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Rough Solutions of Einstein Vacuum Equations in CMCSH Gauge

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Abstract: In this paper, we consider very rough solutions to the Cauchy problem for the Einstein vacuum equations in CMC spatial harmonic gauge, and obtain the local well-posedness result in H^s, $s > 2$. The novelty of our approach lies in that, without resorting to the standard paradifferential regularization over the rough, Einstein metric $\mathbf{g}$, we manage to implement the commuting vector field approach to prove Strichartz estimate for geometric wave equation $\Box_{\mathbf{g}}\phi = 0$ directly.

1. Introduction

In mathematical relativity, a fundamental question is to find a four dimensional Lorentz metric $\mathbf{g}$ that satisfies the vacuum Einstein equations

$$\mathbf{Ric}(\mathbf{g}) = 0. \tag{1.1}$$

Since the equation is diffeomorphic invariant, certain gauges should be fixed before solving it. There exist extensive works on (1.1) under the wave coordinates gauge or the constant mean curvature gauge.

In [2] Andersson and Moncrief consider the vacuum Einstein equation (1.1) under the so-called constant mean curvature and spatial harmonic coordinate (CMCSH) gauge condition. To set up the framework, let Σ be a 3-dimensional compact, connected and orientable smooth manifold, and let $\mathcal{M} := \mathbb{R} \times \Sigma$. Let $t : \mathcal{M} \to \mathbb{R}$ be the projection on the first component and let $\Sigma_t := \{t\} \times \Sigma$ be the level sets of t. One may construct solutions of (1.1) by considering Lorentz metrics $\mathbf{g}$ of the form

$$\mathbf{g} = -n^2 dt \otimes dt + g_{ij}(dx^i + Y^i dt) \otimes (dx^j + Y^j dt)$$

with suitable determination of the scalar function n, the vector field $Y := Y^j \partial_j$ and the Riemannian metric $g := g_{ij} dx^i \otimes dx^j$ on Σ. In order for ∂_t to be time-like, it is necessary to have $n^2 - g_{ij} Y^i Y^j > 0$. Let $\mathbf{T}$ be the time-like unit normal to Σ_t, then

$$\partial_t = n\mathbf{T} + Y.$$

We call n the lapse function and Y the shift vector field.

Let $\hat{g}$ be a fixed smooth Riemannian metric on Σ with Levi-Civita connection $\widehat{\nabla}$ and Christoffel symbol $\hat{\Gamma}_{ij}^k$. Let Γ_{ij}^k denote the Christoffel symbol with respect to g. We may introduce the vector field $U = U^l \partial_l$ with

$$U^l := g^{ij}(\Gamma_{ij}^l - \hat{\Gamma}_{ij}^l).$$

Let k be the second fundamental form of Σ_t in $\mathcal{M}$, i.e. $k_{ij} = -\frac{1}{2}\mathcal{L}_{\mathbf{T}}\mathbf{g}_{ij}$. The solution of (1.1) constructed in [2] is to find the pair (g, k) such that they satisfy the CMCSH condition

$$\mathrm{Tr}k := g^{ij}k_{ij} = t \quad \text{and} \quad U^j = 0 \tag{1.2}$$

and the vacuum Einstein evolution equations

$$\partial_t g_{ij} = -2nk_{ij} + \mathcal{L}_Y g_{ij} \tag{1.3}$$
$$\partial_t k_{ij} = -\nabla_i\nabla_j n + n(R_{ij} + \mathrm{Tr}k k_{ij} - 2k_{im}k_j^m) + \mathcal{L}_Y k_{ij} \tag{1.4}$$

with the constraint equations

$$R - |k|^2 + (\mathrm{Tr}k)^2 = 0 \quad \text{and} \quad \nabla_i \mathrm{Tr}k - \nabla^j k_{ij} = 0. \tag{1.5}$$

It has been shown in [2] that for initial data $(g^0, k^0) \in H^s \times H^{s-1}$ with $s > 5/2$ satisfying the constraint equation (1.5) with $t_0 := \mathrm{Tr}k^0 < 0$, the Cauchy problem for the system (1.2)–(1.5) is locally well-posed. In particular, there is a time $T_* > 0$ depending on $\|g^0\|_{H^s}$ and $\|k^0\|_{H^{s-1}}$ such that the Cauchy problem has a unique solution defined on $[t_0 - T_*, t_0 + T_*] \times \Sigma$. We should mention that, for the solution constructed in this way, the lapse function n and the shift vector field Y satisfy the elliptic equations

$$-\Delta n + |k|^2 n = 1 \tag{1.6}$$

and

$$\Delta Y^i + R_j^i Y^j = \left(-2nk^{jl} + 2\nabla^j Y^l\right) U_{jl}^i + 2\nabla^j n k_j^i - \nabla^i n k_j^j, \tag{1.7}$$

where U_{jl}^i is the tensor defined by

$$U_{jl}^i := \Gamma_{jl}^i - \hat{\Gamma}_{jl}^i. \tag{1.8}$$

It is natural to ask under what minimal regularity on the initial data the CMCSH Cauchy problem (1.2)–(1.5) is locally well-posed. In this paper we prove the following result which shows the well-posedness[1] of the problem when the initial data is in $H^s \times H^{s-1}$ with $s > 2$.

Theorem 1 (Main Theorem). *For any $s > 2$, $t_0 < 0$ and $M_0 > 0$, there exist positive constants T_*, M_1 and M_2 such that the following properties hold true:*

[1] The result of Theorem 1 requires that the initial data can be approximated by a smooth sequence of data satisfying the constraint equation. The issue was settled in [6,7] by using a conformal method.

(i) For any initial data set (g^0, k^0) *satisfying* (1.5) *with* $t_0 := Trk^0 < 0$ *and* $\|g^0\|_{H^s(\Sigma_{t_0})} + \|k^0\|_{H^{s-1}(\Sigma_{t_0})} \le M_0$*, there exists a unique solution* $(g, k) \in C(I_*, H^s \times H^{s-1}) \times C^1(I_*, H^{s-1} \times H^{s-2})$ *to the problem* (1.2)–(1.5)*;*

(ii) There holds

$$\|\widehat{\nabla} g, k\|_{L^2_{I_*} L^\infty_x} + \|\widehat{\nabla} g, k\|_{L^\infty_{I_*} H^{s-1}} \le M_1;$$

(iii) For $2 < r \le s$*, and for each* $\tau \in I_*$ *the linear equation*

$$\begin{cases} \Box_{\mathbf{g}} \psi = 0, & (t, x) \in I_* \times \Sigma \\ \psi(\tau, \cdot) = \psi_0 \in H^r(\Sigma), & \partial_t \psi(\tau, \cdot) = \psi_1 \in H^{r-1}(\Sigma) \end{cases}$$

admits a unique solution $\psi \in C(I_*, H^r) \times C^1(I_*, H^{r-1})$ *satisfying the estimates*

$$\|\psi\|_{L^\infty_t H^r} + \|\partial_t \psi\|_{L^\infty_t H^{r-1}} \le M_2 \|(\psi_0, \psi_1)\|_{H^r \times H^{r-1}}$$

and

$$\|\boldsymbol{D}\psi\|_{L^2_t L^\infty_x} \le M_2 \|(\psi_0, \psi_1)\|_{H^r \times H^{r-1}};$$

where $I_* := [t_0 - T_*, t_0 + T_*]$.

We actually obtain a stronger result than Theorem 1, which is contained in Theorem 2.

1.1. Review and motivation. Since the pioneer work of Choquet-Bruhat [5], there has been extensive work on the well-posedness of quasilinear wave equation

$$\begin{cases} \Box_{g(\phi)} \phi := \partial_t^2 \phi - g^{ij}(\phi) \partial_i \partial_j \phi = N(\phi, \partial\phi), \\ \phi|_{t=0} = \phi_0, \quad \partial_t \phi|_{t=0} = \phi_1 \end{cases} \tag{1.9}$$

in $\mathbb{R}^{n+1}$, where the symmetric matrix $g^{ij}(\phi)$ is positive definite and smooth as a function of ϕ, and the function $N(\phi, \partial\phi)$ is smooth in its arguments and is quadratic in $\partial\phi$. In view of the energy estimate

$$\|\partial\phi(t)\|_{H^{s-1}} \lesssim \|\partial\phi(0)\|_{H^{s-1}} \cdot \exp\left(\int_0^t \|\partial\phi(\tau)\|_{L^\infty_x} d\tau \right), \tag{1.10}$$

the Sobolev embedding and a standard iteration argument, the classical result of Hughes–Kato–Marsden [9] of well-posedness in the Sobolev space H^s follows for any $s > \frac{n}{2} + 1$, where the estimate of $\|\partial\phi\|_{L^\infty_t L^\infty_x}$ is heavily relied on. To improve the classical result, it is crucial to get a good estimate on $\|\partial\phi\|_{L^1_t L^\infty_x}$. This is naturally reduced to deriving the Strichartz estimate for the wave operator $\Box_{g(\phi)}$ which has rough coefficients since $g^{ij}(\phi)$ depend on the solution ϕ and thus at most have as much regularity as ϕ. The first important breakthrough was achieved by Bahouri–Chemin [3,4] and by Tataru [21] using parametrix constructions. They obtained the well-posedness of (1.9) in H^s with $s > \frac{n}{2} + \frac{1}{2} + \frac{1}{4}$ by establishing a Strichartz estimate for solutions to linearized equations of the form

$$\|\partial\phi\|_{L^2_I L^\infty_x} \le c(\|\phi_0\|_{H^{\frac{n}{2}+\frac{1}{2}+\sigma}} + \|\phi_1\|_{H^{\frac{n}{2}-\frac{1}{2}+\sigma}})$$

with a loss of $\sigma > \frac{1}{4}$. This well-posedness result was later improved to $s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6}$ in [23].

The next important progress was made by Klainerman in [10] where a vector field approach was developed to establish the Strichartz estimate. This approach was further developed by Klainerman–Rodnianski in [12] where they successfully improved the local well-posedness of (1.9) in $\mathbb{R}^{3+1}$ to the Sobolev space H^s with $s > 2 + \frac{2-\sqrt{3}}{2}$. Due to the limited regularity of the coefficients, the paradifferential localization procedure in [4, 22,23] was adopted in [12] to consider the Strichartz estimate for solutions of linearized wave equation $\Box_{g_{\le\lambda^a}}\psi = 0$ for some $0 < a \le 1$, where $g_{\le\lambda^a} := S_{\lambda^a}(g(S_{\lambda^a}(\phi)))$ is the truncation of $g(\phi)$ at the frequency level λ^a. Here $S_\lambda := \sum_{\mu\le\lambda} P_\mu$ and P_λ is the Littlewood-Paley projector with frequency $\lambda = 2^k$ defined for any function f by

$$P_\lambda f(x) = f_\lambda(x) = \int e^{ix\cdot\xi}\zeta(\lambda^{-1}\xi)\hat{f}(\xi)d\xi \tag{1.11}$$

with ζ being a smooth function supported in the shell $\{\xi : \frac{1}{2} \le |\xi| \le 2\}$ satisfying $\sum_{k\in\mathbb{Z}}\zeta(2^k\xi) = 1$ for $\xi \ne 0$. We refer to [17,20] for detailed properties of Littlewood-Paley decompositions. With the help of a $\mathcal{T}\mathcal{T}^*$ argument, such Strichartz estimate was reduced to the dispersive estimate for solutions of $\Box_{g_{\le\lambda^a}}\psi = 0$ with frequency localized initial data. It was then further reduced to deriving the boundedness of Morewatz type energy for $\partial\psi$ and its higher derivatives. To derive these energy estimates requires the control of the deformation tensor of Morawetz vector field, which involves the Ricci coefficients relative to the Lorentzian metric $-dt^2 + (g_{\le\lambda^a})_{ij}dx^i \otimes dx^j$. Since **Ric** of the smoothed metric appears crucially in the structure equations for Ricci coefficients, new characteristics techniques were developed to take advantage of the observation that $\mathbf{R}_{44}$, the tangential component of **Ric** along null hypersurfaces, has better structure and the fact that the coefficients g themselves verify equations of the form (1.9). For the Einstein vacuum equation under the wave coordinates gauge, the local well-posedness were obtained in H^s for any $s > 2$ in [13–15]. The core progress which enables the improvement from $s > 2 + \frac{2-\sqrt{3}}{2}$ to $s > 2$ was made in [15] by showing that the Ricci tensor relative to the frequency-truncated metric $\mathbf{h} := \mathbf{g}_{\le\lambda}$ does not deviate from 0 to a harmful level; the decay rate of $\mathbf{Ric}(\mathbf{h})$ and its derivatives were proven to be sufficiently strong in terms of λ. However, similar estimates for $\mathbf{Ric}(\mathbf{h})$ can hardly be obtained for (1.9). The sharp local well-posedness for type (1.9) in H^s with $s > 2$ was achieved by Smith and Tataru in [19] based on a parametrix construction of a solution by using wave packet. The particular structure of $\mathbf{R}_{44}$ observed in [12] also played an important role to control the geometry of null surface. The local well-posedness with $s = 2$ for Einstein vacuum equation was conjectured by Klainerman in [11]. Recently we learned that significant progress has been achieved for this so-called L^2 curvature conjecture [18].

A reduction to consider $\Box_{\mathbf{g}_{\le\lambda^a}}\psi = 0$, with $0 < a \le 1$ appeared in almost all the above mentioned work. This regularization on metric is used to phase-localize the solution, and in most of the works, to balance the differentiability on coefficients required either by parametrix construction or by energy method. Such a regularization on metric, nevertheless, poses major technical baggage, in particular, in carrying out the vector field approach in Einstein vacuum spacetime, since $\mathbf{Ric}(\mathbf{g}_{\le\lambda})$ no longer vanishes. The analysis in [15] on the defected Ricci tensor and its derivatives is a very delicate procedure, which relies crucially on full force of $\partial\mathbf{h}$, hence, on their non-smoothed counter part $\partial\mathbf{g}$ as well. One particular issue tied to CMCSH gauge itself arises due to the lack of control on $\mathbf{D_T}Y$, the time derivative of the shift vector field. By differentiating (1.7), we can obtain an elliptic equation for $\mathbf{D_T}Y$. However, the elliptic equation does not provide a valid control on $\mathbf{D_T}Y$ even in terms of L^2-norm, since the kernel of this elliptic operator is not expected to be trivial. The loss of control over some components of $\partial\mathbf{g}$ becomes a serious hurdle in recovering the decay for $\mathbf{Ric}(\mathbf{h})$ and its derivatives. The potential issue on Ricci defect forces us to abandon the frequency truncation on metric.

The important aspect of our analysis is to implement the vector field approach directly in the non-smoothed Einstein spacetime $(\mathcal{M}, \mathbf{g})$ to establish the Strichartz estimate with an arbitrarily small loss for the linearized problem $\Box_{\mathbf{g}}\psi = 0$. This confirms that, due to the better behavior of $\mathbf{Ric}$, the Einstein metric is in nature "smooth" enough to implement the vector field approach without the truncation on $\mathbf{g}$ in Fourier space, and leads to the H^s well-posedness result with $s > 2$ for Einstein equation.

Compared with the classical approach in [2,9], the risk of carrying out a more direct analysis is expected to arise from lack of $\frac{1}{2}$-derivative. In the heart of the regime of Strichartz estimates contained in [12–15], the main building block is to obtain the dispersive estimate for $P_\lambda \partial_t \psi$ by deriving the bounded Morawetz type energy of derivatives of ψ, with $\Box_{\mathbf{g}_{\le\lambda^a}} \psi = 0$. This procedure relies on H^σ, $\sigma > \frac{1}{2}$ norm of curvature, which is $1/2$ more derivative than the rough Einstein metric could offer. To conquer this difficulty, we firstly manage to derive the dispersive estimate merely by using the Morawetz type energy for ψ itself. The analysis to control such energy is then accomplished based on Proposition 12. The main technical baggage is then reduced to proving (5.179) in Proposition 12, a Strichartz type control over the Ricci coefficients $\hat{\chi}, \zeta$ relative to Einstein metric.

The control of Ricci coefficients consistent with H^2 Einstein metrics has been studied in [16,24,26], where a set of estimates concerning $\mathrm{tr}\chi$, $\hat{\chi}$, ζ, $\underline{\zeta}$ was achieved in terms of curvature flux, combined with flux of k if null hypersurface is foliated by level sets of t. Bearing the flavor of these works, in the situation when $H^{2+\epsilon}$ estimates for $\mathbf{g}$ can be established, we manage to obtain stronger set of estimates on Ricci coefficients in terms of the L^{2+} type flux. This enables us to carry out delicate analysis such as Calderon-Zygmund inequality on null hypersurfaces under rough metric. In this procedure, thanks to working directly in vacuum spacetime, we no longer encounter the technical difficulty in [13–15] posed by the defected $\mathbf{Ric}(\mathbf{h})$. Nevertheless, this set of estimates is much weaker than (5.179). The crucial estimates for $\hat{\chi}$ and ζ will be based on the Hodge systems of $\hat{\chi}$ and ζ, Strichartz estimates on k, $\widehat{\nabla} g$ via a bootstrap argument and Calderon-Zygmund theory. The standard L_x^∞ Calderson-Zygmund inequality ([14, Propositoin 6.20]) would involve the bound of H^σ, $\sigma > 1/2$ for $\widehat{\nabla} k$ and $\widehat{\nabla}^2 g$. One advantage offered by the smoothed metric $\mathbf{g}_{\le\lambda}$ lies in that such a loss of derivative is quantitized to be a log-loss in terms of frequency. Instead of smoothing, we solve this problem by modifying the Calderon-Zygmund inequality and squeezing out an extra bit of differentiability for $\widehat{\nabla} g$, k through Strichartz estimates.

The difficulty coming from $\mathbf{D_T}Y$ still penetrates in key steps in our vector fields approach, where all components of $\partial\mathbf{g}$ were typically involved. We exclude such term by introducing a modified energy current, and by refining $\mathcal{T}\mathcal{T}^*$ argument and curvature decomposition ([27]) into more invariant fashion.

We will divide our work into two parts. In this paper, we establish the Strichartz estimates and close the proof of the main theorem by assuming the estimates of Ricci coefficients contained in Proposition 12. In [27], we will prove Proposition 12.

We emphasize that our approach can be directly applied for reproducing $H^{2+\epsilon}$ result for Einstein equations in wave coordinates gauge. It actually works better under wave coordinates since $\mathbf{D_T}Y$ can be well controlled in this situation. Steps which are involved with getting around this term in CMCSH gauge take simpler and more straightforward form in wave coordinate gauge. The delicate procedures of deforming the actual space-time and controlling the defected Ricci no longer appear in our approach. Our approach gives a vast simplification over the methodology in [13–15].

1.2. Outline of the proof. According to [14,19], in order to complete the proof of Theorem 1 it suffices to show that for any $s > 2$ there exist two positive constants C and T depending on $\|g\|_{H^s(\Sigma_0)}$ and $\|k\|_{H^{s-1}(\Sigma_0)}$ such that

$$\|g\|_{L_t^\infty H^s(I\times\Sigma)} + \|k\|_{L_t^\infty H^{s-1}(I\times\Sigma)} \le C, \tag{1.12}$$

where $I := [t_0 - T, t_0 + T]$. We achieve this by a bootstrap argument. That is, we first make the bootstrap assumption

$$\int_{t_0-T}^{t_0+T} \|\widehat{\nabla} g,\ k,\ \widehat{\nabla} Y,\ \widehat{\nabla} n\|_{L^\infty(\Sigma_t)} dt \le B_1, \tag{BA1}$$

where, for any Σ-tangent tensor F, we will use $\|F\|_{L^\infty(\Sigma_t)}$ to denote its L^∞-norm with respect to the Riemannian metric g on Σ_t. We then show that (BA1) and some auxiliary bootstrap assumptions imply (1.12). We prove these bootstrap assumptions can be improved for small but universal $T > 0$.

We will only work on the time interval $[t_0, t_0 + T]$ since the same procedure applies to the time interval $[t_0 - T, t_0]$ by simply reversing the time. In view of (BA1) and elliptic estimates, we derive in Sect. 2 better estimates for $\widehat{\nabla} Y$ and $\widehat{\nabla} n$. That is, we show that, for any $1 < b < 2$, there holds

$$\|\widehat{\nabla} Y, \widehat{\nabla} n\|_{L^b_{[t_0,t_0+T]} L_x^\infty} \le C$$

which improves the estimates for $\widehat{\nabla} n$ and $\widehat{\nabla} Y$ in (BA1) with T sufficiently small. In order to improve the estimates for $\widehat{\nabla} g$ and k, we establish the core estimates in Theorem 1(ii) by showing that

$$\|\widehat{\nabla} g,\ k\|_{L^2_{[t_0,t_0+T]} L_x^\infty} \le C T^\delta,$$

for some $\delta > 0$. Here we briefly describe the ideas behind the proof.

1.2.1. Step 1. Energy estimates and flux. In Sects. 2 and 3, we derive (1.12) under bootstrap assumptions. We also derive for the scalar solution of homogeneous geometric wave equation $\Box_{\mathbf{g}}\phi = 0$, the energy estimate

$$\|\partial\phi\|_{H^{s-1}} \lesssim \|\phi(0)\|_{H^s} + \|\partial\phi(0)\|_{H^{s-1}}.$$

To obtain (1.12), the typical energy argument is based on considering $\Box_{\mathbf{g}} g$ with bootstrap assumptions on $\|\partial\mathbf{g}\|_{L_t^1 L_x^\infty}$. The second order equations $\Box_{\mathbf{g}} g = \cdots$ contain terms of $\mathbf{D_T}Y$. In view of (1.7), $\mathbf{D_T}Y$ satisfies the elliptic equation

$$\Delta\nabla_{n\mathbf{T}} Y^i + R^i_j \nabla_{n\mathbf{T}} Y^j - 2U_{mp}{}^i \nabla^m \nabla_{n\mathbf{T}} Y^p = -n(\mathrm{curl}\, H)^i_j Y^j + \mathbf{g}\cdot\nabla\tilde{\pi}\cdot\tilde{\pi},$$

where $\tilde{\pi}$ denotes components of $\partial\mathbf{g}$ excluding $\partial_t Y$. Due to the appearance of the term $R^i_j \nabla_{n\mathbf{T}} Y^j$, in general, one cannot show the kernel of the elliptic operator is trivial since the lower bound of R_i^j is not expected to be controlled. This equation is not good enough to provide valid control for $\mathbf{D_T}Y$. The same issue occurs when one considers the elliptic equation $\widehat{\Delta}\mathbf{D_T}Y = \cdots$. In order to avoid the difficulty coming from $\mathbf{D_T}Y$, we manage to employ equations not containing this term at all. In Sects. 2.1 and 3 we derive the energy estimate by considering the first order hyperbolic system,

$$\begin{cases} \partial_t u - \widehat{\nabla}_Y u = nv + F_u \\ \partial_t v - \widehat{\nabla}_Y v = n\hat{\Delta} u + F_v \end{cases} \tag{1.13}$$

for the pairs $(u, v) = (g, -2k)$, (k, E) and $(\phi, e_0\phi)$ with corresponding remainder terms (F_u, F_v), where, for any Σ tangent tensor F,

$$\hat{\Delta} F := g^{ij}\widehat{\nabla}_i\widehat{\nabla}_j F. \tag{1.14}$$

Consistent with these energy estimates, we also obtain the $L_t^\infty H^{s-\frac{1}{2}}$ and $L_t^1 H^s$ estimates of $\widehat{\nabla} n$, $\widehat{\nabla} Y$, $\mathbf{D_T} n$ with the help of elliptic Eqs. (1.6) and (1.7).

However, to derive the flux estimate for $\widehat{\nabla} g$, k, $P_\mu k$, and $P_\mu\widehat{\nabla} g$, we still have to rely on the second order hyperbolic system of $\psi = k$ or $\widehat{\nabla} g$, both of which contain the time derivative of the shift vector field. With a careful manipulation of terms, we observe that the sum of $\Box_{\mathbf{g}}\psi$ with the remainder term $\mathbf{D_T}(n^{-1}F_\psi)$ which contains $\mathbf{D_T}Y$ no longer involves this bad term. In Sect. 3.2, we introduce a modified current $\tilde{P}_\alpha$ to cope with the sum, instead of merely $\Box_{\mathbf{g}}\psi$ via the standard current $P_\alpha = Q_{\alpha\beta}\mathbf{T}^\beta$. This successfully yields the control on flux by divergence theorem. Note that terms of $P_\alpha\mathbf{T}^\alpha$ are almost the same with those in $\tilde{P}_\alpha\mathbf{T}^\alpha$, except that the term $(\mathbf{D_T}\psi)^2$ becomes $(\mathbf{D_T}\psi - n^{-1}F_\psi)^2$ in the latter. $\tilde{P}_\alpha\mathbf{T}^\alpha$ does not give the full energy density, which is not good enough for the purpose of controlling energy. Due to such limitation, the modified current $\tilde{P}_\alpha$ is developed to control flux after the full set of energy estimates on ψ, i.e., (1.12), is established by using the first order system.

As the major technicality to carry out energy estimates in fractional Sobolev space and estimates of dyadic flux, a series of more delicate commutator estimates are established in Appendix (Sect. 6) on the Littlewood Paley projection and the rough metric, particularly to handle the decreased differentiability of coefficients.

1.2.2. Step 2. Reduction to dyadic Strichartz estimates on frequency dependent time intervals. By using the Littlewood-Paley decomposition, it is easy to reduce the proof of Theorem 1 to establishing for sufficiently large λ the estimates

$$\|P_\lambda\widehat{\nabla} g, P_\lambda k\|_{L_I^2 L_x^\infty} \lesssim \lambda^{-\delta}|I|^{\frac{1}{2}-\frac{1}{q}}\|\widehat{\nabla} g, k\|_{H^{s-1}(\Sigma_0)} \tag{1.15}$$

and

$$\|P_\lambda\partial\phi\|_{L_I^2 L_x^\infty} \lesssim \lambda^{-\delta}|I|^{\frac{1}{2}-\frac{1}{q}}\|\widehat{\nabla}\phi, e_0\phi\|_{H^{s-1}(\Sigma_0)} \tag{1.16}$$

for any solution ϕ of the equation $\Box_{\mathbf{g}}\phi = 0$, where $I = [t_0, t_0 + T]$, $q > 2$ is sufficiently close to 2, and $\delta > 0$ is sufficiently close to 0.

We reduce the proof of (1.15) and (1.16) to Strichartz estimates on small time intervals. We pick a sufficiently small $\epsilon_0 > 0$ and partition $[t_0, t_0 + T]$ into disjoint union of subintervals $I_k := [t_{k-1}, t_k]$ of total number $\lesssim \lambda^{8\epsilon_0}$ with the properties that

$$|I_k| \lesssim \lambda^{-8\epsilon_0}T \quad \text{and} \quad \|k, \widehat{\nabla} g, \widehat{\nabla} Y, \widehat{\nabla} n\|_{L_{I_k}^2 L_x^\infty} \le \lambda^{-4\epsilon_0}. \tag{1.17}$$

To explain our approach, we take the derivation of (1.16) as an example. We consider on each I_k the Strichartz norm for $P_\lambda \partial\phi$. By commuting P_λ with $\Box_{\mathbf{g}}$ we have $\Box_{\mathbf{g}} P_\lambda \phi = F_\lambda$, where $F_\lambda = [\Box_{\mathbf{g}}, P_\lambda]\phi$ can be treated as phase-localized at level of λ in certain sense although it is not frequency-localized. We use $W(t, s)$ to denote the operator that sends (f_0, f_1) to the solution of $\Box_{\mathbf{g}}\psi = 0$ satisfying the initial conditions $\psi(s) = f_0$ and $\partial_t\psi(s) = f_1$ at the time s. Using Duhamel principle followed by differentiation, we can represent $P_\lambda\partial\phi$ as

$$P_\lambda \partial\phi(t) = \partial W(t, t_{k-1}) P_\lambda \phi[t_{k-1}] + \int_s^t \partial W(t, s)(0, F_\lambda(s))ds, \tag{1.18}$$

where we used the convention $\phi[t] := (\phi(t), \partial_t\phi(t))$. Running a $\mathcal{T}\mathcal{T}^*$ argument leads to Strichartz estimate for one dyadic piece of $\partial\psi$,

$$\|P_\lambda \partial\psi\|_{L^q_{I_k} L^\infty_x} \lesssim \lambda^{\frac{3}{2}-\frac{1}{q}} \|\psi[0]\|_{H^1}, \tag{1.19}$$

where $q > 2$ is sufficiently close to 2.

A similar procedure was used in [13] for $\Box_{\mathbf{g}_{\le\lambda}}\phi = 0$. Observe that the solution of this homogeneous wave equation is frequency-localized at the level of λ if the data is localized in Fourier space at the dyadic shell $\{\xi : \frac{\lambda}{2} \le |\xi| \le 2\lambda\}$. Therefore, the dyadic Strichartz estimates (1.19) can be applied directly to the representation of $P_\lambda\partial\phi$. Since we will work for the metric $\mathbf{g}$ without frequency truncation, the corresponding operator $W(t, s)$ does not preserve the frequency-localized feature of data. The Strichartz estimate for $\partial W(t, t_{k-1}) P_\lambda\phi[t_{k-1}]$ is no longer expected to be obtained directly from (1.19). We solve this problem in Sect. 4 by modifying (1.18) with the help of the reproducing property of the Littlewood-Paley projections, i.e., $P_\lambda = \tilde{P}_\lambda \tilde{P}_\lambda$, as follows,

$$P_\lambda \partial\phi(t) = \tilde{P}_\lambda \partial W(t, t_{k-1}) \tilde{P}_\lambda \phi[t_{k-1}] + \int_s^t \tilde{P}_\lambda \partial W(t, s)(0, F_\lambda(s))ds. \tag{1.20}$$

This makes it possible to apply (1.19). The effort then goes into piecing together the result of dyadic Strichartz estimates over intervals I_k with the help of (1.17). This trick would have successfully reduced the main estimates to dyadic strichartz estimate for the solution of $\Box_{\mathbf{g}}\phi = 0$ on one sub-interval I_k, had the term of $\mathbf{D_T} Y$ not appeared in F_λ. We then refine (1.20) further by modifying the application of Duhamel principle.

1.2.3. Step 3. Reduction to dispersive estimates and boundedness theorem. By rescaling coordinates as $(t, x) \to ((t - t_{k-1})/\lambda, x/\lambda)$, we need only to consider (1.19) on $[0, t_*] \times \Sigma$ with $t_* \le \lambda^{1-8\epsilon_0} T$. In view of a $\mathcal{T}\mathcal{T}^*$ argument, this essentially relies on the dispersive estimate

$$\|P \mathbf{D_T} W(t, s) I[s]\|_{L^\infty_x} \lesssim \left((1 + |t - s|)^{-\frac{2}{q}} + d(t)\right) \sum_{k=0}^{m} \|\widehat{\nabla}^k I[s]\|_{L^1_x} \tag{1.21}$$

with initial data $I[s] = (\psi(s), \mathbf{D_T}\psi(s))$ for all $0 < s \le t_*$, where m is a positive integer, $d(t)$ is a function satisfying $\|d\|_{L^{\frac{q}{2}}} \lesssim 1$ for $q > 2$ sufficiently close to 2, and P denotes the Littlewood Paley projection $P_{\lambda=1}$.

Let $\{\chi_J\}$ be a suitable partition of unity on Σ supported on balls of radius 1 in rescaled coordinates. We localize the solution of $\Box_{\mathbf{g}}\psi = 0$ by writing $\psi(t, x) = \sum_J \psi_J(t, x)$, where $\psi_J(t,x)$ is the solution of $\Box_{\mathbf{g}}\psi_J=0$ with the initial data $\psi_J[\tau_0]=\chi_J\cdot\psi[\tau_0]$. We then reduce the derivation of (1.21) to proving that

$$\|P\mathbf{D_T}\phi(t)\|_{L_x^\infty}\le\left(\frac{1}{(1+|t-\tau_0|)^{\frac{2}{q}}}+d(t)\right)\sum_{k=0}^{m-2}\|\widehat{\nabla}^k\phi[\tau_0]\|_{L^2},\tag{1.22}$$

with ϕ the solution of $\Box_{\mathbf{g}}\phi=0$ and with data supported within a unit ball at Σ_{τ_0}. It then suffices to consider (1.22) on $\mathcal{J}_0^+$, the causal future of the support of χ_J from $t=\tau_0\approx 1$, where one can introduce optical function u whose level sets are null cones C_u. Thus $\mathcal{J}_0^+$ can be foliated by $S_{t,u}:=C_u\cap\Sigma_t$ and a null frame $\{L,\underline{L},e_1,e_2\}$ can be naturally defined, where e_A, $A=1,2$, are tangent to $S_{t,u}$. Using these vector fields and $\underline{u}=2t-u$, one can introduce the Morawetz vector field $K=\frac{1}{2}n(u^2\underline{L}+\underline{u}^2L)$. Consequently, for any function f, one can introduce the generalized energy

$$\tilde{Q}[f](t):=\int_\Sigma\bar{Q}(K,\mathbf{T})[f],$$

where $\bar{Q}(K,\mathbf{T})[f]$ is defined by applying $X=K$, $Y=\mathbf{T}$, $\Omega=4t$ to

$$\bar{Q}(X,Y)[f]=Q(X,Y)[f]+\frac{1}{2}\Omega fY(f)-\frac{1}{4}f^2Y(\Omega)\tag{1.23}$$

with $Q_{\mu\nu}$ being the standard energy momentum tensor

$$Q_{\mu\nu}:=Q[f]_{\mu\nu}=\partial_\mu f\partial_\nu f-\frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{\alpha\beta}\partial_\alpha f\partial_\beta f).$$

The typical energy method gives

$$\tilde{Q}[f](t)-\tilde{Q}[f](\tau_0)=-\frac{1}{2}\int_{\mathcal{J}_0^+}{}^{(K)}\bar{\pi}_{\alpha\beta}Q[f]_{\alpha\beta}+\int_{\Sigma\times I}\Box_{\mathbf{g}}f\cdot Kf+l.o.t,\tag{1.24}$$

where, for any vector field X, the deformation tensor ${}^{(X)}\pi_{\alpha\beta}:=\mathcal{L}_X\mathbf{g}_{\alpha\beta}$ and ${}^{(K)}\bar{\pi}_{\alpha\beta}:={}^{(K)}\pi_{\alpha\beta}-4t\mathbf{g}_{\alpha\beta}$. By applying (1.24) to $f=\mathbf{D_T}\phi$, we consider bounding generalized energy $\tilde{Q}[\mathbf{D_T}\phi]$ in terms of their initial values at $t=\tau_0\approx 1$. Due to one "bad" term contained in $\Box_{\mathbf{g}}\mathbf{D_T}\phi=[\Box_{\mathbf{g}},\mathbf{D_T}]\phi$, the estimate of $\tilde{Q}[\mathbf{D_T}\phi]$ has to be coupled with $\tilde{Q}[\mathbf{D}_Z\phi]$ with Z either $\underline{L}$ or e_A, for which we need to control $\|\mathbf{D}^{(Z)}\pi\|_{L_t^1L_x^\infty}$ and $\int_0^{t_*}\sup_u\|\mathbf{D}^{(Z)}\pi\|_{L^2(S_{t,u})}dt$. Since $\mathbf{D}^{(Z)}\pi$ contains curvature terms, such estimates relative to non-smoothed metric can only be obtained under the assumption of $H^{\frac{5}{2}+\epsilon}$ on data. A similar regularity issue occurs for the estimates required for $\mathbf{D}^{(\mathbf{T})}\pi$ due to the integration by part argument employed to handle the aforementioned bad term. Therefore, we no longer expect to obtain the boundedness of the conformal energy for any derivative of ϕ, including the one for $\mathbf{D_T}\phi$.

Our strategy is to control $\|P\mathbf{D_T}\phi(t)\|_{L_x^\infty}$ merely in terms of $\tilde{Q}[\phi](t)$, with certain loss of decay rate and with error incorporated into $d(t)$ in (1.22). With ϖ a cut-off function whose support is essentially in a so-called exterior region, our treatment concerning the harder part, $P(\varpi\mathbf{D_T}\phi)$, starts with writing it as $P(\varpi\mathbf{D_T}\phi)=P(\varpi L\phi)-P(\varpi N\phi)$ with N the unit outward normal vector fields on $S_{t,u}\subset\Sigma$. The first term is controlled by the Bernstein inequality and $\tilde{Q}[\phi](t)$. The second term is treated in view of

$$P(\varpi N\phi)=\varpi N^l\partial_lP\phi+[P,\varpi N^l]\partial_l\phi.\tag{1.25}$$

The first term of (1.25) is then related to $\tilde{Q}[\phi]$ with the help of Sobolev embedding and commutator estimates. By using the machinery developed in Sect. 6, the treatment on the commutators involved in both terms in (1.25) is reduced to estimating $\|\partial(\varpi N)\|_{L_x^\infty}$. Note that ∂N can be expressed as $\mathbf{g}\cdot(\chi, \zeta, \widehat{\nabla} g, k)$, thus we need to establish estimates on $L_t^{\frac{q}{2}} L_x^\infty$, $q > 2$ of Ricci coefficients $\hat{\chi}, \zeta$ and L^∞ estimate on $\mathrm{tr}\chi$. The components of ${}^{(K)}\bar{\pi}_{\alpha\beta}$ in (1.24) involve ${}^{(\mathbf{T})}\pi$, χ, ζ and other Ricci coefficients as well. By assuming suitable control on Ricci coefficients, the proof of boundedness theorem is given in Sect. 5. We accomplish this step by showing that (1.22) holds true with $m = 3$.

2. H^2 Estimates

We first derive some preliminary consequences of (BA1) that will be used throughout this paper.

Let X be an arbitrary vector field on Σ. We use $|X|_g$ and $|X|_{\hat{g}}$ to denote the lengths of X measured by g and $\hat{g}$ respectively. It then follows from (1.3) that

$$\partial_t(|X|_g^2) = Y^m \widehat{\nabla}_m g_{ij} X^i X^j - 2nk_{ij} X^i X^j + (g_{im} \widehat{\nabla}_j Y^m + g_{mj} \widehat{\nabla}_i Y^m) X^i X^j.$$

Therefore

$$\left|\partial_t |X|_g^2\right| \le \left(2|\widehat{\nabla} Y| + |Y|_g |\widehat{\nabla} g| + 2n|k|\right) |X|_g^2.$$

In view of (1.6) and the maximum principle, we can derive that $0 < n \le C$, where C is a constant depending only on t_0; see [25, Section 2]. Recall that $|Y|_g \le n$. We thus have

$$\left|\partial_t |X|_g^2\right| \le C\left(|\widehat{\nabla} Y| + |\widehat{\nabla} g| + |k|\right) |X|_g^2.$$

This together with the bootstrap assumption (BA1) implies $C^{-1}|X|_{g(t_0)} \le |X|_{g(t)} \le C|X|_{g(t_0)}$. Since $g(t_0)$ and $\hat{g}$ are always equivalent on compact Σ, we therefore have

$$C^{-1}\hat{g} \le g \le C\hat{g}, \quad \text{on } [t_0, t_0 + T] \times \Sigma \tag{2.26}$$

for some universal constant $C > 0$.[2] This equivalence between g and $\hat{g}$ on each Σ_t gives us the freedom to use g or $\hat{g}$ to measure the length of any Σ-tangent tensor.

Using (2.26) and (BA1), we can follow the arguments in [25, Sections 2 and 3] to derive that

$$C^{-1} < n < C, \quad Q(t) \le C, \quad \|H, E, \mathrm{Ric}\|_{L_x^2} \le C, \quad \|\pi, \mathbf{D}_{n\mathbf{T}} n\|_{H^1} \le C \tag{2.27}$$

$$\|\nabla^3 n\|_{L_x^2} + \|\nabla^2 \mathbf{D}_{n\mathbf{T}} n\|_{L_x^2} \lesssim \|k\|_{L_x^\infty}, \tag{2.28}$$

where π is the deformation tensor of $\mathbf{T}$ with components k and $\nabla \log n$, E and H are the electric and magnetic parts of spacetime curvature defined by $E_{ij} = \mathbf{R}_{0i0j}$ and $H_{ij} = {}^\star\mathbf{R}_{0i0j}$ respectively, and $Q(t)$ is the Bel-Robinson energy defined by

$$Q(t) = \int_\Sigma \left(|E|_g^2 + |H|_g^2\right) d\mu_g.$$

[2] We will always use C to denote a universal constant that depends only on the constant in the bootstrap assumptions, information on $g(t_0)$, $|\Sigma_{t_0}|$ and $\|(g, k)\|_{H^s \times H^{s-1}(\Sigma_{t_0})}$. For two quantities Φ and Ψ we will use $\Phi \lesssim \Psi$ to mean that $\Phi \le C\Psi$ for some universal constant C.

As a consequence of (2.28), we have

$$\|\nabla n, \mathbf{D}_{n\mathbf{T}} n\|_{L_x^\infty} \lesssim 1 + \|k\|_{L_x^\infty}^{\frac{3}{2}-\frac{3}{p}}, \quad 3 < p \le 6. \tag{2.29}$$

Let us fix the convention that $F * G$ denotes contraction by g and $\cdot$ denotes either usual multiplication or contraction by $\hat{g}$.

Lemma 1. *Under the spatial harmonic gauge, the shift vector field Y satisfies the equation*

$$\hat{\Delta} Y = \pi * U + \pi * \pi + g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g \cdot Y + g^3 \cdot \hat{R} \cdot Y \tag{2.30}$$

where $\hat{\Delta}$ is defined in (1.14), U is defined in (1.8), and $\hat{R}$ is the Riemannian curvature with respect to $\hat{g}$.

Proof. Straightforward calculation shows for any vector field Y and tensor F that

$$\nabla_j Y^i = \widehat{\nabla}_j Y^i + U^i_{jq} Y^q, \quad \nabla_j F^i_m = \widehat{\nabla}_j F^i_m + U^i_{jp} F^p_m - U^p_{jm} F^i_p.$$

In view of the spatial harmonic gauge condition $U^i := g^{jl} U^i_{jl} = 0$, we obtain

$$g^{mj} \nabla_m \nabla_j Y^i = g^{mj} \widehat{\nabla}_m \widehat{\nabla}_j Y^i + g^{mj} \widehat{\nabla}_m U^i_{jq} Y^q + 2 g^{mj} U^i_{jq} \widehat{\nabla}_m Y^q + g^{mj} U^i_{mp} U^p_{jq} Y^q.$$

Recall the identity

$$R_{jl} = \hat{R}_{jl} + \widehat{\nabla}_i U^i_{jl} - \widehat{\nabla}_j U^i_{il} + U^p_{jl} U^i_{pi} - U^p_{il} U^i_{pj}, \tag{2.31}$$

which can be checked directly. We can obtain that

$$\begin{aligned} \Delta Y^i + R^i_p Y^p = \hat{\Delta} Y^i &+ \Omega^i_p Y^p + 2 g^{mj} U^i_{jq} \widehat{\nabla}_m Y^q \\ &+ g^{mj} U^i_{mp} U^p_{jq} Y^q + U \cdot U \cdot g \cdot Y + \hat{R} \cdot g \cdot Y. \end{aligned}$$

where $\Omega^i_p = g^{mj} \widehat{\nabla}_m U^i_{jp} + (\widehat{\nabla}_m U^m_{pk} - \widehat{\nabla}_p U^m_{mk}) g^{ki}$.

By using the expression of U, the commutation formula and $U^p = g^{ij} U^p_{ij} = 0$ we have

$$\begin{aligned} \Omega^i_p &= g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g + \frac{1}{2} g^{ki} g^{ml} \left(\widehat{\nabla}_m \widehat{\nabla}_p g_{kl} - \widehat{\nabla}_p \widehat{\nabla}_m g_{kl} \right) \\ &\quad + \frac{1}{2} g^{ki} g^{ml} \left(\widehat{\nabla}_m \widehat{\nabla}_p g_{kl} + \widehat{\nabla}_p \widehat{\nabla}_l g_{mk} - \widehat{\nabla}_p \widehat{\nabla}_k g_{ml} \right) \\ &= g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g + g \cdot g \cdot g \cdot \hat{R}. \end{aligned}$$

Thus

$$\Delta Y^i + R^i_p Y^p = \hat{\Delta} Y^i + (g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g + g^3 \cdot \hat{R}) \cdot Y + 2 g^{mj} U^i_{jq} \widehat{\nabla}_m Y^q + g^{mj} U^i_{mp} U^p_{jq} Y^q.$$

Combining this with (1.7) gives

$$\begin{aligned} &\hat{\Delta} Y^i + 2 g^{mj} U^i_{jq} \widehat{\nabla}_m Y^q + g^{mj} U^i_{mp} U^p_{jq} Y^q \\ &\quad = -2 n k^{mj} U^i_{mj} + 2 \nabla^m Y^j U^i_{mj} + 2 \nabla^m n k^i_m - \nabla^i n k^m_m + g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g \cdot Y + g^3 \cdot \hat{R} \cdot Y. \end{aligned}$$

In view of $\nabla^m Y^l U^i_{ml} = g^{mj} \nabla_j Y^l U^i_{ml} = g^{mj} \widehat{\nabla}_j Y^l U^i_{ml} + g^{mj} U^p_{jq} U^i_{mp} Y^q$, we can obtain the desired equation. $\square$

Lemma 2. *For any Σ-tangent tensor field F, on each Σ_t there holds*

$$\|\widehat{\nabla}^2 F\|_{L^2_x} \lesssim \|\hat{\Delta} F\|_{L^2_x} + \|\widehat{\nabla} g \cdot \widehat{\nabla} F\|_{L^2_x} + \|\widehat{\nabla} F\|_{L^2_x} + \|F\|_{L^2_x}.$$

Proof. Let $d\mu_g$ denote the volume form induced by g on Σ_t. Then, under the spacial harmonic gauge, there holds $\widehat{\nabla}_i(g^{ij} d\mu_g) = 0$ (see [2, Page 3]). Thus, by integration by part, we have

$$\begin{aligned}\int_\Sigma |\widehat{\nabla}^2 F|_g^2 d\mu_g &= \int_\Sigma g^{ij} g^{pq} \widehat{\nabla}_i \widehat{\nabla}_p F^l \widehat{\nabla}_j \widehat{\nabla}_q F_l d\mu_g \\ &= -\int_\Sigma \left(\widehat{\nabla}_p F^l \hat{\Delta} \widehat{\nabla}_q F_l g^{pq} + g^{ij} \widehat{\nabla}_i g^{pq} \widehat{\nabla}_p F^l \widehat{\nabla}_j \widehat{\nabla}_q F_l\right) d\mu_g.\end{aligned}$$

Here and throughout the paper we will use $\hat{g}$ to raise and lower the indices in tensors. It is easy to check the following commutator formula

$$\hat{\Delta} \widehat{\nabla}_q F_l - \widehat{\nabla}_q \hat{\Delta} F_l = g \cdot \widehat{\nabla} g \cdot \widehat{\nabla}^2 F + g \cdot \hat{R} \cdot \widehat{\nabla} F + g \cdot \widehat{\nabla} \hat{R} \cdot F. \tag{2.32}$$

Therefore we can derive that

$$\begin{aligned}&\int_\Sigma |\widehat{\nabla}^2 F|_g^2 d\mu_g \\ &= \int_\Sigma \left(-g^{pq} \widehat{\nabla}_p F^l \widehat{\nabla}_q \hat{\Delta} F_l + g \widehat{\nabla} g \cdot \widehat{\nabla} F \cdot \widehat{\nabla}^2 F + (g \cdot \hat{R} \cdot \widehat{\nabla} F + g \widehat{\nabla} \hat{R} \cdot F) \widehat{\nabla} F\right) d\mu_g\end{aligned}$$

where the first term is $\int_\Sigma \hat{\Delta} F^l \hat{\Delta} F_l d\mu_g$ by integration by part. □

Lemma 3. *On each Σ_t there hold*

$$\|\widehat{\nabla} Y\|_{L^2} \lesssim \|\widehat{\nabla} g\|_{L^2} + 1, \tag{2.33}$$

$$\|\widehat{\nabla}^2 Y\|_{L^2} \lesssim \|(\pi, \widehat{\nabla} Y, \widehat{\nabla} g) \cdot \widehat{\nabla} g\|_{L^2} + \|\widehat{\nabla} g\|_{L^2} + 1, \tag{2.34}$$

$$\begin{aligned}\|\widehat{\nabla}^3 Y\|_{L^2} &\lesssim (\|\widehat{\nabla} Y, \widehat{\nabla} g\|_{H^1} + 1)(\|\widehat{\nabla} g\|_{L^\infty}^{\frac{4}{3}} \|\widehat{\nabla} g\|_{L^2}^{\frac{2}{3}} + \|\widehat{\nabla} g\|_{L^\infty}) \\ &\quad + (\|\widehat{\nabla}^2 g\|_{L^2} + 1) \cdot \|\widehat{\nabla} g, \pi\|_{L^\infty} + \|\widehat{\nabla} g\|_{L^2} + 1.\end{aligned} \tag{2.35}$$

Proof. Consider (2.33) first. By using (2.30) and $\widehat{\nabla}_j(g^{ij} d\mu_g) = 0$, we have

$$\begin{aligned}\|\widehat{\nabla} Y\|_{L^2}^2 &\approx \int g^{ij} \widehat{\nabla}_i Y^l \widehat{\nabla}_j Y_l d\mu_g \\ &= \int -\hat{\Delta} Y^l \cdot Y_l d\mu_g \lesssim \|\pi \cdot \pi\|_{L^1} + \|\widehat{\nabla} g \cdot (\widehat{\nabla} g, \pi)\|_{L^1} + 1.\end{aligned}$$

In view of (2.27), we thus obtain (2.33).

Next, by using (2.30) we have

$$\|\hat{\Delta} Y\|_{L^2} \lesssim \|(\pi, \widehat{\nabla} g) \cdot \widehat{\nabla} g\|_{L^2} + \|\pi\|_{L^4}^2 + 1.$$

It then follows from Lemma 2 that

$$\|\widehat{\nabla}^2 Y\|_{L^2} \lesssim \|(\pi, \widehat{\nabla} Y, \widehat{\nabla} g) \cdot \widehat{\nabla} g\|_{L^2} + \|\widehat{\nabla} Y\|_{L^2} + 1.$$

We thus obtain (2.34) in view of (2.33).

Finally, by writing $\hat{\Delta}\widehat{\nabla}Y = \widehat{\nabla}\hat{\Delta}Y + [\hat{\Delta}, \widehat{\nabla}]Y$, we have from (2.30) and (2.32) that

$$\begin{aligned}\hat{\Delta}\widehat{\nabla}Y &= \widehat{\nabla}\pi \cdot \widehat{\nabla}g + \pi \cdot \widehat{\nabla}^2 g + \pi \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g + \pi \cdot \widehat{\nabla}\pi + \pi \cdot \pi \cdot \widehat{\nabla}g \\ &\quad + \widehat{\nabla}g \cdot \widehat{\nabla}^2 g \cdot Y + \widehat{\nabla}g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g \cdot Y + \widehat{\nabla}g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}Y + g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}^2 Y \\ &\quad + g \cdot \hat{R} \cdot \widehat{\nabla}Y + g \cdot \widehat{\nabla}\hat{R} \cdot Y + \widehat{\nabla}(g^3 \cdot \hat{R} \cdot Y). \end{aligned} \tag{2.36}$$

It is easy to see that L^2 norm of the last three terms involving $\hat{R}$ can be bounded by $1 + \|\widehat{\nabla}g\|_{L^2}$. Note that

$$\|\pi \cdot \pi \cdot \widehat{\nabla}g\|_{L^2} \lesssim \|\widehat{\nabla}g\|_{L^\infty}\|\pi\|_{L^4}^2 \lesssim \|\widehat{\nabla}g\|_{L^\infty},$$
$$\|(\widehat{\nabla}Y, \widehat{\nabla}g, \pi) \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g\|_{L^2} \lesssim (\|\widehat{\nabla}g, \widehat{\nabla}Y\|_{H^1} + \|\pi\|_{L^6})\|\widehat{\nabla}g\|_{L^2}^{\frac{2}{3}}\|\widehat{\nabla}g\|_{L^\infty}^{\frac{4}{3}}.$$

Thus, using Lemma 2, we can obtain (2.35). □

2.1. Energy estimate for $\widehat{\nabla}g$. In order to proceed further, besides (BA1) we also need the following bootstrap assumption

$$\|\widehat{\nabla}g\|_{L^2_{[t_0-T,t_0+T]}L^\infty_x} + \|k\|_{L^2_{[t_0-T,t_0+T]}L^\infty_x} \le B_0. \tag{BA2}$$

which is a stronger version for the corresponding part in (BA1). The verification of (BA1) and (BA2) will be carried out in Sect. 4.

We first introduce some conventions. For any 2-tensors u and v we define

$$\langle u, v\rangle := \hat{g}^{ik}\hat{g}^{jl}u_{ij}v_{kl} \quad \text{and} \quad \langle\widehat{\nabla}u, \widehat{\nabla}v\rangle_g := g^{ij}\langle\widehat{\nabla}_i u, \widehat{\nabla}_j v\rangle.$$

We will use $|u|^2 := \langle u, u\rangle$ and $|\widehat{\nabla}u|_g^2 := \langle\widehat{\nabla}u, \widehat{\nabla}u\rangle_g$.

In the following we will derive some estimates on $\widehat{\nabla}g$ and the derivatives on Y. By using the formula

$$\mathcal{L}_Y u_{ij} = \widehat{\nabla}_Y u_{ij} + u_{im}\widehat{\nabla}_j Y^m + u_{mj}\widehat{\nabla}_i Y^m$$

for any 2-tensor u and the formula under the spatial harmonic gauge,

$$R_{ij} = -\frac{1}{2}\hat{\Delta}g_{ij} + \widehat{\mathrm{R}}_{ij} + g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g \tag{2.37}$$

we can derive from (1.3) and (1.4) that the 2-tensors $u := g$ and $v := -2k$ satisfy the hyperbolic system (1.13) with F_u and F_v given symbolically by

$$F_u = u \cdot \widehat{\nabla}Y \quad \text{and} \quad F_v = 2\nabla^2 n + n \cdot k * k + k * \widehat{\nabla}Y. \tag{2.38}$$

From (1.13) and the commutation formula (2.32), we can derive that

$$\partial_t\widehat{\nabla}u - \widehat{\nabla}_Y\widehat{\nabla}u = n\widehat{\nabla}v + F_{\widehat{\nabla}u} \quad \text{and} \quad \partial_t\widehat{\nabla}v - \widehat{\nabla}_Y\widehat{\nabla}v = n\hat{\Delta}\widehat{\nabla}u + F_{\widehat{\nabla}v}, \tag{2.39}$$

where

$$\begin{aligned} F_{\widehat{\nabla}u} &= \widehat{\nabla}Y \cdot \widehat{\nabla}u + Y \cdot \hat{R} \cdot u + \widehat{\nabla}n \cdot v + \widehat{\nabla}F_u, \\ F_{\widehat{\nabla}v} &= \widehat{\nabla}Y \cdot \widehat{\nabla}v + Y \cdot \hat{R} \cdot v + \widehat{\nabla}n\hat{\Delta}u + \widehat{\nabla}F_v + n(g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}^2 u + g \cdot \hat{R} \cdot \widehat{\nabla}u + g \cdot \widehat{\nabla}\hat{R} \cdot u). \end{aligned}$$

It is straightforward to derive that

$$\begin{cases} \widehat{\nabla}^2 F_u = \widehat{\nabla}^2 Y \cdot \widehat{\nabla} g + g \cdot \widehat{\nabla}^3 Y + \widehat{\nabla} Y \cdot \widehat{\nabla}^2 g, \\ \widehat{\nabla} F_v = g \cdot \widehat{\nabla}^2 Y \cdot k + 2\nabla^3 n + g \cdot \widehat{\nabla} k \cdot \widehat{\nabla} Y + \widehat{\nabla}(ng \cdot k \cdot k) \end{cases}$$

and

$$\begin{aligned} |\widehat{\nabla} F_{\widehat{\nabla} u}, F_{\widehat{\nabla} v}| \le{}& |\widehat{\nabla}^2 Y \cdot (k, \widehat{\nabla} g)| + |\nabla^3 n, \widehat{\nabla}^3 Y| + |(\widehat{\nabla} Y, \nabla n, k, \widehat{\nabla} g) \cdot \widehat{\nabla}(\widehat{\nabla} g, k)| \\ &+ |\widehat{\nabla} n \cdot k \cdot (\widehat{\nabla} g, k)| + |\widehat{\nabla} g \cdot k \cdot \widehat{\nabla} Y| + |\widehat{\nabla} g \cdot k \cdot k| + |\nabla^2 n \cdot k| \\ &+ |k| + |\widehat{\nabla} Y| + |\widehat{\nabla} g| + 1. \end{aligned} \tag{2.40}$$

In order to derive the estimates, we use the energy introduced in [2, Section 2]

$$\mathcal{E}^{(0)}(t) = \mathcal{E}^{(0)}(u, v)(t) := \frac{1}{2} \int_{\Sigma} \left(|u|^2 + |\widehat{\nabla} u|_g^2 + |v|^2 \right) d\mu_g \tag{2.41}$$

with $u = g$ and $v = -2k$.

Proposition 1. *Under the bootstrap assumption* (BA1)*, there holds*

$$\sup_{[t_0, t_0+T]} \|\widehat{\nabla} g\|_{L^2(\Sigma_t)} \le C.$$

Proof. Recall that for any vector fields Z tangent to Σ_t and any scalar function f there holds $\int_{\Sigma_t} \mathcal{L}_Z(f d\mu_g) = \int_{\Sigma_t} \mathrm{div}(fZ) d\mu_g = 0$. Therefore

$$\begin{aligned} \partial_t \mathcal{E}^{(0)}(t) ={}& \frac{1}{2} \int_{\Sigma_t} (\partial_t - \mathcal{L}_Y) \left\{ (|u|^2 + |\widehat{\nabla} u|_g^2 + |v|^2) d\mu_g \right\} \\ ={}& \int_{\Sigma_t} \left[\langle u, \partial_t u - \widehat{\nabla}_Y u \rangle + \langle v, \partial_t v - \widehat{\nabla}_Y v \rangle + g^{ij} \langle \widehat{\nabla}_i u, (\partial_t - \widehat{\nabla}_Y) \widehat{\nabla}_j u \rangle \right] d\mu_g \\ &+ \frac{1}{2} \int_{\Sigma_t} (\partial_t g^{ij} - \mathcal{L}_Y g^{ij}) \langle \widehat{\nabla}_i u, \widehat{\nabla}_j u \rangle d\mu_g \\ &+ \frac{1}{2} \int_{\Sigma_t} (|u|^2 + |\widehat{\nabla} u|_g^2 + |v|^2)(\partial_t - \mathcal{L}_Y)(d\mu_g). \end{aligned}$$

By using (1.3) we have $\partial_t g^{ij} - \mathcal{L}_Y g^{ij} = 2nk^{ij}$ and $(\partial_t - \mathcal{L}_Y)(d\mu_g) = -n\mathrm{Tr} k d\mu_g$. These two identities together with (1.13) and (2.39) give

$$\begin{aligned} \partial_t \mathcal{E}^{(0)}(t) ={}& \int_{\Sigma_t} \left(n\langle u, v \rangle + \langle u, F_u \rangle + \langle v, F_v \rangle + nk^{ij} \widehat{\nabla}_i u \widehat{\nabla}_j u \right) d\mu_g \\ &+ \int_{\Sigma_t} g \cdot \left(\widehat{\nabla} F_u \cdot \widehat{\nabla} u + Y \cdot \hat{R} \cdot u \cdot \widehat{\nabla} u + \widehat{\nabla} Y \cdot \widehat{\nabla} u \cdot \widehat{\nabla} u \right) d\mu_g \\ &- \frac{1}{2} \int_{\Sigma_t} n \mathrm{Tr} k (|u|^2 + |\widehat{\nabla} u|_g^2 + |v|^2) d\mu_g. \end{aligned}$$

In view of the bounds on n, $|Y|$ and g, we can derive that

$$\partial_t \mathcal{E}^{(0)}(t) \lesssim \left(\|k, \widehat{\nabla} Y\|_{L^\infty} + 1 \right) \mathcal{E}^{(0)}(t) + \|\widehat{\nabla} F_u\|_{L^2} \|\widehat{\nabla} u\|_{L^2} + \|v\|_{L^2} \|F_v\|_{L^2}. \tag{2.42}$$

By using (2.27) and Lemma 3 we have

$$\|\widehat{\nabla} F_u\|_{L^2} \le \|\widehat{\nabla} Y\|_{L^\infty}\|\widehat{\nabla} g\|_{L^2} + \|\widehat{\nabla}^2 Y\|_{L^2} \lesssim (\|\widehat{\nabla} Y, \widehat{\nabla} g, \pi\|_{L^\infty} + 1)\|\widehat{\nabla} g\|_{L^2} + 1.$$

and

$$\begin{aligned}\|F_v\|_{L^2} &\lesssim \|\nabla^2 n\|_{L^2} + \|k\|_{L^4}^2 + \|k\|_{L^6}\|\widehat{\nabla} Y\|_{L^3} \lesssim 1 + \|\widehat{\nabla}^2 Y\|_{L^2} + \|\widehat{\nabla} Y\|_{L^2}\\ &\lesssim (\|\pi, \widehat{\nabla} Y, \widehat{\nabla} g\|_{L^\infty} + 1)\|\widehat{\nabla} g\|_{L^2} + 1.\end{aligned}$$

Therefore

$$\partial_t \mathcal{E}^{(0)}(t) \lesssim \left(\|\pi, \widehat{\nabla} Y, \widehat{\nabla} g\|_{L^\infty(\Sigma_t)} + 1\right) \mathcal{E}^{(0)}(t) + 1.$$

This together with the bootstrap assumption (BA1) gives $\mathcal{E}^{(0)}(t) \le \mathcal{E}^{(0)}(t_0) + 1$ for all $t \in [t_0, t_0 + T]$. The proof is thus complete. $\square$

We now consider the energy $\mathcal{E}^{(1)}(t) = \mathcal{E}^{(0)}(\widehat{\nabla} u, \widehat{\nabla} v)$. From (2.42) it follows easily that

$$\partial_t \mathcal{E}^{(1)}(t) \lesssim \left(\|k, \widehat{\nabla} Y\|_{L_x^\infty} + 1\right) \mathcal{E}^{(1)}(t) + \left(\|\widehat{\nabla} F_{\widehat{\nabla} u}\|_{L^2} + \|F_{\widehat{\nabla} v}\|_{L^2}\right) \sqrt{\mathcal{E}^{(1)}(t)}.$$

By taking L^2-norm of (2.40), we can obtain, using (2.27), (2.28), Proposition 1, and (2.33) and (2.35) in Lemma 3, that

$$\begin{aligned}\|\widehat{\nabla} F_{\widehat{\nabla} u}, F_{\widehat{\nabla} v}\|_{L^2} \lesssim &\left(\|\widehat{\nabla} g, k, \widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty} + \||\widehat{\nabla} g\|_{L^\infty}^{\frac{4}{3}}\right)\left(\|\widehat{\nabla}^2 g\|_{L^2} + 1\right)\\ &+ \|\widehat{\nabla} Y\|_{H^1}\left(\|\widehat{\nabla} g, k\|_{L^\infty} + \|\widehat{\nabla} g\|_{L^\infty}^{\frac{4}{3}} + 1\right) + 1.\end{aligned}$$

Using this estimate, $\|\widehat{\nabla}^2 g\|_{L^2} \le \sqrt{\mathcal{E}^{(1)}(t)}$, and the Young's inequality, we obtain

$$\begin{aligned}\partial_t \mathcal{E}^{(1)}(t) \lesssim &\left(1 + \|k, \widehat{\nabla} n, \widehat{\nabla} g, \widehat{\nabla} Y\|_{L^\infty} + \|k, \widehat{\nabla} g\|_{L^\infty}^2\right) \mathcal{E}^{(1)}(t)\\ &+ \|k, \widehat{\nabla} n, \widehat{\nabla} g, \widehat{\nabla} Y\|_{L^\infty} + \|\widehat{\nabla} g\|_{L^\infty}^2 + \|\widehat{\nabla}^2 Y\|_{L_x^2}^3 + 1.\end{aligned}$$

In view of (BA1) and (BA2), it follows easily that $\mathcal{E}^{(1)}(t) \lesssim \mathcal{E}^{(1)}(t_0) + 1 + \|\widehat{\nabla}^2 Y\|_{L_t^3 L_x^2}^3$. This in particular implies that

$$\|\widehat{\nabla}^2 g\|_{L^2(\Sigma_t)} \lesssim 1 + \|\widehat{\nabla}^2 Y\|_{L_t^3 L_x^2}^{3/2}. \tag{2.43}$$

On the other hand, it follows from (2.34), (2.27), and Proposition 1 that

$$\begin{aligned}\|\widehat{\nabla}^2 Y\|_{L_x^2} &\lesssim \|\widehat{\nabla} Y \cdot \widehat{\nabla} g\|_{L^2} + \|\widehat{\nabla} g\|_{L^4}^2 + \|\pi\|_{L^4}^2 + 1\\ &\lesssim \|\widehat{\nabla} Y\|_{L^6}^{\frac{1}{2}}\|\widehat{\nabla} Y\|_{L^2}^{\frac{1}{2}}\|\widehat{\nabla} g\|_{L^6} + \|\widehat{\nabla} g\|_{L^4}^2 + 1.\end{aligned}$$

Using $\|\widehat{\nabla} Y\|_{L^6} \lesssim \|\widehat{\nabla}^2 Y\|_{L^2} + 1$ and (2.33) we can obtain $\|\widehat{\nabla}^2 Y\|_{L^2} \lesssim \|\widehat{\nabla} g\|_{L^6}^2 + 1 \lesssim \|\widehat{\nabla}^2 g\|_{L^2}^2 + 1$. This together with (2.43) gives

$$\|\widehat{\nabla}^2 Y\|_{L_x^2} \lesssim 1 + \|\widehat{\nabla}^2 Y\|_{L_t^3 L_x^2}^3. \tag{2.44}$$

Integrating with respect t over $[t_0, t_0 + T]$ yields

$$\|\widehat{\nabla}^2 Y\|_{L^3_{[t_0,t_0+T]}L^2_x} \lesssim T^{\frac{1}{3}} \left(1 + \|\widehat{\nabla}^2 Y\|^3_{L^3_{[t_0,t_0+T]}L^2_x}\right).$$

Therefore we can choose a small but universal $T > 0$ such that $\|\widehat{\nabla}^2 Y\|_{L^3_{[t_0,t_0+T]}L^2_x} \leq C$ for some universal constant C. Consequently, by using (2.43) and (2.44) we can obtain (2.45) and (2.46) in the following result.

Proposition 2. *Under the bootstrap assumption* (BA1) *and* (BA2), *there hold*

$$\|g\|_{H^2(\Sigma_t)} + \|\widehat{\nabla} k\|_{L^2(\Sigma_t)} \leq C, \tag{2.45}$$

$$\|\widehat{\nabla}^2 Y\|_{L^2(\Sigma_t)} + \|\widehat{\nabla} Y\|_{L^2(\Sigma_t)} \leq C, \tag{2.46}$$

$$\|e_0(\widehat{\nabla} g)\|_{L^2(\Sigma_t)} + \|\partial_t \widehat{\nabla} g\|_{L^2(\Sigma_t)} \leq C \tag{2.47}$$

for all $t \in [t_0, t_0 + T]$ with $T > 0$ being a universal number, where, for any Σ_t-tangent tensor field F, we use the notation $e_0(F) := n^{-1}(\partial_t F - \widehat{\nabla}_Y F)$.

Proof. It remains only to prove (2.47). We use (2.39), (2.45) and (2.46) to deduce that

$$\begin{aligned}\|e_0(\widehat{\nabla} g)\|_{L^2} &\lesssim \|\widehat{\nabla} k\|_{L^2} + \|\widehat{\nabla} Y\|_{L^6}\|\widehat{\nabla} g\|_{L^3} + \|\widehat{\nabla}^2 Y\|_{L^2} + \|\widehat{\nabla} n\|_{L^4}\|k\|_{L^4} \\ &\quad + \|Y\|_{L^\infty}\|g\|_{L^2} \lesssim 1.\end{aligned}$$

Finally, in view of (2.45) we obtain $\|\partial_t \widehat{\nabla} g\|_{L^2} \lesssim \|e_0(\widehat{\nabla} g)\|_{L^2} + \|\widehat{\nabla}^2 g\|_{L^2} \lesssim 1$. □

Lemma 4. *Under the bootstrap assumptions* (BA1) *and* (BA2), *for $3 < p \leq 6$ there hold*

$$\|\widehat{\nabla}^3 Y\|_{L^2} \lesssim \|\widehat{\nabla} g, k\|_{L^\infty} + 1, \tag{2.48}$$

$$\|\widehat{\nabla} Y\|_{L^\infty} \lesssim \|k, \widehat{\nabla} g\|^{3/2-3/p}_{L^\infty} + 1. \tag{2.49}$$

Proof. In view of Proposition 2, (2.27), and Lemma 2, we obtain from (2.36) that

$$\|\widehat{\nabla}^3 Y\|_{L^2} \lesssim \|\widehat{\nabla}\pi, \widehat{\nabla}^2 g\|_{L^2_x}\|\widehat{\nabla} g, \pi\|_{L^\infty_x} + \|\widehat{\nabla}^2 Y \cdot \widehat{\nabla} g\|_{L^2} + \|\widehat{\nabla} g, k\|_{L^\infty_x} + 1.$$

We may write

$$\|\widehat{\nabla}^2 Y \cdot \widehat{\nabla} g\|_{L^2} \lesssim \|\widehat{\nabla}^2 Y\|_{L^3}\|\widehat{\nabla} g\|_{L^6} \lesssim \|\widehat{\nabla}^2 Y\|_{L^3}(\|\widehat{\nabla}^2 g\|_{L^2} + 1).$$

Applying the Sobolev type inequality ([25, Lemma 2.5]) to $\|\widehat{\nabla}^2 Y\|_{L^3}$, and using (2.27), (2.28), (2.45) and (2.46), we can obtain (2.48). Finally we can use the Sobolev embedding given in [25, Lemma 2.6] to conclude (2.49). □

3. $H^{2+\epsilon}$ Estimates

In this section, under the bootstrap assumptions (BA1) and (BA2), we will establish $H^{1+\epsilon}$ type energy estimates for k, $\widehat{\nabla} g$ and $\mathbf{D}\phi$ with ϕ being solutions of homogeneous wave equation $\Box_{\mathbf{g}}\phi = 0$. We will also obtain the $H^{\frac{3}{2}+\epsilon}$ and $H^{2+\epsilon}$ estimates for $\widehat{\nabla}n$, $\widehat{\nabla}Y$, $ne_0(n)$ simultaneously. As the main building block of this section, established in Appendix are a series of product estimates in fractional Sobolev spaces and estimates for commutators between the Littlewood-Paley projections P_μ and the rough coefficients.

For simplicity of exposition, we fix some conventions. We will use $\tilde{\pi}$ to denote any term from the set $\widehat{\nabla}Y$, $\widehat{\nabla}n$, k, $\widehat{\nabla}g$ and $e_0(n)$, where $e_0(n) = n^{-1}(\partial_t n - \widehat{\nabla}_Y n)$ as defined before. It follows from Proposition 2 and (2.27) that $\|\tilde{\pi}\|_{H^1} \le C$. We also introduce the error terms

$$\mathrm{err}_1 = \mathbf{g}\cdot\tilde{\pi}\cdot\widehat{\nabla}\tilde{\pi}, \qquad \mathrm{err}_2 = \mathbf{g}\cdot\tilde{\pi}\cdot\tilde{\pi}\cdot\tilde{\pi}, \tag{3.50}$$

where $\mathbf{g}$ denotes any product of the components of n, g and Y. We denote by $\mathrm{err}(\hat{R})$ any term involving $\hat{R}$ and its derivatives, and satisfying $\|\mathrm{err}(\hat{R})\|_{H^1(\Sigma_t)} \le C$ for all $t \in [t_0, t_0+T]$.

Proposition 3. *For $0<\epsilon<1/2$ there hold*

$$\|\Lambda^{1/2+\epsilon}\big(\widehat{\nabla}^2 n, \widehat{\nabla}(ne_0(n)), \widehat{\nabla}^2 Y\big)\|_{L^2} \lesssim \|\widehat{\nabla}g, k\|_{H^{1+\epsilon}} + 1, \tag{3.51}$$

$$\|\Lambda^{\epsilon}\big(\widehat{\nabla}^3 n, \widehat{\nabla}^2(ne_0(n)), \widehat{\nabla}^3 Y\big)\|_{L^2} \lesssim \|\widehat{\nabla}g, k\|_{L^\infty}\|\widehat{\nabla}g, k\|_{H^{1+\epsilon}} + 1 \tag{3.52}$$

and, for the error type terms defined in (3.50)*, there hold*

$$\|\Lambda^{\epsilon} err_1\|_{L^2} \lesssim \|k, \widehat{\nabla}g\|_{L^\infty}(\|\widehat{\nabla}g, k\|_{H^{1+\epsilon}} + 1) + 1, \tag{3.53}$$

$$\|\Lambda^{\epsilon} err_2\|_{L^2} \lesssim \|\widehat{\nabla}g, k\|_{H^{1+\epsilon}} + 1. \tag{3.54}$$

Proof. For any scalar function f it is easy to derive the commutation formula

$$[\Delta, \nabla_{n\mathbf{T}}]f = -2nk_i^l\nabla_l\nabla^i f - \nabla^i nk_i^l\nabla_l f. \tag{3.55}$$

To obtain the estimates of $\widehat{\nabla}n$ and $ne_0(n)$, we first use (1.6) and (3.55) to derive the identities

$$\hat{\Delta}\widehat{\nabla}n = \widehat{\nabla}(n|k|_g^2) + g\hat{R}\cdot\widehat{\nabla}n, \tag{3.56}$$

$$\hat{\Delta}\big(ne_0(n)\big) = ne_0(n|k|_g^2) - 2nk_a^l\nabla_l\nabla^a n - \nabla^a n\nabla_l nk_a^l. \tag{3.57}$$

In view of (3.56), we have

$$\hat{\Delta}\widehat{\nabla}n = n\widehat{\nabla}k\cdot k\cdot g + (\widehat{\nabla}n, k, \widehat{\nabla}g)^3\cdot(n,g) + \mathrm{err}(\hat{R}). \tag{3.58}$$

It then follows from (6.189) that

$$\|\Lambda^{-1/2+\epsilon}\hat{\Delta}\widehat{\nabla}n\|_{L^2} \lesssim \|\widehat{\nabla}k\|_{H^\epsilon}\|ng\cdot k\|_{H^1} + \|(\widehat{\nabla}n, k, \widehat{\nabla}g)^3\cdot(n,g)\|_{L^2} + \|\mathrm{err}(\hat{R})\|_{L^2}.$$

By using (6.212) and $\|\tilde{\pi}\|_{H^1} \le C$, we can conclude that

$$\|\Lambda^{1/2+\epsilon}\widehat{\nabla}^2 n\|_{L^2} \lesssim \|\widehat{\nabla}k\|_{H^\epsilon} + 1. \tag{3.59}$$

In view of (3.57) and (1.13), we have

$$\hat{\Delta}\big(ne_0(n)\big) = (n\hat{\Delta}g + \nabla^2 n)ng\cdot k + \mathbf{g}\cdot\tilde{\pi}\cdot\tilde{\pi}\cdot\tilde{\pi}. \tag{3.60}$$

Thus, with the help of (6.189) and (3.59), it follows

$$\|\Lambda^{-1/2+\epsilon}\hat{\Delta}(ne_0(n))\|_{L^2} \lesssim \|\widehat{\nabla}^2 g, \widehat{\nabla}^2 n\|_{H^\epsilon}\|\mathbf{g}\cdot k\|_{H^1} + \|\mathbf{g}|\tilde{\pi}|^3\|_{L^2} \lesssim \|\widehat{\nabla}^2 g, \widehat{\nabla} k\|_{H^\epsilon} + 1$$

which implies, in view of (6.212) and $\|\tilde{\pi}\|_{H^1} \le C$, that

$$\|\Lambda^{1/2+\epsilon}\widehat{\nabla}\big(ne_0(n)\big)\|_{L^2} \lesssim \|\widehat{\nabla}^2 g, \widehat{\nabla} k\|_{H^\epsilon} + 1. \tag{3.61}$$

Next we use (2.36) and (6.189) to obtain

$$\|\Lambda^{-1/2+\epsilon}\hat{\Delta}\widehat{\nabla} Y\|_{L^2} \lesssim \|\widehat{\nabla}(\pi, \widehat{\nabla} g, \widehat{\nabla} Y)\|_{H^\epsilon} + \|\mathrm{err}(\hat{R})\|_{L^2}.$$

This together with (3.59), (6.212), $\|\tilde{\pi}\|_{H^1} \le C$ and the interpolation inequality gives

$$\|\Lambda^{1/2+\epsilon}\widehat{\nabla}^2 Y\|_{L^2} \lesssim \|\widehat{\nabla}^2 Y\|_{H^\epsilon} + \|\widehat{\nabla}^2 g, \widehat{\nabla} k\|_{H^\epsilon} + 1 \lesssim \|\widehat{\nabla}^2 g, \widehat{\nabla} k\|_{H^\epsilon} + 1. \tag{3.62}$$

Combining the estimates (3.59), (3.61) and (3.62), we therefore complete the proof of (3.51).

As a byproduct of (3.51), we have

$$\|\Lambda^\epsilon\widehat{\nabla}\tilde{\pi}\|_{L^2} \lesssim \|k, \widehat{\nabla} g\|_{H^{1+\epsilon}} + 1. \tag{3.63}$$

It then follows from (6.190) and (3.63) that

$$\|\Lambda^\epsilon(\tilde{\pi}\cdot\widehat{\nabla}\tilde{\pi})\|_{L^2} \lesssim \|\tilde{\pi}\|_{L^\infty}\|\tilde{\pi}\|_{H^{1+\epsilon}} \lesssim (\|k, \widehat{\nabla} g\|_{H^{1+\epsilon}} + 1)\|\tilde{\pi}\|_{L^\infty}. \tag{3.64}$$

By Lemma 18 and (3.63), we have

$$\|\Lambda^\epsilon(\tilde{\pi}\cdot\tilde{\pi}\cdot\tilde{\pi})\|_{L_x^2} \lesssim \|\tilde{\pi}\|_{H^1}^2\|\tilde{\pi}\|_{H^{1+\epsilon}} \lesssim \|k, \widehat{\nabla} g\|_{H^{1+\epsilon}} + 1. \tag{3.65}$$

To treat the factor $\mathbf{g}$ in the definition (3.50), in view of $\|\mathbf{g}\|_{H^2} \le C$ in Proposition 2, using Lemma 21, and (3.64) and (3.65), we thus obtain (3.53) and (3.54).

Finally, we consider (3.52) with the help of (6.211). Let $F = ne_0(n), \widehat{\nabla} n, \widehat{\nabla} Y$. Then it follows from (2.27) and (2.46) that $\|F\|_{H^1} \lesssim 1$. Moreover, the elliptic equations (3.58), (3.60) and (2.36) can be written symbolically as $\hat{\Delta} F = \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R})$. In view of (3.53), (3.54), and the definition of $\mathrm{err}(\hat{R})$, we thus obtain (3.52). □

3.1. First order hyperbolic systems.

3.1.1. Energy estimates. We consider a pair of tensors (u, v) satisfying the first order hyperbolic system

$$\begin{cases} \partial_t u - \widehat{\nabla}_Y u = nv + F_u \\ \partial_t v - \widehat{\nabla}_Y v = n\hat{\Delta} u + F_v. \end{cases} \tag{3.66}$$

Note that for (u, v) satisfying (3.66), the pair $(U_1, V_1) = (\widehat{\nabla} u, \widehat{\nabla} v)$ satisfies a system of the form (3.66) with

$$\begin{cases} F_{U_1} = \widehat{\nabla} Y^m\widehat{\nabla}_m u + \widehat{\nabla} n\cdot v + \widehat{\nabla} F_u + Y\cdot\hat{R}\cdot u \\ F_{V_1} = \widehat{\nabla} Y^m\widehat{\nabla}_m v + \widehat{\nabla} n\cdot\hat{\Delta} u + \widehat{\nabla} F_v + n\widehat{\nabla} g\cdot\widehat{\nabla}^2 u + n\hat{R}\cdot\widehat{\nabla} u + (\hat{R} Y v + n\widehat{\nabla}(\hat{R}\cdot u)) \end{cases} \tag{3.67}$$

where the last term in F_{U_1} and F_{V_1} can be dropped in case (u, v) is a pair of scalar functions.

We also can check that the pair of functions $(U^\mu, V^\mu) := (P_\mu u, P_\mu v)$ satisfies (3.66) with F_{U^μ} and F_{V^μ} given by[3]

$$\begin{cases} F_{U^\mu} = [P_\mu, Y^m]\partial_m u + [P_\mu, n]v + P_\mu F_u, \\ F_{V^\mu} = [P_\mu, ng]\widehat{\nabla}^2_{ij} u + P_\mu F_v + [P_\mu, Y^m]\partial_m v. \end{cases} \tag{3.68}$$

Thus, it is easy to check that $(U_1^\mu, V_1^\mu) := (P_\mu \widehat{\nabla} u, P_\mu \widehat{\nabla} v)$ satisfies (3.66) with $F_{U_1^\mu}$ and $F_{V_1^\mu}$ given by

$$\begin{cases} F_{U_1^\mu} = [P_\mu, Y^m]\partial_m \widehat{\nabla} u + [P_\mu, n]\widehat{\nabla} v + P_\mu F_{U_1}, \\ F_{V_1^\mu} = [P_\mu, ng]\widehat{\nabla}^2_{ij} \widehat{\nabla} u + [P_\mu, Y^m]\partial_m \widehat{\nabla} v + P_\mu F_{V_1}. \end{cases} \tag{3.69}$$

Lemma 5. *Let* $0 < \epsilon < 1/2$*. Then for* F_{U^μ} *and* F_{V^μ} *defined by* (3.68) *there hold the estimates*

$$\begin{aligned} &\|\mu^{\frac{1}{2}+\epsilon} F_{U^\mu}\|_{l^2_\mu L^2_x} + \|\mu^{-\frac{1}{2}+\epsilon} \widehat{\nabla} F_{U^\mu}\|_{l^2_\mu L^2_x} \\ &\quad \lesssim \|\widehat{\nabla} n, \widehat{\nabla} Y\|_{H^1} \|\widehat{\nabla} u, v\|_{H^\epsilon} + \|\mu^{\frac{1}{2}+\epsilon} P_\mu F_u\|_{l^2_\mu L^2_x}, \end{aligned} \tag{3.70}$$

$$\begin{aligned} &\|\mu^{\epsilon} \widehat{\nabla} F_{U^\mu}\|_{l^2_\mu L^2_x} + \|\mu^{1+\epsilon} F_{U^\mu}\|_{l^2_\mu L^2_x} \\ &\quad \lesssim \|\widehat{\nabla}^2 Y, \widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla} u, v\|_{L^2_x} + \|\widehat{\nabla} n, \widehat{\nabla} Y\|_{L^\infty_x} \|\widehat{\nabla} u, v\|_{H^\epsilon} + \|\mu^\epsilon \widehat{\nabla} P_\mu F_u\|_{l^2_\mu L^2_x}, \end{aligned} \tag{3.71}$$

and

$$\|\mu^\epsilon F_{V^\mu}\|_{l^2_\mu L^2_x} \lesssim \|\widehat{\nabla}(ng), \widehat{\nabla} Y\|_{L^\infty_x} \|\widehat{\nabla} u, v\|_{H^\epsilon} + \|\mu^\epsilon P_\mu F_v\|_{l^2_\mu L^2_x}. \tag{3.72}$$

Proof. (3.71) follows from (6.203), (3.72) follows from (6.196), and (3.70) follows from (6.197). □

Lemma 6. *For* $0 < \epsilon < 1/2$*, there hold*

$$\|\mu^{1+\epsilon} \widehat{\nabla} F_{U^\mu}\|_{l^2_\mu L^2_x} + \|\mu^\epsilon \widehat{\nabla} F_{U_1^\mu}\|_{l^2_\mu L^2_x} \lesssim \|\widehat{\nabla}^2 F_u\|_{H^\epsilon} + \mathcal{I}(\epsilon, u, v), \tag{3.73}$$

$$\begin{aligned} \|\mu^\epsilon F_{V_1^\mu}\|_{l^2_\mu L^2_x} &\lesssim (\|\widehat{\nabla} g\|_{L^\infty_x} + 1)\|\widehat{\nabla}^2 u\|_{H^\epsilon} + \|\widehat{\nabla} F_v\|_{H^\epsilon} + \|\widehat{\nabla} u\|_{L^\infty} \|n \widehat{\nabla} g\|_{H^{1+\epsilon}} \\ &\quad + \mathcal{I}(\epsilon, u, v) + \|\widehat{\nabla} u, v\|_{H^\epsilon}, \end{aligned} \tag{3.74}$$

where

$$\mathcal{I}(\epsilon, u, v) = \|\widehat{\nabla}^2 Y, \widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla}^2 u, \widehat{\nabla} v\|_{L^2_x} + \|\widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty_x} \|\widehat{\nabla} u, v\|_{H^{1+\epsilon}}.$$

Proof. The first part of (3.73) follows from (6.205) and (3.68). In order to prove the second part of (3.73), we may use the same argument for deriving (3.71) to obtain

[3] We remark that the precise form of the first term in F_{V^μ} should be $[P_\mu, ng\widehat{\nabla}^2]u$ which consists of $[P_\mu, ng]\widehat{\nabla}^2 u$ and $ng[P_\mu, \hat{\Gamma}]\partial u$. The latter is of much lower order, which not only can be treated similar to the first term, but also can be done in a much easier way since $\hat{\Gamma}$ is smooth. Thus, we will omit this term for ease of exposition.

$$\|\mu^\epsilon \widehat{\nabla} F_{U_1^\mu}\|_{l_\mu^2 L_x^2} \lesssim \mathcal{I}(\epsilon, u, v) + \|\mu^\epsilon \widehat{\nabla} P_\mu F_{U_1}\|_{l_\mu^2 L_x^2}.$$

In view of (3.67), we apply (6.208) to obtain

$$\begin{aligned}\|\mu^\epsilon \widehat{\nabla} P_\mu F_{U_1}\|_{l_\mu^2 L_x^2} &\lesssim \mathcal{I}(\epsilon, u, v) + \|\mu^\epsilon \widehat{\nabla} P_\mu \widehat{\nabla} F_u\|_{l_\mu^2 L_x^2} + \|\Lambda^\epsilon \widehat{\nabla}(Y \cdot \hat{R} \cdot u)\|_{L^2}\\ &\lesssim \mathcal{I}(\epsilon, u, v) + \|\mu^\epsilon \widehat{\nabla} P_\mu \widehat{\nabla} F_u\|_{l_\mu^2 L_x^2}.\end{aligned}$$

Combining the above two estimates, we therefore obtain the second part of (3.73).

Next we prove (3.74). We first apply Lemma 22 to derive that

$$\|\mu^\epsilon F_{V_1^\mu}\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla}(ng)\|_{L^\infty} \|\widehat{\nabla}^2 u\|_{H^\epsilon} + \|\widehat{\nabla} Y\|_{L^\infty} \|\widehat{\nabla} v\|_{H^\epsilon} + \|\mu^\epsilon P_\mu F_{V_1}\|_{l_\mu^2 L_x^2}. \tag{3.75}$$

By using Lemma 21 and 19 we have

$$\begin{aligned}\|\mu^\epsilon P_\mu F_{V_1}\|_{l_\mu^2 L_x^2} &\lesssim \|\widehat{\nabla}^2 Y\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla} v\|_{L_x^2} + \|\widehat{\nabla} u\|_{L^\infty} \|n \widehat{\nabla} g\|_{H^{1+\epsilon}} + \|\widehat{\nabla} F_v\|_{H^\epsilon}\\ &\quad + \|\widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+\epsilon}} \|\hat{\Delta} u\|_{L_x^2} + \|n \hat{R} \cdot \widehat{\nabla} u\|_{H^\epsilon} + \|Y \hat{R} \cdot v\|_{H^\epsilon}.\end{aligned}$$

With the help of Lemma 21 and $\|\tilde{\pi}\|_{H^1} \le C$, we obtain

$$\begin{aligned}\|\hat{\Delta} u\|_{H^\epsilon} &\lesssim \|\mu^\epsilon g^{ij} P_\mu \widehat{\nabla}_{ij}^2 u\|_{l_\mu^2 L_x^2} + \|\mu^\epsilon [P_\mu, g^{ij}] \widehat{\nabla}_{ij}^2 u\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla}^2 u\|_{H^\epsilon},\\ \|n \hat{R} \cdot \widehat{\nabla} u\|_{H^\epsilon} &\lesssim \|\mu^\epsilon [P_\mu, n \hat{R}] \widehat{\nabla} u\|_{l_\mu^2 L_x^2} + \|\widehat{\nabla} u\|_{H^\epsilon} \lesssim \|\widehat{\nabla} u\|_{H^\epsilon}\end{aligned}$$

Similarly, we have with the help of $\|\widehat{\nabla} Y\|_{H^1} \le C$ that $\|Y \cdot \hat{R} \cdot v\|_{H^\epsilon} \lesssim \|v\|_{H^\epsilon}$. Therefore

$$\|\mu^\epsilon P_\mu F_{V_1}\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla} u\|_{L^\infty} \|n \widehat{\nabla} g\|_{H^{1+\epsilon}} + \|\widehat{\nabla} F_v\|_{H^\epsilon} + \mathcal{I}(\epsilon, u, v) + \|\widehat{\nabla} u, v\|_{H^\epsilon}.$$

Combining this estimate with (3.75) we thus obtain (3.74). $\square$

In the following we will derive the estimates on $\|\widehat{\nabla}^2 g\|_{H^\epsilon}$ and $\|\widehat{\nabla} k\|_{H^\epsilon}$. Recall the energy $\mathcal{E}^{(0)}(u, v)$ defined in (2.41). Let P_μ be the Littlewood-Paley projection with frequency size μ, we can introduce

$$\mathcal{E}_\mu^{(1)}(t) = \mathcal{E}_\mu^{(1)}(u, v) := \mathcal{E}^{(0)}(P_\mu \widehat{\nabla} u, P_\mu \widehat{\nabla} v),$$

and the energy

$$\mathcal{E}^{(1+\epsilon)}(u, v)(t) := \sum_{\mu > 1} \mu^{2\epsilon} \mathcal{E}_\mu^{(1)}(u, v)(t) + \sum_{i=0}^{1} \mathcal{E}^{(i)}(u, v)(t). \tag{3.76}$$

In view of (2.42) we can derive that

$$\partial_t \mathcal{E}_\mu^{(1)}(t) \le \left(\|k, \widehat{\nabla} Y\|_{L^\infty} + 1\right) \mathcal{E}_\mu^{(1)}(t) + \|\widehat{\nabla} F_{U_1^\mu}\|_{L^2} \|\widehat{\nabla} U_1^\mu\|_{L^2} + \|V_1^\mu\|_{L^2} \|F_{V_1^\mu}\|_{L^2}. \tag{3.77}$$

Hence, by the Cauchy-Schwartz inequality, we obtain

$$\partial_t\left(\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t)\right) \le \left(\|k, \widehat{\nabla} Y\|_{L^\infty}+1\right)\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t) + \|\mu^\epsilon \widehat{\nabla} F_{U_1^\mu}\|_{l^2_\mu L^2}\|\mu^\epsilon\widehat{\nabla} U_1^\mu\|_{l^2_\mu L^2} + \|\mu^\epsilon V_1^\mu\|_{l^2_\mu L^2}\|\mu^\epsilon F_{V_1^\mu}\|_{l^2_\mu L^2}. \tag{3.78}$$

We will apply (3.78) with $(u, v) = (g, -2k)$ and (2.38) to derive energy estimates. Lemma 6 will be used to estimate the terms $\|\mu^\epsilon\widehat{\nabla} F_{U_1^\mu}\|_{l^2_\mu L^2}$ and $\|\mu^\epsilon F_{V_1^\mu}\|_{l^2_\mu L^2}$ which involve terms related to F_u and F_v. The following result gives such estimates.

Lemma 7. *For* $0 < \epsilon < 1/2$ *there hold*

$$\|\Lambda^\epsilon \widehat{\nabla}^2 F_u\|_{L^2_x} + \|\Lambda^\epsilon\widehat{\nabla} F_v\|_{L^2_x} \lesssim (\|\widehat{\nabla} g, \widehat{\nabla} Y, k, \widehat{\nabla} n\|_{L^\infty_x}+1)\|\widehat{\nabla} g, k\|_{H^{1+\epsilon}}, \tag{3.79}$$

$$\|\Lambda^{\frac{1}{2}+\epsilon}\widehat{\nabla} F_u\|_{L^2_x} \lesssim \|\widehat{\nabla} g, k\|_{H^{1+\epsilon}} + 1. \tag{3.80}$$

Proof. Recall F_u and F_v from (2.38). By straightforward calculation, symbolically we have

$$\widehat{\nabla}^2 F_u = g\cdot\widehat{\nabla}^3 Y + \mathrm{err}_1 + \mathrm{err}_2, \qquad \widehat{\nabla} F_v = \widehat{\nabla}\nabla^2 n + \mathrm{err}_1 + \mathrm{err}_2.$$

where err_1 and err_2 denote the terms introduced in (3.50). Then (3.79) follows from Proposition 3. Applying (6.208) to $F = g$ and $G = \widehat{\nabla} Y$, and using (3.51) and (2.46) we obtain

$$\|\Lambda^{\epsilon+\frac{1}{2}}\widehat{\nabla}(g\cdot\widehat{\nabla} Y)\|_{L^2_x} \lesssim \|\widehat{\nabla} g\|_{H^{1+\epsilon}}\|\widehat{\nabla} Y\|_{H^1} + \|g\|_{L^\infty}\|\widehat{\nabla}^2 Y\|_{H^{\frac{1}{2}+\epsilon}} \lesssim \|\widehat{\nabla} g, k\|_{H^{1+\epsilon}} + 1$$

which gives (3.80). □

Proposition 4. *For* $0 < \epsilon \le s-2$ *there holds*

$$\|\widehat{\nabla}^2 g(t)\|_{H^\epsilon} + \|\widehat{\nabla} k(t)\|_{H^\epsilon} \le C \tag{3.81}$$

and for any pair (u, v) *satisfying* (3.66) *there holds*

$$\mathcal{E}^{(1+\epsilon)}(t)^{\frac{1}{2}} \lesssim \mathcal{E}^{(1+\epsilon)}(t_0)^{\frac{1}{2}} + \int_{t_0}^t \left(\|\widehat{\nabla} u\|_{L^\infty} + \|\widehat{\nabla} F_u\|_{H^{1+\epsilon}} + \|F_v\|_{H^{1+\epsilon}}\right). \tag{3.82}$$

Proof. Now we consider the energy defined by (3.76) for the pair $(u, v) = (g, -2k)$ by using (3.78). In view of (2.28), Propositions 1 and Proposition 2, we have $\mathcal{E}^{(1)}+\mathcal{E}^{(0)} \le C$. Combining this fact with (3.51), we have

$$\mathcal{I}(\epsilon, u, v) \lesssim \left(\|\widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty}+1\right)\sqrt{\mathcal{E}^{(1+\epsilon)}(u, v)}.$$

This together with Lemma 6, Lemma 7 and (3.78) implies

$$\partial_t\left(\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t)\right) \le \left(\|k, \widehat{\nabla} Y\|_{L^\infty}+1\right)\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t) + \left(\|k, \widehat{\nabla} g, \widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty}+1\right)\mathcal{E}^{(1+\epsilon)}(u, v).$$

which, together with the fact $\mathcal{E}^{(1)} + \mathcal{E}^{(0)} \le C$, gives

$$\partial_t\Big(\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t)\Big) \lesssim \big(\|k, \widehat{\nabla} g, \widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty} + 1\big)\Big(\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t) + 1\Big). \tag{3.83}$$

By the bootstrap assumption (BA1), we obtain $\mathcal{E}^{(1+\epsilon)}(u, v)(t) \lesssim \mathcal{E}^{(1+\epsilon)}(u, v)(t_0) + 1$ for $t_0 \le t \le t_0 + T$ which implies (3.81).

From (3.81), (3.51) and Sobolev embedding, it follows that

$$\|\widehat{\nabla} Y, \widehat{\nabla} n, ne_0(n)\|_{L^\infty} + \|\Lambda^{1/2+\epsilon}(\widehat{\nabla}^2 Y, \widehat{\nabla}^2 n, \widehat{\nabla}(ne_0(n)))\|_{L^2} \lesssim \|\widehat{\nabla}(\widehat{\nabla} g, k)(t_0)\|_{H^\epsilon} + 1. \tag{3.84}$$

Thus for any pair (u, v) satisfying (3.66) there holds $\mathcal{I}(\epsilon, u, v) \lesssim \|\widehat{\nabla} u, v\|_{H^{1+\epsilon}}$ on Σ_t. Now we prove (3.82). We will rely on Lemma 6 to treat $\|\mu^\epsilon \widehat{\nabla} F_{U_1^\mu}\|_{l^2_\mu L^2}$ and $\|\mu^\epsilon F_{V_1^\mu}\|_{l^2_\mu L^2}$ in (3.78). Note that by using (6.208) we can derive $\|n\widehat{\nabla} g\|_{H^{1+\epsilon}} \lesssim \|\widehat{\nabla} g\|_{H^{1+\epsilon}} \lesssim 1$. We then obtain from (3.78) that

$$\begin{aligned}\partial_t\Big(\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t)\Big) &\le \big(\|k, \widehat{\nabla} Y, \widehat{\nabla} g\|_{L^\infty} + 1\big)\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t)\\ &\quad + (\|\widehat{\nabla}(\widehat{\nabla} F_u, F_v)\|_{H^\epsilon} + \|\widehat{\nabla} u\|_{L^\infty} + \|\widehat{\nabla} u, v\|_{H^1})\Big(\sum_\mu \mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t)\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.85}$$

Now we consider lower order energy $\mathcal{E}^{(1)}(u, v)$, by using (3.67) for $(U_1, V_1) = (\widehat{\nabla} u, \widehat{\nabla} v)$ and

$$\|\widehat{\nabla} F_{U^1}, F_{V^1}\|_{L^2_x} \lesssim \|\widehat{\nabla}(\widehat{\nabla} F_u, F_v)\|_{L^2_x} + \|\widehat{\nabla} g\|_{L^\infty}\|\widehat{\nabla}^2 u\|_{L^2} + \|\widehat{\nabla} u, v\|_{H^1} + \|u\|_{L^2_x}$$

which can be derived by using Sobolev embedding and (3.84). By (2.42), we obtain

$$\begin{aligned}\partial_t\mathcal{E}^{(1)}(u, v)(t) &\lesssim \big(\|k, \widehat{\nabla} Y, \widehat{\nabla} g\|_{L^\infty} + 1\big)\mathcal{E}^{(1)}(u, v)(t)\\ &\quad + \|\widehat{\nabla}^2 u, \widehat{\nabla} v\|_{L^2_x}(\|\widehat{\nabla}(\widehat{\nabla} F_u, F_v)\|_{L^2} + \|\widehat{\nabla} u, v\|_{H^1} + \|u\|_{L^2_x}).\end{aligned} \tag{3.86}$$

For $\mathcal{E}^{(0)}(u, v)(t)$, we employ (2.42) again. Combining (3.85), (3.86), (2.42), Lemma 6 and the Gronwall inequality gives (3.82). □

3.1.2. Geometric wave operator. For the pair (u, v) satisfying (3.66), we can show that u satisfies the geometric wave equation

$$n^2\Box_{\mathbf{g}} u = -(nF_v + ne_0(F_u)) + e_0(n)F_u - n^2\pi_{0a}\nabla^a u + n^2\mathrm{Tr}k\, e_0(u). \tag{3.87}$$

Indeed, relative to an orthonormal frame $e_0 := \mathbf{T}, e_j, j = 1, 2, 3$, by straightforward calculation we have

$$\mathbf{g}^{ij}\mathbf{D}^2_{ij}u = \Delta u + \mathrm{Tr}k\mathbf{D_T}u, \quad \mathbf{D_T}\mathbf{D_T}u = e_0(e_0(u)) + \pi_{0j}\nabla^j u.$$

Since $\Box_{\mathbf{g}} u = -\mathbf{D_T}\mathbf{D_T}u + \mathbf{g}^{ij}\mathbf{D}^2_{ij}u$, we obtain

$$n^2\Box_{\mathbf{g}} u = -n^2 e_0(e_0(u)) + n^2\hat{\Delta} u - n^2\pi_{0j}\nabla^j u + n^2\mathrm{Tr}k\,\mathbf{D_T}u. \tag{3.88}$$

In view of (3.66) we have

$$ne_0(ne_0(u)) = ne_0(n) \cdot v + n^2 \hat{\Delta} u + nF_v + ne_0(F_u).$$

Combining this with (3.88) and using (3.66) we obtain (3.87) as desired.

Therefore, if ψ is a solution of the geometric wave equation

$$\Box_{\mathbf{g}} \psi = W, \tag{3.89}$$

we can check by (3.87) that $(u, v) := (\psi, e_0(\psi))$ satisfies the hyperbolic system

$$\begin{aligned} &e_0(u) = v, \quad ne_0(v) = n\hat{\Delta} u + F_v, \\ &F_u = 0, \qquad F_v = -nW - n\pi_{0j} \nabla^j u + nv\mathrm{Tr}k. \end{aligned} \tag{3.90}$$

Lemma 8. *Let ψ be a scalar function satisfying the geometric wave equation $\Box_{\mathbf{g}} \psi = 0$. Then for $0 < \epsilon \le s - 2$ there holds*

$$\|\widehat{\nabla} F_v\|_{H^\epsilon} \lesssim \|\widehat{\nabla}(\widehat{\nabla}\psi, e_0(\psi))\|_{H^\epsilon}. \tag{3.91}$$

Proof. Indeed, in view of (6.208) and $\mathrm{Tr}k = t$ we have

$$\|\widehat{\nabla} F_v\|_{H^\epsilon} \lesssim \|\widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla}^2 u\|_{L_x^2} + \|\widehat{\nabla} n\|_{L_x^\infty} \|\widehat{\nabla}^2 u\|_{H^\epsilon} + \|\widehat{\nabla} v\|_{H^\epsilon}.$$

Since $0 < \epsilon \le s - 2$, we have from (3.84) that $\|\widehat{\nabla} F_v\|_{H^\epsilon} \lesssim \|\widehat{\nabla}^2 u\|_{L_x^2} + \|\widehat{\nabla}(\widehat{\nabla} u, v)\|_{H^\epsilon}$ which gives the estimate. $\Box$

It is standard to derive for ψ satisfying $\Box_{\mathbf{g}} \psi = 0$ that

$$\|\mathbf{D}\psi(t)\|_{L^2} \lesssim \|\mathbf{D}\psi(t_1)\|_{L^2}, \quad t_0 \le t_1 \le t \le t_0 + T. \tag{3.92}$$

We now give the following energy estimate.

Proposition 5. *Let ψ be a scalar function satisfying the geometric wave equation $\Box_{\mathbf{g}} \psi = 0$. Then for any $0 < \epsilon < 1/2$ and $t_0 \le t_1 \le t \le t_0 + T$ there hold the energy estimates*

$$\mathcal{E}^{(i)}(\psi, e_0(\psi))(t) \lesssim \sum_{0 \le j \le i} \mathcal{E}^{(j)}(\psi, e_0(\psi))(t_1), \quad i = 0, 1. \tag{3.93}$$

Under the assumption that

$$\|\widehat{\nabla}\psi\|_{L^1_{[t_0, t_0+T]} L^\infty} \le B_0 \big(\mathcal{E}^{1+\epsilon}(\psi, e_0\psi)(t_0)\big)^{\frac{1}{2}} \tag{BA4}$$

there holds

$$\mathcal{E}^{(1+\epsilon)}(\psi, e_0(\psi))(t) \le C(B_0^2 + 1)\mathcal{E}^{(1+\epsilon)}(\psi, e_0(\psi))(_0).$$

Proof. Since $(u, v) = (\psi, e_0(\psi))$ satisfies (3.90) with $W = 0$, we can easily derive that

$$\|F_v\|_{L_x^2} \lesssim (\|\widehat{\nabla} n\|_{L_x^\infty} + \|\mathrm{Tr}k\|_{L_x^\infty}) \left(\mathcal{E}^{(0)}(u, v)\right)^{\frac{1}{2}}$$

Recall that $F_u = 0$, an application of (2.42) gives (3.93) with $i = 0$. The case $i = 1$ can be proved by employing (2.42) with $(U^1, V^1) := (\widehat{\nabla}\psi, \widehat{\nabla}(e_0(\psi)))$. Indeed, in view of (3.67) and (3.84), we have

$$\|F_{V^1}, \widehat{\nabla} F_{U^1}\|_{L^2_x} \lesssim \|\widehat{\nabla} F_v\|_{L^2_x} + (\mathcal{E}^{(1)}(u,v))^{\frac{1}{2}}(1 + \|\widehat{\nabla} g\|_{L^\infty_x}) + \mathcal{E}^{(0)}(u,v)^{\frac{1}{2}},$$

and

$$\begin{aligned}\|\widehat{\nabla} F_v\|_{L^2_x} &\lesssim (\|\widehat{\nabla}\nabla n\|_{L^3_x} + \|\widehat{\nabla} n\|_{L^\infty_x})\|\widehat{\nabla} u\|_{H^1} + (\|\widehat{\nabla} v\|_{L^2_x} + \|\widehat{\nabla} n\|_{L^\infty_x}\|v\|_{L^2_x})\|\mathrm{Tr} k\|_{L^\infty_x} \\ &\lesssim \left(\mathcal{E}^{(1)}(u,v)(t)\right)^{\frac{1}{2}} + \left(\mathcal{E}^{(0)}(u,v)(t)\right)^{\frac{1}{2}}.\end{aligned}$$

By substituting to (2.42), we can complete the proof of (3.93). Using (3.82), (3.91) and the Gronwall inequality, we can complete the proof of Proposition 5. □

3.2. Flux. In view of (2.39), we can see that $(u, v) := (\widehat{\nabla} g, -2\widehat{\nabla} k)$ satisfies (3.66) with

$$\begin{cases} F_u = g\widehat{\nabla}^2 Y + \widehat{\nabla} Y \cdot \widehat{\nabla} g + Y \cdot \hat{R} \cdot g + \widehat{\nabla} n \cdot k, \\ F_v = \widehat{\nabla}(2\nabla^2 n + nk * k + k * \widehat{\nabla} Y) + \widehat{\nabla} n \hat{\Delta} g + \widehat{\nabla} Y \cdot \widehat{\nabla} k + Y \cdot \hat{R} \cdot k \\ \qquad +n(g \cdot \widehat{\nabla} g \cdot \widehat{\nabla}^2 g + g \cdot \hat{R} \cdot \widehat{\nabla} g + g \cdot \widehat{\nabla} \hat{R} \cdot g). \end{cases} \tag{3.94}$$

By straightforward calculation we have

$$\widehat{\nabla} F_u = g \cdot \widehat{\nabla}^3 Y + \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R}) \quad \text{and} \quad F_v = \widehat{\nabla}^3 n + \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R}).$$

Next we give the first order hyperbolic system for the pair (k, E). Recall that (see [1, (3.11a)])

$$\begin{aligned} n^{-1}(\partial_t - \mathcal{L}_Y)E_{ij} = \mathrm{curl}\, H_{ij} - n^{-1}(\nabla n \wedge H)_{ij} - \frac{5}{2}(E \times k)_{ij} \\ - \frac{2}{3}(E * k)g_{ij} - \frac{1}{2}\mathrm{Tr} k E_{ij}, \end{aligned} \tag{3.95}$$

where $\mathrm{curl}\, F_{ab} = \frac{1}{2}(\epsilon_a{}^{cd}\nabla_d F_{cb} + \epsilon_b{}^{cd}\nabla_d F_{ca})$, for any symmetric 2-tensor F, with $\epsilon_a{}^{cd}$ denoting the components of the volume form of (Σ_t, g). When $\mathrm{div}\, F = 0$ and $\mathrm{Tr} F = t$, symbolically we can obtain the identity

$$\mathrm{curl}\,\mathrm{curl}\, F = -\hat{\Delta} F + \mathrm{Ric} * F + \widehat{\nabla} g \cdot \widehat{\nabla} g \cdot g \cdot F + g \cdot \hat{R} \cdot F + g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} F \tag{3.96}$$

In view of $\mathrm{curl}\, k = -H$, we can use (3.96) with $F = k$ to treat the term $\mathrm{curl}\, H$ in (3.95). Consequently we obtain

$$n^{-1}(\partial_t - \mathcal{L}_Y)E_{ij} = \hat{\Delta} k + \mathrm{Ric} * k + \widehat{\nabla} g \cdot \widehat{\nabla} g \cdot k \cdot g + n^{-1}\nabla n * H + k * k * k. \tag{3.97}$$

By coupling (3.97) with (1.4), we can see that the pair $(u, v) := (k, E)$ satisfies the first order hyperbolic system (3.66) with

$$\begin{cases} F_u = \nabla^2_{ij} n + nk * k + k \cdot \widehat{\nabla} Y, \\ F_v = n\mathrm{Ric} * k + ng \cdot \widehat{\nabla} g \cdot \widehat{\nabla} k + \nabla n * H + E \cdot \widehat{\nabla} Y \\ \qquad + ng \cdot k^3 + ng \cdot (\widehat{\nabla} g)^2 \cdot k + ng \cdot \hat{R} \cdot k. \end{cases} \tag{3.98}$$

Using the Gauss equation $E = Ric + k * k$ and (2.37) to treat E, we have

$$\widehat{\nabla} F_u = \widehat{\nabla}^3 n + \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R}) \quad \text{and} \quad F_v = \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R}).$$

In view of Proposition 3, we obtain

Proposition 6 (Remainder estimates). *Let* (F_u, F_v) *be defined by either* (3.94) *or* (3.98). *Then for any* $0 < \epsilon < 1/2$ *there hold*

$$\begin{cases} \|\widehat{\nabla} F_u\|_{H^\epsilon} + \|F_v\|_{H^\epsilon} \lesssim (\|\widehat{\nabla} g, k\|_{L^\infty} + 1)\|\widehat{\nabla} g, k\|_{H^{1+\epsilon}}, \\ \|F_u\|_{H^{1/2+\epsilon}} \lesssim \|\Lambda^\epsilon \widehat{\nabla}(k, \widehat{\nabla} g)\|_{L^2} \end{cases} \tag{3.99}$$

and

$$\|\widehat{\nabla} F_u\|_{L^2} + \|F_v\|_{L^2} \lesssim (\|\widehat{\nabla} g, k\|_{L^\infty} + 1)\|\widehat{\nabla} g, k\|_{H^1}. \tag{3.100}$$

We now fix a point p in $\Sigma \times I$ and use Γ^+ to denote the time integral curve through p of the forward unit normal $\mathbf{T}$ with $\Gamma^+(t_p) = p$. We use Γ_t to denote the intersection point of Γ^+ with Σ_t. Let u be the outgoing solution of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ satisfying the initial condition $u(\Gamma_t) = t - t_p$ on the time axis. We call this u an optical function. We denote by C_u the level sets of u which are the outgoing null cones with vertex on Γ^+. Let $S_{t,u} = C_u \cap \Sigma_t$ and let $\{e_1, e_2\}$ be an orthonormal frame on $S_{t,u}$. Let N be the outward unit normal along Σ_t to the surface $S_{t,u}$. We define

$$\mathbf{b}^{-1} := \mathbf{T}(u), \quad L := \mathbf{T} + N, \quad \underline{L} := \mathbf{T} - N = 2\mathbf{T} - L \tag{3.101}$$

and call $L, \underline{L}$ the canonical null pair. Then $\{e_1, e_2, e_3 := \underline{L}, e_4 := L\}$ forms a null frame.

For $0 < u \le t_0 + T - t_p$ and $t_p \le t_1 \le t_2 \le t_0 + T$, we introduce the null hypersurface $\mathcal{H} := \cup_{t_1 \le t \le t_2} S_{t,u}$. We will use D^+ to denote the region enclosed by $\mathcal{H}$, Σ_{t_1} and Σ_{t_2}. For any scalar function ψ we introduce the flux

$$\mathcal{F}[\psi] = \int_{\mathcal{H}} \left(|L\psi|^2 + \gamma^{AB} \nabla\!\!\!/_A \psi \nabla\!\!\!/_B \psi\right),$$

where γ is the induced metric on $S_{t,u}$ and $\nabla\!\!\!/$ is the corresponding covariant differentiation. For any scalar functions ϕ and ψ we introduce the energy-momentum tensor

$$Q[\phi, \psi]_{\mu\nu} = \frac{1}{2}(\mathbf{D}_\mu \phi \mathbf{D}_\nu \psi + \mathbf{D}_\nu \phi \mathbf{D}_\mu \psi) - \frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha \phi \mathbf{D}_\beta \psi). \tag{3.102}$$

Let $Q[\psi] := Q[\psi, \psi]$ and define the energy

$$\mathcal{Q}(\psi)(t) := \int_{\Sigma_t} Q[\psi](\mathbf{T}, \mathbf{T}) d\mu_g.$$

It is straightforward to check $\mathcal{F}[\psi] = 2\int_{C_u} Q[\psi](\mathbf{T}, L)$.

For any Σ-tangent tensor field F_μ, we set

$$\nabla_L F_i = e_i^\mu L^\nu \mathbf{D}_\nu F_\mu, \qquad \nabla\!\!\!/_A F_i = e_i^\mu \nabla_A F_\mu \tag{3.103}$$

and introduce the norms

$$|\nabla_L F|_g^2 := g^{ij}\mathbf{D}_L F_i \mathbf{D}_L F_j, \qquad |\nabla\!\!\!/ F|_g^2 := \gamma^{AB} g^{ij} \nabla\!\!\!/_A F_i \nabla\!\!\!/_B F_j.$$

We will drop the subscript g in the definition of norms whenever there occurs no confusion.

Following the same proof in [25, Section 5], we can obtain the following result on tensorial k-flux.

Proposition 7. *Under the bootstrap assumption* (BA1)*, for the tensorial k-flux there holds on the null cone C_u the estimate*

$$\int_{C_u}\left(|\nabla\!\!\!/ k|_g^2+|\nabla_L k|_g^2\right)\le C.$$

The following estimate is the main result of this subsection.

Proposition 8. *Let the bootstrap assumptions* (BA1) *and* (BA2) *hold. Let f be the scalar components of $\widehat{\nabla} g$ and k. Then for $0<\epsilon\le s-2$ there holds*

$$\mathcal{F}^{\frac{1}{2}}[f]+\|\mu^\epsilon\mathcal{F}^{\frac{1}{2}}[P_\mu f]\|_{l_\mu^2}\le C.$$

In the following we will give the proof of Proposition 8. By the standard energy estimate we have

$$\begin{aligned}\mathcal{F}[\psi]&\le|Q(\psi)(t_2)-Q(\psi)(t_1)|\\&\quad+\int_{t_1}^{t_2}Q^{\frac{1}{2}}(\psi)(t')\|\Box_{\mathbf{g}}\psi\|_{L_x^2}+\int_{t_1}^{t_2}C(\|\pi\|_{L_x^\infty}+1)Q(\psi)(t')dt'.\end{aligned}\tag{3.104}$$

Recall that $(\widehat{\nabla}g,-2\widehat{\nabla}k)$ and (k,E) satisfy the first order hyperbolic system (3.66). Thus, for $\psi=\widehat{\nabla}g$ or k, the expression of $\Box_{\mathbf{g}}\psi$ derived from (3.87) contains time derivatives of the shift vector field Y since F_u contains the term $\psi\cdot\widehat{\nabla}Y$ and other terms involving Y. The lack of control on $\mathbf{D_T}Y$ makes it impractical to apply (3.104) directly to $\psi=\widehat{\nabla}g,k$.

To get around the difficulty, we consider the following modified energy current

$$\tilde{P}_\mu=-n^{-1}F_u\mathbf{D}_\mu u+Q_{\mu\nu}\mathbf{T}^\nu+\frac{1}{2}(n^{-1}F_u)^2 n\mathbf{D}_\mu(t).$$

When (u,v) satisfies the system (3.66), u must satisfy (3.87). We claim that

$$\mathbf{D}^\mu\tilde{P}_\mu=\left(-\pi_{0a}\nabla^a u+\mathrm{Tr}ke_0(u)-n^{-1}F_v\right)v-\mathbf{D}^i(n^{-1}F_u)\mathbf{D}_i u+Q_{\mu\nu}{}^{(\mathbf{T})}\pi^{\mu\nu}.\tag{3.105}$$

Indeed, since (3.66) implies $-n^{-1}F_u+\mathbf{D_T}u=v$, we have from the definition of $\tilde{P}_\mu$ that

$$\begin{aligned}\mathbf{D}^\mu\tilde{P}_\mu&=-\mathbf{D}^\mu(n^{-1}F_u)\mathbf{D}_\mu u-n^{-1}F_u\Box_{\mathbf{g}}u+\mathbf{D}^\mu Q_{\mu\nu}\mathbf{T}^\nu+Q_{\mu\nu}{}^{(\mathbf{T})}\pi^{\mu\nu}\\&\quad+(n^{-1}F_u)\mathbf{D}^\mu(n^{-1}F_u)n\mathbf{D}_\mu t\\&=\mathbf{D}_0(n^{-1}F_u)\mathbf{D}_0u-\mathbf{D}^i(n^{-1}F_u)\mathbf{D}_iu-n^{-1}F_u\Box_{\mathbf{g}}u+\Box_{\mathbf{g}}u\mathbf{D_T}u+Q_{\mu\nu}{}^{(\mathbf{T})}\pi^{\mu\nu}\\&\quad+(n^{-1}F_u)\mathbf{D}^\mu(n^{-1}F_u)n\mathbf{D}_\mu t\\&=(-n^{-1}F_u+\mathbf{D_T}u)(\Box_{\mathbf{g}}u+\mathbf{D}_0(n^{-1}F_u))-\mathbf{D}^i(n^{-1}F_u)\mathbf{D}_iu+Q_{\mu\nu}{}^{(\mathbf{T})}\pi^{\mu\nu}\\&=\left(\Box_{\mathbf{g}}u+\mathbf{D}_0(n^{-1}F_u)\right)v-\mathbf{D}^i(n^{-1}F_u)\mathbf{D}_iu+Q_{\mu\nu}{}^{(\mathbf{T})}\pi^{\mu\nu}.\end{aligned}$$

In view of (3.87), we obtain (3.105).

By the divergence theorem we have for $\tilde{P}_\mu$ that

$$\int_{\mathcal{H}}L^\mu\tilde{P}_\mu=\int_{\Sigma_{t_2}\cap\mathcal{D}^+}\tilde{P}_\mu\mathbf{T}^\mu-\int_{\Sigma_{t_1}\cap\mathcal{D}^+}\tilde{P}_\mu\mathbf{T}^\mu-\int_{\mathcal{D}^+}\mathbf{D}^\mu\tilde{P}_\mu.\tag{3.106}$$

Note that

$$\int_{\mathcal{H}} L^\mu \tilde{P}_\mu = \int_{\mathcal{H}} \frac{1}{2}(|\mathbf{D}_L u|^2 + |\nabla\!\!\!/ u|^2 + (n^{-1}F_u)^2) - n^{-1}F_u \mathbf{D}_L u$$
$$= \int_{\mathcal{H}} \frac{1}{2}(\frac{1}{2}|\mathbf{D}_L u|^2 + |\nabla\!\!\!/ u|^2 - (n^{-1}F_u)^2) + (\frac{1}{2}\mathbf{D}_L u - n^{-1}F_u)^2.$$

Thus

$$\int_{\mathcal{H}} \frac{1}{2}(\frac{1}{2}|\mathbf{D}_L u|^2 + |\nabla\!\!\!/ u|^2) \le \int_{\mathcal{H}} L^\mu \tilde{P}_\mu + \frac{1}{2}(n^{-1}F_u)^2.$$

Also using (3.106), we obtain

$$\mathcal{F}[u](\mathcal{H}) \lesssim \int_{\mathcal{H}} \frac{1}{2}(n^{-1}F_u)^2 + \left|\int_{\Sigma_{t_2}\cap\mathcal{D}^+} \tilde{P}_\mu \mathbf{T}^\mu - \int_{\Sigma_{t_1}\cap\mathcal{D}^+} \tilde{P}_\mu \mathbf{T}^\mu\right| + \left|\int_{\mathcal{D}^+} \mathbf{D}^\mu \tilde{P}_\mu\right|. \tag{3.107}$$

Now consider the terms on the right of (3.107). By trace inequality,

$$\int_{\mathcal{H}} (n^{-1}F_u)^2 \lesssim \int_{t_1}^{t_2} \|F_u\|_{H^1} \|F_u\|_{L_x^2}. \tag{3.108}$$

By definition of $\tilde{P}_\mu$ and $C^{-1} < n < C$, for any $0 < t' \le T$,

$$\left|\int_{\Sigma_{t'}\cap\mathcal{D}^+} \mathbf{T}^\mu \tilde{P}_\mu\right| \lesssim \|\mathbf{D}u\|_{L_x^2}^2 + \|F_u\|_{L_x^2}^2. \tag{3.109}$$

For the third term, by (3.105), there holds

$$\left|\int_{\mathcal{D}^+} \mathbf{D}^\mu \tilde{P}_\mu\right| \le \int_{t_0}^{t} \|{}^{(\mathbf{T})}\pi\|_{L_x^\infty} \|\mathbf{D}u\|_{L_x^2} (\|v\|_{L_x^2} + \|\mathbf{D}u\|_{L_x^2})$$
$$+ \int_{\mathcal{D}^+} \left|n^{-1} v F_v + \mathbf{D}^i(n^{-1}F_u)\mathbf{D}_i u\right|. \tag{3.110}$$

Proof (Proof of Proposition 8). We first apply (3.110) to the modified energy current $\tilde{P}_\mu$ corresponding to $(u, v) = (\widehat{\nabla} g, -2\widehat{\nabla} k)$ or (k, E). In view of (3.100), we obtain

$$\left|\int_{\mathcal{D}^+} \mathbf{D}^\mu \tilde{P}_\mu\right| \lesssim \|\widehat{\nabla} g, k\|_{L_t^1 L_x^\infty} \|\widehat{\nabla} g, k\|_{L_t^\infty H^1}^2.$$

By Proposition 2 and (3.100) we have

$$\left|\int_{\Sigma_{t'}\cap\mathcal{D}^+} \mathbf{T}^\mu \tilde{P}_\mu\right| \le C \quad \text{and} \quad \int_{\mathcal{H}} (n^{-1}F_u)^2 \le C.$$

Therefore we can conclude that $\mathcal{F}[\widehat{\nabla} g, k] \le C$.

Recall again that $(u, v) = (\widehat{\nabla} g, -2\widehat{\nabla} k)$ and (k, E) satisfy (3.66) with (F_u, F_v) given by (3.94) and (3.98) respectively. Then the pair $(U^\mu, V^\mu) = (P_\mu u, P_\mu v)$ satisfies (3.66) with (F_{U^μ}, F_{V^μ}) given by in (3.68). In view of (3.70)–(3.72), (3.81), and Proposition 6, we have

$$\|\mu^{\frac{1}{2}+\epsilon} F_{U^\mu}\|_{l_\mu^2 L^2} + \|\mu^{-\frac{1}{2}+\epsilon}\widehat{\nabla} F_{U^\mu}\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla}(\widehat{\nabla} g, k)\|_{H^\epsilon} \lesssim 1, \tag{3.111}$$

$$\|\mu^{\epsilon}\widehat{\nabla} F_{U^\mu}\|_{l_\mu^2 L^2} \lesssim \left(\|k, \widehat{\nabla} g\|_{L_x^\infty} + 1\right)\|\widehat{\nabla}(\widehat{\nabla} g, k)\|_{H^\epsilon} \lesssim \|k, \widehat{\nabla} g\|_{L_x^\infty} + 1, \tag{3.112}$$

$$\|\mu^{\epsilon} F_{V^\mu}\|_{l_\mu^2 L^2} \lesssim \left(\|k, \widehat{\nabla} g\|_{L_x^\infty} + 1\right)\|\widehat{\nabla}(\widehat{\nabla} g, k)\|_{H^\epsilon} \lesssim \|k, \widehat{\nabla} g\|_{L_x^\infty} + 1. \tag{3.113}$$

Define

$$\mathcal{B}_\mu^{(1)} = \mu^{2\epsilon}\|F_{U^\mu}\|_{H^1}\|F_{U^\mu}\|_{L_x^2}. \tag{3.114}$$

Similar to (3.108), we have $\int_{\mathcal{H}} \mu^{2\epsilon}(n^{-1}F_{U^\mu})^2 \lesssim \int_{t_1}^{t_2} \mathcal{B}_\mu^{(1)}$. Using (3.111), it yields

$$\sum_{\mu>1} \mathcal{B}_\mu^{(1)} \lesssim \|\mu^{\epsilon-\frac{1}{2}} F_{U^\mu}\|_{l_\mu^2 H^1}\|\mu^{\epsilon+\frac{1}{2}} F_{U^\mu}\|_{l_\mu^2 L_x^2} \le C \tag{3.115}$$

In view of (3.110), it follows that

$$\sum_{\mu>1}\left|\int_{\mathcal{D}^+} \mu^{2\epsilon}\mathbf{D}^\alpha \tilde{P}_\alpha\right| \lesssim \int_{t_1}^{t_2}\Bigg(\|^{(\mathbf{T})}\pi\|_{L_x^\infty}\sum_{\mu>1}\|\mu^\epsilon(P_\mu\partial u, P_\mu v)\|_{L_x^2}^2$$
$$+\sum_{\mu>1}\|\mu^\epsilon F_{V^\mu}\|_{L_x^2}\|\mu^\epsilon P_\mu v\|_{L_x^2} + \sum_{\mu>1}\|\mu^\epsilon F_{U^\mu}\|_{H_x^1}\|\mu^\epsilon \mathbf{D}_i P_\mu u\|_{L_x^2}\Bigg).$$

Using (3.111)–(3.113), we obtain

$$\sum_{\mu>1}\left|\int_{\mathcal{D}^+} \mu^{2\epsilon}\mathbf{D}^\alpha \tilde{P}_\alpha\right| \lesssim \|^{(\mathbf{T})}\pi, \widehat{\nabla} g, \widehat{\nabla} Y\|_{L_t^1 L_x^\infty}\|\widehat{\nabla}(\widehat{\nabla} g, k)\|_{H^\epsilon}^2 \le C. \tag{3.116}$$

In view of (3.109), (3.111) and (3.81),

$$\sum_{\mu>1}\left|\int_{\Sigma_{t'}\cap\mathcal{D}^+} \mu^{2\epsilon}\mathbf{T}^\alpha \tilde{P}_\alpha\right| \lesssim \sum_{\mu>1}\mu^{2\epsilon}(\|\mathbf{D} P_\mu u\|_{L_x^2}^2 + \|F_{U^\mu}\|_{L_x^2}^2) \le C. \tag{3.117}$$

We conclude in view of (3.115), (3.116) and (3.117) that

$$\sum_{\mu>1}\mu^{2\epsilon}(\mathcal{F}[P_\mu\widehat{\nabla} g] + \mathcal{F}[P_\mu k]) \le C.$$

The proof is thus complete. □

4. Strichartz Estimate and Main Estimates

In this section we will show that (BA1) and (BA2) can be improved. For ease of exposition, we shift the origin of time coordinate to t_0 and consider $[0, T] \times \Sigma$. Now we make the following additional bootstrap assumption: there is a constant B_0 such that

$$\|\mu^\delta P_\mu \widehat{\nabla} g\|_{l^2_\mu L^2_{[0,T]} L^\infty_x} + \|\mu^\delta P_\mu k\|_{l^2_\mu L^2_{[0,T]} L^\infty_x} \le B_0, \tag{BA3}$$

where $0 < \delta < s - 2$ is a sufficiently small number. As an immediate consequence of (BA2), (BA3), and (6.209), there holds

$$\|\mu^\delta P_\mu (g \widehat{\nabla} g)\|_{l^2_\mu L^2_{[0,T]} L^\infty_x} \lesssim B_0.$$

This estimate will always be used together with (BA3). Our goal is to show that the estimates in (BA1), (BA2) and (BA3) can be improved by shrinking the time interval if necessary. We will achieve this by establishing the following main estimates.

Theorem 2 (Main estimates). *Let* (BA2) *and* (BA3) *hold for some sufficiently small number* $0 < \delta < s - 2$. *Then for any number* $q > 2$ *that is sufficiently close to* 2 *there holds*

$$\|\widehat{\nabla} g, k\|_{L^2_{[0,T]} L^\infty_x} + \|\mu^\delta P_\mu (\widehat{\nabla} g, k)\|_{l^2_\mu L^2_{[0,T]} L^\infty_x} \lesssim T^{\frac{1}{2} - \frac{1}{q}}.$$

If ϕ *is a function satisfying* $\Box_{\mathbf{g}} \phi = 0$ *then there holds*

$$\|\partial \phi\|^2_{L^2_I L^\infty_x} + \|\mu^\delta P_\mu \partial \phi\|^2_{l^2_\mu L^2_I L^\infty_x} \le C T^{1 - \frac{2}{q}} \|\widehat{\nabla} \phi, e_0 \phi\|^2_{H^{1+\epsilon}(0)}.$$

4.1. Decay estimate $\Rightarrow$ *Strichartz estimates.* Let us rescale the coordinate $(t, x) \to (\frac{t}{\lambda}, \frac{x}{\lambda})$ for some positive constant λ. We first prove Strichartz estimate by assuming the following decay estimate.

Theorem 3 (Decay estimate). *Let* $0 < \epsilon_0 < s - 2$ *be a given number. There exists a large number* Λ *such that for any* $\lambda \ge \Lambda$ *and any solution* ψ *of the equation*

$$\Box_{\mathbf{g}} \psi = 0 \tag{4.118}$$

on the time interval $I_* = [0, t_*]$ *with* $t_* \le \lambda^{1 - 8\epsilon_0} T$ *there is a function* $d(t)$ *satisfying*

$$\|d\|_{L^{\frac{q}{2}}} \lesssim 1, \text{ for } q > 2 \text{ sufficiently close to } 2 \tag{4.119}$$

such that for any $0 \le t \le t_*$ *there holds*

$$\|P e_0 \psi(t)\|_{L^\infty_x} \le \left(\frac{1}{(1+t)^{\frac{2}{q}}} + d(t) \right) \sum_{m=0}^{3} \|\widehat{\nabla}^m \tilde{\psi}[0]\|_{L^1_x}, \tag{4.120}$$

where $\tilde{\psi}[0] := (\psi(0), n^{-2} \partial_t \psi(0))$ *and* $\|\widehat{\nabla}^m \tilde{\psi}[0]\|_{L^1} := \|\widehat{\nabla}^m \psi(0)\|_{L^1} + \|\widehat{\nabla}^{m-1} (n^{-2} \partial_t \psi(0))\|_{L^1}$.

Using Theorem 3, we can prove the following result.

Theorem 4 (Dyadic Strichartz estimate). *There is a large universal constant* C_0 *such that if on the time interval* $I_* := [0, t_*]$ *there holds*

$$C_0 \|\pi, \widehat{\nabla} g, \widehat{\nabla} Y\|_{L^1_{I_*} L^\infty_x} \le 1, \tag{4.121}$$

then for any ϕ *satisfying the wave equation* $\Box_{\mathbf{g}} \phi = 0$ *and* $q > 2$ *sufficiently close to 2, there holds*

$$\|P \partial \phi\|_{L^q_{I_*} L^\infty_x} \lesssim \|\phi[0]\|_{H^1}, \tag{4.122}$$

where P *denote the Littlewood-Paley projection on the frequency domain* $\{1/2 \le |\xi| \le 2\}$.

We will prove Theorem 4 by adapting a $\mathcal{T}\mathcal{T}^*$ argument from [12,13]. Applying the $\mathcal{T}\mathcal{T}^*$ argument therein directly to our setting requires the control over $\partial \mathbf{g}$ including the undesired quantity $\mathbf{D_T} Y$. To get around this difficulty, we give a careful refinement.

Definition 1. *Let $\omega := (\omega_0, \omega_1) \in H^1(\Sigma) \times L^2(\Sigma)$. We denote by $\phi(t; s, \omega)$ the unique solution of the homogeneous geometric wave equation $\Box_{\mathbf{g}} \phi = 0$ satisfying the initial condition $\phi(s; s, \omega) = \omega_0$ and $\boldsymbol{D}_0 \phi(s; s, \omega) = \omega_1$. We set $\Phi(t; s, \omega) := (\phi(t; s, \omega), \boldsymbol{D}_0 \phi(t; s, \omega))$. By uniqueness we have $\Phi(t; s, \Phi(s; t_0, \omega)) = \Phi(t; t_0, \omega)$.*

We first show that

$$\|P(e_0 \phi)\|_{L^q_{I_*} L^\infty_x} \lesssim \|\phi[0]\|_{H^1}. \tag{4.123}$$

To this end, we let $\mathcal{H} := H^1(\Sigma) \times L^2(\Sigma)$ endowed with the inner product

$$\langle \omega, v \rangle = \int_\Sigma \left(\omega_1 \cdot v_1 + \delta^{ij} \mathbf{D}_i \omega_0 \cdot \mathbf{D}_j v_0 \right)$$

relative to the orthornormal frame $\{e_0 = \mathbf{T}, e_i = 1, 2, 3\}$. Let $I = [t', t_*]$ with $0 \le t' \le t_*$ and let $X = L^q_I L^\infty_x$. Then the dual of X is $X' = L^{q'}_I L^1_x$, where $1/q' + 1/q = 1$. Let $\mathcal{T}(t') : \mathcal{H} \to X$ be the linear operator defined by

$$\mathcal{T}(t') \omega := P \mathbf{D}_0 \phi(t; t', \omega), \tag{4.124}$$

where $\phi := \phi(t; t', \omega)$ is the unique solution of $\Box_{\mathbf{g}} \phi = 0$ satisfying $\phi(t') = \omega_0$ and $D_0 \phi(t') = \omega_1$ with $\omega := (\omega_0, \omega_1)$.

By using the Bernstein inequality for LP projections and the energy estimate it is easy to see that $\mathcal{T}(t') : \mathcal{H} \to X$ is a bounded linear operator, i.e.

$$\|\mathcal{T}(t') \omega\|_X = \|P(e_0 \phi)\|_{L^q_I L^\infty_x} \le C(\lambda) \|\mathbf{D} \phi(t')\|_{L^2_x} \tag{4.125}$$

for some constant $C(\lambda)$ possibly depending on λ. Let $M(t') := \|\mathcal{T}(t')\|_{\mathcal{H} \to X}$. Then $M(t') < \infty$, and for the adjoint $\mathcal{T}(t')^* : X' \to \mathcal{H}$ we have

$$\|\mathcal{T}(t')^*\|_{X' \to \mathcal{H}} = M(t'), \quad \|\mathcal{T}(t') \mathcal{T}(t')^*\|_{X' \to X} = M(t')^2.$$

Note that $M(\cdot)$ is a continuous function on I_*, whose maximum, denoted by M, is achieved at certain $t_0 \in [0, t_*)$. Our goal is to show that M is independent of λ. Our strategy is to show that

$$M^2 \le C + \frac{1}{2} M^2 \tag{4.126}$$

for some universal positive constant C independent of λ. Let us set $I_0 = [t_0, t_*]$ and consider $X = L^q_{I_0} L^\infty_x$, $X' = L^{q'}_{I_0} L^1_x$, and the operators $\mathcal{T}(t_0)$ and $\mathcal{T}(t_0)^*$. For convenience, we drop the t_0 in the notation for operators.

We first calculate $\mathcal{T}^* : X' \to \mathcal{H}$. For any $f \in X'$ and $\omega \in \mathcal{H}$ we have

$$\langle \mathcal{T}^* f, \omega \rangle_{\mathcal{H}} = \langle f, \mathcal{T} \omega \rangle_{X', X} = \int_{I_0 \times \Sigma} f P \mathbf{D}_0 \phi = \int_{I_0 \times \Sigma} (P f) \mathbf{D}_0 \phi(t, t_0, \omega).$$

We introduce the function ψ to be the solution of the initial value problem

$$\begin{cases} \Box_{\mathbf{g}} \psi = -P f, & \text{in } [t_0, t_*) \times \Sigma, \\ \psi(t_*) = \partial_t \psi(t_*) = 0. \end{cases} \tag{4.127}$$

Recall the energy momentum tensor $Q[\phi, \psi]$ introduced in (3.102). For any vector field Z we set $P_\mu := Q[\phi, \psi]_{\mu\nu} Z^\nu$. In view of $\Box_{\mathbf{g}}\phi = 0$, it is easy to check that

$$\mathbf{D}^\beta P_\beta = \frac{1}{2}\left((Z\phi)\Box_{\mathbf{g}}\psi + Q[\phi, \psi]_{\alpha\beta}{}^{(Z)}\pi^{\alpha\beta}\right).$$

By the divergence theorem we have

$$\int_{\Sigma_{t_*}} Q[\phi, \psi]_{\mu\nu} Z^\mu \mathbf{T}^\nu - \int_{\Sigma_{t_0}} Q[\phi, \psi]_{\mu\nu} Z^\mu \mathbf{T}^\nu = \int_{I_0\times\Sigma} \mathbf{D}^\beta P_\beta \tag{4.128}$$

which together with the initial conditions in (4.127) implies that

$$\int_{I_0\times\Sigma} -(Z\phi)\Box_{\mathbf{g}}\psi = 2\int_{\Sigma_{t_0}} \mathbf{T}^\alpha P_\alpha + \int_{I_0\times\Sigma} Q[\phi, \psi]_{\alpha\beta}{}^{(Z)}\pi^{\alpha\beta}. \tag{4.129}$$

Now we take $Z = \mathbf{T}$. Then it follows from (4.129) that

$$\int_{I_0\times\Sigma} -\mathbf{D}_0\phi\Box_{\mathbf{g}}\psi = \int_{\Sigma_{t_0}} \mathbf{D}_0\phi\mathbf{D}_0\psi + \delta^{ij}\mathbf{D}_i\phi\mathbf{D}_j\psi + \int_{I_0\times\Sigma} Q[\phi, \psi]_{\alpha\beta}{}^{(\mathbf{T})}\pi^{\alpha\beta}. \tag{4.130}$$

Therefore

$$\langle \mathcal{T}^* f, \omega\rangle_{\mathcal{H}} = \langle \psi[t_0], \omega\rangle_{\mathcal{H}} + l(\omega), \tag{4.131}$$

where $l(\cdot)$ is a linear functional on $\mathcal{H}$ defined by

$$l(\omega) := \int_{I_0\times\Sigma} Q[\phi, \psi]_{\alpha\beta}{}^{(\mathbf{T})}\pi^{\alpha\beta}.$$

We claim that $l(\cdot)$ is a bounded linear functional on $\mathcal{H}$. To see this, let $\omega \in \mathcal{H}$ with $\|\omega\|_{\mathcal{H}} \le 1$. Then by the energy estimate we have $\|\mathbf{D}\phi\|_{L_t^\infty L_x^2} \le \|\omega\|_{\mathcal{H}} \lesssim 1$. Thus

$$|l(\omega)| \le \|\pi\|_{L_t^1 L_x^\infty}\|\mathbf{D}\phi\|_{L_{I_0}^\infty L_x^2}\|\mathbf{D}\psi\|_{L_{I_0}^\infty L_x^2} \lesssim \|\pi\|_{L_t^1 L_x^\infty}\|\mathbf{D}\psi\|_{L_{I_0}^\infty L_x^2}.$$

Hence, by the Riesz representation theorem we have $l(\omega) = \langle R(f), \omega\rangle_{\mathcal{H}}$ for some $R(f) \in \mathcal{H}$ and there is a universal constant C_1 such that

$$\|R(f)\|_{\mathcal{H}} \le C_1\|\pi\|_{L_t^1 L_x^\infty}\|\mathbf{D}\psi\|_{L_{I_0}^\infty L_x^2}.$$

Moreover, we have from (4.131) that $\mathcal{T}^* f = \psi[t_0] + R(f)$ and hence

$$\mathcal{T}\mathcal{T}^* f = \mathcal{T}\psi[t_0] + \mathcal{T}R(f). \tag{4.132}$$

We claim that there is a universal constant C_2 such that

$$\|\mathbf{D}\psi\|_{L_{I_0}^\infty L_x^2} \le C_2 M\|f\|_{L_{I_0}^{q'} L_x^1}. \tag{4.133}$$

Assuming this claim for a moment, it follows from the definition of M that

$$\|\mathcal{T}R(f)\|_{L_{I_0}^q L_x^\infty} \le C_1 C_2 M^2\|\pi\|_{L_t^1 L_x^\infty}\|f\|_{L_{I_0}^{q'} L_x^1}.$$

Thus, if (4.121) holds with $C_0 \geq 2C_1C_2$, then

$$\|\mathcal{T}R(f)\|_{L^q_{I_0}L^\infty_x} \leq \frac{1}{2}M^2\|f\|_{L^{q'}_{I_0}L^1_x}. \tag{4.134}$$

Next we will estimate $\|\mathcal{T}\psi[t_0]\|_{L^q_{I_0}L^\infty_x}$. We set $F := (0, -nPf)$. By the Duhamel principle we have

$$\psi[t] = \int_{t_*}^t \Phi(t; s, F(s))ds.$$

Then $\psi[t_0] = -\int_{t_0}^{t_*} \Phi(t_0; s, F(s))ds$ and thus

$$\begin{aligned}\mathcal{T}\psi[t_0] &= P\left[e_0\phi\left(t; t_0, -\int_{t_0}^{t_*}\Phi(t_0; s, F(s))\right)\right] = -P\left[e_0\left(\int_{t_0}^{t_*}\Phi(t, s, F(s))ds\right)\right]\\ &= -\int_{t_0}^{t_*} P\left[e_0\Phi(t, s, F(s))\right]ds.\end{aligned}$$

It follows from Theorem 3 that

$$\|P\left[e_0\Phi(t, s, F(s))\right]\|_{L^\infty_x} \leq C\left((1+|t-s|)^{-\frac{2}{q}} + d(|t-s|)\right)\sum_{m=0}^{2}\|\widehat{\nabla}^m Pf(s)\|_{L^1_x} \lesssim \|f\|_{L^1_x}.$$

Thus, in view of the Hardy–Littlewood–Sobolev inequality, (4.119) and Hausdorff Young inequality we obtain

$$\|\mathcal{T}\psi[t_0]\|_{L^q_{I_0}L^\infty_x} \lesssim \|f\|_{L^{q'}_{I_0}L^1_x} + \left\|\int_{t_0}^{t_*} d(|t-s|)\|f(s)\|_{L^1_x}ds\right\|_{L^q_{I_0}} \lesssim \|f\|_{L^{q'}_{I_0}L^1_x}. \tag{4.135}$$

Combining (4.132), (4.134) and (4.135), we therefore obtain (4.126).

It remains to prove (4.133). Let $\tilde{\phi}$ be a solution of $\Box_{\mathbf{g}}\tilde{\phi} = 0$ in I_*. Then for any $t_0 \in [0, t_*]$ there holds the energy estimate $\|\mathbf{D}\tilde{\phi}(t)\|_{L^2(\Sigma)} \lesssim \|\mathbf{D}\tilde{\phi}(t_0)\|_{L^2(\Sigma)}$ for $t \in [t_0, t_*]$. Let $t_0 \leq t' < t_*$. Similar to the derivation of (4.130), we have on $I = [t', t_*]$ that

$$\int_{I\times\Sigma} -\mathbf{D}_0\tilde{\phi}\Box_{\mathbf{g}}\psi = \int_{\Sigma_{t'}} \mathbf{D}_0\phi\mathbf{D}_0\psi + \delta^{ij}\mathbf{D}_i\phi\mathbf{D}_j\psi + \int_{I\times\Sigma} Q[\phi, \psi]_{\alpha\beta}{}^{(\mathbf{T})}\pi^{\alpha\beta}, \tag{4.136}$$

which together with $\Box_{\mathbf{g}}\psi = -Pf$ gives

$$\langle\mathbf{D}\psi, \mathbf{D}\tilde{\phi}\rangle(t') \leq \|Pe_0\tilde{\phi}\|_{L^q_I L^\infty_x}\|f\|_{L^{q'}_I L^1_x} + \|{}^{(\mathbf{T})}\pi\|_{L^1_I L^\infty_x}\|\mathbf{D}\psi\|_{L^\infty_t L^2_x}\|\mathbf{D}\tilde{\phi}\|_{L^\infty_I L^2_x}$$

According to definition of M, we can obtain $\|Pe_0\tilde{\phi}\|_{L^q_I L^\infty_x} \leq M\|\mathbf{D}\tilde{\phi}(t')\|_{L^2_x}$. Thus

$$\langle\mathbf{D}\psi, \mathbf{D}\tilde{\phi}\rangle(t') \lesssim \left(M\|f\|_{L^{q'}_I L^1_x} + \|{}^{(\mathbf{T})}\pi\|_{L^1_I L^\infty_x}\|\mathbf{D}\psi\|_{L^\infty_t L^2_x}\right)\|\mathbf{D}\tilde{\phi}(t')\|_{L^2_x}.$$

Since $\mathbf{D}\tilde{\phi}(t')$ can be arbitrary, there is a universal constant C_3 such that

$$\|\mathbf{D}\psi(t')\|_{L^2_x} \leq C_3 M\|f\|_{L^{q'}_I L^1_x} + C_3\|{}^{(\mathbf{T})}\pi\|_{L^1_I L^\infty_x}\|\mathbf{D}\psi\|_{L^\infty_I L^2_x}.$$

Recall that $t' \in [t_0, t_*)$ is arbitrary. Thus, if (4.121) holds with $C_0 \geq 2C_3$ then

$$\|\mathbf{D}\psi\|_{L^\infty_{[t_0,t_*)}L^2_x} \leq C_3 M \|f\|_{L^{q'}_{[t_0,t_*)}L^1_x} + \frac{1}{2}\|\mathbf{D}\psi\|_{L^\infty_{[t_0,t_*)}L^2_x}.$$

This implies claim (4.133) with $C_2 = 2C_3$. The proof of (4.123) is thus completed. We also have proved for any $t \in I_*$

$$\|\mathbf{D}\psi\|_{L^\infty_{[t,t_*)}L^2_x} \leq C_2 M \|f\|_{L^{q'}_{[t,t_*)}L^1_x}. \tag{4.137}$$

Now we consider $\|P(\partial_m\phi)\|_{L^q_{I_*}L^\infty_x}$. It suffices to estimate

$$\mathcal{I} = \int_{I_*\times\Sigma} f P(\partial_m\phi) = \int_{I_*\times\Sigma} \partial_m\phi P f$$

for any function f satisfying $\|f\|_{L^{q'}_{I_*}L^1_x} \leq 1$. Let ψ be the solution of (4.127), then

$$\mathcal{I} = \int_{I_*\times\Sigma} -\partial_m\phi \Box_{\mathbf{g}}\psi.$$

In view of (4.129), we have with $Z = \partial_m$ that

$$\int_{I_*\times\Sigma} -Z\phi\Box_{\mathbf{g}}\psi = 2\int_{\Sigma_0} \mathbf{T}^\alpha P_\alpha + \int_{I_*\times\Sigma} Q[\phi,\psi]_{\alpha\beta}{}^{(Z)}\pi^{\alpha\beta}.$$

By direct calculation we can see that ${}^{(Z)}\pi = \mathbf{g}\cdot\tilde{\pi}$. Thus it follows from the energy estimate (4.137) and (4.121) that

$$\left|\int_{I_*\times\Sigma} Q[\phi,\psi]_{\alpha\beta}{}^{(Z)}\pi^{\alpha\beta}\right| \lesssim \|\tilde{\pi}\|_{L^1_{I_*}L^\infty_x}\|\mathbf{D}\psi\|_{L^\infty_{I_*}L^2_x}\|\mathbf{D}\phi\|_{L^\infty_{I_*}L^2_x} \lesssim \|\mathbf{D}\phi(0)\|_{L^2_x}\|f\|_{L^{q'}_{I_*}L^1_x}$$

and

$$\left|\int_{\Sigma_0} \mathbf{T}^\alpha P_\alpha\right| \lesssim \|\mathbf{D}\phi(0)\|_{L^2}\|\mathbf{D}\psi\|_{L^\infty_{I_*}L^2_x} \lesssim \|\mathbf{D}\phi(0)\|_{L^2}\|f\|_{L^{q'}_{I_*}L^1_x}.$$

Therefore $|\mathcal{I}| \lesssim \|\mathbf{D}\phi(0)\|_{L^2_x}\|f\|_{L^{q'}_{I_*}L^1_x}$. Hence we can conclude that

$$\|P\partial_m\phi\|_{L^q_{I_*}L^\infty_x} \leq C\|\mathbf{D}\phi(0)\|_{L^2_x}. \tag{4.138}$$

Finally we prove (4.122) for the case that $\partial = \partial_t$, i.e.

$$\|P\partial_t\phi\|_{L^q_{I_*}L^\infty_x} \leq C\|\mathbf{D}\phi(0)\|_{L^2_x}. \tag{4.139}$$

Note that $\partial_t f = n e_0(f) + Y^m\partial_m f$. We can write

$$P\partial_t\phi = nPe_0(\phi) + Y^m P\partial_m\phi + [P,n]e_0\phi + [P,Y^m]\partial_m\phi.$$

By using (6.195), the Bernstein inequality and the finite band property for the Littlewood-Paley projections, we obtain

$$\|[P,n]e_0\phi\|_{L^q_{I_*}L^\infty_x} \lesssim \|\widehat{\nabla} n\|_{L^q_{I_*}L^\infty_x}\|(e_0\phi)_{\le 1}\|_{L^\infty} + \sum_{\ell>1}\|P(n_\ell\cdot(e_0\phi)_\ell)\|_{L^q_{I_*}L^\infty_x}$$
$$\lesssim \|\widehat{\nabla} n\|_{L^q_{I_*}L^\infty_x}\|e_0\phi\|_{L^\infty_{I_*}L^2_x}.$$

By using (2.29) and (BA2) under the rescaling coordinates $\|\widehat{\nabla} n\|_{L^q_{I_*}L^\infty_x} \lesssim \lambda^{-1+1/q}$, also using the energy estimate for ϕ, we can obtain

$$\|[P,n]e_0\phi\|_{L^q_{I_*}L^\infty_x} \lesssim \|\mathbf{D}\phi(0)\|_{L^2_x}.$$

Similarly, we have $\|\widehat{\nabla} Y\|_{L^q_{I_*}L^\infty_x} \lesssim \lambda^{-1+\frac{1}{q}}$ and

$$\|[P,Y^m]\partial_m\phi\|_{L^q_{I_*}L^\infty_x} \lesssim \|\mathbf{D}\phi(0)\|_{L^2_x}.$$

Combining the above two estimates with (4.123) and (4.138) we therefore obtain (4.139). The proof is thus complete.

4.2. Strichartz estimates $\Rightarrow$ main estimates. In this section we will use Theorem 4 to prove Theorem 2. According to the properties of Littlewood-Paley projections, it is easy to derive the desired estimates for the low frequency part. Therefore, to complete the proof of Theorem 2, it suffices to establish the following result.

Proposition 9. *There exists a large number $\Lambda \ge 1$ such that for any $q>2$ sufficiently close to 2 and any $\delta>0$ sufficiently close to 0 there holds on $I=[0,T]$*

$$\sum_{\mu>\Lambda}\left\{\|\mu^\delta P_\mu\partial_m g\|^2_{L^2_I L^\infty_x} + \|\mu^\delta P_\mu\partial_t g\|^2_{L^2_I L^\infty_x}\right\} \lesssim T^{1-\frac{2}{q}}.$$

Moreover for any solution of $\Box_{\mathbf{g}}\phi = 0$, there holds

$$\sum_{\mu>\Lambda}\|\mu^\delta P_\mu\partial\phi\|^2_{L^2_I L^\infty_x} \lesssim T^{1-\frac{2}{q}}\|\widehat{\nabla}\phi, e_0\phi\|^2_{H^{1+\epsilon}(0)}.$$

In order to carry out the proof of Proposition 9, we pick a sufficiently small $\epsilon_0>0$ and for each $\mu>1$ we partition the interval $[0,T]$ into disjoint union of subintervals $I_k=[t_{k-1},t_k)$ with the properties that

$$|I_k| \lesssim \mu^{-8\epsilon_0}T \quad \text{and} \quad \|k, \widehat{\nabla} g, \widehat{\nabla} Y, \mathbf{D}n\|_{L^2_{I_k}L^\infty_x} \le \mu^{-4\epsilon_0}. \tag{4.140}$$

Such partition is always possible. Let κ_μ denote the total number of subintervals in the partition. It is even possible to make $\kappa_\mu \lesssim \mu^{8\epsilon_0}$.

We first consider any pair (u,v) satisfying (3.66). Then $(U^\mu, V^\mu) := (P_\mu u, P_\mu v)$ also satisfies the system (3.66) with F_{U^μ} and F_{V^μ} given by (3.68), i.e.

$$F_{U^\mu} = [P_\mu, Y^m]\partial_m u + [P_\mu, n]v + P_\mu F_u, \tag{4.141}$$
$$F_{V^\mu} = [P_\mu, ng]\widehat{\nabla}^2 u + P_\mu F_v + [P_\mu, Y^m]\partial_m v. \tag{4.142}$$

Consequently, it follows from (3.87) that

$$n^2\Box_{\mathbf{g}} P_\mu u = -n\mathbf{D}_{\mathbf{T}} F_{U^\mu} + n(-F_{V^\mu} - n\pi_{0a}\nabla^a U^\mu + e_0(\ln n)F_{U^\mu} + n\mathrm{Tr}k e_0 U^\mu).$$

Now we will use the Duhamel principle to represent $P_\mu u$. To simplify the notation, we use $W(t,s)$ to denote the operator defined on $\mathcal{H}$ such that, for each $\omega := (\omega_0, \omega_1) \in \mathcal{H}$, $\phi := W(t,s)(\omega)$ is the unique solution of the initial value problem

$$\Box_{\mathbf{g}}\phi = 0, \quad \phi(t; s, x) = \omega_0, \quad \partial_t \phi(t; s, x) = \omega_1. \tag{4.143}$$

Then, by the Duhamel principle, we have for $t \in I_k = [t_{k-1}, t_k]$ that

$$\begin{aligned} P_\mu u(t) = {} & W(t, t_{k-1}) \left(P_\mu u(t_{k-1}), \partial_t P_\mu u(t_{k-1}) - F_{U^\mu}(t_{k-1})\right) \\ & + \int_{t_{k-1}}^t W(t,s)(0, -R_\mu(s)) + W(t,s)(F_{U^\mu}(s), 0) ds. \end{aligned} \tag{4.144}$$

Now we apply P_μ to the both sides and take the spatial derivative. Writing $P_\mu^2 = P_\mu$ by a little abuse of notation, we have

$$\begin{aligned} P_\mu \partial_m u(t) = {} & \int_{t_{k-1}}^t \left\{\partial_m P_\mu W(t,s)(0, -R_\mu(s)) + \partial_m P_\mu W(t,s)(F_{U^\mu}(s), 0)\right\} ds \\ & + \partial_m P_\mu W(t, t_{k-1}) \left(P_\mu u(t_{k-1}), \partial_t P_\mu u(t_{k-1}) - F_{U^\mu}(t_{k-1})\right), \end{aligned} \tag{4.145}$$

where

$$R_\mu = n(-F_{V^\mu} - n\pi_{0a}\nabla^a U^\mu + e_0(\ln n) F_{U^\mu} + n \mathrm{Tr} k e_0 U^\mu) - Y^i \partial_i F_{U^\mu}.$$

By using (4.122) in Theorem 4 with suitable change of coordinates, we have for any one-parameter family of data $\omega(s) := (\omega_0(s), \omega_1(s)) \in \mathcal{H}$ with $s \in I_k := [t_{k-1}, t_k]$ that

$$\mu^{-1+\frac{1}{q}} \|P_\mu \partial W(t,s)(\omega(s))\|_{L^q_{[s,t_k]} L^\infty_x} \lesssim \mu^{\frac{1}{2}} \|\omega(s)\|_{\mathcal{H}}.$$

In view of the Minkowski inequality we then obtain

$$\begin{aligned} \left\| \int_{t_{k-1}}^t P_\mu \partial W(t,s)(\omega(s)) ds \right\|_{L^2_{I_k} L^\infty_x} & \lesssim \int_{t_{k-1}}^{t_k} \|P_\mu \partial W(t,s)(\omega(s))\|_{L^2_{[s,t_k]} L^\infty_x} ds \\ & \lesssim |I_k|^{\frac{1}{2}-\frac{1}{q}} \mu^{\frac{3}{2}-\frac{1}{q}} \int_{I_k} \|\omega(s)\|_{\mathcal{H}} ds. \end{aligned}$$

Since $|I_k| \lesssim T\mu^{-8\epsilon_0}$, it follows that

$$\left\| \int_{t_{k-1}}^t P_\mu \partial W(t,s)(\omega(s)) ds \right\|_{L^2_{I_k} L^\infty_x} \lesssim T^{\frac{1}{2}-\frac{1}{q}} \mu^{(\frac{1}{2}-\frac{1}{q})(1-8\epsilon_0)} \int_{I_k} \mu \|\omega(s)\|_{\mathcal{H}} ds.$$

Applying the above inequality to (4.145) gives, with $\delta_0 := (\frac{1}{2} - \frac{1}{q})(1 - 8\epsilon_0)$, that

$$\begin{aligned} \|P_\mu \partial_m u\|_{L^2_{I_k} L^\infty_x} \lesssim {} & T^{\frac{1}{2}-\frac{1}{q}} \mu^{\delta_0} \left(\|\mu(0, R_\mu)\|_{L^1_{I_k}\mathcal{H}} + \|\mu(F_{U^\mu}, 0)\|_{L^1_{I_k}\mathcal{H}} \right) \\ & + T^{\frac{1}{2}-\frac{1}{q}} \left(B_\mu(t_{k-1}) + C_\mu(t_{k-1})\right), \end{aligned} \tag{4.146}$$

where

$$B_\mu(t) := \mu^{\delta_0} \|\mu \left(P_\mu u(t), \partial_t P_\mu u(t)\right)\|_{\mathcal{H}}, \quad C_\mu(t) := \mu^{\delta_0} \|\mu(0, -F_{U^\mu}(t))\|_{\mathcal{H}}.$$

In the following we will give the estimates on R_μ, F_{U^μ}, $B_\mu(t_{k-1})$ and $C_\mu(t_{k-1})$ separately. Positive indices ϵ_0, q, δ are chosen such that $4\epsilon_0 + \delta_0 + \delta < s - 2$, and $\delta_0 + \delta < 4\epsilon_0$.

4.2.1. Estimates for R_μ, F_{U^μ}.

Lemma 9. *For any $\delta_1 > \delta > 0$ satisfying $b := \delta_0 + \delta_1 < 4\epsilon_0$, there holds*

$$\left(\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}R_\mu\|^2_{L^1_{I_k}L^2_x}\right)^{\frac12} \le T\|\widehat{\nabla}u, v\|_{L^\infty_t H^{1+b}} + \|\mu^{1+b}P_\mu F_u\|_{L^1_I l^2_\mu H^1} + \|\mu^{1+b}P_\mu F_v\|_{L^1_I l^2_\mu L^2_x} + \|\widehat{\nabla}^2 u\|_{L^\infty_I L^2_x}. \tag{4.147}$$

$$\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}(F_{U^\mu}(s),0)\|^2_{L^1_{I_k}\mathcal{H}} \lesssim \left(\|\partial u, v\|_{L^1_I H^{1+b}} + \|\mu^{1+b}P_\mu\partial F_u\|_{L^1_I l^2_\mu L^2_x}\right)^2 \tag{4.148}$$

Proof. We first write $R_\mu = -n[P_\mu, ng]\widehat{\nabla}^2 u - Y^i\partial_i F_{U^\mu} + \check{R}_\mu$, where

$$\check{R}_\mu := n\left(-P_\mu F_v - [P_\mu, Y^m]\partial_m v - n\pi_{0a}\nabla^a U^\mu + e_0(\ln n)F_{U^\mu} + n\mathrm{Tr}k e_0 U^\mu\right).$$

Let us set

$$\mathcal{I}_1 := \sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}[P_\mu, ng]\widehat{\nabla}^2 u\|^2_{L^1_{I_k}L^2_x}, \quad \mathcal{I}_2 := \sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}\check{R}_\mu\|^2_{L^1_{I_k}L^2_x}.$$

It suffices to show that

$$\mathcal{I}_1^{\frac12} \lesssim \|\widehat{\nabla}^2 u\|_{L^\infty_I L^2_x}, \tag{4.149}$$

$$\mathcal{I}_2^{\frac12} \lesssim \|\widehat{\nabla}u, v\|_{L^\infty_t H^{1+b}}T + \|\mu^{1+b}P_\mu F_u\|_{L^1_I l^2_\mu L^2_x} + \|\mu^{1+b}P_\mu F_v\|_{L^1_I l^2_\mu L^2_x}, \tag{4.150}$$

$$\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}\widehat{\nabla}F_{U^\mu}(s)\|^2_{L^1_{I_k}L^2_x} \lesssim \left(\|\partial u, v\|_{L^1_I H^{1+b}} + \|\mu^{1+b}P_\mu\partial F_u\|_{L^1_I l^2_\mu L^2_x}\right)^2. \tag{4.151}$$

By (4.140), we have $\|\widehat{\nabla}(ng)\|_{L^1_{I_k}L^\infty_x} \lesssim \mu^{-8\epsilon_0}$. We can apply Corollary 1 to obtain

$$\begin{aligned}\|\mu^{1+\delta+\delta_0}[P_\mu, ng]\widehat{\nabla}^2 u\|_{L^1_{I_k}L^2_x} &\lesssim \mu^{\delta+\delta_0}\|\widehat{\nabla}(ng)\|_{L^1_{I_k}L^\infty_x}\|\widehat{\nabla}^2 u\|_{L^\infty_t L^2_x}\\ &\lesssim \mu^{\delta+\delta_0-8\epsilon_0}\|\widehat{\nabla}^2 u\|_{L^\infty_t L^2_x}.\end{aligned}$$

Recall also that $\kappa_\mu \lesssim \mu^{8\epsilon_0}$. We can obtain

$$\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta+\delta_0}[P_\mu, ng]\widehat{\nabla}^2 u\|^2_{L^1_{I_k}L^2_x} \le C\mu^{2(\delta+\delta_0-4\epsilon_0)}\|\widehat{\nabla}^2 u\|^2_{L^\infty_t L^2_x}.$$

Since $0 < \delta < \delta_1$ and $b := \delta_0 + \delta_1 < 4\epsilon_0$, we have

$$\mathcal{I}_1 \lesssim \Lambda^{2(b-4\epsilon_0)}\|\widehat{\nabla}^2 u\|_{L^\infty_t L^2_x} \lesssim \|\widehat{\nabla}^2 u\|^2_{L^\infty_t L^2_x}$$

which gives (4.149).

Next we prove (4.150). Since $0 < \delta < \delta_1$, we observe that for any function a_μ there holds

$$\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^\delta a_\mu\|^2_{L^1_{I_k}L^2_x} \le \sum_{\mu>\Lambda}\|\mu^\delta a_\mu\|^2_{L^1_I L^2_x} \le \left(\int_I \sum_{\mu>\Lambda}\|\mu^\delta a_\mu\|_{L^2_x}\right)^2$$
$$\lesssim \left(\int_I \|\mu^{\delta_1} a_\mu\|_{l^2_\mu L^2_x}\right)^2. \tag{4.152}$$

(4.151) and (4.148) can be derived immediately by using (4.152) and (3.73).

In view of (4.152), it suffices to estimate $\int_I \|\mu^{1+b}\check{R}_\mu\|_{l^2_\mu L^2_x}$. From (6.203) it follows that

$$\|\mu^{1+b}[P_\mu, Y^m]\partial_m v\|_{l^2_\mu L^2_x} \lesssim \|\widehat{\nabla} Y\|_{L^\infty_x}\|\widehat{\nabla} v\|_{H^b} + \|\widehat{\nabla}^2 Y\|_{H^{\frac{1}{2}+b}}\|\widehat{\nabla} v\|_{L^2_x}.$$

In view of (4.141), (3.71) and (3.84) we have

$$\begin{aligned}\|\mu^{1+b} e_0 \ln n F_{U^\mu}\|_{l^2_\mu L^2_x} &\lesssim \|e_0(\ln n)\|_{L^\infty}\Big(\|\widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty_x}\|\partial u, v\|_{H^b}\\ &\quad +\|\widehat{\nabla}^2 Y, \widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+b}}\|\partial u, v\|_{L^2_x} + \|\mu^{1+b}P_\mu F_u\|_{l^2_\mu L^2_x}\Big)\\ &\lesssim \|\partial u, v\|_{H^b} + \|\mu^{1+b}P_\mu F_u\|_{l^2_\mu L^2_x}.\end{aligned}$$

Recall that $\|\nabla n, \mathrm{Tr}k\|_{L^\infty_I L^\infty} \le C$, we can derive that

$$\begin{aligned}\|\mu^{1+b}(|\mathrm{Tr}k\, e_0(U^\mu)| + |\nabla_i n \mathbf{D}^i U^\mu|)\|_{l^2_\mu L^2_x} &\lesssim \|\nabla n, \mathrm{Tr}k\|_{L^\infty_x}\|\Lambda^b\widehat{\nabla}(\widehat{\nabla} u, v)\|_{L^2_x}\\ &\lesssim \|\Lambda^b\widehat{\nabla}(\widehat{\nabla} u, v)\|_{L^2_x}\end{aligned}$$

Combining the above three estimates we thus obtain (4.150). □

4.2.2. Estimates for $B_\mu(t_{k-1})$ and $C_\mu(t_{k-1})$.

Lemma 10. *For any $\delta > 0$ satisfying $\alpha := 4\epsilon_0 + \delta_0 + \delta < s - 2$, there holds*

$$\begin{aligned}&\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\mu^{2\delta}B_\mu(t_{k-1})^2\\ &\quad\lesssim \sum_{\mu>\Lambda}\mu^{2\alpha}\sup_{t\in I}\mathcal{E}^{(1)}_\mu(t) + \sup_t\left(\|\widehat{\nabla} u, v\|^2_{H^{\frac{1}{2}+\alpha}} + \|\mu^{\frac{3}{2}+\alpha}P_\mu F_u\|^2_{l^2_\mu L^2_x}\right).\end{aligned}$$

Proof. Since $\kappa_\mu \lesssim \mu^{8\epsilon_0}$, we have from the expression of $B_\mu(t)$ that

$$\begin{aligned}\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\mu^{2\delta}B_\mu(t_{k-1})^2 &\lesssim \sum_{\mu>\Lambda}\sup_{t\in I}(\mu^{4\epsilon_0+\delta}B_\mu(t))^2\\ &\lesssim \sum_{\mu>\Lambda}\mu^{2(1+\alpha)}\sup_{t\in I}\|(P_\mu u(t), \partial_t P_\mu u(t))\|^2_{\mathcal{H}}.\end{aligned}$$

According to the definition of $\mathcal{E}^{(1)}_\mu(t) := \mathcal{E}^{(1)}_\mu(u(t), v(t))$ and the equation for $\partial_t P_\mu u$ we obtain

$$\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\mu^{2\delta}B_\mu(t_{k-1})^2 \lesssim \sum_{\mu>\Lambda}\mu^{2\alpha}\sup_{t\in I}\mathcal{E}_\mu^{(1)}(t)+\sum_{\mu>\Lambda}\sup_{t\in I}\|\mu^{1+\alpha}F_{U^\mu}(t)\|^2_{L_x^2}.$$

With the help of (6.204) we have

$$\|\mu^{1+\alpha}F_{U^\mu}\|^2_{L_x^2}\le \mu^{-1}\sum_\lambda \|\lambda^{\frac{3}{2}+\alpha}F_{U^\lambda}(t)\|^2_{L_x^2}\lesssim \mu^{-1}(\|\widehat{\nabla}u, v\|^2_{H^{\frac{1}{2}+\alpha}}+\|\lambda^{\frac{3}{2}+\alpha}P_\lambda F_u\|^2_{l_\lambda^2 L_x^2}).$$

Plugging this into the above inequality and summing over $\mu>\Lambda$ gives the desired estimate. □

Lemma 11. *For any $\delta_1>\delta>0$ satisfying $b:=\delta_0+\delta_1<4\epsilon_0$ there hold*

$$\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\left(\mu^\delta C_\mu(t_{k-1})\right)^2\le \|\partial_m u, v\|^2_{L_t^\infty H^{4\epsilon_0+b}}+\sup_{t\in I}\sum_{\mu>1}\|\mu^{1+4\epsilon_0+b}P_\mu F_u\|^2_{L_x^2}.$$

Proof. Since $\sum_{k=1}^{\kappa_\mu}\left(\mu^\delta C_\mu(t_{k-1})\right)^2\le \sup_{t\in I}\|\mu^{4\epsilon_0+\delta_0+\delta+1}F_{U^\mu}(t)\|^2_{L_x^2}$, with $0<\delta<\delta_1$, we have

$$\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\left(\mu^\delta C_\mu(t_{k-1})\right)^2\le \sup_t\|\mu^{4\epsilon_0+\delta_0+\delta_1+1}F_{U^\mu}(t)\|^2_{l_\mu^2L_x^2}.$$

In view of (6.204) and (3.84), we complete the proof of Lemma 11. □

In view of (4.146), Lemma 9, Lemma 11, Lemma 10 and writing

$$\sum_{\mu>\Lambda}\|\mu^\delta P_\mu\partial_m u\|^2_{L_I^2L_x^\infty}=\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^\delta P_\mu\partial_m u\|^2_{L_{I_k}^2L_x^\infty},$$

we can obtain the following result.

Proposition 10. *For any $q>2$ sufficiently close to 2 and any $\delta>0$ sufficiently small such that $\alpha:=4\epsilon_0+\delta_0+\delta<s-2$, where $\delta_0:=(\frac{1}{2}-\frac{1}{q})(1-8\epsilon_0)$. Then for any pair (u,v) satisfying (3.66) there holds with $\alpha<\alpha_+\le s-2$ that*

$$\begin{aligned}&\sum_{\mu>\Lambda}\|\mu^\delta P_\mu\partial_m u\|^2_{L_I^2L_x^\infty}\\&\lesssim T^{1-\frac{2}{q}}\left(\|\mu^{2+\alpha}P_\mu F_u\|^2_{L_I^1l_\mu^2L_x^2}+\|\mu^{1+\alpha}P_\mu F_v\|^2_{L_I^1l_\mu^2L_x^2}+\|\mu^{\frac{1}{2}+\alpha}P_\mu\widehat{\nabla}F_u\|^2_{L_I^\infty l_\mu^2L_x^2}\right)\\&\quad+T^{1-\frac{2}{q}}\sup_{t\in I}\mathcal{E}^{(1+\alpha_+)}(u,v)(t).\end{aligned}$$

Now we are ready to derive the estimates on the spatial derivative part in Proposition 9. Recall that $(u,v):=(g,-2k)$ satisfies (1.13). With the help of (3.79), (3.80), (3.81) and (BA1), it follows that $\|\mu^{\frac{1}{2}+\alpha}P_\mu\widehat{\nabla}F_u\|_{L_I^\infty l_\mu^2L_x^2}\lesssim 1$ and then

$$\|\mu^{1+\alpha}P_\mu F_u\|_{L_I^1l_\mu^2L_x^2}+\|\mu^{1+\alpha}P_\mu F_v\|_{L_I^1l_\mu^2L_x^2}\lesssim\|\widehat{\nabla}g, k, \widehat{\nabla}Y, \widehat{\nabla}n\|_{L_I^1L_x^\infty}+1\lesssim 1.$$

Therefore we can obtain from Proposition 10 and (3.81) that

$$\|\partial_m g\|^2_{L^2_I L^\infty_x} + \sum_{\mu>\Lambda} \|\mu^\delta P_\mu \partial_m g\|^2_{L^2_I L^\infty_x} \le C T^{1-\frac{2}{q}}.$$

For a solution ϕ of the equation $\Box_{\mathbf{g}}\phi = 0$, we recall that $(u, v) = (\phi, e_0\phi)$ satisfies (3.90) with $W = 0$. We make the bootstrap assumption (BA4) for ϕ. In view of (3.91) and $F_u = 0$, and $\mathcal{E}^{(1+\alpha_+)}(u, v)(t) \lesssim \mathcal{E}^{(1+\alpha_+)}(u, v)(0)$ in Proposition 5, we may use the same argument as above to conclude that

$$\|\partial_m \phi\|^2_{L^2_I L^\infty_x} + \sum_{\mu>\Lambda} \|\mu^\delta P_\mu \partial_m \phi\|^2_{L^2_I L^\infty_x} \le C T^{1-\frac{2}{q}} (1 + B_0^2) \|\widehat{\nabla}\phi, e_0\phi\|^2_{H^{1+\alpha_+}(0)},$$

which improves assumption of (BA4) for ϕ since T can be chosen sufficiently small and universal.

4.2.3. Estimate for $P_\mu \partial_t u$. From (4.144) it follows that.

$$\begin{aligned} P_\mu \partial_t u(t) &= P_\mu F_{U^\mu}(t) + P_\mu \partial_t W(t, t_{k-1}) \left(P_\mu u(t_{k-1}), \partial_t P_\mu u(t_{k-1}) - F_{U^\mu}(t_{k-1})\right) \\ &\quad + \int_{t_{k-1}}^t P_\mu \{\partial_t W(t, s)(0, -R_\mu(s)) + \partial_t W(t, s)(F_{U^\mu}(s), 0)\} ds. \end{aligned} \tag{4.153}$$

We can use the same argument for dealing with $P_\mu \partial_m u$ to estimate the terms on the right hand side except the first term $P_\mu F_{U^\mu}(t)$.

Lemma 12. *For sufficiently small $\delta > 0$ there holds*

$$\sum_{\mu>\Lambda} \|\mu^\delta P_\mu F_{U^\mu}\|^2_{L^2_I L^\infty_x} \lesssim \sum_{\mu>\Lambda} \|\mu^\delta P_\mu F_u\|^2_{L^2_I L^\infty_x} + T \|\widehat{\nabla}^2 u, \widehat{\nabla} v\|^2_{L^\infty_t H^\delta}.$$

Proof. From (6.210), for $0 < \eta < 1/2$ there holds

$$\mu^\delta \|[P_\mu, Y^m]\partial_m u\|_{L^\infty_x} + \mu^\delta \|[P_\mu, n]v\|_{L^\infty_x} \lesssim \mu^{-\eta} \|\widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty_x} \|\widehat{\nabla}^2 u, \widehat{\nabla} v\|_{H^\delta}.$$

This together with (4.141) implies that

$$\sum_{\mu>\Lambda} \|\mu^\delta P_\mu F_{U^\mu}\|^2_{L^2_I L^\infty_x} \lesssim \sum_{\mu>\Lambda} \|\mu^\delta P_\mu F_u\|^2_{L^2_I L^\infty_x} + T \|\widehat{\nabla} Y, \widehat{\nabla} n\|^2_{L^\infty_I L^\infty_x} \|\widehat{\nabla}^2 u, \widehat{\nabla} v\|^2_{L^\infty_I H^\delta}.$$

In view of Proposition 3, we therefore obtain the desired estimate. □

By using (4.153), Lemma 12 and Proposition 5 for the solution ϕ of $\Box_{\mathbf{g}}\phi = 0$, in view of $F_u = 0$ we derive that

$$\|\partial_t \phi\|^2_{L^2_I L^\infty_x} + \sum_{\mu>\Lambda} \|\mu^\delta P_\mu \partial_t \phi\|^2_{L^2_I L^\infty_x} \le C T^{1-\frac{2}{q}} \|\widehat{\nabla}\phi, e_0\phi\|^2_{H^{1+\epsilon}(0)}.$$

Next we consider $(u, v) = (g, -2k)$ which satisfies (1.13). Recall that $F_u = \widehat{\nabla} Y \cdot g$ in (1.13), we have

$$\|\mu^\delta P_\mu F_u\|_{L^2_I L^\infty_x} \lesssim \|\mu^\delta [P_\mu, g]\widehat{\nabla} Y\|_{L^2_t L^\infty_x} + \|\mu^\delta P_\mu \widehat{\nabla} Y\|_{L^2_t L^\infty_x}.$$

By using (6.209) we have with $0 < \eta < 1/2$ that

$$\|\mu^{\delta}[P_{\mu}, g]\widehat{\nabla} Y\|_{L_x^{\infty}} \lesssim \mu^{-\eta}\|\widehat{\nabla} Y\|_{L_x^{\infty}}\|\widehat{\nabla}^2 g\|_{H^{\delta}}.$$

Therefore

$$\sum_{\mu > \Lambda} \|\mu^{\delta} P_{\mu} F_u\|^2_{L_I^2 L_x^{\infty}} \lesssim T\left(\|\widehat{\nabla}^2 Y\|^2_{L_t^{\infty} H^{\frac{1}{2}+\delta}} + \|\widehat{\nabla}^2 g, \widehat{\nabla} k\|^2_{L_t^{\infty} H^{\delta}}\right)$$

From this, (4.153), (3.81) and (3.84), we conclude that $\sum_{\mu > \Lambda} \|\mu^{\delta} P_{\mu} \partial_t g\|^2_{L_I^2 L_x^{\infty}} \leq CT^{1-\frac{2}{q}}$.

4.3. Boundedness theorem $\Rightarrow$ decay estimates. In this subsection we give the proof of Theorem 3 under the rescaled coordinates. A time interval $I = [0, T]$ becomes $I_* = [0, \lambda T]$ after rescaling. Let τ_* denote a number such that $t_* \leq \tau_* \leq \lambda T$ and let t_0 be certain number satisfying $1 \approx t_0 < \tau_*$. We may take a sequence of balls $\{B_J\}$ of radius $1/2$ such that their union covers Σ_{t_0} and any ball in this collection intersect at most 10 other balls. Let $\{\chi_J\}$ be a partition of unity subordinate to the cover $\{B_J\}$. We may assume that $\sum_{m=1}^{3} |\widehat{\nabla}^m \chi_J|_{L_x^{\infty}} \leq C_1$ uniformly in J. By using this partition of unity and a standard argument we can reduce the proof of Theorem 3 by establishing the following dispersive estimate result with initial data supported on a ball of radius $1/2$.

Proposition 11. *There exists a large constant Λ such that for any $\lambda \geq \Lambda$ and any solution ψ of*

$$\Box_{\mathbf{g}} \psi = 0$$

on the time interval $[0, \tau_*]$ with $\tau_* \leq \lambda T$, with certain $t_0 \in [1, C]$ and any initial data $\psi[t_0] = (\psi(t_0), \partial_t \psi(t_0))$ supported in the geodesic ball $B_{1/2}$ of radius $\frac{1}{2}$, there is a function $d(t)$ satisfying

$$\|d\|_{L^{\frac{q}{2}}[0,\tau_*]} \lesssim 1, \quad \text{for } q > 2 \text{ sufficiently close to } 2 \tag{4.154}$$

such that for all $t_0 \leq t \leq \tau_*$,

$$\|Pe_0\psi(t)\|_{L_x^{\infty}} \leq \left(\frac{1}{(1+|t-t_0|)^{\frac{2}{q}}} + d(t)\right)(\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L^2}). \tag{4.155}$$

where $\|\psi[t_0]\|_{H^1} := \|\widehat{\nabla}\psi(t_0)\|_{L^2} + \|\partial_t \psi(t_0)\|_{L^2}$.

Proof (Proof of Theorem 3). To derive Theorem 3, we apply the above result to ψ_I,

$$\Box_{\mathbf{g}} \psi_I = 0, \quad \psi_I(0) = \chi_I \cdot \psi(0), \quad \partial_t \psi_I(0) = \chi_I \cdot \partial_t \psi(0),$$

where ψ is the solution of (4.118) with initial data $\psi[0] := (\psi(0), \partial_t \psi(0))$. For $0 < t < t_0$, it follows immediately from Bernstein inequality and (3.92) that

$$\|Pe_0\psi(t)\|_{L_x^{\infty}} \lesssim \|e_0\psi(t)\|_{L_x^2}. \tag{4.156}$$

By (4.155), we have for $t_0 \le t \le \tau_*$

$$\|Pe_0\psi_I(t)\|_{L_x^\infty} \le \left(\frac{1}{(1+|t-t_0|)^{\frac{2}{q}}} + d(t)\right)(\|\psi_I[t_0]\|_{H^1} + \|\psi_I(t_0)\|_{L^2}). \quad (4.157)$$

By applying (3.93) to the solution ψ_I

$$\|\psi_I[t]\|_{H^1} + \|\psi_I(t)\|_{L^2} \lesssim \|\psi_I[0]\|_{H^1} + \|\psi_I(0)\|_{L^2}.$$

Then by combining (4.157) with (4.156) and using the above energy estimate, we have

$$\|Pe_0\psi_I(t)\|_{L_x^\infty} \le \left(\frac{1}{(1+|t-t_0|)^{\frac{2}{q}}} + d(t)\right)(\|\tilde{\psi}_I[0]\|_{H^1} + \|\psi_I(0)\|_{L^2}). \quad (4.158)$$

where $\tilde{\psi}[0] = (\psi(0), n^{-2}\partial_t\psi(0))$ and we employed the fact that there exists $C > 0$ such that $C^{-1} < n < C$. (4.120) then follows by applying Sobolev embedding also in view of $\psi(t,x) = \sum_I \psi_I(t,x)$. $\square$

We will prove Proposition 11 by establishing boundedness theorem for conformal energy. For this purpose, we introduce the setup and notation. We denote by Γ^+ the portion in $[0, \lambda T]$ of the integral curve of $\mathbf{T}$ passing through the center of $B_{\frac{1}{2}}$. We define the optical function u to be the solution of eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ with $u = t$ on Γ^+. We denote the outgoing null cone initiating from Γ^+ by C_u with $0 \le u \le \lambda T$. Let $S_{t,u} = C_u \cap \Sigma_t$. Let us set $\mathcal{D}_0^+ = \cup_{\{t\in[t_0,\tau_*], 0\le u\le t\}} S_{t,u}$ and $\mathcal{D}^+ = \cup_{\{t\in[0,\tau_*],0\le u\le t\}} S_{t,u}$. We denote the exterior region on Σ_t, $t \ge t_0$ by $\mathrm{Ext}_t = \{0 \le u \le 3t/4\}$. By $C^{-1} < n < C$, we can always choose $t_0 \in [1, 2C]$ such that $B_{\frac{1}{2}} \subset (\mathcal{D}_0^+ \cap \Sigma_{t_0})$.

Next we extend the time axis $\Gamma^+ : u = t$ backward by following the integral curve of $\mathbf{T}$ to $t = -\lambda T$. Let us denote the extended portion of the integral curve of $\mathbf{T}$ by Γ^-. Let C_u be the outgoing null cone initiating from vertex $p(t) \in \Gamma^-$ with $u = t$. We also foliate the null hypersurfaces by time foliation, $C_u = \cup_{u\le t\le\tau_*} S_{t,u}$.

Let $\underline{\varpi}$ and ϖ be smooth cut-off functions depending only on two variables t, u. For $t > 0$, they are defined as follows

$$\underline{\varpi} = \begin{cases} 1 & \text{on } 0 \le u \le t \\ 0 & \text{on } u \le -\frac{t}{4} \end{cases} \qquad \varpi = \begin{cases} 1 & \text{on } 0 \le \frac{u}{t} \le \frac{1}{2} \\ 0 & \text{if } \frac{u}{t} \ge \frac{3}{4} \text{ or } u \le -\frac{t}{4} \end{cases}.$$

We also suppose ϖ and $\underline{\varpi}$ coincide in the region $\cup_{\{t\in[t_0,\tau_*],-\frac{t}{4}<u\le 0\}} S_{t,u}$.

Let us denote by N the outward unit normal of $S_{t,u} \in \Sigma_t$. Define $\theta_{AB} = \langle \mathbf{D}_A N, e_B\rangle$ and $\chi_{AB} = \langle \mathbf{D}_{e_A} L, e_B\rangle$. We decompose χ as $\chi_{AB} = \hat{\chi}_{AB} + \frac{1}{2}\mathrm{tr}\chi\gamma_{AB}$.

$$\nabla\!\!\!/_N N = -(\nabla\!\!\!/ \log \mathbf{b})e_A, \qquad \nabla\!\!\!/_A N_B = \theta_{AB} e_B, \qquad \chi_{AB} = \theta_{AB} - k_{AB} \quad (4.159)$$

We recall some useful results in Proposition 7, Proposition 8 and those established in [25, Sections 4 and 8]. Under (BA1) and (BA2),

(i) There exists $\delta_* > 0$ depending only on B_1 and the norm of initial data $\|(g,k)\|_{H^2\times H^1(\Sigma_0)}$ such that if $T \le \delta_*$ then the outgoing null radius of injectivity satisfies $i_*(p) > T - t(p)$ for any $p \in [-T, T] \times \Sigma$.

And under (BA1), (BA2) and the rescaled coordinate,

(ii) Let $\mathcal{N}^+(p)$ be an outgoing null cone initiating from $p \in [-\lambda T, \lambda T] \times \Sigma$ and contained therein. Then on every $\mathcal{N}^+(p)$ there holds $\mathcal{F}^{\frac{1}{2}}[\widehat{\nabla} g] \lesssim \lambda^{-\frac{1}{2}}$, the curvature flux $\mathcal{R}$ together with flux type norm of components of π on $\mathcal{N}^+(p)$ satisfies (for definition we refer to [25,26].)

$$\mathcal{R} + \mathcal{N}_1[\not{\pi}] \lesssim \lambda^{-\frac{1}{2}}. \tag{4.160}$$

(iii) For $0 \leq T \leq \delta_*$, consider $C_u \subset [-\lambda T, \lambda T] \times \Sigma$, with $S_{t,u} = C_u \cap \Sigma_t$ and $r(t,u) = \sqrt{\frac{|S_{t,u}|}{4\pi}}$. As a consequence of (4.160) and $C^{-1} < n < C$ the metric $\gamma_{t,u}$ on $\mathbb{S}^2$, obtained by restricting the metric g on Σ_t to $S_{t,u}$ and then pulling it back to $\mathbb{S}^2$ by the exponential map $\mathcal{G}(t,u,\cdot)$, verifies with small quantity $0 < \epsilon < 1/2$ that

$$|r^{-2}\gamma_{t,u}(X,X) - \gamma_{\mathbb{S}^2}(X,X)| < \epsilon\gamma_{\mathbb{S}^2}(X,X), \quad \forall X \in T\mathbb{S}^2,$$

where $\gamma_{\mathbb{S}^2}$ is the standard metric on $\mathbb{S}^2$; there holds $(t-u) \approx r(t,u)$. There hold

$$\left|\frac{\mathbf{b}}{n} - 1\right| \leq \frac{1}{2}, \qquad r\mathrm{tr}\chi \approx 1, \qquad v_{t,u} := \sqrt{\frac{|\gamma_{t,u}|}{|\gamma_{\mathbb{S}^2}|}} \approx r^2 \tag{4.161}$$

$$\|\tilde{\pi}, \hat{\chi}, \not\nabla \log \mathbf{b}\|_{L^4(S_{t,u})} + \|r^{-\frac{1}{2}}(\tilde{\pi}, \hat{\chi}, \not\nabla \log \mathbf{b})\|_{L^2(S_{t,u})} \lesssim \lambda^{-\frac{1}{2}}.$$

We will constantly employ the following result, where all the constants suppressed in $\lesssim$ are independent of frequency λ.

Lemma 13. *For any Σ-tangent tensor field F there hold for $-\tau_*/2 \leq u < t$*

$$\int_{S_{t,u}} |F|^2 \lesssim \|F\|_{H^1(\Sigma_t)}\|F\|_{L^2(\Sigma_t)}, \qquad \|F\|_{L^4(S_{t,u})} + \|r^{-\frac{1}{2}}F\|_{L^2(S_{t,u})} \lesssim \|F\|_{H^1(\Sigma_t)}.$$

Proof. This is [25, Proposition 7.5]. □

Now we prove a commutator estimate for P, the Littlewood Paley projection with frequency 1. This estimate is slightly more general than needed.

Lemma 14. *Let $b \geq 2$. For scalar function f and G, with* $\max\{b, 3\} < p \leq \infty$ *there holds*

$$\|[P,G]\partial_m f\|_{W^{1,b}} + \|[P,G]\partial_m f\|_{L^\infty_x} \lesssim \|\widehat{\nabla} G\|_{L^p_x}\|\widehat{\nabla} f\|_{L^2_x}.$$

Proof. The L^∞ estimate follows by Sobolev embedding and $W^{1,b}$ estimates. Now we consider the $W^{1,b}$ estimate. In view of (6.195), we can write

$$[P,G]\partial_m f = [P,G](\partial_m f)_{\leq 1} + \sum_{\ell > 1} P\left(G_\ell \cdot P_\ell(\partial_m f)\right). \tag{4.162}$$

Using Corollary 1, we obtain with $\frac{1}{p} + \frac{1}{b*} = \frac{1}{b}$ that

$$\|[P,G](\partial_m f)_{\leq 1}\|_{L^b_x} \lesssim \|\widehat{\nabla} G\|_{L^p_x}\|\widehat{\nabla} f_{\leq 1}\|_{L^{b*}_x} \lesssim \|\widehat{\nabla} G\|_{L^p_x}\|\widehat{\nabla} f\|_{L^2_x}. \tag{4.163}$$

Consider $I = \widehat{\nabla}[P,G](\partial_m f)_{\leq 1}$. Apply (6.198) to $(G, \partial_m f)$ and $\mu = 1$ we can obtain

$$I = \int \widehat{\nabla} M_1(x-y)(x-y)^j \int_0^1 \partial_j G(\tau y + (1-\tau)x) d\tau (\partial_m f)_{\leq 1}(y) dy$$
$$- \int M_1(x-y)\widehat{\nabla} G(x)(\partial_m f)_{\leq 1}(y) dy. \tag{4.164}$$

Similar to Corollary 1, we derive that

$$\|I\|_{L_x^b} \lesssim \|\widehat{\nabla} G\|_{L_x^p} \|\partial f_{\le 1}\|_{L_x^{b*}}.$$

Now consider $J = \sum_{\ell>1} \widehat{\nabla} P\,(G_\ell P_\ell(\partial_m f))$. By finite band property, with $\frac{1}{p}+\frac{1}{b*}=\frac{1}{2}$ we have

$$\begin{aligned}\|J\|_{L_x^b} &\lesssim \sum_{\ell>1} \ell^{-1}\|\widehat{\nabla} G\|_{L_x^p}\|P_\ell(\partial_m f)\|_{L_x^{b*}} \lesssim \sum_{\ell>1} \ell^{-1+\frac{3}{p}}\|\partial_m f\|_{L_x^2}\|\widehat{\nabla} G\|_{L_x^p}\\ &\lesssim \|\widehat{\nabla} G\|_{L_x^p}\|\widehat{\nabla} f\|_{L_x^2}.\end{aligned}$$

Combining(4.162) with the estimates for I and J , we can complete the proof. □

For ease of exposition, let us introduce the first version of conformal energy and state its boundedness theorem whose proof occupies the rest of the paper.

Theorem 5 (Boundedness theorem). *Let ψ be a solution of $\Box_{\mathbf{g}}\psi = 0$ whose initial data is supported in $B_{\frac{1}{2}} \subset (\mathcal{D}_0^+ \cap \Sigma_{t_0})$. In the region $\mathcal{D}_0^+$,*

$$C[\psi](t) := \int_{\Sigma_t} \{t^2(|\nabla\!\!\!/\psi|^2 + |\nabla\!\!\!/_L\psi|^2) + u^2|\nabla\psi|^2 + (\frac{t^2}{(t-u)^2}+1)\psi^2\} d\mu_g \qquad (4.165)$$

under (BA1)–(BA3), there holds for $t \in [t_0, \tau_*]$, $C[\psi](t) \lesssim \|\psi[t_0]\|_{H^1}^2 + \|\psi(t_0)\|_{L^2(\Sigma)}^2$.

Lemma 15. *Let $q > 2$ and $0 < \delta \le 1 - \frac{2}{q}$ be two numbers. Assuming (BA2) and*

$$\|\varpi(\hat{\chi}, \nabla\!\!\!/ \log \mathbf{b})\|_{L^2[0,\tau_*]L_x^\infty} \lesssim \lambda^{-\frac{1}{2}}, \quad \text{with } \tau_* \le \lambda T, \qquad (4.166)$$

for any solution ψ of $\Box_{\mathbf{g}}\psi = 0$, there holds

$$\|[P, \varpi N^m]\partial_m\psi\|_{L_x^\infty} + \|[\varpi\nabla\!\!\!/, P]\psi\|_{H^1} \lesssim \tilde{d}(t)(\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L_x^2}),$$

where

$$(1+t)^\delta \tilde{d}(t) \lesssim (1+t)^{-\frac{2}{q}} + d(t)$$

with $d(t)$ being a function satisfying (4.154).

The condition (4.166) is incorporated in (5.179) in Proposition 12 and is proved in [27].

Proof. We first claim for $t > t_0$ there hold

$$\|\widehat{\nabla}(\varpi N)\|_{L_x^\infty} \lesssim \|\varpi(\hat{\chi}, \nabla\!\!\!/ \log \mathbf{b}), k, \widehat{\nabla} g\|_{L^\infty} + (1+t)^{-1}. \qquad (4.167)$$

Indeed, for $t \ge t_0$, on the support of ϖ, i.e. $\cup_{\{-\frac{t}{4}\le u\le\frac{3t}{4}\}} S_{t,u}$, the radius r of $S_{t,u}$ within the support of ϖ satisfies $r \approx (1+t)$. (4.167) follows by using (4.159), (4.161) and the fact that $\|\widehat{\nabla}\varpi\|_{L_x^\infty} \lesssim (1+t)^{-1}$.

Let $\Pi_{ij} = g_{ij} - N_i N_j$ denote the projection tensor on Σ. Then for any scalar function f, we have $\nabla\!\!\!/_j f = \Pi_j^i \partial_i f$ and

$$[P, \varpi\nabla\!\!\!/_j]f = -(\varpi N^i N_j)P\partial_i f + P((\varpi N_j N^i)\partial_i f) = [P, \varpi N_j N^i]\partial_i f. \quad (4.168)$$

Applying Lemma 14 to $(G, f) = (\varpi N^i, \psi)$, $(\varpi N_j N^i, \psi)$, and using Proposition 5 for ψ, we have

$$\|[P, G]\partial_i f\|_{L_x^\infty} + \|[P, G]\partial_i f\|_{H^1} \lesssim \|\widehat{\nabla} G\|_{L_x^\infty}(\|\mathbf{D}\psi(t_0)\|_{L_x^2} + \|\psi(t_0)\|_{L_x^2}).$$

Now for $q > 2$, we set $\tilde{d}(t) = \|\widehat{\nabla} G\|_{L_x^\infty}$. We have from (4.167) that

$$\tilde{d}(t) \lesssim (1+t)^{-1} + \|\varpi(\hat{\chi}, \nabla\!\!\!/ \log \mathbf{b}), \widehat{\nabla} g, k\|_{L_x^\infty} = (1+t)^{-1} + \tilde{d}^{(2)}(t).$$

By using (4.166) and Hölder inequality, we have

$$\|\tilde{d}^{(2)}(t)\|_{L^{\frac{q}{2}}} \lesssim \lambda^{\frac{2}{q}-1} T^{\frac{2}{q}-\frac{1}{2}}.$$

Thus, with $0 < \delta \leq 1 - \frac{2}{q}$ and $d(t) = (1+t)^\delta \tilde{d}^{(2)}(t)$, we can complete the proof. $\square$

Lemma 16. *(i) Let $S_t = \Sigma_t \cap \mathcal{N}^+(p)$, with $p \in [-\lambda T, \lambda T] \times \Sigma$. For S_t tangent tensor F, there holds*

$$\|r^{1-2/q} F\|_{L^q(S_t)} \lesssim \|r \nabla\!\!\!/ F\|_{L^2(S_t)}^{1-2/q} \|F\|_{L^2(S_t)}^{2/q} + \|F\|_{L^2(S_t)}, \quad 2 \leq q < \infty. \tag{4.169}$$

(ii) For any $\delta \in (0, 1)$, any $q \in (2, \infty)$ and any scalar function f there hold

$$\sup_{S_{t,u}} |f| \lesssim r^{\frac{2\delta(q-2)}{2q+\delta(q-2)}} \left(\int_{S_{t,u}} \left(|\nabla\!\!\!/ f|^2 + r^{-2}|f|^2 \right) \right)^{\frac{1}{2} - \frac{\delta q}{2q+\delta(q-2)}} \times \left(\int_{S_{t,u}} \left(|\nabla\!\!\!/ f|^q + r^{-q}|f|^q \right) \right)^{\frac{2\delta}{2q+\delta(q-2)}}.$$

Proof. This is [12, Theorem 5.2] $\square$

Now we are ready to complete the proof of Proposition 11.

Proof (Proof of Proposition 11). We first claim that

$$\|\varpi P\psi\|_{L^\infty(\Sigma_t)} \lesssim \left(\frac{1}{(1+|t-t_0|)^{\frac{2}{q}}} + d(t) \right) (\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L^2}). \tag{4.170}$$

Since ϖ vanishes outside the region $\{-t/4 \leq u < 3t/4\}$, this claim is trivial there. Thus we may restrict our consideration to the region $\{-t/4 \leq u < 3t/4\}$. In view of $r \approx t - u$, we thus have $r \approx t$ for $t > 0$. Recall that ϖ is constant on each $S_{t,u}$, from Lemma 16 (ii), we can obtain

$$\sup_{S_{t,u}} |\varpi P\psi|^2 \lesssim r^\delta \left(\int_{S_{t,u}} \left(|\varpi \nabla\!\!\!/ P\psi|^2 + r^{-2}|\varpi P\psi|^2 \right) \right)^{1-\delta} \times \left(\int_{S_{t,u}} \left(|\varpi \nabla\!\!\!/ P\psi|^4 + r^{-4}|\varpi P\psi|^4 \right) \right)^{\frac{1}{2}\delta}.$$

Applying Lemma 13 and the finite band property, we then obtain

$$\sup_{S_{t,u}} |\varpi P\psi|^2 \lesssim r^\delta \left(\int_{S_{t,u}} \left(|P(\varpi \nabla\!\!\!/ \psi)|^2 + r^{-2}|\varpi P\psi|^2 + |[P, \varpi \nabla\!\!\!/]\psi|^2 \right) \right)^{1-\delta}$$

$$\times \left(\int_{S_{t,u}} \left(|P(\varpi \nabla\!\!\!/ \psi)|^4 + r^{-4}|\varpi P\psi|^4 + |[P, \varpi \nabla\!\!\!/]\psi|^4 \right) \right)^{\frac{1}{2}\delta}$$

$$\lesssim r^\delta \left(t^{-2} C[\psi](t) + \|[P, \varpi \nabla\!\!\!/]\psi\|^2_{H^1} \right).$$

By letting $0 < \delta \leq 2(1 - \frac{2}{q})$, (4.170) then follows from Theorem 5 and Lemma 15.

In order to derive the estimate on $P(e_0(\psi))$, we can write $\|P\,(e_0\psi)\,\|_{L_x^\infty} \leq \|P(\varpi e_0\psi)\|_{L_x^\infty} + \|P\left((\underline{\varpi} - \varpi)e_0\psi\right)\|_{L_x^\infty}$. By Bernstein inequality and Theorem 5, we have

$$\|P\left((\underline{\varpi} - \varpi)e_0\psi\right)\|_{L_x^\infty} \lesssim \|(\underline{\varpi} - \varpi)e_0\psi\|_{L_x^2} \lesssim (1+t)^{-1} C[\psi]^{\frac{1}{2}}(t)$$

$$\lesssim (1+t)^{-1}(\|\psi[t_0]\|_{H^1(\Sigma)} + \|\psi(t_0)\|_{L^2(\Sigma)})$$

and

$$\|P(\varpi e_0\psi)\|_{L_x^\infty} \lesssim \|P(\varpi L\psi)\|_{L_x^\infty} + \|P(\varpi N\psi)\|_{L_x^\infty} \lesssim \|\varpi L\psi\|_{L_x^2} + \|P(\varpi N\psi)\|_{L_x^\infty}.$$

From Theorem 5 we have

$$\|\varpi L\psi\|_{L_x^2} \lesssim (1+t)^{-1} C[\psi]^{\frac{1}{2}}(t) \lesssim (1+t)^{-1}(\|\psi[t_0]\|_{H^1(\Sigma)} + \|\psi(t_0)\|_{L_\Sigma^2}).$$

Moreover

$$\|P(\varpi N\psi)\|_{L_x^\infty} \leq \|[P, \varpi N^l]\partial_l\psi\|_{L_x^\infty} + \|\varpi N^l P\partial_l\psi\|_{L_x^\infty}. \tag{4.171}$$

The first term in (4.171) can be estimated by Lemma 15. By using (4.170), the second term in (4.171) can be estimated as

$$\|\varpi N^l P\partial_l\psi(t)\|_{L_x^\infty} \leq \|\varpi \tilde{P}\psi(t)\|_{L_x^\infty} \lesssim \left((1+t)^{-\frac{2}{q}} + d(t) \right) (\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L_\Sigma^2}),$$

where $\tilde{P}$ denotes a Littlewood Paley projection with frequency 1 associated to a different symbol. Putting the above estimates together completes the proof. □

5. Boundedness Theorem for Conformal Energy

In this section we will present the proof of Theorem 5 under the bootstrap assumptions (BA1)–(BA3). We will work under the rescaled coordinates. Let $M_* = [0, \tau_*] \times \Sigma$, where $\tau_* \leq \lambda T$, where $\lambda \geq \Lambda$ and Λ is a sufficiently large number.

Recall the definition of the optical function u. We will set $\underline{u} := 2t - u$ and introduce the Morawetz vector field $K := \frac{1}{2}n(u^2\underline{L} + \underline{u}^2 L)$. Associated to K we introduce the modified energy density

$$\bar{Q}(K, \mathbf{T}) = \bar{Q}[\psi](K, \mathbf{T}) = Q[\psi](K, \mathbf{T}) + 2t\psi\mathbf{T}\psi - \psi^2\mathbf{T}(t),$$

and the total conformal energy

$$\bar{Q}[\psi](t) = \int_{\Sigma_t} \bar{Q}[\psi](K, \mathbf{T}).$$

Definition 2. *We define $\mathcal{C}[F]$ for a scalar function F with supp$F \subseteq \mathcal{D}_0^+$, by*

$$\mathcal{C}[F](t) = \mathcal{C}[F]^{(i)}(t) + \mathcal{C}[F]^{(e)}(t)$$

where $t_0 \le t \le \tau_*$, and

$$\mathcal{C}[F]^{(i)}(t) = \int_{\Sigma_t} (\underline{\varpi} - \varpi)\left(t^2|\boldsymbol{D}F|^2 + \left(1 + \frac{t^2}{(t-u)^2}\right)|F|^2\right) d\mu_g,$$
$$\mathcal{C}[F]^{(e)}(t) = \int_{\Sigma_t} \varpi\left(\underline{u}^2|\boldsymbol{D}_L F|^2 + u^2|\boldsymbol{D}_{\underline{L}} F|^2 + \underline{u}^2|\nabla\!\!\!/ F|^2 + |F|^2\right) d\mu_g.$$

From the definition it is easy to see that $\bar{Q}[\psi](t) \lesssim \mathcal{C}[\psi](t)$ and $C[\psi](t) \lesssim \mathcal{C}[\psi](t)$. We will prove the following results.

Theorem 6 (Comparison Theorem). *$T > 0$ can be chosen appropriately small but depending on universal constants, such that for any function ψ supported in $\mathcal{D}_0^+$ and any $1 \le t \le \tau_*$ there holds*

$$\mathcal{C}[\psi](t) \approx \bar{Q}[\psi](t).$$

Theorem 7 (Boundedness theorem). *There exists a large universal number Λ and a small universal number $T > 0$ such that for any $\lambda > \Lambda$, $\tau_* \le \lambda T$ and any function ψ satisfying the geometric wave equation*

$$\Box_{\mathbf{g}}\psi = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha(\mathbf{g}^{\alpha\beta}\sqrt{|\mathbf{g}|}\partial_\beta\psi) = 0 \quad in\ [0, \tau_*] \times \mathbb{R}^3 \tag{5.172}$$

with initial data $\psi[t_0]$ supported on the ball $B_{1/2}(0)$ there holds

$$\bar{Q}[\psi](t) \lesssim \bar{Q}[\psi](t_0) \qquad \forall\, t_0 \le t \le \tau_*.$$

5.0.1. Canonical null pair L, $\underline{L}$. Recall that in (3.101) we have introduced along the null hypersurface C_u the canonical null frame $\{L, \underline{L}, e_1, e_2\}$, where $\{e_1, e_2\}$ is an orthonormal frame on $S_{t,u}$. Let $e_3 = \underline{L} = \mathbf{T} - N$ and $e_4 = L = \mathbf{T} + N$. Then from (3.101) it follows that

$$\mathbf{D}_3 u = 2\mathbf{b}^{-1}, \quad \mathbf{D}_3\underline{u} = 2(n^{-1} - \mathbf{b}^{-1}), \quad \mathbf{D}_4\underline{u} = 2n^{-1}, \quad \mathbf{D}_4 u = 0.$$

Associated to this canonical null frame, we define on each null cone $S_{t,u}$ the Ricci coefficients

$$\chi_{AB} = \langle \mathbf{D}_A e_4, e_B\rangle, \underline{\chi}_{AB} = \langle \mathbf{D}_A e_3, e_B\rangle$$
$$\zeta_A = \frac{1}{2}\langle \mathbf{D}_3 e_4, e_A\rangle, \underline{\zeta}_A = \frac{1}{2}\langle \mathbf{D}_4 e_3, e_A\rangle$$
$$\xi_A = \frac{1}{2}\langle \mathbf{D}_3 e_3, e_A\rangle.$$

It is well-known ([8,13]) that there hold the identities

$$\underline{\chi}_{AB} = -\chi_{AB} - 2k_{AB}, \quad \underline{\zeta}_A = -k_{AN} + \nabla\!\!\!/_A \log n,$$
$$\xi_A = k_{AN} - \zeta_A + \nabla\!\!\!/_A \log n, \quad \zeta_A = \nabla\!\!\!/_A \log \mathbf{b} + k_{AN}$$

and the frame equations

$$\begin{aligned} \mathbf{D}_4 e_4 &= -(k_{NN} + \pi_{0N}) e_4, & \mathbf{D}_A e_4 &= \chi_{AB} e_B - k_{AN} e_4, \\ \mathbf{D}_A e_3 &= \underline{\chi}_{AB} e_B + k_{AN} e_3, & \mathbf{D}_4 e_3 &= 2\underline{\zeta}_A e_A + (k_{NN} + \pi_{0N}) e_3, \\ \mathbf{D}_3 e_4 &= 2\zeta_A e_A + (k_{NN} - \pi_{0N}) e_4, & \mathbf{D}_3 e_3 &= 2\xi_A e_A - (k_{NN} - \pi_{0N}) e_3, \\ \mathbf{D}_4 e_A &= \nabla\!\!\!/_4 e_A + \underline{\zeta}_A e_4, & \mathbf{D}_3 e_A &= \nabla\!\!\!/_3 e_A + \zeta_A e_3 + \xi_A e_4. \end{aligned}$$

We rely on the following result to prove the boundedness theorem. (5.173) and estimates of ζ, k in (5.175) can be seen in (4.161). (5.174) consists of (BA2) and the $L_t^2 L_x^\infty$ estimates on $e_0 n$, $\widehat{\nabla} n$, $\widehat{\nabla} Y$ that can be derived immediately by using (BA2), (2.29) and (2.49), under the rescaled coordinates. In [27], we will prove (5.175)-(5.179).

Proposition 12. *Under the bootstrap assumptions* (BA1)*,* (BA2) *and* (BA3)*, on* $\mathcal{D}^+ \subset [0, \tau_*] \times \Sigma$ *there hold the estimates*

$$(t-u) tr\chi \approx 1 \tag{5.173}$$

$$\|\tilde{\pi}\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2} \tag{5.174}$$

$$\|z, \zeta\|_{L^4(S_{t,u})} + \|r^{-1/2}(z, \zeta, k)\|_{L^2(S_{t,u})} \lesssim \lambda^{-1/2} \tag{5.175}$$

$$\|\Omega\|_{L_t^2 L_x^\infty} + \|\Omega\|_{L^4(S_{t,u})} + \|(t-u)^{-1/2} \Omega\|_{L^2(S_{t,u})} \lesssim \lambda^{-1/2} \tag{5.176}$$

$$\|z\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2}, \tag{5.177}$$

$$\|r^{3/2} \nabla\!\!\!/ z, r^{3/2} \underline{L} z\|_{L_t^\infty L_u^\infty L_\omega^p} \lesssim \lambda^{-1/2} \tag{5.178}$$

$$\|\hat{\chi}, \zeta\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2}, \tag{5.179}$$

where $p > 2$ *is such that* $0 < 1 - 2/p < s - 2$*, and* $z = tr\chi - \frac{2}{n(t-u)}$, $\Omega = \frac{\mathbf{b}^{-1} - n^{-1}}{t-u}$.

Let ${}^{(K)}\pi$ denote the deformation tensor of K and let ${}^{(K)}\bar{\pi} := {}^{(K)}\pi - 4t\mathbf{g}$. Then we have

$$\begin{aligned} &{}^{(K)}\bar{\pi}_{44} = -2u^2 n(\nabla_L \log n + k_{NN} + \pi_{0N}), \quad {}^{(K)}\bar{\pi}_{4A} = u^2 n(\underline{\zeta}_A - k_{AN} - \nabla\!\!\!/_A \log n), \\ &{}^{(K)}\bar{\pi}_{34} = -4un(\mathbf{b}^{-1} - n^{-1}) + nu^2(k_{NN} - \pi_{0N} - \mathbf{D}_3 \log n) + n\underline{u}^2(k_{NN} + \pi_{0N} - \mathbf{D}_4 \log n), \\ &{}^{(K)}\bar{\pi}_{33} = -8\underline{u}n(n^{-1} - \mathbf{b}^{-1}) - 2n\underline{u}^2(k_{NN} - \pi_{0N} + \mathbf{D}_3 \log n), \\ &{}^{(K)}\bar{\pi}_{3A} = n\underline{u}^2(\zeta_A + k_{AN} - \nabla\!\!\!/_A \log n) + nu^2 \xi_A, \\ &{}^{(K)}\bar{\pi}_{AB} = -2nu^2 \hat{k}_{AB} - nu^2 \mathrm{tr} k \delta_{AB} + 4tn(t-u)\hat{\chi}_{AB} \\ &\qquad + 2tn(t-u)\left(\mathrm{tr}\chi - \frac{2}{n(t-u)}\right)\delta_{AB}. \end{aligned}$$

For simplicity of presentation, we will drop the superscript K in ${}^{(K)}\bar{\pi}$. In view of (5.173), as an immediate consequence of Proposition 12, we have

Proposition 13. *Under the conditions in Proposition* 12 *we have on* $[0, \tau_*] \times \Sigma$ *that*

$$\begin{aligned} &\|u^{-2}\, \bar{\pi}_{44}\|_{L_t^2 L_x^\infty} + \|(u\underline{u})^{-1}\, \bar{\pi}_{34}\|_{L_t^2 L_x^\infty} + \|\underline{u}^{-2}\, \bar{\pi}_{33}\|_{L_t^2 L_x^\infty} \\ &\quad + \|u^{-2}\, \bar{\pi}_{4A}\|_{L_t^2 L_x^\infty} + \|\underline{u}^{-2}\, \bar{\pi}_{3A}\|_{L_t^2 L_x^\infty} + \|\underline{u}^{-2}\, \bar{\pi}_{AB}\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2}. \end{aligned}$$

Now we are ready to give the proof of Theorem 7 and Theorem 6.

5.1. Proof of Theorem 7. By calculating $\partial_t \bar{Q}[\psi]$ and integrating over the interval $[t_0, t]$, we have

$$\bar{Q}[\psi](t) = \bar{Q}[\psi](t_0) - \frac{1}{2}\mathcal{J}_1 + \mathcal{J}_2,$$

where

$$\mathcal{J}_1 = \int_{[t_0,t]\times\Sigma} Q^{\alpha\beta}[\psi]\ {}^{(K)}\bar{\pi}_{\alpha\beta} \quad \text{and} \quad \mathcal{J}_2 = \int_{[t_0,t]\times\Sigma} \psi^2 \Box_{\mathbf{g}} t.$$

It is easy to see that

$$\begin{aligned}\mathcal{J}_1 = \int_{[t_0,t]\times\Sigma} \Big(&\frac{1}{4}\bar{\pi}_{33}(L\psi)^2 + \frac{1}{4}\bar{\pi}_{44}(\underline{L}\psi)^2 + \frac{1}{2}\bar{\pi}_{34}|\nabla\!\!\!/\psi|^2 - \bar{\pi}_{4A}\underline{L}\psi\nabla\!\!\!/_A\psi \\ &-\bar{\pi}_{3A}L\psi\nabla\!\!\!/_A\psi + \bar{\pi}_{AB}\nabla\!\!\!/_A\psi\nabla\!\!\!/_B\psi + \frac{1}{2}\mathrm{tr}\bar{\pi}\left(\underline{L}\psi L\psi - |\nabla\!\!\!/\psi|^2\right)\Big).\end{aligned}$$

Observe that

$$\mathrm{tr}\bar{\pi} = \delta^{AB}\bar{\pi}_{AB} = 4tn(t-u)(\mathrm{tr}\chi - \frac{2}{n(t-u)}) - 2u^2 n\mathrm{tr}k.$$

It is easy to derive from (5.174) that

$$\int_{[t_0,t]\times\Sigma} |u^2 n\mathrm{tr}k L\psi\underline{L}\psi| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

Thus, by letting

$$\mathcal{B} = \int_{[t_0,t]\times\Sigma} 2t'n(t'-u)\left(\mathrm{tr}\chi - \frac{2}{n(t'-u)}\right)\underline{L}\psi L\psi,$$

we have from Proposition 13 that

$$|\mathcal{J}_1 - \mathcal{B}| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t). \tag{5.180}$$

Since $\Box_{\mathbf{g}} t = -e_0(n^{-1}) + n^{-1}\mathrm{Tr}k$, we can conclude from (5.174) that

$$|\mathcal{J}_2| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t)$$

In the following we will show that

$$|\mathcal{B}| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t) \tag{5.181}$$

We can write $\mathcal{B} = \mathcal{B}^i + \mathcal{B}^e$, where

$$\begin{aligned}\mathcal{B}^e &= \int_{[t_0,t]\times\Sigma} 2t'n(t'-u)\left(\mathrm{tr}\chi - \frac{2}{n(t'-u)}\right)L\psi\underline{L}\psi\varpi, \\ \mathcal{B}^i &= \int_{[t_0,t]\times\Sigma} 2t'n(t'-u)\left(\mathrm{tr}\chi - \frac{2}{n(t'-u)}\right)L\psi\underline{L}\psi(\underline{\varpi} - \varpi).\end{aligned}$$

In view of (5.177), we have

$$\mathcal{B}^i \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

We still need to estimate $\mathcal{B}^e$. Using [13, p. 1162] and the integration by part we have

$$\frac{1}{2}\mathcal{B}^e = -I_1 + I_2 + I_3 - I_4,$$

where

$$\begin{aligned}
I_1 &= \int_{[t_0,t]\times\Sigma} \varpi n t'(t'-u) z (\underline{L} L\psi)\psi,\\
I_2 &= \int_{[t_0,t]\times\Sigma} \left(-\underline{L}(\varpi n t'(t'-u)z) + (\mathrm{tr}\theta + N\log n - \mathrm{Tr}k - \operatorname{div} Y\right)\varpi n t'(t'-u)z) L\psi\,\psi,\\
I_3 &= \int_{\Sigma_t} \varpi n t'(t'-u) z L\psi\,\psi, \qquad I_4 = \int_{\Sigma_{t_0}} \varpi n t'(t'-u) z L\psi\,\psi.
\end{aligned}$$

Recall that in the exterior region $\{0 \le u \le 3t'/4\}$ we have $r(t', u) \approx t'$. Thus with the help of the Sobolev inequality (4.169) on $S_{t',u}$, for any function ψ there holds

$$\int_{0\le u\le 3t'/4} \|t'^{1-2/q}\psi\|^2_{L^q(S_{t',u})} du \lesssim \mathcal{C}[\psi](t'), \quad 2 \le q < \infty. \tag{5.182}$$

Therefore, by using (5.175) and (5.182), for the boundary term I_3 and I_4 we have the estimate

$$|I_3| + |I_4| \lesssim \|tL\psi\|_{L_t^\infty L_\Sigma^2} \|r^{\frac{1}{2}}\psi\|_{L_t^\infty L_u^2 L_x^4} \sup_{t, 0<u<\frac{3t}{4}} \|r^{\frac{1}{2}} z\|_{L^4(S_{t,u})} \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

Now we consider I_2. We write $I_2 = I_2^{(1)} + I_2^{(2)}$, where

$$\begin{aligned}
I_2^{(1)} &= \int_{[t_0,t]\times\Sigma} \underline{L}(\varpi n t'(t'-u)z) L\psi\,\psi,\\
I_2^{(2)} &= \int_{[t_0,t]\times\Sigma} (\mathrm{tr}\theta + N\log n - \mathrm{Tr}k - \operatorname{div} Y)\varpi n t'(t'-u) z L\psi\,\psi.
\end{aligned}$$

In view of (5.173), (5.174) and (5.177) in Proposition 12, by Hölder inequality we have

$$|I_2^{(2)}| \lesssim (\tau_* \|\pi, \nabla Y, z\|_{L_t^2 L_x^\infty} + \tau_*^{\frac{1}{2}}) \|z\|_{L_t^2 L_x^\infty} \sup_t \mathcal{C}[\psi](t) \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

Observe that

$$|\underline{L}\varpi| \lesssim r^{-1}, \quad \underline{L}t = n^{-1}, \quad \underline{L}u = 2\mathbf{b}^{-1}. \tag{5.183}$$

Let $p > 2$ be close to 2 such that $0 < 1 - 2/p < s - 2$ and let $1/p + 1/q = 1/2$. Then it follows from (5.178) that

$$\begin{aligned}
|I_2^{(1)}| \lesssim &\left(\|(|(t'-u)\underline{L}\varpi| + 1)z\|_{L_t^1 L_x^\infty} + \tau_* \|\underline{L}n\|_{L_t^2 L_x^\infty} \|z\|_{L_t^2 L_x^\infty} \right) \|t' L\psi\|_{L_t^\infty L_\Sigma^2} \|\psi\|_{L_t^\infty L_\Sigma^2}\\
&+ \|t'^{\frac{2}{q}} \varpi \underline{L} z\|_{L_t^1 L_u^\infty L_x^p} \sup_{t'} \left(\int_{0\le u\le \frac{3t'}{4}} \|t'^{1-\frac{2}{q}}\psi\|^2_{L^q(S_{t',u})} du \right)^{\frac{1}{2}} \|t' L\psi\|_{L_t^\infty L_\Sigma^2}.
\end{aligned}$$

In view of (5.182), (5.174), (5.177) and (5.178), we obtain

$$|I_2^{(1)}| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t)$$

Finally we will use (5.172) to estimate I_1. We first rewrite (5.172) as

$$\Box_{\mathbf{g}}\psi = -\underline{L}L\psi + \Delta\!\!\!/\,\psi + 2\zeta_A \nabla\!\!\!/_A\psi - \frac{1}{2}\mathrm{tr}\chi\underline{L}\psi - (\frac{1}{2}\mathrm{tr}\underline{\chi} + \nu)L\psi. \tag{5.184}$$

Then I_1 can be written as $I_1 = I_{11} + I_{12} + I_{13}$, where

$$\begin{aligned}
I_{11} &= \int_{[t_0,t]\times\Sigma} \varpi n t'(t'-u) z \Delta\!\!\!/\,\psi\psi\\
I_{12} &= -\frac{1}{2}\int_{[t_0,t]\times\Sigma} \varpi n t'(t'-u) z \mathrm{tr}\chi\psi\underline{L}\psi\\
I_{13} &= \int_{[t_0,t]\times\Sigma} \varpi n t'(t'-u) z \left(2\zeta_A\nabla\!\!\!/_A\psi - \left(\frac{1}{2}\mathrm{tr}\underline{\chi} + \nu\right)L\psi\right)\psi.
\end{aligned}$$

Recall that for any vector field X tangent to $S_{t,u}$ there holds

$$\int_{\Sigma_t} F\,\mathrm{div}\,X = -\int_{\Sigma_t}\{\nabla\!\!\!/ + (\zeta + \underline{\zeta})\}F\cdot X. \tag{5.185}$$

In view of (5.185) we have

$$I_{11} = -\int_{[t_0,t]\times\Sigma} \nabla\!\!\!/(\varpi n t'(t'-u)z\psi)\nabla\!\!\!/\psi + (\zeta + \underline{\zeta})\varpi n t'(t'-u)z\psi\cdot\nabla\!\!\!/\psi.$$

Now we introduce the following types of terms:

$$\begin{aligned}
&\mathrm{Er}_1 = nt'(t'-u)\left(\pi, z, \frac{\mathbf{b}^{-1} - n^{-1}}{(t-u)}\right)\cdot(L\psi, \nabla\!\!\!/\psi)\cdot z,\\
&\mathrm{Er}_2 = nzt'L\psi, \qquad \mathrm{Er}_3 = nt'(t'-u)\nabla\!\!\!/ z\cdot\nabla\!\!\!/\psi,\\
&\mathrm{Er}_4 = nt'z((t'-u)\nabla\!\!\!/\psi, \psi), \qquad \mathrm{Er}_5 = nt'(t'-u)\zeta\cdot\nabla\!\!\!/\psi\cdot z.
\end{aligned}$$

Then, symbolically we can write I_{11} and I_{13} as

$$|I_{11}| + |I_{13}| = \left|\int_{[t_0,t]\times\Sigma} \varpi(\mathrm{Er}_1 + \mathrm{Er}_2 + \mathrm{Er}_3 + \mathrm{Er}_5)\psi\right| + \left|\int_{[t_0,t]\times\Sigma} \varpi\,\mathrm{Er}_4\cdot\nabla\!\!\!/\psi\right|.$$

By using (5.174), (5.177), (5.176) and Hölder inequality, we can derive

$$\begin{aligned}
\left|\int_{[t_0,t]\times\Sigma} |\varpi\,\mathrm{Er}_1\cdot\psi|\right| &\lesssim \|z\|_{L_t^2 L_x^\infty}\left\|z, \pi, \frac{\mathbf{b}^{-1} - n^{-1}}{n(t'-u)}\right\|_{L_t^2 L_x^\infty} \tau_* \sup_t \mathcal{C}[\psi](t)\\
&\lesssim T\sup_t \mathcal{C}[\psi](t)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\int_{[t_0,t]\times\Sigma} |\varpi\,\mathrm{Er}_2\psi| + |\varpi\,\mathrm{Er}_4\nabla\!\!\!/\psi|\right|\\
&\quad\lesssim \|z\|_{L_t^1 L_x^\infty}\sup_t\left\{\|t(|L\psi| + |\nabla\!\!\!/\psi|)\|_{L_\Sigma^2}(\|\psi\|_{L_\Sigma^2} + \|t\nabla\!\!\!/\psi\|_{L_\Sigma^2})\right\} \lesssim T^{\frac{1}{2}}\sup_t \mathcal{C}[\psi](t).
\end{aligned}$$

By using (5.178) and (5.182) with $0 < 1 - 2/p < s - 2$ and $1/q + 1/p = \frac{1}{2}$, we have

$$\begin{aligned}
&\left|\int_{[t_0,t]\times\Sigma} |\varpi \mathrm{Er}_3 \cdot \psi|\right| \\
&\lesssim \sup_t \|t^{\frac{2}{q}} \varpi \not\nabla z\|_{L_t^1 L_u^\infty L_x^p} \sup_{t'} \left\{ \mathcal{C}[\psi](t')^{\frac{1}{2}} \left(\int_{0\le u\le \frac{3t'}{4}} \|t'^{1-\frac{2}{q}} \psi\|^2_{L^q(S_{t',u})} du \right)^{\frac{1}{2}} \right\} \\
&\lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).
\end{aligned}$$

With the help of (5.177), (5.182) with $q = 4$ and (5.175), we obtain

$$\begin{aligned}
&\left|\int_{[t_0,t]\times\Sigma} |\varpi \mathrm{Er}_5 \psi|\right| \\
&\lesssim \int_{t_0}^t \sup_u \|r^{\frac{1}{2}} \varpi z \cdot \zeta\|_{L_x^4} \cdot \sup_{t'} \left\{ \|t' \not\nabla \psi\|_{L^2_\Sigma} \left(\int_{0\le u\le \frac{3t'}{4}} \|r'^{\frac{1}{2}} \psi\|^2_{L^4(S_{t',u})} du \right)^{\frac{1}{2}} \right\} \\
&\lesssim \|r^{\frac{1}{2}} \zeta\|_{L_t^\infty L_u^\infty L_x^4} \|z\|_{L_t^1 L_x^\infty} \sup_t \mathcal{C}[\psi](t) \lesssim T \sup_t \mathcal{C}[\psi](t).
\end{aligned}$$

Now we consider I_{12}. By integration by part we can obtain $I_{12} := I_{12}^{(1)} + I_{12}^{(2)} + I_{12}^{(3)}$, where

$$\begin{aligned}
I_{12}^{(1)} &= \frac{1}{4} \int_{[t_0,t]\times\Sigma} -\underline{L}(\varpi n t'(t'-u) z \mathrm{tr}\chi) \psi^2, \\
I_{12}^{(2)} &= \frac{1}{4} \int_{[t_0,t]\times\Sigma} (\mathrm{tr}\theta + N \log n - \mathrm{Tr}k - \mathrm{div}\, Y)\, \varpi n t'(t'-u) z \mathrm{tr}\chi\, \psi^2, \\
I_{12}^{(3)} &= \frac{1}{4} \int_{\Sigma_t} \varpi n t'(t'-u) z \mathrm{tr}\chi\, \psi^2 - \frac{1}{4} \int_{\Sigma_0} \varpi n t'(t'-u) z \mathrm{tr}\chi\, \psi^2.
\end{aligned}$$

Using (5.173), (5.174) and (5.177), the term $I_{12}^{(2)}$ can be bounded by

$$\begin{aligned}
|I_{12}^{(2)}| &\lesssim \int_{[t_0,t]\times\Sigma} \left| \varpi (z + \pi + \nabla Y + \frac{1}{n(t-u)}) z \right| t' n \psi^2 \\
&\lesssim \|z\|_{L_t^2 L_x^\infty} \||\psi|^2\|_{L_t^\infty L^1_\Sigma} (\tau_*^{1/2} + \tau_* \|\pi, \nabla Y, z\|_{L_t^2 L_x^\infty}) \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).
\end{aligned}$$

For the term $I_{12}^{(3)}$, using (5.173), (5.175) and (5.182) with $q = 4$, we have

$$|I_{12}^{(3)}| \lesssim \sup_{t,u} \|z\|_{L^2(S_{t,u})} \int_{0<u\le \frac{3t}{4}} \|t'|\psi|^2\|_{L^2(S_{t,u})} \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

It remains only to consider the term $I_{12}^{(1)}$. We have $|I_{12}^{(1)}| \lesssim J_1 + J_2$, where

$$\begin{aligned}
J_1 &:= \Big| \int_{[t_0,t]\times\Sigma} \Big(\underline{L}\varpi n t'(t'-u) \mathrm{tr}\chi + \varpi n \mathrm{tr}\chi \underline{L}(t'(t'-u)) \\
&\qquad + \varpi \mathrm{tr}\chi \underline{L} n t'(t'-u) + \varpi \underline{L}(\frac{1}{n(t'-u)}) n t'(t'-u) \Big) z \psi^2 \Big|, \\
J_2 &:= \int_{[t_0,t]\times\Sigma} |\varpi \underline{L} z n t'(t'-u)| (|\mathrm{tr}\chi| + |z|) \psi^2.
\end{aligned}$$

Since $r \approx (t-u)$ and $t(|\mathrm{tr}\chi|+|z|) \approx 1$ on Ext_t, with $\frac{1}{p}+\frac{1}{q}=1$ and p is slightly greater than 2, also using Hölder inequality, we obtain

$$\begin{aligned} J_2 &\lesssim \|\varpi r^{1-\frac{2}{p}} \underline{L} z\|_{L_t^1 L_u^\infty L_x^p} \sup_{t'} \Big(\int_{0\le u\le \frac{3t'}{4}} \|r^{\frac{2}{p}}|\psi|^2\|_{L^q(S_{t',u})} du\Big) \\ &\lesssim \|\varpi r^{1-\frac{2}{p}} \underline{L} z\|_{L_t^1 L_u^\infty L_x^p} \sup_{t'} \Big(\int_{0\le u\le \frac{3t'}{4}} \|r^{1-\frac{2}{q_1}}\psi\|_{L^{q_1}(S_{t',u})} \|r^{1-\frac{2}{q_2}}\psi\|_{L^{q_2}(S_{t',u})} du\Big) \end{aligned}$$

where $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Using (5.182) and (5.178) we obtain

$$J_2 \lesssim T^{\frac{1}{2}} \sup_{t'} \mathcal{C}[\psi](t').$$

To estimate J_1, in view of (5.183) and

$$\underline{L}\Big(\frac{1}{n(t-u)}\Big) = \frac{n\mathbf{b}^{-1}}{n^2(t-u)^2} - \frac{\underline{L}\log n}{n(t-u)} - \frac{n(n^{-1}-\mathbf{b}^{-1})}{n^2(t-u)^2},$$

using (5.177), (5.174) and (5.176), we have

$$\begin{aligned} J_1 &\lesssim \|z\psi^2\|_{L_t^1 L_\Sigma^1} + \|\underline{L}n, \frac{\mathbf{b}^{-1}-n^{-1}}{n(t-u)}\|_{L_t^2 L_\Sigma^\infty} \|z\|_{L_t^2 L_\Sigma^\infty} \tau_* \sup_t \int_{0\le u\le \frac{3t}{4}} \|\psi^2\|_{L^1(S_{t,u})} \\ &\lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t). \end{aligned}$$

The proof is therefore complete.

5.2. Proof of comparison theorem. We will adapt the argument in [13] to prove Theorem 6. For simplicity, we use Θ to denote any term from the collection

$$\left\{\mathrm{tr}\chi - \frac{2}{n(t-u)},\ \ \mathrm{Tr}k,\ \ \frac{\mathbf{b}^{-1}-n^{-1}}{t-u},\ \ \hat{k}_{NN}\right\}.$$

According to (5.175) and (5.176) in Proposition 12 we have

$$\|r^{-\frac{1}{2}}\Theta\|_{L^2(S_{t,u})} \lesssim \lambda^{-\frac{1}{2}}. \tag{5.186}$$

By following the argument in [13, Section 6] we can derive

$$\begin{aligned} \bar{Q}[\psi](t) &\gtrsim \int_\Sigma \left(\underline{u}^2(L\psi)^2 + u^2(\underline{L}\psi)^2 + (u^2+\underline{u}^2)|\nabla\!\!\!\!/\,\psi|^2 + \left(1+\frac{t^2}{(t-u)^2}\right)\psi^2\right) \\ &\quad - \int_\Sigma \left(1+\frac{t^2}{(t-u)^2}\right)\psi^2 (t-u)\Theta \\ &\gtrsim \mathcal{C}[\psi](t) - \int_\Sigma \left(1+\frac{t^2}{(t-u)^2}\right)\psi^2(t-u)\Theta. \end{aligned}$$

By using (5.186) and the inequality, which can be derived in view of (4.169),

$$\|t\psi\|_{L_u^2 L_\omega^4} \lesssim \|t\nabla\!\!\!\!/\,\psi\|_{L_x^2} + \|\frac{t}{t-u}\psi\|_{L_x^2} \lesssim \mathcal{C}^{\frac{1}{2}}[\psi](t),$$

we can obtain

$$\int_\Sigma \frac{t^2}{t-u}\Theta\psi^2 \lesssim \|r^{\frac{1}{2}}\Theta\|_{L_u^\infty L_\omega^2}\|r^{\frac{1}{2}}t^2\psi^2\|_{L_u^1 L_\omega^2} \lesssim \lambda^{-\frac{1}{2}}\tau_*^{\frac{1}{2}}\|t^2\psi^2\|_{L_u^1 L_\omega^2} \lesssim T^{\frac{1}{2}}\mathcal{C}[\psi](t).$$

Similarly we have

$$\int_\Sigma \psi^2(t-u)\Theta \lesssim \lambda^{-\frac{1}{2}}\tau_*^{\frac{1}{2}}\|r^2\psi^2\|_{L_u^1 L_\omega^2} \lesssim T^{\frac{1}{2}}\mathcal{C}[\psi](t).$$

Therefore, there is a universal constant $C_0 > 0$ such that

$$\mathcal{C}[\psi](t) \le C_0\bar{Q}[\psi](t) + C_0 T^{\frac{1}{2}}\mathcal{C}[\psi](t).$$

This implies the desired conclusion by taking T to be small universal constant.

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6. Appendix: Commutator estimates

In this section we derive various commutator estimates involving the LP projections P_λ in fractional Sobolev spaces that are extensively used in this paper, where P_λ is defined by (1.11). One can refer to [17,20] for various properties of LP projections. In view of the LP decomposition, the norm in the Sobolev space H^ϵ with $0 \le \epsilon < 1$ is defined by

$$\|F\|_{H^\epsilon} := \|F\|_{L^2} + \left(\sum_{\lambda>1}\lambda^{2\epsilon}\|P_\lambda F\|_{L^2}^2\right)^{1/2}$$

for any scalar function F. For any nonnegative integer m and $0 \le \epsilon < 1$, we define $\|F\|_{H^{m+\epsilon}} := \|F\|_{H^m} + \|\widehat{\nabla}^m F\|_{H^\epsilon}$. For simplicity of exposition, we will write $F_\lambda := P_\lambda F$, $F_{\le\lambda} := \sum_{\mu\le\lambda} P_\mu F$, and $\|\Lambda^r F\|_{L^2} := \left(\sum_{\lambda>1}\lambda^{2r}\|P_\lambda F\|_{L^2}^2\right)^{1/2}$. For any sequence (a_λ) we will use $\|a_\lambda\|_{l_\lambda^2}^2$ to denote $\sum_{\lambda\ge1}|a_\lambda|^2$.

6.1. Product estimates. We first derive some useful product estimates. According to the Littlewood-Paley (LP) decomposition, one has, for any scalar functions F and G, the trichotomy law which schematically says that

$$P_\mu(F\cdot G) = P_\mu(F_{\le\mu}\cdot G_\mu) + P_\mu(F_\mu\cdot G_{\le\mu}) + \sum_{\lambda>\mu} P_\mu(F_\lambda\cdot G_\lambda). \tag{6.187}$$

We will use this decomposition repeatedly.

Lemma 17. *For any* $0 < \epsilon < 1$ *and any scalar functions* F *and* G *there hold*

$$\|\Lambda^\epsilon(F\cdot G)\|_{L^2} \lesssim \|F\|_{H^{1/2+\epsilon}}\|G\|_{H^1} + \|G\|_{H^{1/2+\epsilon}}\|F\|_{H^1}, \tag{6.188}$$

$$\|\mu^{-1/2+\epsilon}P_\mu(F\cdot G)\|_{l_\mu^2 L^2} \lesssim \|G\|_{H^\epsilon}\|F\|_{H^1}. \tag{6.189}$$

Proof. We first prove (6.188). By using the Bernstein inequality and the finite band property of the LP projections, we have

$$\mu^{\epsilon}\|P_\mu(F_{\le\mu}\cdot G_\mu)\|_{L^2}\lesssim \mu^{\epsilon}\sum_{\lambda\le\mu}\|F_\lambda\|_{L^\infty}\|G_\mu\|_{L^2}$$
$$\lesssim \sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{1-\epsilon}\|\lambda^{1/2+\epsilon}F_\lambda\|_{L^2}\|\widehat{\nabla}G_\mu\|_{L^2}.$$

Therefore

$$\|\mu^{\epsilon}P_\mu(F_{\le\mu}\cdot G_\mu)\|_{l_\mu^2L_x^2}\lesssim \|F\|_{H^{1/2+\epsilon}}\|\widehat{\nabla}G\|_{L^2}.$$

Similarly we have

$$\|\mu^{\epsilon}P_\mu(F_\mu\cdot G_{\le\mu})\|_{l_\mu^2L^2}\lesssim \|G\|_{H^{1/2+\epsilon}}\|\widehat{\nabla}F\|_{L^2}.$$

Moreover, we have

$$\mu^{\epsilon}\|P_\mu(F_\lambda\cdot G_\lambda)\|_{L^2}\lesssim \mu^{\frac{1}{2}+\epsilon}\|F_\lambda\|_{L^2}\|G_\lambda\|_{L^6}\lesssim \left(\frac{\mu}{\lambda}\right)^{1/2+\epsilon}\|\lambda^{1/2+\epsilon}F_\lambda\|_{L^2}\|\widehat{\nabla}G_\lambda\|_{L^2}.$$

This implies that

$$\Big\|\mu^{\epsilon}\sum_{\lambda>\mu}P_\mu(F_\lambda\cdot G_\lambda)\Big\|_{l_\mu^2L^2}\lesssim \|\Lambda^{1/2+\epsilon}F\|_{L^2}\|\widehat{\nabla}G\|_{L^2}.$$

Combining the above estimates and using the trichotomy law (6.187) we obtain (6.188).

Next we prove (6.189). Using the properties of the LP projections we have

$$\|\mu^{-1/2+\epsilon}P_\mu(F_{\le\mu}\cdot G_\mu)\|_{l_\mu^2L^2}\lesssim \Big\|\sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{1/2}\|\mu^{\epsilon}G_\mu\|_{L^2}\|\widehat{\nabla}F_\lambda\|_{L^2}\Big\|_{l_\mu^2}$$
$$\lesssim \|F\|_{H^1}\|G\|_{H^\epsilon},$$
$$\|\mu^{-1/2+\epsilon}P_\mu(F_\mu\cdot G_{\le\mu})\|_{l_\mu^2L^2}\lesssim \Big\|\sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{3/2-\epsilon}\|\widehat{\nabla}F\|_{L^2}\|\lambda^{\epsilon}G_\lambda\|_{L^2}\Big\|_{l_\mu^2}$$
$$\lesssim \|\widehat{\nabla}F\|_{L^2}\|G\|_{H^\epsilon}$$

and

$$\Big\|\sum_{\lambda>\mu}\mu^{-1/2+\epsilon}P_\mu(F_\lambda\cdot G_\lambda)\Big\|_{l_\mu^2L^2}\lesssim \Big\|\mu^{1/2+\epsilon}\sum_{\lambda\ge\mu}\|F_\lambda\|_{L^3}\|G_\lambda\|_{L^2}\Big\|_{l_\mu^2}$$
$$\lesssim \Big\|\sum_{\lambda\ge\mu}\left(\frac{\mu}{\lambda}\right)^{1/2+\epsilon}\|\widehat{\nabla}F_\lambda\|_{L^2}\|\lambda^{\epsilon}G_\lambda\|_{L^2}\Big\|_{l_\mu^2}\lesssim \|F\|_{H^1}\|G\|_{H^\epsilon}.$$

Combining the above estimates with (6.187) yields (6.189). □

Lemma 18. *For any* $\epsilon>0$ *and any scalar functions* G_1, G_2 *and* G_3 *there holds*

$$\big\|\Lambda^{\epsilon}(G_1G_2G_3)\big\|_{L^2}\lesssim \sum_{j=1}^{3}\left(\|G_j\|_{H^{1+\epsilon}}\prod_{l\ne j}\|G_l\|_{H^1}\right).$$

Proof. Using the facts $\|(G_1)_{\le\mu}\|_{L^\infty} \lesssim \mu^{1/2}\|\widehat{\nabla} G_1\|_{L^2}$ and $\|(G_2G_3)_\mu\|_{L^2} \lesssim \mu^{-1}\|P_\mu \widehat{\nabla} (G_2G_3)\|_{L^2}$ together with (6.189) in Lemma 17, we have

$$\begin{aligned}\left\|\mu^\epsilon P_\mu((G_1)_{\le\mu}(G_2G_3)_\mu)\right\|_{l^2_\mu L^2} &\lesssim \|\widehat{\nabla} G_1\|_{L^2}\left\|\mu^{-1/2+\epsilon}P_\mu\widehat{\nabla}(G_2G_3)\right\|_{l^2_\mu L^2}\\ &\lesssim \|G_1\|_{H^1}\left(\|G_2\|_{H^{1+\epsilon}}\|G_3\|_{H^1}+\|G_3\|_{H^{1+\epsilon}}\|G_2\|_{H^1}\right),\end{aligned}$$

In view of $\|(G_1)_\mu\|_{L^3} \lesssim \mu^{-1/2}\|\widehat{\nabla}(G_1)_\mu\|_{L^2}$ and $\|(G_2G_3)_{\le\mu}\|_{L^6} \lesssim \mu^{1/2}\|G_2G_3\|_{L^3}$, we obtain

$$\begin{aligned}\left\|\mu^\epsilon P_\mu\left((G_1)_\mu(G_2G_3)_{\le\mu}\right)\right\|_{l^2_\mu L^2} &\lesssim \left\|\mu^\epsilon\widehat{\nabla}(G_1)_\mu\right\|_{l^2_\mu L^2}\|G_2\|_{L^6}\|G_3\|_{L^6}\\ &\lesssim \|G_1\|_{H^{1+\epsilon}}\|G_2\|_{H^1}\|G_3\|_{H^1}.\end{aligned}$$

Furthermore, by using $\|(G_1)_\lambda\|_{L^6} \lesssim \|\widehat{\nabla}(G_1)_\lambda\|_{L^2}$, we have

$$\begin{aligned}\left\|\sum_{\lambda>\mu}\mu^\epsilon P_\mu\left((G_1)_\lambda(G_2G_3)_\lambda\right)\right\|_{l^2_\mu L^2} &\lesssim \left\|\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^\epsilon\|\lambda^\epsilon\widehat{\nabla}(G_1)_\lambda\|_{L^2}\right\|_{l^2_\mu}\|G_2G_3\|_{L^3}\\ &\lesssim \|G_1\|_{H^{1+\epsilon}}\|G_2\|_{H^1}\|G_3\|_{H^1}.\end{aligned}$$

In view of the trichotomy law (6.187) and the above estimates, we thus complete the proof. □

Lemma 19. *For any* $0<\epsilon<1$ *there hold*

$$\|\Lambda^\epsilon(F\cdot\widehat{\nabla} G)\|_{L^2} \lesssim \|F\|_{L^\infty}\|G\|_{H^{1+\epsilon}} + \|G\|_{L^\infty}\|F\|_{H^{1+\epsilon}}. \tag{6.190}$$

Proof. We use the properties of the LP projections to obtain

$$\begin{aligned}&\|\mu^\epsilon P_\mu(F_{\le\mu}\cdot\widehat{\nabla} G_\mu)\|_{l^2_\mu L^2} \lesssim \|F\|_{L^\infty}\|\mu^\epsilon(\widehat{\nabla} G)_\mu\|_{l^2_\mu L^2} \lesssim \|F\|_{L^\infty}\|\Lambda^\epsilon\widehat{\nabla} G\|_{L^2},\\ &\|\mu^\epsilon P_\mu(F_\mu\cdot\widehat{\nabla} G_{\le\mu})\|_{l^2_\mu L^2} \lesssim \|\mu^{1+\epsilon}F_\mu\|_{l^2_\mu L^2}\|G\|_{L^\infty} \lesssim \|\Lambda^\epsilon\widehat{\nabla} F\|_{L^2}\|G\|_{L^\infty}\end{aligned}$$

and

$$\begin{aligned}\left\|\mu^\epsilon\sum_{\lambda>\mu}P_\mu(F_\lambda\cdot\widehat{\nabla} G_\lambda)\right\|_{l^2_\mu L^2} &\lesssim \|F\|_{L^\infty}\left\|\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^\epsilon\|\lambda^\epsilon\widehat{\nabla} G_\lambda\|_{L^2}\right\|_{l^2_\mu}\\ &\lesssim \|F\|_{L^\infty}\|\Lambda^\epsilon\widehat{\nabla} G\|_{L^2}.\end{aligned}$$

Combining the above three estimates, we obtain (6.190) using the trichotomy law (6.187). □

6.2. Commutator estimates. In this subsection we will derive various estimates related to the commutators $[P_\mu, F]G$. We first consider the general setting. Let $m(\xi)$ define a multiplier

$$\mathcal{P}f(x) = \int e^{ix\xi}m(\xi)\hat{f}(\xi)d\xi. \tag{6.191}$$

By introducing the function $M(x)$ defined by

$$M(x) = \int e^{ix\cdot\xi}m(\xi)d\xi = \hat{m}(-x), \tag{6.192}$$

then for any scalar functions F and G we can write

$$[\mathcal{P}, F]G(x) = \int M(x-y)(F(y)-F(x))G(y)dy$$
$$= \int M(x-y)(x-y)^j \int_0^1 \partial_j F(\tau y + (1-\tau)x)d\tau G(y)dy$$
$$= \int M(h)h^j \int_0^1 \partial_j F(x-\tau h)d\tau G(x-h)dh. \quad (6.193)$$

By taking the L^q-norm with $1 \le q \le \infty$ and using the Minkowski inequality we obtain

$$\|[\mathcal{P}, F]G\|_{L^q} \le \int |M(h)||h| \int_0^1 \|\partial F(\cdot - \tau h)G(\cdot - h)\|_{L^q} d\tau dh \quad (6.194)$$

An application of the Hölder inequality gives the following result whose special case with $p = \infty$ and $q = r = 2$ is [12, Lemma 8.2].

Lemma 20. *Let $\mathcal{P}$ be the multiplier operator defined by* (6.191) *and let M be the function given by* (6.192)*. Then, for any $1 \le p, q, r \le \infty$ satisfying $1/p + 1/r = 1/q$ and any scalar functions F and G, there holds*

$$\|[\mathcal{P}, F]G\|_{L^q} \le \|\partial F\|_{L^p} \|G\|_{L^r} \int |x||M(x)|dx.$$

Recall that the LP projection P_μ is a multiplier operator with $m(\xi) = \Psi(\mu^{-1}\xi)$, where Ψ is a mollifier with support on $\{1/2 < |\xi| < 2\}$. Observe that $M(x) = \mu^3 \widehat{\Psi}(-\mu x)$. We have $\int |x||M(x)|dx \lesssim \mu^{-1}$. Therefore, from Lemma 20 we obtain the following commutator estimate.

Corollary 1. *For any $1 \le p, q, r \le \infty$ satisfying $1/p + 1/r = 1/q$ and any scalar functions F and G there holds*

$$\|[P_\mu, F]G\|_{L^q} \lesssim \mu^{-1} \|\widehat{\nabla} F\|_{L^p} \|G\|_{L^r}.$$

In the following we will give further estimates related to the commutator $[P_\mu, F]G$ for any scalar functions F and G. We can write $[P_\mu, F]G = [P_\mu, F]G_{\le 2\mu} + [P_\mu, F]G_{>2\mu}$. By the orthogonality of the LP projections, we have

$$[P_\mu, F]G_{>2\mu} = \sum_{\mu_1 > 2\mu} P_\mu(F \cdot G_{\mu_1}) = \sum_{\mu_1 > 2\mu} \sum_{\frac{\mu_1}{2} \le \mu_2 \le 2\mu_1} P_\mu(F_{\mu_2} \cdot G_{\mu_1}).$$

Thus, schematically we can write

$$[P_\mu, F]G = [P_\mu, F]G_{\le \mu} + \sum_{\lambda > \mu} P_\mu(F_\lambda G_\lambda) \quad (6.195)$$

which is not quite accurate but harmless to derive estimates.

Lemma 21. *For $0 < \epsilon < 1$ there holds*

$$\|\mu^\epsilon [P_\mu, F]G\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{H^{\frac{1}{2}+\epsilon}} \|G\|_{L^2}.$$

Proof. First we have

$$\Big\|\mu^\epsilon \sum_{\lambda>\mu} P_\mu(F_\lambda\cdot G_\lambda)\Big\|_{l_\mu^2 L^2} \lesssim \|G\|_{L^2}\Big\|\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^\epsilon \|\lambda^\epsilon F_\lambda\|_{L^\infty}\Big\|_{l_\mu^2} \lesssim \|\widehat{\nabla} F\|_{H^{\frac{1}{2}+\epsilon}}\|G\|_{L^2}.$$

Now we decompose

$$[P_\mu, F]G_{\le\mu}(x) = \sum_{\lambda\le\mu}[P_\mu, F_{\le\lambda}]G_\lambda + \sum_{\lambda\le\mu}[P_\mu, F_{>\lambda}]G_\lambda.$$

By using Corollary 1, $\|\widehat{\nabla} F_\ell\|_{L^\infty} \lesssim \ell^{1-\epsilon}\|\widehat{\nabla} F_\ell\|_{H^{\frac{1}{2}+\epsilon}}$ and $\|G_\lambda\|_{L^\infty} \lesssim \lambda^{3/2}\|G_\lambda\|_{L^2}$, we obtain

$$\begin{aligned}
\mu^\epsilon\|[P_\mu, F]G_{\le\mu}\|_{L^2} &\lesssim \mu^{-1+\epsilon}\sum_{\lambda\le\mu}\|\widehat{\nabla} F_{\le\lambda}\|_{L^\infty}\|G_\lambda\|_{L^2} + \mu^{-1+\epsilon}\sum_{\lambda\le\mu}\|\widehat{\nabla} F_{>\lambda}\|_{L^2}\|G_\lambda\|_{L^\infty}\\
&\lesssim \|\widehat{\nabla} F\|_{H^{\frac{1}{2}+\epsilon}}\sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{1-\epsilon}\|G_\lambda\|_{L^2}\\
&\quad+\sum_{\lambda\le\mu}\sum_{\lambda'>\lambda}\left(\frac{\lambda}{\mu}\right)^{1-\epsilon}\left(\frac{\lambda}{\lambda'}\right)^{\frac{1}{2}+\epsilon}\|\lambda'^{\frac{1}{2}+\epsilon}\widehat{\nabla} F_{\lambda'}\|_{L^2}\|G_\lambda\|_{L^2}.
\end{aligned}$$

Taking the l_μ^2-norm gives

$$\|\mu^\epsilon[P_\mu, F]G_{\le\mu}\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{H^{1/2+\epsilon}}\|G\|_{L^2}.$$

In view of (6.195), the proof is therefore complete. □

Lemma 22. *For* $0<\epsilon<1$, *there holds*

$$\mu^\epsilon\|[P_\mu, F]\widehat{\nabla} G\|_{L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty}\Big(\sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{1-\epsilon}\|\lambda^\epsilon G_\lambda\|_{L^2} + \sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^\epsilon\|\lambda^\epsilon G_\lambda\|_{L^2}\Big). \tag{6.196}$$

Proof. By using Corollary 1 we have

$$\|[P_\mu, F]\widehat{\nabla} G_{\le\mu}\|_{L^2} \lesssim \mu^{-1}\|\widehat{\nabla} F\|_{L^\infty}\sum_{\lambda\le\mu}\|\widehat{\nabla} G_\lambda\|_{L^2} \lesssim \mu^{-1}\|\widehat{\nabla} F\|_{L^\infty}\sum_{\lambda\le\mu}\lambda\|G_\lambda\|_{L^2}.$$

On the other hand, by using the properties of the LP projections we have

$$\sum_{\lambda>\mu}\|P_\mu(F_\lambda\cdot\widehat{\nabla} G_\lambda)\|_{L^2} \lesssim \sum_{\lambda>\mu}\|F_\lambda\|_{L^\infty}\|\widehat{\nabla} G_\lambda\|_{L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty}\sum_{\lambda>\mu}\|G_\lambda\|_{L^2}.$$

Combining these two estimates and using the decomposition (6.195) we complete the proof. □

Lemma 23. *For* $0<\epsilon<1/2$ *and* $\mu\ge1$ *there hold*

$$\begin{aligned}
&\mu^{-\frac{1}{2}+\epsilon}\|\widehat{\nabla}[P_\mu, F]G\|_{L^2} + \mu^{\frac{1}{2}+\epsilon}\|[P_\mu, F]G\|_{L^2}\\
&\quad\lesssim \|\widehat{\nabla} F\|_{L^6}\Big(\sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{1/2-\epsilon}\|\lambda^\epsilon G_\lambda\|_{L^2} + \sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^{1/2+\epsilon}\|\lambda^\epsilon G_\lambda\|_{L^2}\Big)
\end{aligned}\tag{6.197}$$

Proof. From the we can obtain

$$\widehat{\nabla}[P_\mu, F]G_{\le\mu} = \int \widehat{\nabla} M_\mu(x-y)(x-y)^j \int_0^1 \partial_j F(\tau y + (1-\tau)x) d\tau G_{\le\mu}(y) dy \\ + \int M_\mu(x-y)\widehat{\nabla} F(x) G_{\le\mu}(y) dy, \quad (6.198)$$

where M_μ is given by (6.192) with $m(\xi) = \Psi(\mu^{-1}\xi)$. One can check $\int |M_\mu(x)|dx \lesssim 1$ and $\int |x||\widehat{\nabla} M_\mu(x)|dx \lesssim 1$. Thus, it follows from the Minkowski inequality that

$$\|\widehat{\nabla}[P_\mu, F]G_{\le\mu}\|_{L^2} \lesssim \|\widehat{\nabla} F\|_{L^6}\|G_{\le\mu}\|_{L^3} \lesssim \|\widehat{\nabla} F\|_{L^6} \sum_{\lambda\le\mu} \lambda^{1/2}\|G_\lambda\|_{L^2}.$$

On the other hand, by the properties of LP projections, we have

$$\sum_{\lambda>\mu} \|\widehat{\nabla} P_\mu(F_\lambda \cdot G_\lambda)\|_{L^2} \lesssim \mu \sum_{\lambda>\mu} \|F_\lambda\|_{L^6}\|G_\lambda\|_{L^3} \lesssim \mu \|\widehat{\nabla} F\|_{L^6} \sum_{\lambda>\mu} \lambda^{-1/2}\|G_\lambda\|_{L^2}.$$

In view of the decomposition (6.195) we thus obtain the estimate for the first term on the left hand side of (6.197). Next we estimate the second term. By using Corollary 1 we obtain

$$\|\mu^{\frac{1}{2}+\epsilon}[P_\mu, F]G_{\le\mu}\|_{L^2} \lesssim \mu^{-\frac{1}{2}+\epsilon}\|\widehat{\nabla} F\|_{L^6}\|G_{\le\mu}\|_{L^3} \lesssim \mu^{-\frac{1}{2}+\epsilon}\|\widehat{\nabla} F\|_{L^6} \sum_{\lambda\le\mu} \lambda^{\frac{1}{2}}\|G_\lambda\|_{L^2},$$

while by using the properties of the LP projections we have

$$\mu^{\frac{1}{2}+\epsilon} \sum_{\lambda>\mu} \|P_\mu(F_\lambda \cdot G_\lambda)\|_{L^2} \lesssim \mu^{1+\epsilon} \sum_{\lambda>\mu} \|F_\lambda \cdot G_\lambda\|_{L^{\frac{3}{2}}} \lesssim \mu^{1+\epsilon}\|\widehat{\nabla} F\|_{L^6} \sum_{\lambda>\mu} \lambda^{-1}\|G_\lambda\|_{L^2}.$$

The proof is thus complete using (6.195). □

Lemma 24. *For* $0 < \epsilon < 3/2$ *there holds*

$$\|\mu^{-1/2+\epsilon}[P_\mu, F]\widehat{\nabla} G\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla} F\|_{H^1}\|G\|_{H^\epsilon}. \quad (6.199)$$

Proof. We use the decomposition (6.195). We first consider $\sum_{\lambda>\mu} \mu^{-1/2+\epsilon} P_\mu(F_\lambda \cdot \widehat{\nabla} G_\lambda)$. By the Bernstein inequality and the finite band property of LP projections, we have

$$\mu^{-1/2+\epsilon}\|P_\mu(F_\lambda \cdot \widehat{\nabla} G_\lambda)\|_{L^2} \lesssim \mu^\epsilon \|F_\lambda \cdot \widehat{\nabla} G_\lambda\|_{L^{3/2}} \lesssim \mu^\epsilon \|F_\lambda\|_{L^6}\|\widehat{\nabla} G_\lambda\|_{L^2} \\ \lesssim \mu^\epsilon \|\widehat{\nabla} F_\lambda\|_{L^6}\|G_\lambda\|_{L^2} \lesssim \left(\frac{\mu}{\lambda}\right)^\epsilon \|\lambda^\epsilon G_\lambda\|_{L^2}\|\widehat{\nabla} F\|_{H^1}.$$

Therefore

$$\left\| \sum_{\lambda>\mu} \mu^{-1/2+\epsilon} P_\mu(F_\lambda \cdot \widehat{\nabla} G_\lambda) \right\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{H^1}\|\Lambda^\epsilon G\|_{L^2}. \quad (6.200)$$

Next we consider the term $\mu^{-1/2+\epsilon}[P_\mu, F]\widehat{\nabla} G_{\le\mu}$. By using (6.194) and setting $F_{h,\tau}(x) = F(x - \tau h)$, we obtain

$$\|[P_\mu, F]\widehat{\nabla} G_{\le\mu}\|_{L^2} \lesssim \mu^{-1} \sup_{h,\tau} \|\widehat{\nabla} F_{h,\tau} \cdot \widehat{\nabla} G_{\le\mu}\|_{L^2}.$$

By using the orthogonality of the LP projections, we can write $\widehat{\nabla} F_{h,\tau} \cdot \widehat{\nabla} G_{\le\mu} = a_\mu + b_\mu$, where

$$a_\mu = \sum_{\lambda>\mu} P_\lambda((\widehat{\nabla} F_{h,\tau})_\lambda \cdot \widehat{\nabla} G_{\le\mu}) \quad \text{and} \quad b_\mu = \sum_{\lambda\le\mu} P_\lambda(\widehat{\nabla} F_{h,\tau} \cdot \widehat{\nabla} G_{\le\mu}).$$

By the finite band property and the Bernstein inequality of LP projections we have

$$\begin{aligned}\|a_\mu\|_{L^2} &\lesssim \sum_{\lambda>\mu} \|(\widehat{\nabla} F_{h,\tau})_\lambda\|_{L^2} \|\widehat{\nabla} G_{\le\mu}\|_{L^\infty} \lesssim \sum_{\lambda>\mu, \lambda'\le\mu} \lambda^{-1} \lambda'^{5/2} \|\widehat{\nabla} F_{h,\tau}\|_{H^1} \|G_{\lambda'}\|_{L^2} \\ &\lesssim \|\widehat{\nabla} F\|_{H^1} \sum_{\lambda'\le\mu} \mu^{-1} \lambda'^{5/2} \|G_{\lambda'}\|_{L^2}\end{aligned}$$

Therefore

$$\|\mu^{-3/2+\epsilon} a_\mu\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{H^1} \|G\|_{H^\epsilon}. \tag{6.201}$$

Next we consider b_μ. By using the orthogonality of the LP projections we can write

$$\begin{aligned} b_\mu = &\sum_{\lambda\le\mu} \sum_{\lambda'\le\lambda} P_\lambda \left((\widehat{\nabla} F_{h,\tau})_\lambda \cdot \widehat{\nabla} G_{\lambda'} + (\widehat{\nabla} F_{h,\tau})_{\lambda'} \cdot \widehat{\nabla} G_\lambda \right) \\ &+ \sum_{\lambda\le\mu} \sum_{\lambda<\lambda'\le\mu} P_\lambda \left((\widehat{\nabla} F_{h,\tau})_{\lambda'} \cdot \widehat{\nabla} G_{\lambda'} \right). \end{aligned}$$

By using the Bernstein inequality and the finite band property, we obtain

$$\begin{aligned} \|b_\mu\|_{L^2} &\lesssim \sum_{\lambda\le\mu} \sum_{\lambda'\le\lambda} \left(\lambda^{-1} \|(\widehat{\nabla} F_{h,\tau})_\lambda\|_{H^1} \|\widehat{\nabla} G_{\lambda'}\|_{L^\infty} + \lambda'^{\frac12} \|\widehat{\nabla} (\widehat{\nabla} F_{h,\tau})_{\lambda'}\|_{L_x^2} \|\widehat{\nabla} G_\lambda\|_{L_x^2} \right) \\ &\quad + \sum_{\lambda\le\mu} \sum_{\lambda<\lambda'\le\mu} \lambda \|(\widehat{\nabla} F_{h,\tau})_{\lambda'}\|_{L^2} \|\widehat{\nabla} G_{\lambda'}\|_{L^3} \\ &\lesssim \|\widehat{\nabla} F\|_{H^1} \sum_{\lambda\le\mu} \Big(\sum_{\lambda'\le\lambda\le\mu} (\lambda^{-1} \lambda'^{5/2} + \lambda'^{\frac12} \lambda) \|G_{\lambda'}\|_{L^2} + \sum_{\lambda<\lambda'\le\mu} \lambda \lambda'^{1/2} \|G_{\lambda'}\|_{L^2} \Big) \\ &\lesssim \|\widehat{\nabla} F\|_{H^1} \sum_{\lambda\le\mu} \sum_{\lambda'\le\mu} \lambda'^{1/2} \lambda \|G_{\lambda'}\|_{L^2}. \end{aligned}$$

Therefore

$$\|\mu^{-3/2+\epsilon} b_\mu\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{H^1} \|G\|_{H^\epsilon}. \tag{6.202}$$

Combining (6.201) and (6.202) yields

$$\|\mu^{-1/2+\epsilon} [P_\mu, F] \widehat{\nabla} G_{\le\mu}\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{H^1} \|G\|_{H^\epsilon}$$

which together with (6.200) gives the desired estimate. □

Lemma 25. *For* $0 < \epsilon < 1$ *there holds*

$$\|\mu^{1+\epsilon} [P_\mu, F] G\|_{l_\mu^2 L^2} + \|\mu^\epsilon \widehat{\nabla} [P_\mu, F] G\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla}^2 F\|_{H^{\frac12+\epsilon}} \|G\|_{L^2} + \|\widehat{\nabla} F\|_{L^\infty} \|G\|_{H^\epsilon} \tag{6.203}$$

$$\|\mu^{1+\epsilon} [P_\mu, F] G\|_{l_\mu^2 L^2} \lesssim (\|\widehat{\nabla}^2 F\|_{H^{\frac12}} + \|\widehat{\nabla} F\|_{L^\infty}) \|G\|_{H^\epsilon} \tag{6.204}$$

Proof. We first use (6.195) to write $\mu^{1+\epsilon}[P_\mu, F]G = a_\mu + b_\mu$, where

$$a_\mu := \mu^{1+\epsilon} \sum_{\lambda>\mu} P_\mu(F_\lambda \cdot G_\lambda) \quad \text{and} \quad b_\mu := \mu^{1+\epsilon} \sum_{\lambda\le\mu} [P_\mu, F]G_\lambda.$$

In view of $\|F_\lambda\|_{L^\infty} \lesssim \lambda^{-1}\|\widehat{\nabla} F\|_{L^\infty}$, it follows that

$$\|a_\mu\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty} \Big\| \sum_{\lambda>\mu} \left(\frac{\mu}{\lambda}\right)^{1+\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2} \Big\|_{l^2_\mu} \lesssim \|\widehat{\nabla} F\|_{L^\infty} \|\Lambda^\epsilon G\|_{L^2}.$$

To estimate b_μ, we introduce $M_\mu(x) = \int e^{ix\cdot\xi} m_\mu(\xi) d\xi$ with $m_\mu(\xi) = \Psi(\mu^{-1}\xi)$. It is easy to see that $\int |x|^q |M_\mu(x)| dx \lesssim \mu^{-q}$ for any $q > -3$. It follows from (6.193) that

$$[P_\mu, F]G_\lambda(x) = A_{\mu,\lambda}(x) + B_{\mu,\lambda}(x) + C_{\mu,\lambda}(x),$$

where

$$A_{\mu,\lambda}(x) = \partial_j F(x) \int M_\mu(x-y)(x-y)^j G_\lambda(y) dy,$$

$$B_{\mu,\lambda}(x) = \int M_\mu(x-y)(x-y)^j \int_0^1 \left[\partial_j F_{\le\lambda}(x-\tau(x-y)) - \partial_j F_{\le\lambda}(x)\right] d\tau G_\lambda(y) dy,$$

$$C_{\mu,\lambda}(x) = \int M_\mu(x-y)(x-y)^j \int_0^1 \left[\partial_j F_{\ge\lambda}(x-\tau(x-y)) - \partial_j F_{\ge\lambda}(x)\right] d\tau G_\lambda(y) dy.$$

For the term $A_{\mu,\lambda}(x)$, it is nonzero only if μ and λ are at the same magnitude since both M_μ and G_λ are frequency localized at the level μ and λ respectively. Thus

$$\sum_{\lambda\le\mu} \|A_{\mu,\lambda}\|_{L^2} \lesssim \mu^{-1}\|\widehat{\nabla} F_{\le\mu}\|_{L^\infty} \|G_\mu\|_{L^2} \lesssim \mu^{-1}\|G_\mu\|_{L^2} \|\widehat{\nabla} F\|_{L^\infty}.$$

For the term $B_{\mu,\lambda}$ we write

$$B_{\mu,\lambda}(x) = \int M_\mu(x-y)(x-y)^j (x-y)^l \int_0^1 \int_0^1 - \tau \partial_l \partial_j F_{\le\lambda}\left(x - \tau\tau'(x-y)\right) d\tau d\tau' G_\lambda(y) dy.$$

Thus, by the Minkowski inequality we obtain

$$\|B_{\mu,\lambda}\|_{L^2} \lesssim \mu^{-2}\|\widehat{\nabla}^2 F_{\le\lambda}\|_{L^\infty} \|G_\lambda\|_{L^2} \lesssim \mu^{-2}\lambda \|G_\lambda\|_{L^2} \|\widehat{\nabla} F\|_{L^\infty}.$$

By the similar argument as above, we can obtain

$$\|C_{\mu,\lambda}\|_{L^2} \lesssim \mu^{-2}\|\widehat{\nabla}^2 F_{>\lambda}\|_{L^2} \|G_\lambda\|_{L^\infty} \lesssim \mu^{-2}\lambda^{3/2} \|\widehat{\nabla}^2 F_{>\lambda}\|_{L^2} \|G_\lambda\|_{L^2}$$

Therefore

$$\|b_\mu\|_{L^2} \lesssim \sum_{\lambda\le\mu} \sum_{\lambda'>\lambda} \left(\frac{\lambda}{\mu}\right)^{1-\epsilon} \left(\frac{\lambda}{\lambda'}\right)^{1/2+\epsilon} \|\lambda'^{1/2+\epsilon} \widehat{\nabla}^2 F_{\lambda'}\|_{L^2} \|G_\lambda\|_{L^2}$$
$$+ \|\widehat{\nabla} F\|_{L^\infty} \Big(\mu^\epsilon \|G_\mu\|_{L^2} + \sum_{\lambda\le\mu} \left(\frac{\lambda}{\mu}\right)^{1-\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2} \Big)$$

which implies that $\|b_\mu\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty}\|G\|_{H^\epsilon} + \|\widehat{\nabla}^2 F\|_{H^{1/2+\epsilon}}\|G\|_{L^2}$. This together with the estimates on a_μ and b_μ gives the estimate for the first term on the left hand of (6.203). (6.204) may follow by a slight modification of the estimate of $\|b_\mu\|_{l^2_\mu L^2}$.

Next we derive the estimate for the second term on the left hand of (6.203). By using (6.195) we can write $\mu^\epsilon \widehat{\nabla}[P_\mu, F]G = I_\mu + J_\mu$, where

$$I_\mu := \mu^\epsilon \sum_{\lambda>\mu} \widehat{\nabla} P_\mu(F_\lambda \cdot G_\lambda) \quad \text{and} \quad J_\mu := \mu^\epsilon \widehat{\nabla}[P_\mu, F]G_{\leq\mu}.$$

The same treatment as for a_μ implies

$$\|I_\mu\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty}\|\Lambda^\epsilon G\|_{L^2}.$$

In order to estimate J_μ, we write

$$\begin{aligned} J_\mu(x) &= \mu^\epsilon \int \widehat{\nabla}_x \left(M_\mu(x-y)(F(x)-F(y))\right) G_{\leq\mu}(y)dy \\ &= \mu^\epsilon \widehat{\nabla} F(x) \int M_\mu(x-y)G_{\leq\mu}(y)dy \\ &\quad + \mu^\epsilon \int \widehat{\nabla} M_\mu(x-y)(F(x)-F(y))G_{\leq\mu}(y)dy. \end{aligned}$$

By writing $F(x) - F(y) = (x-y)^j \int_0^1 \partial_j F(x - \tau(x-y))d\tau$, we can decompose J_μ as $J_\mu = J_\mu^{(1)} + J_\mu^{(2)} + J_\mu^{(3)}$, where

$$\begin{aligned} J_\mu^{(1)} &= \mu^\epsilon \widehat{\nabla} F(x) \int M_\mu(x-y)G_{\leq\mu}(y)dy, \\ J_\mu^{(2)} &= \mu^\epsilon \partial_j F(x) \int \widehat{\nabla} M_\mu(x-y)(x-y)^j G_{\leq\mu}(y)dy, \\ J_\mu^{(3)} &= \mu^\epsilon \int \widehat{\nabla} M_\mu(x-y)(x-y)^j \int_0^1 \left[\partial_j F(x-\tau(x-y)) - \partial_j F(x)\right] d\tau G_{\leq\mu}(y)dy. \end{aligned}$$

By using $\int |x||\widehat{\nabla} M_\mu(x)|dx \lesssim 1$ and the frequency localization of M_μ and G_λ we can obtain

$$\|J_\mu^{(1)}\|_{L^2} + \|J_\mu^{(2)}\|_{L^2} \lesssim \mu^\epsilon \|G_\mu\|_{L^2}\|\widehat{\nabla} F\|_{L^\infty}.$$

For the term $J_\mu^{(3)}$, we can write

$$\begin{aligned} &J_\mu^{(3)}(x) \\ &= \mu^\epsilon \sum_{\lambda\leq\mu} \int \widehat{\nabla} M_\mu(x-y)(x-y)^j \int_0^1 \left[\partial_j F_{\leq\lambda}(x-\tau(x-y)) - \partial_j F_{\leq\lambda}(x)\right] d\tau G_\lambda(y)dy \\ &\quad + \mu^\epsilon \sum_{\lambda\leq\mu} \int \widehat{\nabla} M_\mu(x-y)(x-y)^j \int_0^1 \left[\partial_j F_{>\lambda}(x-\tau(x-y)) - \partial_j F_{>\lambda}(x)\right] d\tau G_\lambda(y)dy. \end{aligned}$$

By using the same treatment for $B_{\mu,\lambda}$ and $C_{\mu,\lambda}$ in the proof of the first part, we derive

$$\|J_\mu^{(3)}\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla}^2 F\|_{H^{1/2+\epsilon}}\|G\|_{L^2} + \|\widehat{\nabla} F\|_{L^\infty}\|G\|_{H^\epsilon}.$$

This together with the estimates on $J_\mu^{(1)}$ and $J_\mu^{(2)}$ gives

$$\|J_\mu\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty}\|G\|_{H^\epsilon} + \|\widehat{\nabla}^2 F\|_{H^{1/2+\epsilon}}\|G\|_{L^2}.$$

Combining this with the estimate on I_μ completes the proof of the second part of (6.203). □

Proposition 14. *For* $0<\epsilon<1/2$ *there holds*

$$\|\mu^{1+\epsilon}\widehat{\nabla}[P_\mu, F]G\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty}\|G\|_{H^{1+\epsilon}} + \|\widehat{\nabla}^2 F\|_{H^{\frac{1}{2}+\epsilon}}\|G\|_{H^1}. \tag{6.205}$$

Proof. As can be seen from the proof of the second part of (6.203), it suffices to estimate the term $a_\mu := \mu J_\mu^{(3)}$. We can write $a_\mu = a_\mu^{(1)} + a_\mu^{(2)} + a_\mu^{(3)}$, where

$$\begin{aligned}
a_\mu^{(1)} &= \mu^{1+\epsilon}\int \widehat{\nabla} M_\mu(x-y)(x-y)^j(x-y)^l \\
&\quad\times \int_0^1\int_0^1 -\tau\partial_{jl}^2 F_{\geq\mu}\left(x-\tau\tau'(x-y)\right) d\tau d\tau' G_{\leq\mu}(y)dy \\
a_\mu^{(2)} &= \mu^{1+\epsilon}\partial_{jl}^2 F_{<\mu}(x)\int \widehat{\nabla} M_\mu(x-y)(x-y)^j(x-y)^l G_{\leq\mu}(y)dy \\
a_\mu^{(3)} &= \mu^{1+\epsilon}\int \widehat{\nabla} M_\mu(x-y)(x-y)^j(x-y)^l \\
&\quad\times \int_0^1\int_0^1 -\tau\left[\partial_{jl}^2 F_{<\mu}(x-\tau\tau'(x-y)) - \partial_{jl}^2 F_{<\mu}(x)\right] d\tau d\tau' G_{\leq\mu}(y)dy.
\end{aligned}$$

By using the properties of the LP projections, it is easy to derive that

$$\begin{aligned}
\|a_\mu^{(1)}\|_{L^2} &\lesssim \mu^\epsilon \sum_{\lambda\geq\mu}\|\widehat{\nabla}^2 F_\lambda\|_{L^2}\|G_{\leq\mu}\|_{L^\infty} \lesssim \sum_{\lambda\geq\mu}\left(\frac{\mu}{\lambda}\right)^{1/2+\epsilon}\|\lambda^{1/2+\epsilon}\widehat{\nabla}^2 F_\lambda\|_{L^2}\|\widehat{\nabla} G\|_{L^2}, \\
\|a_\mu^{(2)}\|_{L^2} &\lesssim \mu^\epsilon\|\widehat{\nabla}^2 F_{<\mu}\|_{L^\infty}\|G_\mu\|_{L^2} \lesssim \mu^{1+\epsilon}\|G_\mu\|_{L^2}\|\widehat{\nabla} F\|_{L^\infty}, \\
\|a_\mu^{(3)}\|_{L^2} &\lesssim \mu^{-1+\epsilon}\|\widehat{\nabla}^3 F_{<\mu}\|_{L^2}\|G_{\leq\mu}\|_{L^\infty} \lesssim \sum_{\lambda<\mu}\left(\frac{\lambda}{\mu}\right)^{1/2-\epsilon}\|\lambda^{1/2+\epsilon}\widehat{\nabla}^2 F_\lambda\|_{L^2}\|\widehat{\nabla} G\|_{L^2}.
\end{aligned}$$

Therefore, by taking the l_μ^2-norm, we obtain

$$\|a_\mu^{(1)}\|_{l_\mu^2 L^2} + \|a_\mu^{(3)}\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla}^2 F\|_{H^{1/2+\epsilon}}\|G\|_{H^1}, \quad \|a_\mu^{(2)}\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty}\|G\|_{H^{1+\epsilon}}.$$

The proof is thus complete. □

Lemma 26. *For any* $\epsilon>0$ *and any scalar functions* F *and* G, *there holds*

$$\|\mu^\epsilon P_\mu(\widehat{\nabla} F\cdot G)\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{H^{\frac{1}{2}+\epsilon}}\|G\|_{H^1} + \|F\|_{L^\infty}\|G\|_{H^{1+\epsilon}}. \tag{6.206}$$

Proof. By the trichotomy law (6.187), we can write

$$P_\mu(\widehat{\nabla} F\cdot G) = P_\mu((\widehat{\nabla} F)_\mu\cdot G_{\leq\mu}) + \Big(P_\mu(\widehat{\nabla} F_{\leq\mu}\cdot G_\mu) + \sum_{\lambda>\mu} P_\mu(\widehat{\nabla} F_\lambda\cdot G_\lambda)\Big) =: a_\mu + b_\mu.$$

For the terms b_μ, it is easy to derive that $\|\mu^\epsilon b_\mu\|_{l^2_\mu L^2} \lesssim \|F\|_{L^\infty}\|G\|_{H^{1+\epsilon}}$. For the term a_μ, we can write

$$a_\mu = [P_\mu, G_{\le\mu}]\widehat{\nabla} F_\mu + G_{\le\mu} \cdot P_\mu(\widehat{\nabla} F)_\mu. \tag{6.207}$$

By the Bernstein inequality for LP projections, it is easy to obtain

$$\|\mu^\epsilon G_{\le\mu} P_\mu(\widehat{\nabla} F)_\mu\|_{L^2} \lesssim \mu^\epsilon \|G_{\le\mu}\|_{L^6}\|P_\mu(\widehat{\nabla} F)_\mu\|_{L^3} \lesssim \|G\|_{L^6}\|\mu^{\frac{1}{2}+\epsilon}(\widehat{\nabla} F)_\mu\|_{L^2},$$

while by using Corollary 1 we have

$$\begin{aligned}\|\mu^\epsilon[P_\mu, G_{\le\mu}](\widehat{\nabla} F)_\mu\|_{L^2} &\lesssim \mu^{\epsilon-1}\|\widehat{\nabla} G_{\le\mu}\|_{L^\infty}\|(\widehat{\nabla} F)_\mu\|_{L^2}\\ &\lesssim \sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{\frac{3}{2}}\|\widehat{\nabla} G_\lambda\|_{L^2}\|\mu^{\frac{1}{2}+\epsilon}(\widehat{\nabla} F)_\mu\|_{L^2}.\end{aligned}$$

Therefore $\|\mu^\epsilon a_\mu\|_{l^2_\mu L^2_x} \lesssim \|\widehat{\nabla} F\|_{H^{\frac{1}{2}+\epsilon}}\|G\|_{H^1}$. The combination of the estimates for a_μ and b_μ gives (6.206). □

By using Lemma 21 and (6.206), we can derive the following product estimate.

Lemma 27. *For* $0 < \epsilon < 1$ *there holds*

$$\|\Lambda^\epsilon \widehat{\nabla}(F \cdot G)\|_{L^2} \lesssim \|\widehat{\nabla} F\|_{H^{\frac{1}{2}+\epsilon}}\|G\|_{H^1} + \|F\|_{L^\infty}\|G\|_{H^{1+\epsilon}} \tag{6.208}$$

Proof. We observe that

$$\begin{aligned}\|\Lambda^\epsilon \widehat{\nabla}(F \cdot G)\|_{L^2} &\lesssim \|\mu^\epsilon P_\mu(\widehat{\nabla} F \cdot G)\|_{L^2_\mu L^2}\\ &\quad + \|\mu^\epsilon[P_\mu, F]\widehat{\nabla} G\|_{l^2_\mu L^2} + \|F\|_{L^\infty_x}\|\mu^\epsilon P_\mu \widehat{\nabla} G\|_{l^2_\mu L^2}\end{aligned}$$

(6.208) then follows by using Lemma 21 and (6.206). □

Lemma 28. *Let* $0 < \epsilon < 1/2$. *For any* $\mu \ge 1$ *and any scalar functions* F *and* G, *there hold*

$$\begin{aligned}\|[P_\mu, F]G\|_{L^\infty} &\lesssim \mu^{-\frac{1}{2}-\epsilon}\|G\|_{L^\infty}\Big(\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}+\epsilon}\|\lambda^\epsilon \widehat{\nabla}^2 F_\lambda\|_{L^2}\\ &\quad + \sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\epsilon}\|\lambda^\epsilon \widehat{\nabla}^2 F_\lambda\|_{L^2}\Big)\end{aligned} \tag{6.209}$$

$$\begin{aligned}\|[P_\mu, F]G\|_{L^\infty} &\lesssim \mu^{-\frac{1}{2}-\epsilon}\|\widehat{\nabla} F\|_{L^\infty}\Big(\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^{2+\epsilon}\|\lambda^\epsilon \widehat{\nabla} G_\lambda\|_{L^2}\\ &\quad + \sum_{\lambda\le\mu}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\epsilon}\|\lambda^\epsilon \widehat{\nabla} G_\lambda\|_{L^2}\Big)\end{aligned} \tag{6.210}$$

Proof. In view of (6.195), we can write $[P_\mu, F]G = a_\mu + b_\mu + c_\mu$, where

$$a_\mu = \sum_{\lambda>\mu} P_\mu(F_\lambda G_\lambda), \quad b_\mu = \sum_{\lambda\le\mu}[P_\mu, F_\lambda]G_{\le\mu}, \quad c_\mu = \sum_{\lambda>\mu}[P_\mu, F_\lambda]G_{\le\mu}.$$

It is easy to derive that

$$\|a_\mu\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon} \sum_{\lambda>\mu} \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}+\epsilon} \|\lambda^\epsilon \widehat{\nabla}^2 F_\lambda\|_{L_x^2} \|G_\lambda\|_{L^\infty}.$$

By using Corollary 1 and the Bernstein inequality, we also have

$$\|b_\mu\|_{L^\infty} \lesssim \|G\|_{L^\infty} \sum_{\lambda\le\mu} \mu^{-1}\|\widehat{\nabla} F_\lambda\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon}\|G\|_{L^\infty} \sum_{\lambda\le\mu} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\epsilon} \|\lambda^\epsilon \widehat{\nabla}^2 F_\lambda\|_{L^2}.$$

By the trichotomy law, c_μ can be simplified as $c_\mu = F_{\ge\mu} \cdot G_\mu$. Consequently

$$\|c_\mu\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon}\|G_\mu\|_{L^\infty} \sum_{\lambda>\mu} \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}+\epsilon} \|\lambda^\epsilon \widehat{\nabla}^2 F_\lambda\|_{L^2}.$$

Thus we complete the proof of (6.209). Next we use the properties of LP projections and Corollary 1 to derive that

$$\|a_\mu\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon}\|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda>\mu} \left(\frac{\mu}{\lambda}\right)^{2+\epsilon} \|\lambda^\epsilon \widehat{\nabla} G_\lambda\|_{L^2},$$

$$\|[P_\mu, F]G_{\le\mu}\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon}\|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda\le\mu} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\epsilon} \|\lambda^\epsilon \widehat{\nabla} G_\lambda\|_{L^2}.$$

In view of (6.195), the proof of (6.210) is completed. □

6.3. H^ϵ elliptic estimates.

Lemma 29. *For $0 < \epsilon < 1/2$ and any Σ-tangent tensor field F there hold*

$$\|\widehat{\nabla}^2 F\|_{\dot{H}^\epsilon} \lesssim \|\hat{\Delta} F\|_{\dot{H}^\epsilon} + \|F\|_{H^1}, \tag{6.211}$$

$$\|\widehat{\nabla} F\|_{\dot{H}^{1/2+\epsilon}} \lesssim \|\Lambda^{\epsilon-1/2}\hat{\Delta} F\|_{L^2} + \|F\|_{H^1}. \tag{6.212}$$

Proof. Consider (6.211) first. We will use err_μ to denote any error term satisfying $\|\mathrm{err}_\mu\|_{L^2} \lesssim \mu^{-1}\|F\|_{H^1}$. Using Corollary 1, we can obtain

$$P_\mu \widehat{\nabla}^2 F = \widehat{\nabla}^2 P_\mu F + \mathrm{err}_\mu. \tag{6.213}$$

Recall from Lemma 2 that

$$\|\widehat{\nabla}^2 P_\mu F\|_{L^2} \lesssim \|\hat{\Delta} P_\mu F\|_{L^2} + \|\widehat{\nabla} P_\mu F\|_{L^2} + \|P_\mu F\|_{L^2} \tag{6.214}$$

Recall also that $\hat{\Delta} = g^{ij}\widehat{\nabla}_i\widehat{\nabla}_j$, we have

$$\hat{\Delta} P_\mu F = P_\mu \hat{\Delta} F + [P_\mu, g^{ij}](\partial_i\partial_j F - \hat{\Gamma}_{ij}^k \partial_k F) + \mathrm{err}_\mu. \tag{6.215}$$

By Lemma 24 and 21, we have

$$\|\mu^\epsilon[P_\mu, g^{ij}]\partial_i\partial_j F\|_{l^2_\mu L^2} \lesssim \|\partial g\|_{H^1}\|\Lambda^{\epsilon+1/2}\partial F\|_{L^2}, \tag{6.216}$$

$$\|\mu^\epsilon[P_\mu, g^{ij}](\hat{\Gamma}^k_{ij}\partial_k F)\|_{l^2_\mu L^2} \lesssim \|\partial g\|_{H^{\frac{1}{2}+\epsilon}}\|\partial F\|_{L^2}. \tag{6.217}$$

This together with (6.213)–(6.215) and interpolation give (6.211).

To prove (6.212) for the case of scalar function, we first use the equivalence between g and $\hat{g}$ and the integration by parts to obtain

$$\|P_\mu\widehat{\nabla}F\|^2_{L^2} \approx \int_\Sigma g^{ij}P_\mu\partial_i F P_\mu\partial_j F d\mu_g = \int_\Sigma P_\mu F\hat{\Delta}P_\mu F d\mu_g. \tag{6.218}$$

In view of (6.199), we can obtain

$$\begin{aligned}\sum_\mu \mu^{1+2\epsilon}\left|\int_\Sigma P_\mu F[P_\mu, g^{ij}]\partial_i\partial_j F d\mu_g\right| &\lesssim \sum_\mu \mu^{3/2+\epsilon}\|P_\mu F\|_{L^2}\|\mu^{-1/2+\epsilon}[P_\mu, g^{ij}]\partial_i\partial_j F\|_{L^2}\\ &\lesssim \|\widehat{\nabla}F\|_{H^{1/2+\epsilon}}\|\widehat{\nabla}g\|_{H^1}\|\Lambda^\epsilon\partial F\|_{L^2},\end{aligned}$$

and in view of (6.197)

$$\begin{aligned}\sum_\mu \mu^{1+2\epsilon}\left|\int P_\mu F\cdot[P_\mu, g\hat{\Gamma}]\partial F\right| &\lesssim \|\mu^{\frac{1}{2}+\epsilon}P_\mu F\|_{l^2_\mu L^2}\|\widehat{\nabla}(g\cdot\hat{\Gamma})\|_{L^6}\|\partial F\|_{H^\epsilon}\\ &\lesssim \|F\|_{H^{\frac{1}{2}+\epsilon}}\|\partial F\|_{H^\epsilon}\|g\|_{H^2}.\end{aligned}$$

In view of (6.215) and (6.218), we have with $p = \epsilon/(1/2+\epsilon)$ that

$$\begin{aligned}\sum_\mu \mu^{1+2\epsilon}\|P_\mu\widehat{\nabla}F\|^2_{L^2} &\lesssim (\| F\|_{\dot{H}^{\frac{1}{2}+\epsilon}} + \|\partial F\|_{H^{1/2+\epsilon}})\|\partial F\|^p_{H^{1/2+\epsilon}}\|\partial F\|^{1-p}_{L^2}\|g\|_{H^2}\\ &\quad + \sum_\mu \|\mu^{3/2+\epsilon}P_\mu F\|_{L^2}\|\mu^{-1/2+\epsilon}P_\mu\hat{\Delta}F\|_{L^2}.\end{aligned}$$

By the fact $\|g\|_{H^2} \lesssim 1$ and the Young's inequality, we obtain (6.212).

To prove (6.212) for the vector field case, we note that

$$P_\mu\widehat{\nabla}_i F^m = P_\mu\widehat{\nabla}_i(F^m) + [P_\mu, \hat{\Gamma}]F + \hat{\Gamma}\cdot P_\mu F.$$

Using Corollary 1, then there holds $P_\mu\widehat{\nabla}_i F^m = P_\mu\widehat{\nabla}_i(F^m) + \mathrm{err}_\mu$, hence we can obtain

$$\|\widehat{\nabla}F^m\|_{H^{1/2+\epsilon}} \lesssim \|\widehat{\nabla}(F^m)\|_{H^{1/2+\epsilon}} + \|F\|_{H^1}.$$

Now we can use (6.212) for the scalar function case to derive

$$\|\widehat{\nabla}F^m\|_{H^{1/2+\epsilon}} \lesssim \|\Lambda^{\epsilon-1/2}\hat{\Delta}(F^m)\|_{L^2} + \|F\|_{H^1}. \tag{6.219}$$

In view of (6.215), by deriving similar estimates as (6.216) and (6.217) we can obtain

$$\|\Lambda^{\epsilon-1/2}(\hat{\Delta}F^m)\|_{L^2} \lesssim \|\Lambda^{\epsilon-1/2}(\hat{\Delta}F)^m\|_{L^2} + \|\widehat{\nabla}F\|_{L^2}.$$

Combining this with (6.219) completes the proof. □

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