ANTI-DE SITTER SPACE AT FINITE TEMPERATURE

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We consider a conformally invariant scalar field at finite temperature in anti-de Sitter space, and find the symmetric two-point function. Since it is meromorphic and it has both a real-time and imaginary-time periodicity, it is an elliptic function. From it, the expectation values of ϕ^2 and the stress–energy tensor are calculated exactly, and then compared to a Tolman-redshifted radiation gas, and to Page's "optical" approximation The total energy of the radiation is finite.

1. Introduction. Anti-de Sitter space (ADS) is a hyperbolic sheet with topology $S^1 \times R^3$ [1]. It is a maximally symmetric spacetime with constant negative curvature, which arises naturally in gauged supergravity theories, and whose properties have been extensively studied [2–4].

This paper considers the effects of filling the spacetime with a gas of thermal radiation. In effect the field is coupled to an imaginary "external" heat bath and then allowed to come to equilibrium.

ADS contains closed timelike curves and is not globally hyperbolic. For the latter reason one must impose boundary conditions at spatial infinity in order to solve the Cauchy problem [2]. These boundary conditions are also required by supersymmetry [3]. They have the effect of reflecting any flux of energy–momentum that reaches spatial infinity back into the space. Thus these boundary conditions conserve energy by preventing escape across the timelike infinity. There are two possible choices of these reflecting boundary conditions, here called Neumann (N) and Dirichlet (D).

Because of the boundary conditions, ADS acts like a perfectly reflecting box, and can be filled with a radiation gas [4]. In equilibrium, the local temperature of this radiation is proportional, via the Tolman relation, to $(g_{00})^{-1/2}$ [5]. This has the effect of breaking anti-de Sitter invariance: the location of the imaginary heat bath singles out a prefered center in the spacetime. In this way SO(2,3) invariance is broken to $SO(2)\times SO(3)$ by the presence of the temperature.

For a bosonic field in equilibrium at temperature $T=1/\beta$, the n-point correlation functions are periodic in imaginary time, with period $i\beta$ [6]. Now ADS space is *already* periodic in real time. Thus the correlation functions, considered as complex functions in the complex-time plane are doubly periodic. This fact permits one to find the two-point symmetric function with virtually no effort. This is because it is an elliptic function, and is thus completely determined once its two periods are known, and the locations and residues of its poles in one fundamental cell are specified (see fig. 1) [7]. In section 2 we use this technique to find the symmetric two-point function for a conformally invariant scalar field ϕ.

The symmetric two-point function contains all the information about the non-interacting quantum field. In particular, the stress–energy tensor can be found from it. Essentially it is determined by the short-distance behavior of the propagator. In section 3, we find the stress–energy tensor in this way. We then compare it to the

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fourth power of the local "Tolman" temperature. In section four, we obtain a finite expression for the total energy by integrating the energy density $T^0{}_0$ over a spatial surface. This result can also be obtained directly from the partition function of the radiation.

2. *The symmetric two-point function.* In this section, we use the double-periodicity of the two-point function to express it as an elliptic function.

The metric of ADS is [1]

$$ds^2 = a^{-2} \sec^2\rho \, [-dt^2 + d\rho^2 + \sin^2\rho \, (d\theta^2 + \sin^2\theta \, d\varphi^2)], \tag{2.1}$$

where a is a constant, and the coordinate ranges are $t \in]-\infty, +\infty[$, $\rho \in [0, \pi]$, $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. We use the curvature conventions of ref. [1], and set $\hbar = k = c = 1$. The Ricci tensor is $R_{ab} = -3a^2 g_{ab}$.

For zero temperature, the propagator for a conformally invariant scalar field obeying $(-\Box + \frac{1}{6}R)\phi = 0$ is [2]

$$D^{(1)}(x, x') = \langle \phi(x)\phi(x') + \phi(x')\phi(x) \rangle = -[a^2/(2\pi)^2][(1-Z)^{-1} \pm (1+Z)^{-1}]. \tag{2.2}$$

Here the O(2,3) invariant biscalar function $Z(x, x')$ is

$$Z(x, x') = \frac{\cos(t-t') - \sin\rho \sin\rho' \cos\Omega}{\cos\rho \cos\rho'}, \tag{2.3}$$

where

$$\cos\Omega = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi'). \tag{2.4}$$

Since Z is a function of $\tau = t - t'$, we will write (2.3) as $Z(\tau; \rho, \rho'; \Omega)$. Note also that $Z(\tau + \pi; -\rho, \rho'; \Omega) = -Z(\tau; \rho, \rho'; \Omega)$.

In formula (2.2), and elsewhere in this paper, the upper sign refers to Dirichlet boundary conditions, and the lower sign to Neumann boundary conditions [2,3]. These boundary conditions are also called, respectively, regular and irregular boundary conditions.

If we now demand that the point $\rho = 0$ be held at temperature $T = 1/\beta$ the propagator becomes periodic in $a^{-1}\tau$ with period $i\beta$. Thus letting $G_\beta^{(1)}$ denote the finite temperature propagator, we find

$$G_\beta^{(1)}(x, x') = -[a^2/(2\pi)^2][J_\beta(\tau; \rho, \rho') \pm J_\beta(\tau + \pi; -\rho, \rho')], \tag{2.5}$$

where

$$J_\beta(\tau; \rho, \rho') = \sum_{n=-\infty}^{+\infty} [1 - Z(\tau + ia\beta n; \rho, \rho'; \Omega)]^{-1}. \tag{2.6}$$

The function J_β is a doubly periodic function of the complex variable τ and we will now show that it is an elliptic function [7].

The periodicity properties of J_β are as follows:

(1) $J_\beta(\tau; \rho, \rho') = J_\beta(\tau + 2\pi, \rho, \rho')$,

(2) $J_\beta(\tau; \rho, \rho') = J_\beta(\tau + ia\beta; \rho, \rho')$,

as can be easily seen from (2.3) and (2.6). Property (1) reflects the real-time periodicity of ADS, and property (2) is due to the thermal imaginary-time periodicity. Within a fundamental cell shown in fig. 1 ($-\pi < \mathrm{Re}\,\tau < \pi$ and $-a\beta/2 < \mathrm{Im}\,\tau < a\beta/2$), the function J_β has two poles of opposite strength. The poles are located at $\tau = \pm\eta$, where

$$\eta(\rho, \rho'; \Omega) = \cos^{-1}[\cos\rho \cos\rho' + \sin\rho \sin\rho' \cos\Omega], \tag{2.7}$$

and they have residues $\pm\cos\rho \cos\rho' \csc\eta$, respectively. The function J_β is meromorphic in the complex τ-plane and is thus an elliptic function.

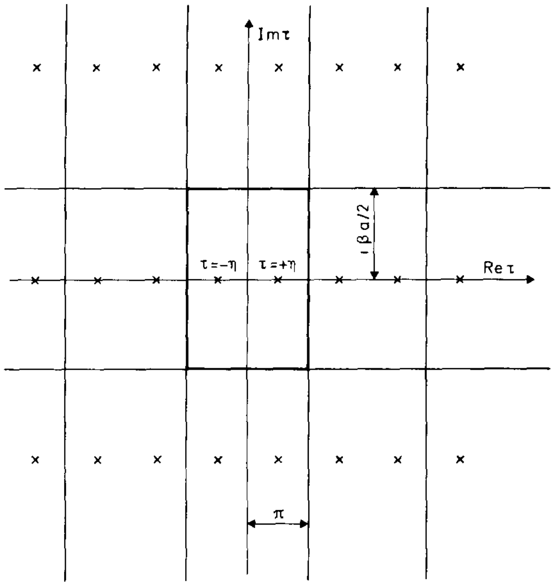

Fig 1. Fundamental cell in the complex τ-plane. The propagator is a doubly periodic elliptic function, with two poles in its fundamental cell, at $\tau=\pm\eta$. The real-time periodicity of the space-time provides the real period, the thermal heat bath at temperature β^{-1} provides the imaginary period

From a fundamental theorem of elliptic functions (see p. 474 of ref. [7]) such a function is completely determined by its two periods and the locations and strengths of its poles in a fundamental cell. Thus we find

$$J_\beta(\tau;\rho,\rho')=(\cos\rho\cos\rho'/\sin\eta)[(\mathrm{d}/\mathrm{d}\tau)\ln\theta_1(\tfrac{1}{2}(\tau-\eta),\exp(-\tfrac{1}{2}a\beta))-(\mathrm{d}/\mathrm{d}\tau)\ln\theta_1(\tfrac{1}{2}(\tau+\eta),\exp(-\tfrac{1}{2}a\beta))]. \tag{2.8}$$

Here $\theta_1(Z,q)$ is an elliptic theta function as defined in ref. [7]. A useful formula for it is

$$(\mathrm{d}/\mathrm{d}Z)\ln\theta_1(Z,q)=\cot Z+4\sum_{k=1}^{\infty}(q^{-2k}-1)^{-1}\sin 2kZ. \tag{2.9}$$

There is in addition an additive constant in (2.8), which can be set to zero. We can now use (2.5), (2.7), (2.8) and (2.9) to find the following form for the propagator:

$$G^{(1)}_\beta(\tau;\rho,\rho';\Omega)=-\frac{a^2}{(2\pi)^2}\Bigg[(1-Z)^{-1}\pm(1+Z)^{-1}$$

$$-4\cos\rho\cos\rho'\sum_{k=1}^{\infty}\left(\frac{\sin k\eta(\rho,\rho';\Omega)}{\sin\eta(\rho,\rho';\Omega)}\pm(-1)^k\frac{\sin k\eta(-\rho,\rho';\Omega)}{\sin\eta(-\rho,\rho';\Omega)}\right)\frac{\cos k\tau}{\exp(ka\beta)-1}\Bigg]. \tag{2.10}$$

It is easy to see that the zero-temperature limit $\beta\to\infty$ reduces to the correct function (2.2), and that the flat-space limit at finite temperature is also correct.

3. $\langle\phi^2\rangle$ *and the stress tensor.* Now that the exact form of the symmetric function $G^{(1)}_\beta$ is known, it is straightforward to calculate the expectation value of $\phi^2(x)$, the expectation value of the stress tensor $T^{ab}(x)$ and the total energy, which is the integral of the energy density over a spacelike surface.

To do this, we will utilise the Hadamard development of the symmetric function $G^{(1)}_\beta$. It can be put in the form

$$G^{(1)}_\beta=(2\pi)^{-2}\,(\Delta^{1/2}/\sigma+W), \tag{3.1}$$

where σ is one-half the square of the geodesic distance and Δ is the Van Vleck determinant. In ADS they are

$$\sigma(x,x')=6R^{-1}[\cos^{-1} Z(x,x')]^2 \tag{3.2}$$

and

$$\Delta=(\tfrac{1}{6}R\sigma)^{3/2}\csc^3(\tfrac{1}{6}R\sigma)^{1/2}. \tag{3.3}$$

The symmetric biscalar $W(x,x')$ can be developed in a short distance expansion of the form

$$W(x,x')=\omega(x)-\tfrac{1}{2}\omega_a(x)\sigma^{;a}+\tfrac{1}{2}\omega_{ab}(x)\sigma^{;a}\sigma^{;b}+\dots, \tag{3.4}$$

where

$$\omega(x)=\lim_{x'\to x} W(x,x'),\quad \omega_a(x)=\omega(x)_{,a},\quad \omega_{ab}(x)=\lim_{x'\to x} W(x,x')_{,ab}. \tag{3.5}$$

If ω and ω_{ab} are known, then the expectation values of ϕ^2 and the stress tensor T^{ab} are given in ADS by the following formulae [8,9]:

$$\langle\phi^2\rangle=(8\pi^2)^{-1}\omega, \tag{3.6}$$

$$\langle T_{ab}\rangle=(8\pi^2)^{-1}[-(\omega_{ab}-\tfrac{1}{2}g_{ab}w_c{}^c)+(-\tfrac{1}{12}g_{ab}\Box+\tfrac{1}{3}\nabla_a\nabla_b+\tfrac{1}{2}a^2 g_{ab})\omega-\tfrac{1}{30}a^4 g_{ab}]. \tag{3.7}$$

One can now calculate these quantities from the exact two-point function which we have found in the previous section.

To express ω_{ab} and $\langle T_{ab}\rangle$ in a geometric form, one can introduce unit length timelike and radial vector fields, which we call t^a and ρ^a respectively. They are given in coordinates (t,ρ,θ,φ) of the metric (2.1) by

$$t_a=a^{-1}\sec\rho\,(1,0,0,0),\quad \rho_a=a^{-1}\sec\rho\,(0,1,0,0), \tag{3.8}$$

and are normalised so that $t_a t^a=-1$ and $\rho_a\rho^a=1$.

Using the formula for $G_\beta^{(1)}$ (2.10) and the above definitions, we find after a long and tedious calculation, that ω and ω_{ab} are given by

$$\omega=a^2\{[\tfrac{1}{3}+4\cos^2\rho\, f_1(a\beta)]\pm[-\tfrac{1}{2}+2\cot\rho\, S_0(a\beta,\rho)]\} \tag{3.9}$$

and

$$\begin{aligned}\omega_{ab}=a^4\{&[-\tfrac{17}{120}+\tfrac{4}{3}(\cos^2\rho-3)\cos^2\rho\, f_1(a\beta)-\tfrac{4}{3}\cos^4\rho\, f_3(a\beta)]g_{ab}\\&+[-\tfrac{8}{3}\cos^4\rho\, f_1(a\beta)-\tfrac{16}{3}\cos^4\rho\, f_3(a\beta)]\,t_a t_b+[2\sin^2 2\rho\, f_1(a\beta)]\,\rho_a\rho_b\}\\&\pm a^4\{[\tfrac{1}{4}-\tfrac{1}{2}\cot\rho\csc^2\rho(1+2\sin^2\rho)S_0(a\beta,\rho)+\cot^2\rho\, C_1(a\beta,\rho)]\,g_{ab}\\&+[-\tfrac{1}{2}\cot\rho\csc^2\rho\, S_0(a\beta,\rho)+\cot^2\rho\cos 2\rho\, C_1(a\beta,\rho)-2\cot\rho\cos^2\rho\, S_2(a\beta,\rho)]\,t_a t_b\\&+[\tfrac{3}{2}\cot\rho\csc^2\rho\, S_0(a\beta,\rho)-\cot^2\rho(2\sin^2\rho+3)C_1(a\beta,\rho)-2\cot\rho\cos^2\rho\, S_2(a\beta,\rho)]\,\rho_a\rho_b\}.\end{aligned} \tag{3.10}$$

The functions f_m, S_m and C_m are given by the following formulae:

$$f_m(x)=\sum_{n=1}^{\infty} n^m(e^{nx}-1)^{-1}, \tag{3.11a}$$

$$S_m(x,\rho)=\sum_{n=1}^{\infty} n^m(-1)^n(e^{nx}-1)^{-1}\sin 2n\rho, \tag{3.11b}$$

$$C_m(x,\rho)=\sum_{n=1}^{\infty} n^m(-1)^n(e^{nx}-1)^{-1}\cos 2n\rho. \tag{3.11c}$$

Now using the Hadamard definitions (3.6) and (3.7) of the expectation values of ϕ^2 and the stress–energy tensor, we find

$$\langle\phi^2\rangle=a^2(8\pi^2)^{-1}\{[\tfrac{1}{3}+4\cos^2\rho\, f_1(a\beta)]\pm[-\tfrac{1}{2}+2\cot\rho\, S_0(a\beta,\rho)]\}, \tag{3.12}$$

and

$$\begin{aligned}\langle T_{ab}\rangle=&\,a^4(8\pi^2)^{-1}\{[-\tfrac{1}{120}+\tfrac{4}{3}\cos^4\rho\, f_3(a\beta)]\, g_{ab}+[\tfrac{16}{3}\cos^4\rho\, f_3(a\beta)]\, t_a t_b\}\\ &\pm a^4(8\pi^2)^{-1}\{[-\tfrac{1}{6}\csc^2\rho\cos 2\rho\, S_0(a\beta,\rho)+\tfrac{1}{3}\cot\rho\, C_1(a\beta,\rho)+\tfrac{2}{3}\cos^2\rho\, S_2(a\beta,\rho)]\, g_{ab}\\ &+[\tfrac{1}{6}(3-\cot^2\rho)\, S_0(a\beta,\rho)+\cot\rho(1-\tfrac{2}{3}\cos^2\rho)\, C_1(a\beta,\rho)+2\cos^2\rho\, S_2(a\beta,\rho)]\, t_a t_b\\ &+[\tfrac{1}{6}(3\csc^2\rho-4)\, S_0(a\beta,\rho)+\cot\rho\,(\tfrac{2}{3}\sin^2\rho-1)\, C_1(a\beta,\rho)-\tfrac{2}{3}\cos^2\rho\, S_2(a\beta,\rho)]\,\rho_a\rho_b\}.\end{aligned} \tag{3.13}$$

As before the upper sign is with Dirichlet (regular) boundary conditions, and the lower sign is with Neumann (irregular) boundary conditions. Before continuing, we note that the stress tensor consists of a zero-temperature trace anomaly, which is proportional to the metric and is independent of the boundary conditions, and of a temperature-dependent part which vanishes at low temperatures and dominates at high temperatures.

We now examine several limits, in particular the limits $\rho\to 0$ and $\rho\to\frac{1}{2}\pi$ and the limits $\beta\to 0$ and $\beta\to\infty$.

In the limit $\rho=0$ we find that the stress tensor becomes

$$\langle T_{ab}\rangle(\rho=0)=-(a^4/960\pi^2)\, g_{ab}+(a^4/16\pi^2)\, A(a\beta)(g_{ab}+4t_a t_b), \tag{3.14}$$

where

$$A(x)=\sum_{n=1}^{\infty}(e^x-1)^{-1}n[3n^2\pm(-1)^n(2n^2+1)]. \tag{3.15}$$

Note that the termal structure of the stress tensor at $\rho=0$. The temperature dependent contribution is proportional to the radiation stress tensor $(-3, 1, 1, 1)$.

In the limit $\rho=\frac{1}{2}\pi$ (spatial infinity) the stress–energy tensor reduces to the trace-anomaly term. It is easy to understand why the thermal part vanishes. It is because the redshift is infinite at $\rho=\frac{1}{2}\pi$ and thus any quanta that manage to get all the way out to spatial infinity have zero energy upon their arrival. Because the thermal part of the stress–energy tensor vanishes at spatial infinity, it is clear that there can be no flux of energy–momentum across spatial infinity, the "box" does not leak!

In the low-temperature limit $\beta\to\infty$ the series that appear in $\langle T_{ab}\rangle$ are well approximated by their first terms. Hence

$$\langle T_{ab}\rangle \underset{\beta\to\infty}{\approx} -\frac{a^4}{960\pi^2}\, g_{ab}. \tag{3.16}$$

Thus only the trace anomaly survives in the zero-temperature limit.

Finally in the high-temperature limit $\beta\to 0^+$ one can use the Mellin transform

$$\int_0^\infty x^{s-1}(e^{\beta x}-1)^{-1}dx=\beta^{-1}\Gamma(s)\xi(s) \tag{3.17}$$

to approximate f_3. The sums S_i and C_i can be approximated also and give, up to exponentially small terms of the form $e^{-1/\beta}$,

$$S_0(a\beta,\rho)\cong(\pi/2a\beta)[(\cosh Z)/\sinh Z-1/Z],$$

$$C_1(a\beta,\rho)\cong -\tfrac{1}{2}(\pi/a\beta)^2(1/\sinh^2 Z-1/Z^2),$$

$$S_2(a\beta,\rho)\cong -(\pi/a\beta)^3[(\cosh Z)/\sinh^3 Z-1/Z^3], \tag{3.18}$$

where we have introduced the variable $Z=(2\pi/a\beta)(\rho-\tfrac{1}{2}\pi)$. Note that it is only when Z is of order 1 that S_2 (for example) is of order β^{-3}; otherwise S_2 is small. For Z to be of order 1, ρ must be close to $\pi/2-\beta a$. Thus at high temperature these "boundary terms" only contribute very close to spatial infinity. Nevertheless because the volume of a spatial surface is infinite, we will see shortly that the boundary terms still contribute a term of order β^{-3} to the total energy.

Thus at high temperature, the stress–energy tensor takes the approximate form

$$\langle T_{ab}\rangle = -(a^4/960\pi^2)\, g_{ab}+\tfrac{1}{90}\pi^2\beta^{-4}[g_{ab}+4t_a t_b]\cos^4\rho$$

$$\mp \tfrac{1}{12}\pi a\beta^{-3}\frac{\cos^3\rho}{\sin\rho}\left[\frac{\cosh[(2\pi/a\beta)(\rho-\frac{1}{2}\pi)]}{\sinh^3[(2\pi/a\beta)(\rho-\frac{1}{2}\pi)]}-\left(\frac{a\beta}{2\pi(\rho-\frac{1}{2}\pi)}\right)^3\right](g_{ab}+3t_a t_b-\rho_a\rho_b). \tag{3.19}$$

This formula is interesting for two reasons. First, notice that the boundary terms are of odd order in β^{-1}, as is to be expected from the contribution of boundary terms to the asymptotic form of the heat kernel [10]. Second the leading term proportional to β^{-4} is exactly what would be expected from the Tolman formula [5] discussed in the introduction. That relation is

$$T_{\rm L}(\rho)/T_{\rm L}(0)=[g_{tt}(0)/g_{tt}(\rho)]^{1/2}, \tag{3.20}$$

where $T_{\rm L}(\rho)$ is the local temperature at radial distance ρ. The term in (3.19) proportional to β^{-4} corresponds to a local energy density $\frac{1}{30}\pi^2 T_{\rm L}(\rho)^4$ – exactly what one would expect from the Tolman relation.

4. *The total energy.* In the previous section we found an exact expression for the stress–energy tensor $\langle T_{ab}\rangle$. One now easily calculates the total energy at temperature β^{-1},

$$E(\beta)=\int T^{ab}K_a\, d\Sigma_b. \tag{4.1}$$

Here K_a is the timelike Killing vector (normalized to unit length at the "heat source" located at $\rho=0$) and $d\Sigma_b$ is the volume element of the spacelike surface $t=$constant,

$$K_a=(\cos\rho)^{-1}t_a, \tag{4.2}$$

$$d\Sigma_b=a^3 t_b(\cos\rho)^{-3}\sin^2\rho\sin\theta\, d\rho\, d\theta\, d\varphi. \tag{4.3}$$

Because the trace anomaly gives an infinite contribution to the energy integral, one must compare the energy at two different temperatures, by calculating $E(\beta)-E(\infty)$. This is the purely thermal part of the energy density, in which the vacuum energy (zero-point energy) has been left out.

To calculate the energy one substitutes the stress–energy tensor (3.10) into the integral (4.1). Because of the spherical symmetry, the integral over the angular variables θ, φ gives 4π. The remaining integrals over ρ can all be reduced (via integration by parts) to the basic integral

$$\int_0^{\pi/2}\frac{\sin\rho\sin 2n\rho}{\cos\rho}\, d\rho=-\tfrac{1}{2}\pi(-1)^n,\quad n\geqslant 1. \tag{4.4}$$

One thus obtains the simply result

$$E(\beta)-E(\infty)=a\sum_{n=1}^{\infty}\tfrac{1}{2}n^2(n\mp 1)[\exp(a\beta n)-1]^{-1}. \tag{4.5}$$

We will now show that this result can also be obtained in a much simpler way, with a partition function.

The energy levels for the two reflecting boundary conditions are given in ref. [2]. They are

$$E_{nml}=a(2n+l+2)\quad \text{Dirichlet,} \tag{4.6a}$$

$$E_{nml}=a(2n+l+1)\quad \text{Neumann.} \tag{4.6b}$$

Here the quantum numbers n, m, l take the values

$$n=0,1,2,\ldots,\quad l=0,1,2,\ldots,\quad m=-l,\ldots,+l. \tag{4.7}$$

Using the method of Allen and Davis [11] one can easily show that the partition function $Z[\beta]$ is

$$\ln Z[\beta]=-\sum_{n,m,l}\ln[1-\exp(-\beta E_{nml})]=-\sum_{b=1}^{\infty}\tfrac{1}{2}k(k\mp 1)\ln[1-\exp(-a\beta k)]. \tag{4.8}$$

Thus the total energy $E=-(\partial/\partial\beta)\ln Z$ is exactly the same as in the previous calculation (4.5). (Note that the zero-point energy is automatically excluded from the definition of the partition function (4.8). This is explained in ref. [12].) The properties of the partition function are discussed in more detail in ref. [13].

5. *Conclusion.* In this paper, the double periodicity of the propagator at finite temperature in ADS was exploited to find an exact formula for it in terms of elliptic functions. This permitted an exact evaluation of the finite-temperature stress–energy tensor T_{ab}. At high temperature, the stress tensor is the sum of a trace anomaly and a term proportional to the fourth power of the local temperature. Our work gives a more refined description of a heat bath in ADS than that originally given in ref. [4].

One might wonder if the Page approximation [14,15] to T_{ab} is exact as is the case in de Sitter space. Unfortunately it is not. Page's formula reproduces the correct result a zero temperature only if T is given the imaginary value $T=(2\pi)^{-1}(\frac{1}{3}\Lambda)^{1/2}$. At finite temperature the Page formula gives a non-real result. However, Page's method can probably be modified as discussed in ref. [15], by adding the contributions from multiple spatial geodesics. This would probably give the high-temperature limit of the "direct" term of our calculations, i.e., the average of the Neumann/Dirichlet results. However, this approximation could not be expected to reproduce the boundary-condition dependent terms, because it relies on the Schwinger–DeWitt expansion which is purely local.

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