \\
&\leq C(t), \qquad (100)
\end{aligned}$$

where the Hölder inequality, energy bound (96) and lemma 13 have been used.

Now,

$$\begin{aligned}
\left|\int_{S^1} U(t,\theta)d\theta\right| &= \left|\int_{t_i}^t \int_{S^1} \dot{U}(t,\theta)d\theta ds + C\right| \\
&\leq \int_{t_i}^t \int_{S^1} \left|\dot{U}(t,\theta)\right| d\theta ds + |C| \\
&\leq \int_{t_i}^t \left(\int_{S^1} \alpha^{1/2} d\theta\right)^{1/2} \left(\int_{S^1} \alpha^{-1/2}\dot{U}^2 d\theta\right)^{1/2} ds + |C|, \qquad (101)
\end{aligned}$$

where the Hölder inequality has been used. Since α is monotonically decreasing function along increasing time t, the right-hand-side of the above inequality can be bounded. Thus,

$$\left|\int_{S^1} U(t,\theta)d\theta\right| \leq C(t), \qquad (102)$$

for some uniformly bounded function $C(t)$.

Finally, we obtain a uniform bound on U. We have the following identity:

$$2\pi \max_{S^1} U(t,\theta) = \int_{S^1} U(t,\theta)d\theta + \int_{S^1} \left(\max_{S^1} U(t,\theta) - U(t,\theta)\right) d\theta. \qquad (103)$$

The right-hand-side of this identity is bounded from above since (100) and (102) hold. . For $\min_{S^1} U(t,\theta)$, one can use the same argument and then, $\min_{S^1} U(t,\theta)$ is bounded from below. Thus, U is uniformly bounded on $[t_i, T_+) \times S^1$.

We can obtain uniform boundedness for ϕ by replacing U with ϕ in the above argument. □

Lemma 15 *The functions γ is bounded on $[t_i, T_+) \times S^1$.*

Proof. (cf. Step 1 of Section 6 in [BCIM]). From the constraint equation (49) for γ, we have two inequalities:

$$\dot{\gamma} \leq te, \qquad (104)$$

and

$$\dot{\gamma} \geq -\frac{1}{4} t Q^2 e^{2(\gamma+\lambda\phi-U)}. \qquad (105)$$

From the inequality (104), we have

$$\int_{S^1} \gamma(t,\theta)d\theta - \int_{S^1} \gamma(t_i,\theta)d\theta = \int_{t_i}^t \frac{d}{ds}\left(\int_{S^1} \gamma(s,\theta)d\theta\right) ds$$

$$
\begin{aligned}
&\leq \sup_{S^1} \sqrt{\alpha}(t_i,\theta) \int_{t_i}^{t} sE(s)ds \\
&\leq C \int_{t_i}^{t} sE(t_i)ds \\
&= \frac{CE(t_i)}{2}(t^2 - t_i^2),
\end{aligned} \tag{106}
$$

which controls $\int_{S^1} \gamma(t,\theta)d\theta$ from above. Now, we have the following identity:

$$
\int_{S^1} \gamma(t,\theta)d\theta = 2\pi \max_{S^1} \gamma + \int_{S^1} \left(\gamma - \max_{S^1} \gamma \right) d\theta. \tag{107}
$$

By the equation (50) of γ and a basic inequality, we have

$$
\int_{S^1} |\gamma'| \, d\theta \leq tE(t_i). \tag{108}
$$

Then,

$$
|\gamma(t,\theta_2) - \gamma(t,\theta_1)| = \left| \int_{\theta_1}^{\theta_2} \gamma' d\theta \right| \leq \int_{\theta_1}^{\theta_2} |\gamma'| \, d\theta \leq \int_{S^1} |\gamma'| \, d\theta \leq tE(t_i). \tag{109}
$$

Therefore, combining (107) and (109), we have the upper bound for γ:

$$
\max_{S^1} \gamma \leq C(t), \tag{110}
$$

where $C(t)$ is a bounded function of $t \in [t_i, T_+)$.

From the inequalities (105) and (110), and lemma 14, if the coupling constant λ is non-negative,

$$
\dot{\gamma} \geq -\frac{1}{4} t Q^2 e^{2(\gamma+\lambda\phi-U)} \geq Ct \exp\left[2 \left(\max_{S^1} \gamma + \lambda \max_{S^1} \phi - \min_{S^1} U \right) \right] \geq Cte^{c(t)}, \tag{111}
$$

for some bounded function $c(t)$ of $t \in [t_i, T_+)$ and $C < 0$. If λ is negative, $\max_{S^1} \phi$ is replaced by $\min_{S^1} \phi$. Thus, $\dot{\gamma}$ is controlled into the future, so we have upper and lower bounds for γ on $[t_i, T_+) \times S^1$. $\square$

Corollary 2 *$\dot{\alpha}\alpha^{-1}$ (hence $\ln \alpha$ and α), η and $\dot{\alpha}$ are bounded on $[t_i, T_+) \times S^1$.*

Proof. The constraint equation (51) can be written as

$$
\frac{\dot{\alpha}}{\alpha} = -tQ^2 e^{2(\gamma+\lambda\phi-U)}. \tag{112}
$$

With boundedness of γ (lemma 15), ϕ and U (lemma 14), $\dot{\alpha}\alpha^{-1} = \partial_t (\ln \alpha)$ is bounded on $[t_i, T_+) \times S^1$. As immediate results, $\ln \alpha$ and α are bounded on $[t_i, T_+) \times$

S^1. Since $\eta = \gamma - \frac{1}{2}\ln\alpha$, η is bounded on $[t_i, T_+) \times S^1$. Using these results to the constraint equation (51), we have a conclusion that $\dot{\alpha}$ is bounded on $[t_i, T_+) \times S^1$. $\square$

Once the boundedness of $\dot{\alpha}\alpha^{-1}$ has been obtained, the following arguments are similar with ones of the past direction because key lemmas (lemma 2 and lemma 3) can be used and the arguments do not depend on time directions.

Lemma 16 *The functions U, A, ϕ, σ and their first derivatives are bounded on $[t_i, T_+) \times S^1$.*

Proof. From lemma 2 and corollary 2, we have the boundedness for $\mathcal{E}$ on $[t_i, T_+) \times S^1$. The proof is the same with one of lemma 9. $\square$

Lemma 17 *The functions $\dot{\eta}$, η', α', $\dot{\alpha}'$ and $\ddot{\alpha}$ are bounded on $[t_i, T_+) \times S^1$.*

Proof. Since the constraint equation (9) is described in terms of bounded functions and t, we have bounds on $\dot{\eta}$. From the constraint equation (50), we have bounds on γ' and then, boundedness for α', $\dot{\alpha}'$ and $\ddot{\alpha}$ can be obtained by the same argument with the proof of lemma 10. Combining this result, boundedness of γ' and definition of γ, we have that η' is bounded. $\square$

Lemma 18 *The second derivatives of U, A, ϕ and σ are bounded on $[t_i, T_+) \times S^1$.*

Proof. By lemma 3 we have the boundedness for $\tilde{\mathcal{E}}$ on $[t_i, T_+) \times S^1$. The proof is the same with one of lemma 11. $\square$

Lemma 19 *α'', $\ddot{\eta}$, $\dot{\eta}'$ and η'' are bounded on $[t_i, T_+) \times S^1$.*

Proof. The argument is the same to lemma 12. $\square$

3.3 Proof of theorem 4

We can continue to obtain bounds on higher derivatives of the fields by repeating the above arguments. Fortunately, to apply the global existence theorem in [MA], it is enough to get C^2 bounds of all functions. Thus, it has been shown that the functions η, α, U, A, ϕ and σ extend to $t \to 0$ into the past direction and to $t \to \infty$ into the future direction. $\square$

4 Comments

We should like to comment concerning the TCC and the condition for coupling constant λ. Note that these conditions are needed to prove theorem 4 into only the past direction. It is expected that the TCC is satisfied near initial singularities because strong focusing effect by gravity is dominant than repulsing one by a positive potential (cosmological constant) there. Note that spacetimes described by our AVTD solutions satisfy the TCC. Also, it is possible to expand in acceleration of the spacetimes into the future direction since the TCC does not hold necessarily there and the positive potential would become dominant. Thus, theorem 4 does not deny paradigm-A.

The condition $|\lambda| \leq \bar{\lambda} < 1/2$ admits $\lambda = 0$, which means that there is a positive cosmological constant. Thus, our theorem is applicable to not only theories with dilaton coupling but also ones with a pure cosmological constant. Now, let us discuss $\bar{\lambda}$. It is known that there is a critical value λ_C in n-dimensional homogeneous and isotropic spacetimes [TP, WMNR]. In our notation with $n = 4$, $|\lambda_C| = 1/2$. Here, "critical" means the boundary whether late-time attractor solutions indicate accelerated expansion or not. Roughly speaking, λ describes steepness of the potential. Therefore, for $\lambda^2 > \lambda_C^2$, the dilaton field falls down the potential hill soon and then decelerating expansion solutions with transient accelerating one are obtained, while we have attractor solutions with eternal accelerating expansion if $\lambda^2 < \lambda_C^2$. It is believed that such critical values exist for generic spacetimes, although we do not know λ_C for spacetimes we considered here, in particular, our results give us no information about relation between λ_C and $\bar{\lambda}$. Thus, it is not clear that the solution obtained in theorem 3 is consistent with paradigm-A at the intermediate- and late-time. To answer this question, we need to analyze future asymptotic behavior (e.g. see [RA04]), which is left for future research.

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A Local existence and uniqueness for smooth case

Let us consider the smooth version of the initial-value problem for our non-standard setup formulated in section 2. A key idea is to construct a symmetric-hyperbolic system by introducing a new variable $\alpha' := Z_{14}$ [IK]. Let us define $\vec{Z} := Z_i = (U, \dot{U}, U', A, \dot{A}, A', \phi, \dot{\phi}, \phi', \sigma, \dot{\sigma}, \sigma', \alpha, \alpha', \eta)$. Here, i runs from 1 to 15. The system consisting in the effective evolution equations (9), (11), (13)-(16) becomes the following first-order symmetric-hyperbolic one:

$$\mathcal{A}_0 \partial_t \vec{Z} = \mathcal{A}_1 \partial_\theta \vec{Z} + F(t, \theta, \vec{Z}), \tag{113}$$

where

$$\mathcal{A}_0 = \mathrm{diag}(1, 1, \alpha, 1, 1, \alpha, 1, 1, \alpha, 1, 1, \alpha, 1, 1, 1), \tag{114}$$

and

$$\mathcal{A}_1 = \begin{pmatrix} \mathcal{A}_2 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{A}_2 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{A}_2 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{A}_2 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{A}_3 \end{pmatrix}, \tag{115}$$

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{116}$$

Thus, we have a unique solution to the effective evolution equations by prescribing the smooth initial data for $t = t_0 > 0$ if the constraint equations (10) and $\alpha' = Z_{14}$ hold for any t.

Now, as the analytic case, to assure the local existence and uniqueness of the initial-value problem, it is enough to show that the constraints propagate. Let us set

$$N_1 := \eta' - 2DUU' - \frac{e^{4U}}{2t^2}DAA' - \frac{1}{2}D\phi\phi' - \frac{e^{2\lambda\phi}}{2}D\sigma\sigma' + \frac{Z_{14}}{2\alpha}, \tag{117}$$

and

$$N_2 := Z_{14} - \alpha'. \tag{118}$$

By direct calculation, we have the following linear, homogeneous ODE system:

$$(D + \mathcal{B})\vec{N} = 0, \tag{119}$$

where $\vec{N} := (N_1, N_2)$ and

$$\mathcal{B} = \frac{D\alpha}{2\alpha^2} \begin{pmatrix} \alpha & -1 \\ 4\alpha^2 & -2\alpha \end{pmatrix}. \tag{120}$$

Thus, the uniqueness theorem for ODE systems guarantees that $\vec{N}$ is identically zero for any time t if we set initial data for $t = t_0$ such that $\vec{N}(t_0) = 0$. Thus, the local existence and uniqueness of the initial-value problem for our case has been shown in the smooth case.

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