Prepared for submission to JHEP

On quantum compatibility of counterterm deformations and duality symmetries in $\mathcal{N} \geq 5$ supergravities

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Abstract: In $\mathcal{N} = 5, 6, 8$ supergravities there are hidden symmetries of equations of motion, described by duality groups $SU(1,5)$, $SO^*(12)$, $E_{7(7)}$ respectively. UV divergences and known candidate counterterms violate the deformed duality symmetry current conservation. Extra higher derivative terms in the action are required to restore duality. We study the effect of a two-vector part of the counterterm for $\mathcal{N} \geq 5$ supergravities using the universality of the symplectic structure of extended supergravities. We construct a compact form of a deformed action with infinite number of higher derivative terms and restored duality symmetry with deformation parameter λ. We find, in λ^2 approximation, that the $SU(\mathcal{N})$ symmetry of the deformed theory is restored on shell.

Contents

1 Introduction

All classical extended supergravities with $\mathcal{N}$ local supersymmetries have duality symmetry, as shown by Gaillard and Zumino [1]. These symmetries rotate equations of motion into Bianchi identities and are consistent with local extended supersymmetry of the classical action.

The local UV divergences at the loop level can be eliminated (absorbed into a redefinition of parameters) if the classical action of extended supergravities are deformed, to preserve a duality symmetry in presence of higher derivative terms. The issue of compatibility of the deformed duality symmetric extended supergravity with a global $\mathcal{N}$-extended supersymmetry of the on-shell amplitudes will be addressed here.

A deformation of $\mathcal{N}=8$ supergravity by the candidate counterterms (CTs) [2], [3] leads to a violation of the duality current conservation [4], [5], unless the consistent procedure of the deformation of the twisted selfduality condition [6],[7] can be implemented. In its general form proposed in [7] it has been already applied for Born-Infeld models with higher derivatives with $U(1)$ duality in [8]. Other examples of the restoration of duality current conservation with rigid $\mathcal{N}=2$ supersymmetry and $U(1)$ duality were presented in [9]. The procedure of [7] was however not explicitly applied to extended supergravities.

We will solve the first part of the problem here, in a particular sector of the theory: we will construct a deformed bosonic action of $\mathcal{N}$-extended supergravity where a two-vector part of the CT is added to the classical action. All higher order terms with higher and higher derivatives will be identified, so that the deformed actions in $\mathcal{N}=5,6,8$ supergravities in the two-vector sector have restored $SU(1,5)$, $SO^*(12)$, $E_{7(7)}$ duality symmetry, respectively. We will investigate the properties of the deformed bosonic action here and study the supersymmetric embedding of the deformed action and the superamplitudes.

A consistent reduction of $\mathcal{N}=8$ to all pure extended supergravities allows us to work with all $\mathcal{N}\geq 4$ models. The $\mathcal{N}=4$ pure supergravity has a $U(1)$ duality anomaly [10], [11], which might have caused the four-loop UV divergence [12]. It was suggested recently in [13] that the one-loop anomalous amplitudes in this theory can be cancelled by a finite local counterterm. It remains an interesting open problem to understand the consequences of this (and perhaps higher-loop) counterterm(s) on the four-loop divergence of the four-graviton amplitude.

Meanwhile, the four-loop UV divergence in $\mathcal{N}=5$ supergravity is absent, [14]. Moreover, it has been recently established in [15] that $\mathcal{N}\geq 5$ supergravities do not exhibit $U(1)$ duality anomalies in their one-loop amplitudes, of the kind known to be present in $\mathcal{N}=4$ case [11].

The first relevant prediction of a UV finiteness of $\mathcal{N}=8$ supergravity due to $E_{7(7)}$ symmetry in [4] was based on an observation that the Lorentz and SU(8) covariant, $E_{7(7)}$ invariant unitarity constraint expressing the 56-dimensional $E_{7(7)}$ doublet via 28 independent vectors, and consistent with supersymmetry, is unique [1]. This argument is easy to extended to other cases of $\mathcal{N} \geq 5$ supergravities since it is based on the geometric nature of the $\frac{\mathcal{G}}{\mathcal{H}}$ coset space where scalars are coordinates. Later on in [17] the argument was given that for all $\mathcal{N} \geq 5$ supergravities the procedure of restoration of the duality current conservation broken by the CT is not available. The argument in [17] was based on the properties of invariants of the groups of type E_7, which are duality groups in $\mathcal{N} \geq 5$ supergravities [18]. It suggested that a deformation of the twisted selfduality condition for groups of the type E_7, consistent with supersymmetry, is not possible. Additional reasons for an obstruction to $E_{7(7)}$deformations in $\mathcal{N} = 8$ supergravity based on superconformal $SU(2,2|8)$ algebra were developed in [19].

It is important to stress here that the conjectured breaking of continuous $E_{7(7)}$ to discrete $E_{7(7)}(\mathbb{Z})$ would be a non-perturbative effect, whereas we are analyzing here only perturbative supergravity. The perturbative quantization of $\mathcal{N} = 8$ supergravity is studied in [20] in a formulation where its $E_{7(7)}$ symmetry is realized off-shell, but Lorentz invariance is no longer manifest.

Relying on the cancellation of $SU(8)$ current anomalies it is shown there that there are no anomalies for the non-linearly realized $E_{7(7)}$ either. As a consequence, the $E_{7(7)}$ Ward identities can be consistently implemented and imposed at all orders in perturbation theory, and therefore potential divergent counterterms must respect the full non-linear $E_{7(7)}$ symmetry.

In view of the highly non-trivial cancellation of the UV divergences in $\mathcal{N} = 5$ supergravity in four loops discovered in [14] and the fact that no new explanations of this fact, besides the one in [17], have been suggested we would like to revisit and clarify the status of the duality conservation arguments in [4], [5] and [6].

The UV finiteness of $\mathcal{N} = 5$ supergravity in four loops established in [14] may shed some light on the UV properties of the maximal $\mathcal{N} = 8$ supergravity, if there exists a universal formalism describing all $\mathcal{N}$-extended supergravities. Such a formalism is, indeed, available for $\mathcal{N} \geq 2$ and it was constructed to describe the supersymmetric black hole universality [21], [22]. In $\mathcal{N} = 2$ the special geometry is represented by a symplectic section [23], [24]. The symplectic sections for higher $\mathcal{N}$ have been constructed in [25], [26].

Our purpose here is to make an analysis using the relatively simple two-vector sector of the theory. After the deformed bosonic action with duality symmetry will be presented we will study its supersymmetric embedding.

[1] In case of $\mathcal{N} = 8$ supergravity the direct and simple finiteness argument is based on the absence of light-cone supersymmetric invariant counterterm candidates [16]. We are grateful to L. Brink for a recent reminder that light-cone CT's are still not available, despite a significant effort. But since here we are interested also in $5 \leq \mathcal{N} < 8$ supergravity we cannot rely on light-cone superspace, which is known only for $\mathcal{N} = 8$ supergravity. We will work in the Lorentz covariant approach and try to use the duality/supersymmetry argument.

2 Twisted selfduality constraint and its deformation in $\mathcal{N} \geq 5$ supergravities

The models of $\mathcal{N} = 5, 6, 8$ supergravities are reviewed in detail in Appendix A, based on [25], [26]. The scalars are coordinates of the $\frac{\mathcal{G}}{\mathcal{H}}$ cosets, see Table 1; the notation is universal for all of them. Their duality groups $\mathcal{G}$ are $SU(1,5)$, $SO^*(12)$, $E_{7(7)}$ respectively. The isotropy groups $\mathcal{H}$ are $U(5), U(6), SU(8)$, respectively.

We are looking at the bosonic part of the two-vector sector of the CT [2], [3], which has a manifest duality symmetry as well as a supersymmetry, under condition that all fields in the CT satisfy classical equations of motion, $\frac{\delta S_{\rm cl}}{\delta \phi} = 0$. But once such a CT is added to the action with some constant λ in front of it, the new equation of motion has a correction

$$\frac{\delta S_{\rm deformed}}{\delta \phi} = \frac{\delta S_{\rm cl}}{\delta \phi} + \lambda \frac{\delta S_{CT}}{\delta \phi} = 0. \tag{2.1}$$

In particular, the λ-dependent terms break duality current conservation [4], [5] at order $\mathcal{O}(\lambda^2)$. New terms of $\mathcal{O}(\lambda^2)$ are therefore necessary to correct this issue; they, in turn, push the non-conservation of duality current to $\mathcal{O}(\lambda^3)$, etc. The current conservation is restored with an infinite number of higher order terms, which also have an infinite number of higher derivatives, [6],[7].

We will now study dualities in $\mathcal{N} \geq 5$ supergravities, [25], [26]; the field content of these theories is given only by the corresponding gravitational multiplet. In the case of $\mathcal{N} = 4$ supergravity the duality symmetry is anomalous, [10], [11] but $\mathcal{N} \geq 5$ are anomaly-free [15]. These theories contain in the bosonic sector the metric, a number n_v of vectors and m of (real) scalar fields, see Table 1. The relevant classical vector and scalar part of action has the following general form:

$$\mathcal{L}_{vec} = \mathrm{i} \left[\bar{\mathcal{N}}_{\Lambda\Sigma} F^{-\Lambda}_{\mu\nu} F^{-\Sigma|\mu\nu} - \mathcal{N}_{\Lambda\Sigma} F^{+\Lambda}_{\mu\nu} F^{+\Sigma|\mu\nu} \right] + \frac{1}{2}\, g_{rs}(\Phi) \partial_\mu \Phi^r \partial^\mu \Phi^s \,, \tag{2.2}$$

where $g_{rs}(\Phi)$ $(r, s, \cdots = 1, \cdots, m)$ is the scalar metric on the scalar manifold $\mathcal{M}_{scalar}$ of real dimension m and the vectors kinetic matrix $\mathcal{N}_{\Lambda\Sigma}(\Phi)$ is a complex, symmetric, $n_v \times n_v$ matrix depending on the scalar fields, see Table 1. $F^{\pm\Lambda}$ are self-dual and anti-self-dual combinations of the vectors field strengths (see Appendix A for details).

The formalism of symplectic sections [25], [26] corresponds to a particular parametrization of the coset representative. It allows a better way to study duality symmetry of extended supergravities for the case of a general $\mathcal{N}$. The details are in Appendix A and we give examples of symplectic sections in $\mathcal{N} = 5$ and $\mathcal{N} = 8$ supergravity in Appendix C. Instead of a metric $\mathcal{N}_{\Lambda\Sigma}(\Phi)$ in the vector space, in eq. (2.2) one can introduce duality doublets – referred to as a symplectic section – depending on scalars of the theory

$$\begin{pmatrix} f^{\Lambda}{}_{AB} \\ h_{\Lambda AB} \end{pmatrix} \tag{2.3}$$

Table 1. *Scalar Manifolds of $\mathcal{N} \geq 4$ Extended Supergravities*

N	Duality group $\mathcal{G}$	isotropy $\mathcal{H}$	$\mathcal{M}_{scalar}$	n_v	m
4	$SU(1,1) \otimes SO(6)$	$U(4)$	$\frac{SU(1,1)}{U(1)}$	6	2
5	$SU(1,5)$	$U(5)$	$\frac{SU(1,5)}{S(U(1)\times U(5))}$	10	10
6	$SO^\star(12)$	$U(6)$	$\frac{SO^\star(12)}{U(1)\times SU(6)}$	16	30
7, 8	$E_{7(7)}$	$SU(8)$	$\frac{E_{7(7)}}{SU(8)}$	28	70

In the table, n_v is the number of vectors and m is the number of real scalar fields. In all the cases the duality group $\mathcal{G}$ is embedded in $Sp(2\,n_v, \mathbb{R})$.

so that the kinetic matrix $\mathcal{N}$ can be written in terms of the sub-blocks $\mathbf{f}$, $\mathbf{h}$ as $\mathcal{N} = \mathbf{h}\,\mathbf{f}^{-1}$ or component-by-component as

$$\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda AB}\,(f^{-1})^{AB}{}_{\Sigma}\ . \tag{2.4}$$

The vector doublet is defined by the vector field strength $F^{\Lambda}_{\mu\nu} \equiv \frac{1}{2}\left(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu\right)$ and by the derivative of the action over it, namely, ${}^\star G_{\Lambda|\mu\nu} \equiv \frac{1}{2}\frac{\partial \mathcal{L}}{\partial F^\Lambda_{\mu\nu}}$

$$\mathcal{F} \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}. \tag{2.5}$$

The only way to construct $\mathcal{G}$-invariants is by contracting the symplectic doublets. For example, the graviphoton – the $\mathcal{N}(\mathcal{N}-1)/2$-component supersymmetric partner of the $\mathcal{N}$-component gravitino ψ_A – is defined as

$$T^{\pm}_{AB} = (f^{\Lambda}{}_{AB}, h_{\Lambda AB}) \begin{pmatrix} 0 & -1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix} \begin{pmatrix} F^{\pm\Lambda} \\ G^{\pm}_{\Lambda} \end{pmatrix}. \tag{2.6}$$

Here $F^{\mp\Lambda}_{\mu\nu}$ are the Maxwell field strength in the action in eq. (2.2) , whereas $G^{\mp}_{\Lambda|\mu\nu}$ are defined as derivatives of the action over $F^{\mp\Lambda}_{\mu\nu}$,

$$G^{\mp}_{\Lambda|\mu\nu} \equiv \mp\frac{\mathrm{i}}{2}\frac{\partial\mathcal{L}}{\partial F^{\mp\Lambda}_{\mu\nu}}\,. \tag{2.7}$$

Note that the graviphoton is a $\mathcal{G}$-invariant and are covariant under the $\mathcal{H}$-symmetry, the $U(\mathcal{N})$ for $6 \geq \mathcal{N} \geq 4$ and $SU(8)$ for $\mathcal{N} = 8$.

We consider a two-vector part of the CT in $\mathcal{N} \geq 5$ supergravities, [2], [3]. The relevant expression – a supersymmetric partner of R^4 – depends on the graviphoton $T^-_{\mu\nu AB}$ and its conjugate defined in eqs. (A.34), (A.35):

$$\mathcal{L}_{CT} = \lambda\, T^-_{AB}\,\Delta\,\bar{T}^{-\,AB}\ . \tag{2.8}$$

In spinor notation the CT is

$$\mathcal{L}_{CT} = \lambda \mathcal{T}^{\alpha\beta\gamma\delta\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \nabla_{\alpha\dot{\delta}} T_{\beta\gamma AB} \nabla_{\delta\dot{\alpha}} \bar{T}^{AB}_{\dot{\beta}\dot{\gamma}} \tag{2.9}$$

with $\mathcal{T}^{\alpha\beta\gamma\delta\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \lambda\, C^{\alpha\beta\gamma\delta}\bar{C}^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ being the Bel-Robinson tensor in spinor notation and $\nabla_{\alpha\dot{\delta}}$ an $\mathcal{H}$-covariant space-time derivative. The differential operator in eq. (2.8) acting on two-forms $f_{\mu\nu}$ is defined as follows

$$(\Delta(f))_{\mu\nu} \equiv \Delta_{\mu\nu}{}^{\rho\sigma} f_{\rho\sigma} := \nabla_\kappa \mathcal{T}_{[\mu}{}^{\kappa\lambda[\sigma} \nabla_\lambda \delta^{\rho]}_{\nu]} f_{\rho\sigma}\ ; \tag{2.10}$$

it maps a self-dual 2-form into an anti-self-dual one and vice versa. Here the Bel–Robinson tensor is given in the vector form

$$\mathcal{T}^{\mu\nu\sigma\rho} \equiv C^{\mu\kappa\sigma\lambda} C^{\nu}{}_{\kappa}{}^{\rho}{}_{\lambda} - \frac{3}{2} g^{\mu[\nu} C^{\kappa\lambda]\sigma\vartheta} C_{\kappa\lambda}{}^{\rho}{}_{\vartheta}\,, \tag{2.11}$$

with the Weyl tensor $C_{\mu\nu\sigma\rho}$. The $\mathcal{H}$-covariant field strength of the graviphoton is $T_{\alpha\beta\, AB} \equiv \sigma^{\mu\nu}_{\alpha\beta} T^{AB}_{\mu\nu}$ and its complex conjugate is $\bar{T}_{\dot{\alpha}\dot{\beta}AB}$.

Since the dual field strength G is defined in terms of the field strength F through (A.5), to carry out perturbative calculation it is necessary, on the one hand, to express explicitly G in terms of F. On the other, adding a deformation such as (2.9), depending on both F and G, to the classical action defines the action implicitly, by relating it to its partial derivatives with respect to F. Thus, to carry out perturbative calculations with the deformed action it is necessary to solve this differential equation; the solution will generically exhibit arbitrarily-high powers of the deformation parameter λ. An alternative approach, which we will carry our in the next section and Appendix B, is to determine G by solving a deformed twisted self-duality constraint. The deformation of the classical twisted self-duality constraint is chosen such that the leading (i.e. $\mathcal{O}(\lambda)$) term reproduces the CT deformation of the classical action. There are many such deformations of the classical twisted self-duality constraint, which differ by terms of order $\mathcal{O}(\lambda^{n\geq 2})$. In the discussion in the next section and Appendix B we shall assume that no such higher-order terms are present.

Adding more derivatives, corresponding to superpartners of $D^{2k}R^4$, will not change the general structure of the two-vector vector (and hence its duality properties), but will change the dimension of the CT and the number of loops were it might be generated. In the context of the four-graviton amplitude it corresponds to an insertion of a dimension-increasing function of Mandelstam variables $f(s,t,u)$. The operator Δ in such case will have additional derivatives compared with the expression shown in (2.10).

3 Complete two-vector deformed action with duality symmetry

The *twisted nonlinear selfduality constraint* in classical supergravity at $\lambda = 0$ was proposed in [27], [28]. In $\mathcal{H}$-covariant form it states that there are only n_v physical vectors. The constraint is

$$T^{+}_{\mu\nu\,AB} = h_{\Lambda AB}\,F^{+\Lambda}_{\mu\nu} - f^{\Lambda}{}_{AB}\,G^{+}_{\mu\nu\,\Lambda} = 0\,, \tag{3.1}$$

together with its complex conjugate. If instead of using the $\mathcal{H}$-covariant constraint we would like to use the $\mathcal{G}$-covariant one, we can multiply the equation on f^{-1} so that

$$G^{+}_{\mu\nu\,\Lambda} - (f^{-1}h)_{\Lambda\Sigma}\,F^{+\Sigma}_{\mu\nu} = 0 \qquad \Rightarrow \qquad G^{+}_{\mu\nu\,\Lambda} - \mathcal{N}_{\Lambda\Sigma}\,F^{+\Sigma}_{\mu\nu} = 0\,. \tag{3.2}$$

A non-vanishing deformation on the right-hand side of these equations, which would also be Lorentz and $\mathcal{H}$-covariant, was presented in eq. (5.7) in [7]. It can be derived, following the proposal in [6] to use the manifestly duality invariant source of deformation. In this case it depends on a duality doublet $\mathcal{F} = (F, G)$; that is, the classical twisted self-duality constraint (3.2) is not valid and we propose that its right-hand side is given by the source of deformation

$$\mathcal{I} = \lambda\,T^{-}_{AB}\,\Delta\,\bar{T}^{-\,AB} = \lambda(h_{\Sigma AB}F^{-\Sigma} - f^{\Sigma}{}_{AB}\,G^{-}_{\Sigma})\Delta(\bar{h}_{\Lambda}{}^{AB}\,F^{+\Lambda} - \bar{f}^{\Lambda AB}\,G^{+}_{\Lambda})\ . \tag{3.3}$$

It leads to a constraint of the type given in eq. (5.7) in [7]

$$T^{+}_{AB} + \lambda\Delta T^{-}_{AB} = 0 \tag{3.4}$$

where the $\mathcal{H}$ covariant differential operator Δ is defined in eq. (2.10). In fact all results below are valid in a more general case when Δ depends also on scalars and gravitons. For the subsequent analysis it is convenient to switch to a $\mathcal{G}$-covariant form of equations

$$(f^{-1})^{AB}{}_{\Lambda}\Big(T^{+}_{AB} + \lambda\Delta T^{-}_{AB}\Big) = 0\ , \tag{3.5}$$

which will give us the following (we skip indices, they are easy to restore)

$$[G^{+} - \mathcal{N}\,F^{+} + X(G^{-} - \mathcal{N}F^{-})]_{\Lambda} = 0\,, \tag{3.6}$$

and the complex conjugate is

$$[G^{-} - \bar{\mathcal{N}}\,F^{-} + \bar{X}(G^{+} - \bar{\mathcal{N}}F^{+})]_{\Lambda} = 0\,. \tag{3.7}$$

Here the differential operators X and $\bar{X}$ are

$$X = \lambda f^{-1}\Delta f, \qquad \bar{X} = \lambda\bar{f}^{-1}\bar{\Delta}\bar{f}\ . \tag{3.8}$$

We may substitute G^{-} from (3.7) into (3.6) and we get

$$G^{+}_{\Lambda} = \Big[(1 - X\bar{X})^{-1}[X(\mathcal{N} - \bar{\mathcal{N}})F^{-} + (\mathcal{N} - X\bar{X}\bar{\mathcal{N}})F^{+}]\Big]_{\Lambda}\ . \tag{3.9}$$

This can be integrated to produce the deformed action, so that the derivative of the action over F^{+} will produce the value of G^{+} in (3.9). The result is

$$\mathcal{L}_{\text{def}} = -iF^{+}(1 - X\bar{X})^{-1}X(\mathcal{N} - \bar{\mathcal{N}})F^{-} - iF^{+}(1 - X\bar{X})^{-1}(\mathcal{N} - X\bar{X}\bar{\mathcal{N}})F^{+} + \text{h.c.}\ . \tag{3.10}$$

The integrability condition requires that

$$\frac{\delta G^+_\Lambda}{\delta F^{+\Sigma}} = \frac{i}{2}\frac{\delta^2 S}{\delta F^{+\Lambda}\delta F^{+\Sigma}}\,, \qquad \frac{\delta G^+_\Lambda}{\delta F^{-\Sigma}} = \frac{i}{2}\frac{\delta^2 S}{\delta F^{+\Lambda}\delta F^{-\Sigma}} = -\frac{\delta G^-_\Sigma}{\delta F^{+\Sigma}}\,. \tag{3.11}$$

We test the integrability condition in the appendix B and show that the action (3.10) leads to (3.9). And since every term in the expression for G is linear in F, it is easy to present a nice and simple form of the vector-dependent part of the action, it is given in the form

$$\mathcal{L}_{\rm def} = F\tilde{G}. \tag{3.12}$$

In conclusion of this section, we have derived a deformed action (3.10), (3.12) for $\mathcal{N} \geq 5$ supergravity, with terms with higher derivatives of an infinite order, which has a duality current conservation. The first deformation term, a CT, is proportional to $X = \lambda f^{-1}\Delta f$ and has 8 derivatives, other terms with X^n are of the order $\lambda^n \partial^{2n}$. Since now $G^+ = \frac{i}{2}\frac{\delta \mathcal{L}_{\rm def}}{\delta F^+}$, we find that deformed equations of motion for the F-field become exact Bianchi identity for the G-field.

4 Duality restoration in an example: λ^2 approximation, no scalars

Our deformed (bosonic) action is given in eq. (3.10). We are interested in vector-dependent terms which are independent, linear and quadratic in $X \sim \lambda$

$$\begin{aligned}\mathcal{L}^{0+1+2} =& -iF^+\mathcal{N}F^+ - iF^+X(\mathcal{N}-\bar{\mathcal{N}})F^- - iF^+X\bar{X}(\mathcal{N}-\bar{\mathcal{N}})F^+\\ &+ iF^-\bar{\mathcal{N}}F^- + iF^+(\bar{\mathcal{N}}-\mathcal{N})\bar{X}F^- + iF^-(\bar{\mathcal{N}}-\mathcal{N})\bar{X}XF^-\,.\end{aligned} \tag{4.1}$$

At the base point of the coset space we will take $\mathcal{N} = -i$, $\mathcal{N} - \bar{\mathcal{N}} = -2i$, $f = 1/\sqrt{2}$, $X = \lambda f^{-1}\Delta f = \lambda\Delta$ and we take $\Delta = \Delta^\dagger$

$$\mathcal{L}^{0+1+2}_{\rm base} = -\Big[(F^+)^2 + (F^-)^2\Big] - 4\lambda F^+\Delta F^- - 2\lambda^2 F^+\Delta^2 F^+ - 2\lambda^2 F^-\Delta^2 F^-\,. \tag{4.2}$$

In such case we defined the dual field strength as

$$\tilde{G} = \frac{1}{2}\frac{\delta\mathcal{L}}{\delta F}\,. \tag{4.3}$$

To check directly that the current conservation, broken due to terms λ [4], [5] and restored by the terms of order λ^2 in the action we need to compute the B component of the duality current conservation $\partial_\mu J^{\mu\Lambda\Sigma}B_{\Lambda\Sigma}$. The B component of the Gaillard-Zumino duality current $J^\mu_{\rm GZ}B = \tilde{G}^{\mu\nu}B\mathcal{B}_\nu$, corresponding to the transformation $F^{\Lambda'} = A^\Lambda{}_\Sigma F^\Sigma + B^{\Lambda\Sigma}G_\Sigma$, can be defined only in the presence of the equation of motion $dG = 0$, i.e. in the presence of the dual vector $\mathcal{B}_\nu$ such that $G = d\mathcal{B}$.

Here we will just check that, in absence of scalars, $\partial_\mu J^{\mu\Lambda\Sigma} B_{\Lambda\Sigma}$ vanishes through $\mathcal{O}(\lambda^2)$.

This component of the duality current[2]

$$\partial_\mu J^{\mu\Lambda\Sigma} B_{\Lambda\Sigma} = -\Big(\frac{\delta S}{\delta F^{+\Lambda}}\frac{\delta S}{\delta F^{+\Sigma}} - \frac{\delta S}{\delta F^{-\Lambda}}\frac{\delta S}{\delta F^{-\Sigma}}\Big) B_{\Lambda\Sigma} = G^+_\Lambda B^{\Lambda\Sigma} G^+_\Sigma - G^-_\Lambda B^{\Lambda\Sigma} G^-_\Sigma = 2i G_\Lambda B^{\Lambda\Sigma} \tilde{G}_\Sigma \ , \tag{4.4}$$

with the self-dual and anti-self-dual dual field strengths given by

$$-\frac{\delta S^{0+1+2}_{\rm base}}{\delta F^{+\Lambda}} = 2F^{+\Lambda} + 4\lambda\Delta\mathcal{F}^{-\Lambda} + 4\lambda^2\Delta^2\mathcal{F}^{+\Lambda}, \tag{4.5}$$

$$-\frac{\delta S^{0+1+2}_{\rm base}}{\delta F^{-\Lambda}} = 2F^{-\Lambda} + 4\lambda\Delta F^{+\Lambda} + 4\lambda^2\Delta^2 F^{-\Lambda} \ . \tag{4.6}$$

Eq. (4.4) becomes then

$$\begin{aligned} G^+_\Lambda G^+_\Sigma - G^-_\Lambda G^-_\Sigma &= (F^+_\Lambda + 2\lambda\Delta F^-_\Lambda + 2\lambda^2\Delta^2 F^+_\Lambda)(F^+_\Sigma + 2\lambda\Delta F^-_\Sigma + 2\lambda^2\Delta^2 F^+_\Sigma) \\ &- (F^-_\Lambda + 2\lambda\Delta F^+_\Lambda + 2\lambda^2\Delta^2 F^-_\Lambda)(F^-_\Sigma + 2\lambda\Delta F^+_\Sigma + 2\lambda^2\Delta^2 F^-_\Sigma) \end{aligned} \tag{4.7}$$

which, up to terms of order $\mathcal{O}(\lambda^3)$ is a total divergence

$$G^+_\Lambda G^+_\Sigma - G^-_\Lambda G^-_\Sigma = F^+_\Lambda F^+_\Sigma - F^-_\Lambda F^-_\Sigma + \mathcal{O}(\lambda^3) \ . \tag{4.8}$$

This supports and illustrates at the λ^2 level the general proof in section 2 that our deformed action has a duality current conservation.

5 $SU(\mathcal{N})$ restoration from $SO(\mathcal{N})$ in the six-point amplitude example

Note that the deformed action in (4.2) has terms with $SO(\mathcal{N})$ symmetry, for example using indices we have $\lambda^2(F^{+AB})\Delta^2(F^{+AB})$ as well as terms with $SU(\mathcal{N})$ symmetry, like $\lambda F^{+AB}\Delta F^-_{AB}$. In classical theory there are also $SO(\mathcal{N})$ invariant terms, like $(F^{+AB})^2$, however, the on shell action is known to have an $SU(\mathcal{N})$ symmetry. Here we will find out if the presence of the new $SU(\mathcal{N})$ symmetry breaking terms, like $\lambda^2(F^{+AB})\Delta^2(F^{+AB})$, affects the on shell symmetry of the theory. For this purpose we will compute all contributions to the λ^2 amplitude, the one from the single $\lambda^2(F^{+AB})\Delta^2(F^{+AB})$ vertex and the one from the tree diagram with two vertices $\lambda F^{+AB}\Delta F^-_{AB}$, as shown in Figure 1.

In this section we will treat the parameter λ as independent of the gravitational coupling. To test the on-shell symmetry properties of the deformed action (4.2) it therefore suffices to analyze tree-level amplitudes with λ-dependent vertices. Our strategy will thus be to concentrate on the simplest possible non-trivial tree amplitude involving the correction term

[2] The position of duality indices Λ was not specified strictly at the level of [1]-[29], as it becomes later when in symplectic sections upper component was taken with the duality index up, and lower component with the index down.

in lowest order, with four gravitons and two vectors on the external legs, which is such that no other (of the infinitely many) higher order vertices can contribute. To this aim we start from (4.2) where all dependence on the scalar fields has been stripped off. Introducing the chiral projectors

$$P_{\pm}{}^{\mu_1\nu_1}_{\rho_1\sigma_1} = \frac{1}{4}\big(\delta^{\mu_1}_{\rho_1}\delta^{\nu_1}_{\sigma_1} - \delta^{\nu_1}_{\rho_1}\delta^{\mu_1}_{\sigma_1} \mp i\epsilon^{\mu_1\nu_1}{}_{\rho_1\sigma_1}\big) \tag{5.1}$$

onto the self-dual and anti-self-dual components of 2-forms we can schematically represent the operator Δ in the form

$$\begin{aligned} \Delta &= P_+(Xhh)P_- \,+\, P_-(Xhh)P_+ + \mathcal{O}(h^3)\,, \\ \Delta^2 &= P_+(Xhh)P_-(Xhh)P_+ \,+\, P_-(Xhh)P_+(Xhh)P_- + \mathcal{O}(h^5)\,, \end{aligned} \tag{5.2}$$

where Xhh is the leading term in the expansion of the fourth order differential operator Δ to lowest (quadratic) order in the metric fluctuations. With this the action (4.2) contains the following pieces up to and including second order in λ (still omitting internal indices)

$$\begin{aligned} -4\lambda F^+\Delta F^- &= -4\lambda(\partial A)P_+(Xhh)P_-(\partial A) + \mathcal{O}(h^3)\,, \\ -2\lambda^2 F^+\Delta^2 F^+ &= -2\lambda^2(\partial A)P_+(Xhh)P_-(Xhh)P_+(\partial A) + \mathcal{O}(h^5)\,, \\ -2\lambda^2 F^-\Delta^2 F^- &= -2\lambda^2(\partial A)P_-(Xhh)P_+(Xhh)P_-(\partial A) + \mathcal{O}(h^5)\,. \end{aligned} \tag{5.3}$$

For the computation of the scattering amplitude we must saturate these vertices with the polarization states $\epsilon^{\pm\pm}_{\mu\nu}(p)$ for the gravitons, and $\epsilon^{\pm}_{\mu}(p)$ for the vectors (with the usual on-shell conditions $p^2 = 0$ and $p^\mu\epsilon^{\pm\pm}_{\mu\nu}(p) = p^\mu\epsilon^{\pm}_{\mu}(p) = 0$). Putting back the internal indices we recall that the vector fields of $\mathcal{N}$-extended supergravity transform in the adjoint of $SO(\mathcal{N})$ (with an extra singlet vector for $\mathcal{N} = 6$); the vector polarizations therefore carry an extra $SO(\mathcal{N})$ label $[AB]$. When applied to the field strength this $SO(\mathcal{N})$ label becomes elevated to an $(S)U(\mathcal{N})$ index pair, where we must now distinguish between upper and lower positions of the indices $[AB]$. For instance, for $\mathcal{N} = 8$ supergravity this results in the substitutions

$$\begin{aligned} F^{+AB}_{\mu\nu} &\to ip_{[\mu}\epsilon^{+AB}_{\nu]} \\ F^{-}_{\mu\nu AB} &\to ip_{[\mu}\epsilon^{-}_{\nu]AB} \end{aligned} \tag{5.4}$$

where $F^{+AB}_{\mu\nu}$ transforms in the $\mathbf{28}$ of $SU(8)$, while $F^{-}_{\mu\nu AB}$ transforms in the $\overline{\mathbf{28}}$ of $SU(8)$, with independent polarizations $\epsilon^{\pm AB}_{\mu}$ for all vectors. These $SU(8)$ assignments are furthermore consistent with the relations

$$\begin{aligned} P_{+\mu\nu}{}^{\rho\sigma}(ip_\rho\epsilon^{+AB}_{\sigma}) &= ip_{[\mu}\epsilon^{+AB}_{\nu]} \quad , \quad P_{-\mu\nu}{}^{\rho\sigma}(ip_\rho\epsilon^{-}_{\sigma\,AB}) = ip_{[\mu}\epsilon^{-}_{\nu]AB}\,, \\ P_{+\mu\nu}{}^{\rho\sigma}(ip_\rho\epsilon^{-}_{\sigma\,AB}) &= 0 \qquad\qquad\quad , \quad P_{-\mu\nu}{}^{\rho\sigma}(ip_\rho\epsilon^{+AB}_{\sigma}) = 0\,. \end{aligned} \tag{5.5}$$

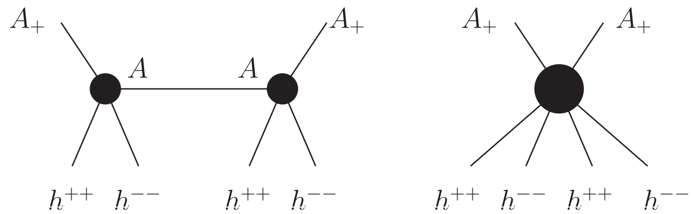

Figure 1. Graphs contributing to the amplitude $\mathcal{A}(A_+^{AB}(p_1)A_+^{CD}(p_2)h_{++}(p_3)h_{++}(p_4)h_{--}(p_5)h_{--}(p_6))$.

At order $\mathcal{O}(\lambda^2)$ the amplitude

$$\mathcal{A}^{AB,CD}(p_1,\ldots,p_6) = \Big\langle A_+^{AB}(p_1)A_+^{CD}(p_2)h_{++}(p_3)h_{++}(p_4)h_{--}(p_5)h_{--}(p_6)\Big\rangle \tag{5.6}$$

will thus receive two contributions, namely one from the square of the quadratic vertex (first line in (5.3)) with two vectors contracted, and the other from the sextic vertex (second and third line in (5.3)); these two contributions are depicted in Figure 1. We note that in this amplitude both index pairs $[AB]$ and $[CD]$ are in the upper position because of the positive helicities of the external spin-one states. Since one cannot form an $SU(\mathcal{N})$ singlet with four upper indices for $\mathcal{N} \geq 5$, a non-vanishing result for this amplitude would indicate a breakdown of the $SU(\mathcal{N})$ R symmetry. However, we will now show that this amplitude indeed vanishes.

To proceed we first consider the square of the $\mathcal{O}(\lambda)$ vertex: not forgetting a factor 1/2 from the expansion of the exponential this leads to

$$\frac{1}{2}\big(-4i\lambda(\partial A)P_+(Xhh)P_-(\partial\, \underbracket{A)\big)\big(-4i\lambda(\partial A}\,)P_-(Xhh)P_+(\partial A)\big) \tag{5.7}$$

where the underbracket denotes the contraction (= vector propagator in a convenient gauge)

$$\underbracket{A_\mu^{AB}(k)A_\nu^{CD}}(-k) \;=\; -\frac{i}{k^2}\,\eta_{\mu\nu}\delta^{B[A}\delta^{C]D} \tag{5.8}$$

and where the positive helicity vectors are left uncontracted as they will be dressed with positive helicity polarizations in accord with (5.5). Now using the relation

$$P_{-}{}_{\rho_2\sigma_2}^{\mu_2\nu_2}k_{\mu_2}\eta_{\nu_2\bar\nu_2}P_{-}{}_{\bar\rho_2\bar\sigma_2}^{\bar\mu_2\bar\nu_2}k_{\bar\mu_2} = \frac{1}{4}k^2 P_{-\,\rho_2\sigma_2;\bar\rho_2\bar\sigma_2}\,, \tag{5.9}$$

with the momentum $k = p_1 + p_3 + p_5$ $(= -p_2 - p_4 - p_6)$ on the internal line we see that the propagator factor is cancelled, and we end up with an effective *local* vertex

$$+\,2i\lambda^2(\partial A)P_+(Xhh)P_-(Xhh)P_+(\partial A)$$

which is the same as the contact interaction in the second line in (5.3). Therefore the two contributions exactly cancel at the order of λ^2. Thus, the deformed all order higher derivatives action, which has a duality current conservation, yields a six-point on-shell amplitudes at the λ^2 order which does exhibit the expected $SU(\mathcal{N})$ demanded by $\mathcal{N}$-extended supergravity. Using the vertices in eq (5.3) it is not difficult to show that the eight-point $\mathcal{O}(\lambda^3)$ $SU(\mathcal{N})$-breaking amplitude also vanishes.

To conclude, the fact that the bosonic duality-symmetric action with higher derivatives does not break the $SU(\mathcal{N})$ symmetry of the six- and eight-point on shell amplitudes to $SO(\mathcal{N})$ means that this amplitude might be a part of a supersymmetric amplitude.

6 Discussion

In this paper we have constructed a complete deformed action of the two-vector sector of the candidate UV divergence serving as the seed of deformation of $\mathcal{N} \geq 5$ supergravities. We have solved perturbatively the twisted non-linear constraint equation (3.4) and identified the dual field strength $G^+(F, \phi)$ to all orders in λ, presented in (3.9). We have also found the complete all order in λ action (3.10) such that the corresponding duality current is conserved.

Our deformed action, when expanded near the base point of the moduli space $\frac{\mathcal{G}}{\mathcal{H}}$ has terms which break $SU(\mathcal{N})$ symmetry down to $SO(\mathcal{N})$ symmetry. This feature, if it would persist on shell, would prevent our deformed action from being consistent with supersymmetry. We have therefore computed the six-point amplitude, as shown in Figure 1, and we have found that the contribution from the $SU(\mathcal{N})$ symmetry violating λ^2 vertex in the deformed action is precisely cancelled by the tree diagram with two λ vertices. These examples indicate that an analogous cancellation and restoration of $SU(\mathcal{N})$ symmetry in scattering amplitudes will take place at all higher orders in λ and for all n-point amplitudes.

Our conclusion here is the following. When using the two-vector sector of the candidate counterterm as a seed for deformation of the action we do not find an inconsistency between the requirement of duality current conservation and supersymmetry of the deformed action. It does not mean that our deformed action has a supersymmetric embedding, but there is also no obvious obstruction to it: the six-point tree amplitude based on deformed action has an $SU(\mathcal{N})$ symmetry, which is necessary but not sufficient condition for supersymmetry.

Our analysis here does not explain why $\mathcal{N} = 5$ supergravity in four loops is UV finite, [14]. We will continue with analogous investigation of more general sectors of the deformation of the theory in Part II of this project. We will take into account the one-vector and the four-vector sectors, in addition to the two-vector sector we have studied here. Ultimately, the goal is to either construct a supersymmetric deformed action of $\mathcal{N} \geq 5$ supergravity, or to find that it is not available.

Acknowledgments: We are grateful to Z. Bern, L. Brink, J.J. Carrasco, D. Freedman, M.

Green, M. Duff, S. Ferrara, H. Elvang, Y.-T. Huang, P. Townsend, A. Tseytlin and A. Van Proeyen for stimulating discussions. The work of RK and YY is supported by SITP and by the US National Science Foundation grant PHY-1720397. The work of HN is suppoted in part by the European Research Council under the European Union's Horizon 2020 research and innovation program (grant agreement No 740209). The work of RR is supported in part by the US Department of Energy under grant DE-SC0013699. RK and HN are grateful to the organizers of SUSY 2016 conference in Melbourne where this work was initiated. RR is grateful to SITP at Stanford and RK to Max Planck Institute of Gravitational Physics in Potsdam for the hospitality when a part of this work was performed. We are grateful to the participants of the conference 'Hawking 75' in Cambridge for the interest to our work and important discussions.

A A review of classical $\mathcal{N}$-extended supergravities

We start by recalling [3] the main features of four dimensional pure $\mathcal{N}$-extended supergravities, $\mathcal{N} \geq 5$.

These theories contain in the bosonic sector, besides the metric, a number n_v of vectors and m of (real) scalar fields. The relevant classical bosonic vector and scalar part of action is known to have the following general form:

$$\mathcal{S} = \int \sqrt{-g}\, d^4x \left(-\frac{1}{2}\, R + \mathrm{Im}\mathcal{N}_{\Lambda\Gamma} F^{\Lambda}_{\mu\nu} F^{\Gamma|\mu\nu} + \frac{1}{2\sqrt{-g}} \mathrm{Re}\mathcal{N}_{\Lambda\Gamma} \epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Gamma}_{\rho\sigma} + \right. \\ \left. + \frac{1}{2}\, g_{rs}(\Phi) \partial_\mu \Phi^r \partial^\mu \Phi^s \right) . \tag{A.1}$$

The vector-scalar part of this action was presented in (2.2) and notations explained there. Duality rotations and symplectic covariance of these theories were uncovered in [1].

We consider a theory of n_v abelian gauge fields A^{Λ}_μ, in a $D = 4$ space-time with Lorentz signature (which we take to be mostly minus). They correspond to a set of n_v differential 1-forms

$$A^\Lambda \equiv A^\Lambda_\mu\, dx^\mu \qquad (\Lambda = 1, \ldots, n_v) . \tag{A.2}$$

The corresponding field strengths and their Hodge duals are defined by [4]

$$\begin{aligned} F^\Lambda &\equiv d\, A^\Lambda \equiv F^\Lambda_{\mu\nu}\, dx^\mu \wedge dx^\nu , \\ F^\Lambda_{\mu\nu} &\equiv \frac{1}{2} \left(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu \right) , \\ ({}^\star F^\Lambda)_{\mu\nu} &\equiv \frac{\sqrt{-g}}{2} \varepsilon_{\mu\nu\rho\sigma}\, F^{\Lambda|\rho\sigma} . \end{aligned} \tag{A.3}$$

[3] This is a shortened version of the corresponding review in [26], which focuses on the details important to our case.

[4] We use, for the ϵ tensor, the convention: $\epsilon_{0123} = -1$.

The dynamics of a system of abelian gauge fields coupled to scalars in a gravity theory is encoded in the bosonic action (A.1). Introducing self-dual and anti-self-dual combinations

$$F^{\pm} = \frac{1}{2}\left(F \pm \mathrm{i}\,{}^{\star}F\right) \quad , \qquad {}^{\star}(F^{\pm}) = \mp \mathrm{i} F^{\pm} \; , \tag{A.4}$$

the vector part of the Lagrangian defined by (A.1) can be rewritten in the form given in (2.2) We introduce new tensors

$${}^{\star}G_{\Lambda|\mu\nu} \equiv \frac{1}{2}\frac{\partial\mathcal{L}}{\partial F^{\Lambda}_{\mu\nu}} = \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\, F^{\Sigma}_{\mu\nu} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\, {}^{\star}F^{\Sigma}_{\mu\nu} \quad \longleftrightarrow \quad G^{\mp}_{\Lambda|\mu\nu} \equiv \mp\frac{\mathrm{i}}{2}\frac{\partial\mathcal{L}}{\partial F^{\mp\Lambda}_{\mu\nu}} \, , \tag{A.5}$$

the Bianchi identities and field equations associated with the Lagrangian (A.1) can be written as

$$\nabla^{\mu\star}F^{\Lambda}_{\mu\nu} = 0 \quad , \qquad \nabla^{\mu\star}G_{\Lambda|\mu\nu} = 0 \tag{A.6}$$

or equivalently

$$\nabla^{\mu}\mathrm{Im}F^{\pm\Lambda}_{\mu\nu} = 0 \quad , \qquad \nabla^{\mu}\mathrm{Im}G^{\pm}_{\Lambda|\mu\nu} = 0 \; . \tag{A.7}$$

Introducing the $2n_v$-component column vector

$${}^{\star}\mathcal{F} \equiv \begin{pmatrix} {}^{\star}F^{\Lambda} \\ {}^{\star}G_{\Lambda} \end{pmatrix} \; , \tag{A.8}$$

a general duality rotation is any general linear transformations on such a vector,

$$\begin{pmatrix} {}^{\star}F \\ {}^{\star}G \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} {}^{\star}F \\ {}^{\star}G \end{pmatrix} . \tag{A.9}$$

For any constant matrix $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n_v, \mathbb{R})$ the transformed vector of magnetic and electric field-strengths ${}^{\star}\mathcal{F}' = \mathcal{S}\cdot{}^{\star}\mathcal{F}$ satisfies the same equations (A.6) as the original one. In a condensed notation we can write

$$\partial\,{}^{\star}\mathcal{F} = 0 \quad \Longleftrightarrow \quad \partial\,{}^{\star}\mathcal{F}' = 0. \tag{A.10}$$

Separating the self-dual and anti-self-dual parts

$$F = F^{+} + F^{-} \quad ; \qquad G = G^{+} + G^{-} \tag{A.11}$$

and taking into account that F and G are related by (A.5),

$$G^{+} = \mathcal{N}F^{+} \qquad ; \qquad G^{-} = \bar{\mathcal{N}}F^{-} \; , \tag{A.12}$$

the duality rotation in eq. (A.9) can be rewritten as

$$\begin{pmatrix} F^+ \\ G^+ \end{pmatrix}' = \mathcal{S} \begin{pmatrix} F^+ \\ \mathcal{N} F^+ \end{pmatrix} \qquad ; \qquad \begin{pmatrix} F^- \\ G^- \end{pmatrix}' = \mathcal{S} \begin{pmatrix} F^- \\ \bar{\mathcal{N}} F^- \end{pmatrix}. \tag{A.13}$$

The kinetic matrix $\mathcal{N} = \mathcal{N}(\Phi)$ transforms under a duality rotation such that the definition of $G^\mp$ as a variation of the Lagrangian continues to hold:

$$G'^{+}_{\Lambda} = (C + D\mathcal{N})_{\Lambda\Sigma} F^{+\Sigma} \equiv -\frac{\mathrm{i}}{2} \frac{\partial \mathcal{L}'}{\partial F'^{+\Lambda}} = (A + B\mathcal{N})^{\Delta}{}_{\Sigma} \mathcal{N}'_{\Lambda\Delta} F^{+\Sigma} \tag{A.14}$$

that

$$\mathcal{N}'_{\Lambda\Sigma}(\Phi') = \left[(C + D\mathcal{N}) \cdot (A + B\mathcal{N})^{-1} \right]_{\Lambda\Sigma}. \tag{A.15}$$

The condition that the matrix $\mathcal{N}$ is symmetric both before and after the duality transformation implies that

$$\mathcal{S} \in Sp(2n_v, \mathbb{R}) \subset GL(2n_v, \mathbb{R}), \tag{A.16}$$

that is:

$$\mathcal{S}^T \, \mathbb{C} \, \mathcal{S} = \mathbb{C}, \tag{A.17}$$

where $\mathbb{C}$ is the symplectic invariant $2n_v \times 2n_v$ matrix:

$$\mathbb{C} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{A.18}$$

It is useful to rewrite the symplectic condition (A.17) in terms of the $n_v \times n_v$ blocks defining $\mathcal{S}$:

$$A^T \, C - C^T \, A = B^T \, D - D^T \, B = 0 \;\; ; \;\; A^T \, D - C^T \, B = \mathbb{1}. \tag{A.19}$$

In $\mathcal{N} \geq 5$ models the fields are in some representation of the isometry group $\mathcal{G}$ of the scalar manifold or of its maximal compact subgroup $\mathcal{H}$. [5] All the properties of supergravity theories for $\mathcal{N} \geq 5$ are completely fixed in terms of the geometry of the coset $\mathcal{G}/\mathcal{H}$; they can be formulated in terms of the coset representatives L satisfying by

$$L(\Phi') = g L(\Phi) h(g, \Phi). \tag{A.20}$$

Here $g \in \mathcal{G}$, $h \in \mathcal{H}$ and $\Phi' = \Phi'(\Phi)$, Φ being the coordinates of $\mathcal{G}/\mathcal{H}$. Note that the scalar fields in $\mathcal{G}/\mathcal{H}$ can be assigned, in the linearized theory, to linear representations $R_{\mathcal{H}}$ of the local isotropy group $\mathcal{H}$ so that dim $R_{\mathcal{H}}$ = dim $\mathcal{G}$ − dim $\mathcal{H}$ (in the full theory, $R_{\mathcal{H}}$ is the representation which the vielbein of $\mathcal{G}/\mathcal{H}$ belongs to).

[5] This group is also the isotropy group of the scalar manifold and it is also isomorphic to the R-symmetry group; we use these names interchangeably when referring to $\mathcal{H}$.

Fermions in extended supergravities form representations the isotropy subgroup $\mathcal{H}$ rather than of the isometry group $\mathcal{G}$ of the scalar manifold. For example, there is the graviphoton – 2-form T_{AB} – appearing in the supersymmetry transformation law of the gravitino 1-form

$$\delta\psi_A = \nabla\epsilon_A + \alpha T_{AB|\mu\nu}\gamma^a\gamma^{\mu\nu}\epsilon^B V_a + \cdots \,. \tag{A.21}$$

Here ∇ is the covariant derivative in terms of the space-time spin connection and the composite connection of $\mathcal{H}$, α is a coefficient fixed by supersymmetry, V^a is the space-time vielbein, $A = 1, \cdots, \mathcal{N}$ is the index acted on by the automorphism group $\mathcal{H}$ in the fundamental representation. Here and in the following the ellipsis denote trilinear fermion terms. The 2-form field strength T_{AB} will be constructed by dressing the bare field strengths F^Λ with the coset representative $L(\Phi)$ of $\mathcal{G}/\mathcal{H}$, Φ denoting a set of coordinates of $\mathcal{G}/\mathcal{H}$. The same field strength T_{AB} which appears in the gravitino transformation law is also present in the dilatino transformation law

$$\delta\chi_{ABC} = P_{ABCD,\ell}\partial_\mu\phi^\ell\gamma^\mu\epsilon^D + \beta T_{[AB|\mu\nu}\gamma^{\mu\nu}\epsilon_{C]} + \cdots \,. \tag{A.22}$$

Here $P_{ABCD} = P_{ABCD,\ell}d\phi^\ell$ is the vielbein of the scalar manifold, β is a coefficient fixed by supersymmetry.

In order to give the explicit dependence on scalars of T_{AB} it is necessary to recall that, according to the Gaillard–Zumino construction, the isometry group G of the scalar manifold acts on the vector $(F^{-\Lambda}, G^-_\Lambda)$ (or its complex conjugate) as a subgroup of $Sp(2n_v, \mathbb{R})$ (n_v is the number of vector fields) with duality transformations interchanging electric and magnetic field strengths, as shown in (A.13)

Let now $L(\Phi)$ be the coset representative of $\mathcal{G}/\mathcal{H}$ in the symplectic representation, namely as a $2\,n_v \times 2\,n_v$ matrix belonging to $Sp(2n_v, \mathbb{R})$ and therefore, in each theory, it can be described in terms of $n_v \times n_v$ blocks A_L, B_L, C_L, D_L satisfying the same relations (A.19) as the corresponding blocks of the generic symplectic transformation $\mathcal{S}$.

Since the fermions of supergravity theories transform in a complex representation of the R-symmetry group $\mathcal{H} \subset \mathcal{G}$, it is useful to introduce a complex basis in the vector space of $Sp(2n_v, \mathbb{R})$, defined by the action of following unitary matrix:

$$\mathcal{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & i\,\mathbb{1} \\ \mathbb{1} & -i\,\mathbb{1} \end{pmatrix},$$

and to introduce a new matrix $\mathbf{V}(\Phi)$ obtained by complexifying the right index of the coset representative $L(\Phi)$, so as to make its transformation properties under right action of $\mathcal{H}$ manifest:

$$\mathbf{V}(\Phi) = \begin{pmatrix} \mathbf{f} & \bar{\mathbf{f}} \\ \mathbf{h} & \bar{\mathbf{h}} \end{pmatrix} = L(\Phi)\mathcal{A}^\dagger\,, \tag{A.23}$$

where:

$$\mathbf{f} = \frac{1}{\sqrt{2}}(A_L - \mathrm{i}B_L)\ ;\ \ \mathbf{h} = \frac{1}{\sqrt{2}}(C_L - \mathrm{i}D_L)\,.$$

From the properties of $L(\Phi)$ as a symplectic matrix, it is easy to derive the following properties for $\mathbf{V}$:

$$\mathbf{V}\,\eta\,\mathbf{V}^\dagger = -i\mathbb{C}\ \ ;\quad \mathbf{V}^\dagger\,\mathbb{C}\,\mathbf{V} = i\eta\,, \tag{A.24}$$

where the symplectic invariant matrix $\mathbb{C}$ and η are defined as follows:

$$\mathbb{C} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\ ;\ \ \eta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}\,, \tag{A.25}$$

and, as usual, each block is an $n_v \times n_v$ matrix. The above relations imply on the matrices $\mathbf{f}$ and $\mathbf{h}$ the following properties:

$$\begin{aligned} \mathrm{i}(\mathbf{f}^\dagger\mathbf{h} - \mathbf{h}^\dagger\mathbf{f}) &= \mathbb{1} \\ (\mathbf{f}^t\mathbf{h} - \mathbf{h}^t\mathbf{f}) &= 0. \end{aligned} \tag{A.26}$$

The $n_v \times n_v$ blocks $\mathbf{f}$, $\mathbf{h}$ of $\mathbf{V}$ acquire the following form

$$\begin{aligned} \mathbf{f} &= f^\Lambda{}_{AB}\,, \\ \mathbf{h} &= h_{\Lambda AB}\,, \end{aligned} \tag{A.27}$$

where AB are indices in the two-index antisymmetric representation of $H = SU(\mathcal{N}) \times U(1)$ or $SU(8)$ in $\mathcal{N} = 8$ case. Upper $SU(\mathcal{N})$ indices label objects in the complex conjugate representation of $SU(\mathcal{N})$:

$$(f^\Lambda{}_{AB})^* = \bar{f}^{\Lambda AB} \tag{A.28}$$

etc. Thus we have another symplectic section depending on scalars of the theory and transforming as follows

$$\begin{pmatrix} f^\Lambda{}_{AB} \\ h_{\Lambda AB} \end{pmatrix}' = \mathcal{S}\begin{pmatrix} f^\Lambda{}_{AB} \\ h_{\Lambda AB} \end{pmatrix}. \tag{A.29}$$

The kinetic matrix $\mathcal{N}$ can be written in terms of the sub-blocks $\mathbf{f}$, $\mathbf{h}$, and turns out to be:

$$\mathcal{N} = \mathbf{h}\,\mathbf{f}^{-1}, \qquad \mathcal{N} = \mathcal{N}^t\,, \tag{A.30}$$

transforming projectively under $Sp(2n_v,\mathbb{R})$ duality rotations as already shown in the previous section. By using (A.26)and (A.30) we find that

$$(\mathbf{f}^t)^{-1} = \mathrm{i}(\mathcal{N} - \bar{\mathcal{N}})\bar{\mathbf{f}}\,, \tag{A.31}$$

that is

$$(\mathbf{f}^{-1})^{AB}{}_{\Lambda} = \mathrm{i}(\mathcal{N} - \bar{\mathcal{N}})_{\Lambda\Sigma}\bar{f}^{\Sigma\, AB}\,. \tag{A.32}$$

For the symplectic product in general $\langle\ |\ \rangle$, one can use the convention

$$\langle \mathcal{A} \mid \mathcal{B} \rangle \equiv \mathcal{B}^{\Lambda}\mathcal{A}_{\Lambda} - \mathcal{B}_{\Lambda}\mathcal{A}^{\Lambda}\,. \tag{A.33}$$

In particular, a symplectic invariant can be constructed using one symplectic section depending on field strength and its dual $(F^{\pm}, G^{\pm})$ and the other one depending on scalars (f, h)

$$T^{\pm}_{AB} = (f^{\Lambda}{}_{AB}, h_{\Lambda AB}) \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} F^{\pm\Lambda} \\ G^{\pm}_{\Lambda} \end{pmatrix}. \tag{A.34}$$

Here $T^{\pm}_{AB}$ is a $\mathcal{G}$-invariant since $\mathcal{S}^T\,\mathbb{C}\,\mathcal{S} = \mathbb{C}$, but it transforms under the group $\mathcal{H}$. Thus, the graviphoton and its conjugate are

$$\begin{aligned} T^{-}_{AB} &= h_{\Lambda AB}\, F^{-\Lambda} - f^{\Lambda}{}_{AB}\, G^{-}_{\Lambda}\,, \\ \bar{T}^{-AB} &= (T^{-}_{AB})^{*} = \bar{h}_{\Lambda}{}^{AB}\, F^{+\Lambda} - \bar{f}^{\Lambda AB}\, G^{+}_{\Lambda}\,. \end{aligned} \tag{A.35}$$

Note that, in classical supergravity, the graviphoton satisfies the constraint shown in eq. (3.1) as a consequence of eqs. (A.30), (A.12). It is an $\mathcal{H}$-covariant form of what is known as a twisted selfduality constraint, covariant under $\mathcal{G}$ transformations.

The constraint eq. (3.1) is known as *linear twisted self-duality constraint*. It can be given in the following form. We can use a 56-dimensional real symplectic vector of field strengths

$$\mathcal{F} \equiv \begin{pmatrix} F^{\Lambda} \\ G_{\Lambda} \end{pmatrix}, \tag{A.36}$$

that transforms in the **56** of $E_{7(7)} \subset Sp(56, \mathbb{R})$. The scalars of the theory are described by the symplectic section

$$\mathcal{V}_{AB} \equiv \begin{pmatrix} f^{\Lambda}{}_{AB} \\ h_{\Lambda\, AB} \end{pmatrix}. \tag{A.37}$$

The period matrix is defined by the property

$$h_{\Lambda\, AB} = \mathcal{N}_{\Lambda\Sigma} f^{\Sigma}{}_{AB}\,. \tag{A.38}$$

This relation of the components of the section $\mathcal{V}_{IJ}$ with the components of the symplectic $E_{7(7)}/SU(8)$ coset representative imply the constraints

$$\langle \mathcal{V}_{AB} \mid \overline{\mathcal{V}}^{CD} \rangle = -2i\delta_{AB}{}^{CD}\,, \qquad \langle \mathcal{V}_{AB} \mid \mathcal{V}_{CD} \rangle = 0\,. \tag{A.39}$$

The graviphoton field strength is defined by

$$T_{AB} \equiv \langle \mathcal{V}_{AB} \mid \mathcal{F} \rangle \,, \tag{A.40}$$

and its self- and anti-self-dual parts are

$$T_{AB}{}^{\pm} \equiv \langle \mathcal{V}_{AB} \mid \mathcal{F}^{\pm} \rangle \,. \tag{A.41}$$

They all transform under compensating $SU(8)$ transformations only. Since the $\mathcal{H}$-tensor T_{AB} is complex, we have

$$T^{AB\,\pm} = \overline{(T_{AB}{}^{\mp})} \,. \tag{A.42}$$

Finally, the linear twisted self-duality constraint eq. (A.12), is equivalent to the vanishing of

$$\overline{T}^{AB\,-} = \overline{(T_{AB}{}^{+})} = 0 \,. \tag{A.43}$$

We are now able to derive some differential relations using the Maurer–Cartan equations obeyed by the scalars through the embedded coset representative $\mathbf{V}$. Indeed, let $\Gamma = \mathbf{V}^{-1}d\mathbf{V}$ be the $Sp(2n_v, \mathbb{R})$ Lie algebra left invariant one form satisfying:

$$d\Gamma + \Gamma \wedge \Gamma = 0 \,. \tag{A.44}$$

In terms of $(\mathbf{f}, \mathbf{h})$, Γ has the following form:

$$\Gamma \equiv \mathbf{V}^{-1}d\mathbf{V} = \begin{pmatrix} \mathrm{i}(\mathbf{f}^\dagger d\mathbf{h} - \mathbf{h}^\dagger d\mathbf{f}) & \mathrm{i}(\mathbf{f}^\dagger d\bar{\mathbf{h}} - \mathbf{h}^\dagger d\bar{\mathbf{f}}) \\ -\mathrm{i}(\mathbf{f}^t d\mathbf{h} - \mathbf{h}^t d\mathbf{f}) & -\mathrm{i}(\mathbf{f}^t d\bar{\mathbf{h}} - \mathbf{h}^t d\bar{\mathbf{f}}) \end{pmatrix} \equiv \begin{pmatrix} \Omega^{(H)} & \bar{\mathcal{P}} \\ \mathcal{P} & \bar{\Omega}^{(H)} \end{pmatrix} , \tag{A.45}$$

where the $n_v \times n_v$ sub-blocks $\Omega^{(H)}$ and $\mathcal{P}$ embed the $\mathcal{H}$-connection and the vielbein of $\mathcal{G}/\mathcal{H}$ respectively. This identification follows from the Cartan decomposition of the $Sp(2n_v, \mathbb{R})$ Lie algebra.

From (A.23) and (A.45), we obtain the $(n_v \times n_v)$ matrix equation:

$$\begin{aligned} D(\Omega)\mathbf{f} &= \bar{\mathbf{f}}\,\mathcal{P} \,, \\ D(\Omega)\mathbf{h} &= \bar{\mathbf{h}}\,\mathcal{P} \,, \end{aligned} \tag{A.46}$$

together with their complex conjugates. The $\mathcal{H}$-connection is

$$\Omega^{(H)} = \mathrm{i}[\mathbf{f}^\dagger(D\mathbf{h} + \mathbf{h}\omega) - \mathbf{h}^\dagger(D\mathbf{f} + \mathbf{f}\omega)] = \omega \mathbb{1} \,, \tag{A.47}$$

where we have used:

$$D\mathbf{h} = \bar{\mathcal{N}} D\mathbf{f}; \quad \mathbf{h} = \mathcal{N}\mathbf{f} \,, \tag{A.48}$$

which follow from (A.46) and the fundamental identity (A.26). Furthermore, using the same relations, the embedded vielbein $\mathcal{P}$ can be written as follows

$$\mathcal{P} = -\mathrm{i}(\mathbf{f}^t D\mathbf{h} - \mathbf{h}^t D\mathbf{f}) = \mathrm{i}\mathbf{f}^t(\mathcal{N} - \bar{\mathcal{N}})D\mathbf{f} \,, \tag{A.49}$$

and

$$D(\omega)f^{\Lambda}{}_{AB} = \frac{1}{2}\bar{f}^{\Lambda CD}P_{ABCD}. \tag{A.50}$$

For $N > 4$, $\mathcal{P}$ coincides with the vielbein P_{ABCD} of the relevant $\mathcal{G}/\mathcal{H}$.

This equation is a part of the Maurer-Cartan equation

$$D\mathcal{V}_{AB} = \tfrac{1}{2}\mathcal{P}_{ABCD}\overline{\mathcal{V}}^{CD}, \tag{A.51}$$

where D is the $\mathcal{H}$-covariant derivative and $\mathcal{P}_{ABCD}$ the vielbein 1-form on the scalar manifold. Using the definition of the graviphoton field strength (A.40) we also find that

$$DT_{AB} = \tfrac{1}{2}\mathcal{P}_{ABCD} \wedge \overline{T}^{CD}, \tag{A.52}$$

and its complex conjugate.

It is useful in the context of black holes to define the central charges, as integrals over the dressed, scalar dependent graviphoton, Z_{AB} and $\bar{Z}^{AB}$ and symplectic doublet charges Q which are integrals over field strength's F and G which are scalar independent. These are related as follows

$$\frac{1}{2}Z_{AB}\bar{Z}^{AB} = -\frac{1}{2}Q^t\mathcal{M}(\mathcal{N})Q, \tag{A.53}$$

where $\mathbb{C}$ is the symplectic metric while $\mathcal{M}(\mathcal{N})$ and Q are:

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} \mathbb{1} & -\mathrm{Re}\mathcal{N} \\ 0 & \mathbb{1} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Im}\mathcal{N} & 0 \\ 0 & \mathrm{Im}\mathcal{N}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1} & 0 \\ -\mathrm{Re}\mathcal{N} & \mathbb{1} \end{pmatrix} = \mathbb{C}\,\mathbf{V}\,\mathbf{V}^\dagger\,\mathbb{C}, \tag{A.54}$$

$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}. \tag{A.55}$$

More useful relations follow

$$\begin{aligned} \mathbf{f}\,\mathbf{f}^\dagger &= -\mathrm{i}\left(\mathcal{N} - \bar{\mathcal{N}}\right)^{-1}, \\ \mathbf{h}\,\mathbf{h}^\dagger &= -\mathrm{i}\left(\bar{\mathcal{N}}^{-1} - \mathcal{N}^{-1}\right)^{-1} \equiv -\mathrm{i}\mathcal{N}\left(\mathcal{N} - \bar{\mathcal{N}}\right)^{-1}\bar{\mathcal{N}}, \\ \mathbf{h}\,\mathbf{f}^\dagger &= \mathcal{N}\mathbf{f}\,\mathbf{f}^\dagger, \\ \mathbf{f}\,\mathbf{h}^\dagger &= \mathbf{f}\,\mathbf{f}^\dagger\bar{\mathcal{N}}. \end{aligned} \tag{A.56}$$

B Integrability of the deformed twisted self-duality in $\mathcal{N} \geq 5$ models

In this appendix we use matrix-like notation and omit the Lorentz, $\mathcal{G}$, $\mathcal{H}$ indices.

$$\mathbf{f}^{-1}\bar{\mathbf{f}}^{-1} = \bar{\mathbf{f}}^{-1}\mathbf{f}^{-1} = i(\mathcal{N} - \bar{\mathcal{N}}), \tag{B.1}$$

$$X\bar{X} = \lambda^2\mathbf{f}^{-1}\Delta\mathbf{f}\bar{\mathbf{f}}^{-1}\bar{\Delta}\bar{\mathbf{f}} = \lambda^2\mathbf{f}^{-1}\Delta\mathbf{M}\bar{\Delta}\bar{\mathbf{M}}\mathbf{f}, \tag{B.2}$$

$$(X\bar{X})^n = \lambda^{2n}\mathbf{f}^{-1}(\Delta\mathbf{M}\bar{\Delta}\bar{\mathbf{M}})^n\mathbf{f}, \tag{B.3}$$

$$X(\mathcal{N} - \bar{\mathcal{N}}) = \lambda\mathbf{f}^{-1}\Delta\mathbf{f}(\mathcal{N} - \bar{\mathcal{N}}) = -i\lambda\mathbf{f}^{-1}\Delta\mathbf{f}(\mathbf{f}^{-1}\bar{\mathbf{f}}^{-1}) = -i\lambda\mathbf{f}^{-1}\Delta\bar{\mathbf{f}}^{-1}. \tag{B.4}$$

We consider the following action,

$$\mathcal{L} = \alpha F^+ (1 - X\bar{X})^{-1} X(\mathcal{N} - \bar{\mathcal{N}}) F^- + \beta F^+ (1 - X\bar{X})^{-1} (\mathcal{N} - X\bar{X}\bar{\mathcal{N}}) F^+ + \text{h.c.}, \qquad \text{(B.5)}$$

where α and β are complex constants. We rewrite this action by using the identities. The first term can be rewritten as

$$\begin{aligned}
&\alpha F^+ (1 - X\bar{X})^{-1} X(\mathcal{N} - \bar{\mathcal{N}}) F^- \\
&= \alpha F^+ \sum_{n=0} (X\bar{X})^n X(\mathcal{N} - \bar{\mathcal{N}}) F^- \\
&= -i\alpha F^+ \sum_{n=0} \lambda^{2n+1} \mathbf{f}^{-1} (\mathbf{\Delta M \bar{\Delta} \bar{M}})^n \mathbf{f} \times \mathbf{f}^{-1} \mathbf{\Delta} \bar{\mathbf{f}}^{-1} F^- \\
&= -i\alpha F^+ \sum_{n=0} \lambda^{2n+1} \mathbf{f}^{-1} (\mathbf{\Delta M \bar{\Delta} \bar{M}})^n \mathbf{\Delta} \bar{\mathbf{f}}^{-1} F^-. \qquad \text{(B.6)}
\end{aligned}$$

The hermitian conjugate of this action is

$$\begin{aligned}
&i\bar{\alpha} F^- \sum_{n=0} \lambda^{2n+1} \bar{\mathbf{f}}^{-1} (\mathbf{\bar{\Delta} \bar{M} \Delta M})^n \bar{\mathbf{\Delta}} \mathbf{f}^{-1} F^+ \\
&= i\bar{\alpha} F^+ \sum_{n=0} \lambda^{2n+1} \mathbf{f}^{-1} (\mathbf{\Delta M \bar{\Delta} \bar{M}})^n \mathbf{\Delta} \bar{\mathbf{f}}^{-1} F^- + \text{tot.div}, \qquad \text{(B.7)}
\end{aligned}$$

where we have used the partial integral, and note that $\bar{\Delta}$ becomes Δ by raising and lowering Lorentz indices, which do not change the sign.

The second term becomes

$$\begin{aligned}
&\beta F^+ (1 - X\bar{X})^{-1} (\mathcal{N} - X\bar{X}\bar{\mathcal{N}}) F^+ \\
&= \beta F^+ (1 - X\bar{X})^{-1} \{(\mathcal{N} - \bar{\mathcal{N}}) - (1 - X\bar{X})\bar{\mathcal{N}})\} F^+ \\
&= \beta F^+ \mathcal{N} F^+ + \beta F^+ \sum_{n=1} (X\bar{X})^n (\mathcal{N} - \bar{\mathcal{N}}) F^+ \\
&= \beta F^+ \mathcal{N} F^+ + \beta F^+ \sum_{n=1} \lambda^{2n} \mathbf{f}^{-1} (\mathbf{\Delta M \bar{\Delta} \bar{M}})^n \mathbf{f} (\mathcal{N} - \bar{\mathcal{N}}) F^+ \\
&= \beta F^+ \mathcal{N} F^+ + \beta F^+ \sum_{n=1} \lambda^{2n} \mathbf{f}^{-1} (\mathbf{\Delta M \bar{\Delta} \bar{M}})^{n-1} \mathbf{\Delta M \bar{\Delta} \bar{M} f} (\mathcal{N} - \bar{\mathcal{N}}) F^+ \\
&= \beta F^+ \mathcal{N} F^+ + \beta F^+ \sum_{n=1} \lambda^{2n} \mathbf{f}^{-1} (\mathbf{\Delta M \bar{\Delta} \bar{M}})^{n-1} \mathbf{\Delta M \bar{\Delta} \bar{f}} (\mathcal{N} - \bar{\mathcal{N}}) F^+ \\
&= \beta F^+ \mathcal{N} F^+ - i\beta F^+ \mathbf{f}^{-1} \sum_{n=1} \lambda^{2n} (\mathbf{\Delta M \bar{\Delta} \bar{M}})^{n-1} \mathbf{\Delta M \bar{\Delta}} \mathbf{f}^{-1} F^+. \qquad \text{(B.8)}
\end{aligned}$$

Therefore the Lagrangian becomes

$$\begin{aligned}\mathcal{L} =& \beta F^+ \mathcal{N} F^+ - i\beta F^+ \mathbf{f}^{-1} \sum_{n=1} \lambda^{2n} (\Delta\mathbf{M}\bar{\Delta}\bar{\mathbf{M}})^{n-1} \Delta\mathbf{M}\bar{\Delta}\mathbf{f}^{-1} F^+ \\ &+ 2(\mathrm{Im}\alpha) F^+ \sum_{n=0} \lambda^{2n+1} \mathbf{f}^{-1} (\Delta\mathbf{M}\bar{\Delta}\bar{\mathbf{M}})^n \Delta\bar{\mathbf{f}}^{-1} F^- \\ &+ \bar{\beta} F^- \bar{\mathcal{N}} F^- + i\bar{\beta} F^- \bar{\mathbf{f}}^{-1} \sum_{n=1} \lambda^{2n} (\bar{\Delta}\bar{\mathbf{M}}\Delta\mathbf{M})^{n-1} \bar{\Delta}\bar{\mathbf{M}}\Delta\bar{\mathbf{f}}^{-1} F^- + \text{tot.div.}\end{aligned} \tag{B.9}$$

Let us recall the form of dual tensor G, and rewrite it with identities.

$$\begin{aligned}G^+ &= (1 - X\bar{X})^{-1} [X(\mathcal{N} - \bar{\mathcal{N}}) F^- + (\mathcal{N} - X\bar{X}\bar{\mathcal{N}}) F^+] \\ &= -i \sum_{n=0} \lambda^{2n+1} \mathbf{f}^{-1} (\Delta\mathbf{M}\bar{\Delta}\bar{\mathbf{M}})^n \Delta\bar{\mathbf{f}}^{-1} F^- + \mathcal{N} F^+ - i\mathbf{f}^{-1} \sum_{n=1} \lambda^{2n} (\Delta\mathbf{M}\bar{\Delta}\bar{\mathbf{M}})^{n-1} \Delta\mathbf{M}\bar{\Delta}\mathbf{f}^{-1} F^+.\end{aligned} \tag{B.10}$$

On the other hand,

$$G^+ = \frac{i}{2} \frac{\partial S}{\partial F^+} \tag{B.11}$$

$$\begin{aligned}&= i(\mathrm{Im}\alpha) \sum_{n=0} \lambda^{2n+1} \mathbf{f}^{-1} (\Delta\mathbf{M}\bar{\Delta}\bar{\mathbf{M}})^n \Delta\bar{\mathbf{f}}^{-1} F^- + i\beta \mathcal{N} F^+ \\ &\quad + \beta \mathbf{f}^{-1} \sum_{n=1} \lambda^{2n} (\Delta\mathbf{M}\bar{\Delta}\bar{\mathbf{M}})^{n-1} \Delta\mathbf{M}\bar{\Delta}\mathbf{f}^{-1} F^+.\end{aligned} \tag{B.12}$$

Note that we have performed partial integrals, lowering and raising operations in the above. Thus, by choosing $\alpha = -i$ and $\beta = -i$, we can reproduce the deformed dual tensor G from the action.

We would like to check the integrability condition (3.11) more carefully. As a simple example consider the first (i.e. $\mathcal{O}(\lambda^2)$) correction discussed in sec. 4, but dressed with scalars:

$$\mathcal{L} = -iF^+ X\bar{X}(\mathcal{N} - \bar{\mathcal{N}}) F^+ + \text{h.c.} = -i\lambda^2 F^+ \mathbf{f}^{-1} \Delta\mathbf{M}\bar{\Delta}\mathbf{f}^{-1} F^+ + \text{h.c.}. \tag{B.13}$$

Here

$$f^{\Sigma}{}_{AB} (\bar{f}^{-1})_{CD\Sigma} = M_{ABCD}. \tag{B.14}$$

From the identity, the conjugate to eq.(A.32) we get

$$(\bar{f}^{-1})_{CD\Sigma} = i(\mathcal{N} - \bar{\mathcal{N}})_{\Sigma\Lambda} f^{\Lambda}{}_{CD}, \tag{B.15}$$

and

$$M_{ABCD} = if^{\Sigma}{}_{AB} (\mathcal{N} - \bar{\mathcal{N}})_{\Sigma\Lambda} f^{\Lambda}{}_{CD}. \tag{B.16}$$

Then, the part of the action can be written as

$$iF^{+}(f^{-1})^{AB}\Delta f^{\Sigma}{}_{AB}(\mathcal{N}-\bar{\mathcal{N}})_{\Sigma\Lambda'}f^{\Lambda'}{}_{CD}\bar{\Delta}(f^{-1})^{CD}F^{+} \tag{B.17}$$

and

$$F^{+}(f^{-1})^{AB}\,\Delta\,M_{AB\,CD}\,\bar{\Delta}\,(f^{-1})^{CD}F^{+}. \tag{B.18}$$

Note that on an $\mathcal{H}$-invariant the covariant derivative is a simple one:

$$DS = dS. \tag{B.19}$$

For $S = K_{AB}\bar{K}^{AB}$ we find that

$$DK_{AB} = dK_{AB} + B_{[A}{}^{C}K_{CB]} \tag{B.20}$$

and

$$\bar{D}\bar{K}^{AB} = d\bar{K}_{AB} + \bar{B}^{[A}{}_{C}\bar{K}^{CB]}. \tag{B.21}$$

To agree with $DS = dS$ we need the $\mathcal{H}$-connection to be antihermitian

$$B = -B^{\dagger}. \tag{B.22}$$

Now we present (B.18) as follows

$$\tilde{F}^{AB}\,\overrightarrow{\Delta}\,\tilde{M}_{AB} \tag{B.23}$$

where we have defined

$$\tilde{M}_{AB} \equiv M_{AB\,CD}\,\bar{\Delta}\,(f^{-1})^{CD}F^{+} \qquad \tilde{F}^{AB} \equiv iF^{+}(f^{-1})^{AB} \tag{B.24}$$

since we are only interested in $\mathcal{H}$-covariant properties. We perform partial integration in (B.23) and use the fact that d becomes $-d$ and our Δ has 2 factors $d+B$, each becomes $-d+B^{T}$ to act to the left. We use the antihermitian property of B and replace it by $-d-\bar{B}$. Since Δ has 2 of these factors we find that

$$\tilde{F}^{AB}\,\overrightarrow{\Delta}\,\tilde{M}_{AB} = \tilde{F}^{AB}\,\overleftarrow{\bar{\Delta}}\,\tilde{M}_{AB}. \tag{B.25}$$

The action acquires a form

$$iF^{+}(f^{-1})^{AB}\overleftarrow{\bar{\Delta}}\,M_{AB\,CD}\,\overrightarrow{\bar{\Delta}}\,(f^{-1})^{CD}F^{+}. \tag{B.26}$$

This is a confirmation of a consistency condition at this level. In the linear approximation it gives a local amplitude which has at least 6 points

$$\langle h^{++}\,h^{++}\,h^{--}\,h^{--}\,v^{+}\,v^{+}\rangle + \text{h.c.} \tag{B.27}$$

and more. But is also seems to hint towards some kind of $U(1)$ anomaly

$$\langle h^{++}\, h^{++}\, h^{--}\, h^{--}\, v^{+}\, v^{+}\rangle - \text{h.c.}. \tag{B.28}$$

We know from [10] that the $U(1)$ subgroup in $H = U(5)$ in $\mathcal{N} = 5$ and $H = U(6)$ in $\mathcal{N} = 6$ are anomaly free, in $H = SU(8)$ in $\mathcal{N} = 8$ there is no $U(1)$ subgroup. Moreover, it was established more recently that there is no one-loop anomaly in $\mathcal{N} \geq 5$ supergravities.

C Examples of symplectic sections (f, h)

The action and supersymmetry rules of $\mathcal{N} = 5$ supergravity were given in [30]. The symplectic sections were presented in [31], and we refer to notations and details in [31]. The theory has 5 complex scalars z^i, and $\Lambda = ij$ and the symplectic section is:

$$f^{ij}{}_{AB} = \left(e_1 \delta^{ij}_{AB} + \frac{e_1}{2}\epsilon^{ijABm} z_m + 2e_2 \delta^{[A}_{[i} z^{B]} z_{j]}\right)\,, \qquad i = 1,2,3,4,5\,, \tag{C.1}$$

$$h_{ij|AB} = \mathcal{N}_{ij|mn} f^{mn}{}_{AB}, \tag{C.2}$$

$$\mathcal{N}_{ij|kl} = -\frac{i}{1-(z_m)^2}\left(\frac{1}{2}\left[1+(z_n)^2\right]\delta^{ij}_{kl} - \frac{1}{2}\epsilon^{ijklp} z_p - 2\delta_{[i\,[k} z_{l]} z_{j]}\right)\,, \tag{C.3}$$

$$h_{ij|AB} = -i\left[\frac{e_1}{2}\delta^{ij}_{AB} - \frac{e_1}{4}\epsilon^{ijABk} z_k + e_2 \delta^{[A}_{[i} z^{B]} z_{j]}\right]. \tag{C.4}$$

Here $e_1^2 \equiv \frac{1}{1-|z|^2}$, $e_2 \equiv \frac{1-e_1}{|z|^2}$.

The action and supersymmetry rules of $\mathcal{N} = 8$ supergravity were given in [27] and in [28]. Here we are using the ones in [28], in $SL(8, \mathbb{R})$-basis. The translation between between the symplectic formalism for extended supergravities reviewed in [26] and the original formulation of $\mathcal{N} = 8$ supergravity of [28], (including the more recent analysis of the gauge-fixing local $SU(8)$ in [29]), which was presented in [32].

The coset representative for $E_{7(7)}/SU(8)$ was parametrized in [28] as follows

$$\mathcal{V} = \begin{pmatrix} u_{ij}^{IJ} & v_{ijKL} \\ v^{klIJ} & u^{kl}_{KL} \end{pmatrix}. \tag{C.5}$$

The sub-matrices u and v carry indices of both $E_{7(7)}$ and $SU(8)$ ($I = 1, \ldots, 8$, $I = 1, \ldots, 8$) but one can choose a *suitable $SU(8)$ gauge for the fields, and then retain only manifest invariance with respect to the rigid diagonal subgroup of $E_{7(7)} \times SU(8)$, without distinction* *among the two types of indices.* Comparing the notation of [28] (in particular the appendix B) with the symplectic formalism of [1, 26], we can identify

$$\begin{cases} \phi_0 \equiv u & \qquad u_{ij}{}^{kl} = (P^{-1/2})_{ij}{}^{kl} \,, \\ \phi_1 \equiv v & \qquad v^{ijkl} = -(\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl} \end{cases}$$

so that

$$\begin{cases} f = \frac{1}{\sqrt{2}}(\phi_0 + \phi_1) = \frac{1}{\sqrt{2}}(u+v) \\ ih = \frac{1}{\sqrt{2}}(\phi_0 - \phi_1) = \frac{1}{\sqrt{2}}(u-v) \end{cases} . \tag{C.6}$$

Since sections are sub-matrices of the symplectic representation, relatively to electric and magnetic subgroups, their explicit indices components are given by

$$\begin{aligned} f_{ij}{}^{kl} &= \frac{1}{\sqrt{2}}\left((P^{-1/2})_{ij}{}^{kl} - (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}\right) \,, \\ h_{ij,kl} &= \frac{-i}{\sqrt{2}}\left((P^{-1/2})_{ij}{}^{kl} + (\bar{P}^{-1/2})^{ij}{}_{mn}\bar{y}^{mnkl}\right) \,, \end{aligned} \tag{C.7}$$

where, in matrix notation,

$$P = 1 - YY^\dagger \,, \qquad Y = B\frac{\tanh\sqrt{B^\dagger B}}{\sqrt{B^\dagger B}} \,, \qquad B_{ij,kl} = -\frac{1}{2\sqrt{2}}\phi_{ijkl} \,, \tag{C.8}$$

the last definition coming from the choice of the symmetric gauge for the coset representative in eq. (B.1) of [28]. If one defines

$$\tilde{P} = 1 - Y^\dagger Y \,, \tag{C.9}$$

and uses the identity

$$(\tilde{P}^{-1/2})Y^\dagger = Y^\dagger(P^{-1/2}) \,, \tag{C.10}$$

the following simple expressions for **f** and **h** are finally achieved:

$$f = \frac{1}{\sqrt{2}}\left[P^{-1/2} - (\tilde{P}^{-1/2})Y^\dagger\right] = \frac{1}{\sqrt{2}}[1 - Y^\dagger]\frac{1}{\sqrt{1-YY^\dagger}} \,, \tag{C.11}$$

$$h = -\frac{i}{\sqrt{2}}\left[P^{-1/2} + (\tilde{P}^{-1/2})Y^\dagger\right] = -\frac{i}{\sqrt{2}}[1 + Y^\dagger]\frac{1}{\sqrt{1-YY^\dagger}} \,. \tag{C.12}$$

The above notations are such that

$$\begin{aligned} P^{1/2} &= \sqrt{1-YY^\dagger} \qquad & P_{ij}{}^{kl} &= \delta_{ij}^{kl} - y_{ijmn}\bar{y}^{mnkl} \\ \tilde{P}^{1/2} &= \sqrt{1-Y^\dagger Y} \qquad & \bar{P}^{kl}{}_{ij} &= \delta_{ij}^{kl} - \bar{y}^{klmn}y_{mnij}. \end{aligned} \tag{C.13}$$

It is easily checked that the symplectic sections satisfy the usual relations

$$
\begin{aligned}
i(\mathbf{f}^\dagger \mathbf{h} - \mathbf{h}^\dagger \mathbf{f}) &= 1 \ , \\
\mathbf{h}^T \mathbf{f} - \mathbf{f}^T \mathbf{h} &= 0 \ .
\end{aligned}
\tag{C.14}
$$

These are obtained writing the symplectic sections as in (C.11) and (C.12), and using the identity

$$
Y \tilde{P}^{-1} = P^{-1} Y \ . \tag{C.15}
$$

The kinetic matrix is given in terms of the symplectic sections by [26]

$$
\mathcal{N} = \mathbf{h}\mathbf{f}^{-1} \ . \tag{C.16}
$$

Therefore, eqs. (C.11) and (C.12) yield

$$
\begin{aligned}
\mathcal{N} &= -i\,[1+Y^\dagger]\frac{1}{\sqrt{1-YY^\dagger}}\sqrt{1-YY^\dagger}\frac{1}{1-Y^\dagger} = \\
&= -i\,\frac{1+Y^\dagger}{1-Y^\dagger}
\end{aligned}
$$

or, component-by-component,

$$
\mathcal{N}_{ij|kl} = -i(\delta^{kl}_{mn} + \bar{y}^{mnkl})(\delta^{mn}_{ij} - \bar{y}^{ijmn})^{-1} \ . \tag{C.17}
$$

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