CONJUGATE COMPLEX HOMOGENEOUS SPACES WITH NON-ISOMORPHIC FUNDAMENTAL GROUPS

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ABSTRACT. Let $X = G/\Gamma$ be the quotient of a connected reductive algebraic $\mathbf{C}$-group G by a finite subgroup Γ. We describe the topological fundamental group of the homogeneous space X, which is nonabelian when Γ is nonabelian. Further, we construct an example of a homogeneous space X and an automorphism σ of $\mathbf{C}$ such that the topological fundamental groups of X and of the conjugate variety σX are not isomorphic.

RÉSUMÉ. *Espaces homogènes complexes conjugués avec groupes fondamentaux non isomorphes.* Soit $X = G/\Gamma$ le quotient d'un $\mathbf{C}$-groupe algébrique réductif connexe G par un sous-groupe fini Γ. On décrit le groupe fondamental topologique de l'espace homogène X, qui est non abélien quand Γ est non abélien. Puis on construit un exemple d'espace homogène X et d'automorphisme σ de $\mathbf{C}$ tels que les groupes fondamentaux topologiques de X et de la variété conjuguée σX ne sont pas isomorphes.

ABRIDGED FRENCH VERSION

Soit X une variété algébrique pointée définie sur le corps $\mathbf{C}$ des nombres complexes, supposée irréductible et quasi-projective. L'espace topologique pointé $X(\mathbf{C})$ est alors connexe ; on désigne par $\pi_1(X) := \pi_1^{\text{top}}(X(\mathbf{C}))$ son groupe fondamental, appelé groupe fondamental topologique de X. Soit σ un automorphisme du corps $\mathbf{C}$ (pas forcément continu). En appliquant σ aux coefficients des polynômes définissant X, on obtient une variété σX sur $\mathbf{C}$, dite variété conjuguée. Les complétés profinis des groupes $\pi_1(X)$ et $\pi_1(\sigma X)$ sont canoniquement isomorphes (comme groupes topologiques), car ils s'identifient naturellement au groupe fondamental étale de X. En revanche, les groupes $\pi_1(X)$ et $\pi_1(\sigma X)$ ne sont pas toujours isomorphes, par un résultat de Serre [Se]. Les exemples de Serre comprennent des surfaces projectives lisses. D'autres exemples ont été obtenus plus récemment : des variétés de Shimura dans [MS, R], et des surfaces projectives dans [BCG, GJ] pour des choix très généraux de l'automorphisme σ (dans [GJ] pour tout σ dont la restriction à $\overline{\mathbf{Q}}$ diffère de l'identité et de la conjugaison complexe).

Dans cette note, nous donnons un exemple des *espaces homogènes* conjugués avec groupes fondamentaux topologiques non isomorphes. Le plan de la note est le suivant. Nous considérons, dans le §2, les groupes fondamentaux de certains espaces homogènes topologiques de la forme G/Γ, où G est un groupe de Lie réel connexe et $\Gamma \subset G$ est un sous-groupe discret. Nous en déduisons, dans le §3, une formule explicite pour décrire

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le groupe fondamental $\pi_1(G/\Gamma)$ dans le cas où G est un groupe algébrique linéaire connexe défini sur $\mathbf{C}$, et Γ est un sous-groupe fini de G. En utilisant cette formule, nous construisons dans le §4 un exemple d'espace homogène affine $X = G/\Gamma$ défini sur $\mathbf{C}$ et un automorphisme σ de $\mathbf{C}$ tels que les groupes fondamentaux topologiques $\pi_1(\sigma X)$ et $\pi_1(X)$ ne sont pas isomorphes. Précisément, on choisit $G = \mathrm{SL}(n,\mathbf{C}) \times \mathbf{C}^*$ avec $n \geq 5$, et Γ un sous-groupe non abélien fini d'ordre 55. L'inclusion de Γ dans G est donnée par un plongement arbitraire de Γ dans $\mathrm{SL}(n,\mathbf{C})$ et par un homomorphisme non trivial de Γ dans $\mathbf{C}^*$. Notre formule permet de vérifier que $\pi_1(X)$ et isomorphe à $(\mathbf{Z}/11\mathbf{Z}) \rtimes_4 \mathbf{Z}$, où la notation signifie que le générateur 1 de $\mathbf{Z}$ agit sur $\mathbf{Z}/11\mathbf{Z}$ par multiplication par 4, tandis que pour σ envoyant $\zeta = \exp 2\pi i/5$ sur ζ^2, le groupe fondamental $\pi_1(\sigma X)$ de la variété conjuguée est isomorphe à $(\mathbf{Z}/11\mathbf{Z}) \rtimes_9 \mathbf{Z}$. Un argument simple permet de vérifier que ces deux groupes ne sont pas isomorphes.

1. Introduction

Let X be a pointed algebraic variety defined over $\mathbf{C}$. We assume that X is irreducible and quasi-projective. The pointed topological space $X(\mathbf{C})$ is then connected, and we denote by $\pi_1(X)$ the topological fundamental group of $X(\mathbf{C})$, i.e., $\pi_1(X) := \pi_1^{\mathrm{top}}(X(\mathbf{C}))$. Let σ be a field automorphism of $\mathbf{C}$, not necessarily continuous. On applying σ to the coefficients of the polynomials defining X, we obtain a conjugate algebraic variety σX over $\mathbf{C}$. Though the profinite completions of $\pi_1(X)$ and $\pi_1(\sigma X)$ are isomorphic, the groups $\pi_1(X)$ and $\pi_1(\sigma X)$ themselves are not necessarily isomorphic. Serre [Se] obtained the first examples of conjugate varieties X and σX with $\pi_1(\sigma X) \not\simeq \pi_1(X)$. Serre's examples include smooth projective surfaces. More examples were obtained recently: Shimura varieties in [MS] and [R], and smooth projective surfaces in [BCG] and [GJ] for a very general choice of σ (in [GJ] for any σ whose restriction to $\overline{\mathbf{Q}}$ differs from the identity and the complex conjugation).

In this note we give an example of conjugate *homogeneous spaces* with non-isomorphic topological fundamental groups. The outline of the note is as follows. In Section 2 we consider topological homogenous spaces of the form G/Γ, where G is a connected real Lie group and $\Gamma \subset G$ is a discrete subgroup. In Section 3 we write an explicit formula for $\pi_1(G/\Gamma)$ when G is a complex linear algebraic group and $\Gamma \subset G$ is a finite subgroup. In Section 4 using this formula we construct an example of an affine homogeneous space $X = G/\Gamma$ over $\mathbf{C}$ and an automorphism σ of $\mathbf{C}$ such that $\pi_1(\sigma X)$ is not isomorphic to $\pi_1(X)$. In our example $G = \mathrm{SL}(n,\mathbf{C}) \times \mathbf{C}^*$ with $n \geq 5$, and Γ is a nonabelian finite subgroup of order 55.

2. The quotient of a Lie group by a discrete subgroup

Let

$$1 \to S \xrightarrow{i} G \xrightarrow{\tau} T \to 1$$

be a short exact sequence of connected real Lie groups. Let $\Gamma \subset G$ be a discrete subgroup such that the projection $\Lambda = \tau(\Gamma) \subset T$ is discrete. Our goal is to describe $\pi_1(G/\Gamma)$, where G/Γ is viewed as a pointed manifold with base point the image of 1.

Set $\Gamma_S = \Gamma \cap S$. The homomorphism $\tau\colon G \to T$ induces a fibration $G/\Gamma \to T/\Lambda$ with fiber S/Γ_S, which gives rise to an exact sequence in homotopy groups

$$\pi_1(S/\Gamma_S) \xrightarrow{i_*} \pi_1(G/\Gamma) \xrightarrow{\tau_*} \pi_1(T/\Lambda) \to 1.$$

The fibration $G \to G/\Gamma$ with fiber Γ gives rise to an exact sequence in homotopy groups

$$1 \to \pi_1(G) \to \pi_1(G/\Gamma) \xrightarrow{f} \Gamma \to 1,$$

where f is a homomorphism by Lemma 2.2 below. Considering the above fibrations and also the fibrations $S \to S/\Gamma_S$, $T \to T/\Lambda$ and $G \to T$, we obtain the following commutative diagram of groups and homomorphisms with exact rows and columns:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S) \arrow[r] \arrow[d] & \pi_1(S/\Gamma_S) \arrow[r] \arrow[d, "i_*"] & \Gamma_S \arrow[r] \arrow[d, "i"] & 1 \\
1 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d] & \pi_1(G/\Gamma) \arrow[r, "f"] \arrow[d, "\tau_*"] & \Gamma \arrow[r] \arrow[d, "\tau"] & 1 \\
1 \arrow[r] & \pi_1(T) \arrow[r] \arrow[d] & \pi_1(T/\Lambda) \arrow[r, "f_T"] \arrow[d] & \Lambda \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

From this diagram we obtain homomorphisms

$$\chi\colon \pi_1(S) \to \pi_1(S/\Gamma_S) \xrightarrow{i_*} \pi_1(G/\Gamma) \quad \text{and} \quad \phi\colon \pi_1(G/\Gamma) \to \pi_1(T/\Lambda) \underset{\Lambda}{\times} \Gamma,$$

where the fiber product $\pi_1(T/\Lambda) \times_\Lambda \Gamma$ is the group of pairs $(x,\gamma) \in \pi_1(T/\Lambda) \times \Gamma$ such that $f_T(x) = \tau(\gamma)$. The homomorphism ϕ takes $y \in \pi_1(G/\Gamma)$ to the pair $(\tau_*(y), f(y)) \in \pi_1(T/\Lambda) \times_\Lambda \Gamma$.

Theorem 2.1. *With the above notation, the sequence*

$$\pi_1(S) \xrightarrow{\chi} \pi_1(G/\Gamma) \xrightarrow{\phi} \pi_1(T/\Lambda) \underset{\Lambda}{\times} \Gamma \to 1$$

is exact. In particular, if S is simply connected, then ϕ is an isomorphism.

Proof. We prove the theorem by diagram chasing. Clearly $\phi \circ \chi = 1$. We show that $\ker\phi \subset \operatorname{im}\chi$. Let $y \in \ker\phi \subset \pi_1(G/\Gamma)$, then $f(y) = 1$ and $\tau_*(y) = 1$. Then y comes from some element $z \in \pi_1(G)$, whose image in $\pi_1(T)$ is 1. Hence z comes from some element $u \in \pi_1(S)$. We see that $y = \chi(u)$, as required.

We show that ϕ is surjective. Let $(x,\gamma) \in \pi_1(T/\Lambda) \times_\Lambda \Gamma$, i.e., $x \in \pi_1(T/\Lambda)$, $\gamma \in \Gamma$, and $f_T(x) = \tau(\gamma)$. We can lift x to some element $y \in \pi_1(G/\Gamma)$, then $\tau(f(y)) = \tau(\gamma)$. Set $z = f(y)\gamma^{-1}$, then $\tau(z) = 1$, hence z comes from some element of Γ_S and from some element u of $\pi_1(S/\Gamma_S)$. Set $y' = i_*(u)^{-1}y \in \pi_1(G/\Gamma)$, then $f(y') = \gamma$ and $\tau_*(y') = \tau_*(y) = x$. We see that $(x,\gamma) = \phi(y')$, as required. □

The following lemma, which we used above, is well-known.

Lemma 2.2. *Let G be a connected Lie group, $\Gamma \subset G$ be a (closed) Lie subgroup, not necessarily connected. Then the connecting map $f\colon \pi_1(G/\Gamma) \to \pi_0(\Gamma)$ in the exact sequence*

$$\pi_1(\Gamma) \to \pi_1(G) \to \pi_1(G/\Gamma) \xrightarrow{f} \pi_0(\Gamma) \to 1$$

is a homomorphism.

Proof. Denote by $\lambda : \Gamma \to \pi_0(\Gamma)$ the canonical epimorphism. Consider two based loops $\theta_i\colon [0,1] \to G/\Gamma$ in G/Γ ($i = 1,2$). Let $\tilde{\theta}_i\colon [0,1] \to G$ be a path lifting the loop θ_i to G with $\tilde{\theta}_i(0) = 1$, and set $\gamma_i = \tilde{\theta}_i(1) \in \Gamma$. By definition $f([\theta_i]) = \lambda(\gamma_i) \in \pi_0(\Gamma)$, where $[\theta_i]$ denotes the class of the based loop θ_i in $\pi_1(G/\Gamma)$. Then $\gamma_1\tilde{\theta}_2$ is a path in G from γ_1 to $\gamma_1\gamma_2$ mapping in G/Γ to the loop θ_2, hence the concatenation of $\tilde{\theta}_1$ and $\gamma_1\tilde{\theta}_2$ is a path in G from 1 to $\gamma_1\gamma_2$ mapping in G/Γ to the loop obtained by concatenation of θ_1 and θ_2. Thus $f([\theta_1]\cdot[\theta_2]) = \lambda(\gamma_1\gamma_2) = \lambda(\gamma_1)\,\lambda(\gamma_2) = f([\theta_1])\,f([\theta_2])$, as required. □

3. THE QUOTIENT OF A COMPLEX ALGEBRAIC GROUP BY A FINITE SUBGROUP

Let G be a connected linear algebraic group over $\mathbf{C}$. Let $\Gamma \subset G$ be a finite subgroup. Set $X = G/\Gamma$. We wish to compute the topological fundamental group $\pi_1(X)$.

Let U denote the unipotent radical of G, then $G' := G/U$ is reductive. The canonical epimorphism $\rho\colon G \to G'$ induces a fibration $G/\Gamma \to G'/\Gamma'$ with fiber U, where $\Gamma' = \rho(\Gamma)$, and hence, the induced homomorphism $\rho_*\colon \pi_1(G/\Gamma) \to \pi_1(G'/\Gamma')$ is an isomorphism. Therefore, we may and shall assume that G is reductive. Replacing the reductive group G by a finite cover and Γ by its inverse image, we may and shall assume that the semisimple group $S := [G,G]$ is simply connected. Let Λ denote the image of Γ in the algebraic torus $T := G/S$, then T/Λ is also an algebraic torus, hence $\pi_1(T/\Lambda)$ is a free abelian group isomorphic to $\mathbf{Z}^{\dim T}$. The next corollary, which follows immediately from Theorem 2.1, describes $\pi_1(G/\Gamma)$ in terms of Γ and the free abelian group $\pi_1(T/\Lambda)$.

Corollary 3.1. *Let G be a connected reductive algebraic group over $\mathbf{C}$ such that the commutator subgroup S of G is simply connected. Set $T = G/S$. Let $\Gamma \subset G$ be a finite subgroup, and let Λ denote the image of Γ in T. Then there is a canonical isomorphism*

$$\pi_1(G/\Gamma) \xrightarrow{\sim} \pi_1(T/\Lambda) \underset{\Lambda}{\times} \Gamma,$$

where $\pi_1(T/\Lambda) \times_\Lambda \Gamma$ is the fiber product with respect to the epimorphism $\pi_1(T/\Lambda) \to \Lambda$ of Lemma 2.2 and the canonical epimorphism $\Gamma \to \Lambda$.

4. EXAMPLE

Let $A = \mathbf{Z}/m\mathbf{Z}$, the additive group of residues modulo m. Let $B \subset (\mathbf{Z}/m\mathbf{Z})^*$ be a *cyclic* subgroup of some order r in the multiplicative group of invertible residues modulo m. The group B acts naturally on A by multiplication: an element $b \in B \subset (\mathbf{Z}/m\mathbf{Z})^*$ acts by $a \mapsto ba$. Set

$$H = A \rtimes B$$

(the semidirect product). We regard B as a subgroup of H. Consider an embedding $\varphi\colon B \hookrightarrow \mathbf{C}^*$, then $\varphi(B) = \mu_r \subset \mathbf{C}^*$, the group of r-th roots of unity.

Choose an embedding $\alpha\colon H \hookrightarrow \mathrm{SL}(n,\mathbf{C})$ for some natural number n. Set

$$G = \mathrm{SL}(n,\mathbf{C}) \times \mathbf{C}^*.$$

For $(a,b) \in A \rtimes B = H$ set

$$\psi(a,b) = (\alpha(a,b), \varphi(b)) \in \mathrm{SL}(n,\mathbf{C}) \times \mathbf{C}^*.$$

We obtain an embedding $\psi = \psi_{\alpha,\varphi}\colon H \hookrightarrow G$. Set $\Gamma = \psi(H)$, $X = X_{\alpha,\varphi} = G/\Gamma$. Then X is an affine algebraic variety over $\mathbf{C}$.

Let $b \in B$. Write $A \rtimes_b \mathbf{Z}$ for the semidirect product of A and $\mathbf{Z}$, where the generator 1 of $\mathbf{Z}$ acts on A by multiplication by b. Set $\zeta = \exp 2\pi i/r \in \mu_r$.

Proposition 4.1. $\pi_1(X_{\alpha,\varphi}) \simeq (\mathbf{Z}/m\mathbf{Z}) \rtimes_{\varphi^{-1}(\zeta)} \mathbf{Z}$.

Proof. Set $S = \mathrm{SL}(m,\mathbf{C})$, $T = \mathbf{C}^*$. Let $\tau\colon G = S \times T \to T$ denote the projection, then $\tau(\psi(a,b)) = \varphi(b)$ for $(a,b) \in A \rtimes B = H$. Set $\Lambda = \tau(\Gamma) = \tau(\psi(H)) \subset T$, then $\Lambda = \varphi(B) = \mu_r \subset \mathbf{C}^* = T$.

Consider the following universal covering of $T = \mathbf{C}^*$:

$$\varepsilon\colon \mathbf{C} \to \mathbf{C}^* = T, \quad z \mapsto \exp 2\pi i z \ \text{ for } \ z \in \mathbf{C},$$

it induces a universal covering of T/Λ:

$$\mathbf{C} \xrightarrow{\varepsilon} \mathbf{C}^* \to \mathbf{C}^*/\mu_r = T/\Lambda \simeq \mathbf{C}^*.$$

We identify $\pi_1(T/\Lambda)$ with $\varepsilon^{-1}(\mu_r) = \frac{1}{r}\mathbf{Z} \subset \mathbf{C}$, then the homomorphism $\pi_1(T/\Lambda) \to \Lambda = \mu_r$ of Lemma 2.2 is the restriction of ε to $\frac{1}{r}\mathbf{Z}$, hence it takes the generator $\frac{1}{r} \in \frac{1}{r}\mathbf{Z} = \pi_1(T/\Lambda)$ to the element $\varepsilon(\frac{1}{r}) = \zeta \in \mu_r$.

Since $S = \mathrm{SL}(n,\mathbf{C})$ is simply connected, by Corollary 3.1 we have

$$\pi_1(X_{\alpha,\varphi}) = \pi_1(G/\Gamma) = \pi_1(T/\Lambda) \underset{\Lambda}{\times} \Gamma \simeq \tfrac{1}{r}\mathbf{Z} \underset{\mu_r}{\times} H,$$

where the homomorphism $\frac{1}{r}\mathbf{Z} \to \mu_r$ takes $\frac{1}{r}$ to ζ and the homomorphism $H \to \mu_r$ takes $(a,b) \in H$ to $\tau(\psi(a,b)) = \varphi(b)$. Since $\frac{1}{r}\mathbf{Z}$ is a free abelian group, the group extension

$$1 \to \{0\} \times A \to \tfrac{1}{r}\mathbf{Z} \underset{\mu_r}{\times} H \to \tfrac{1}{r}\mathbf{Z} \to 1$$

splits, hence $\pi_1(X_{\alpha,\varphi}) \simeq A \rtimes \frac{1}{r}\mathbf{Z}$. The action of $\frac{1}{r}\mathbf{Z}$ on A in this semidirect product decomposition is the canonical action of the quotient group $\frac{1}{r}\mathbf{Z}$ of $\frac{1}{r}\mathbf{Z}\times_{\mu_r} H$ on the normal abelian subgroup A. Since the element $\frac{1}{r} \in \frac{1}{r}\mathbf{Z}$ has image ζ in μ_r, which lifts to $\varphi^{-1}(\zeta) \in B \subset H$, we see that $\frac{1}{r} \in \frac{1}{r}\mathbf{Z}$ lifts to $(\frac{1}{r}, \varphi^{-1}(\zeta)) \in \frac{1}{r}\mathbf{Z}\times_{\mu_r} B \subset \frac{1}{r}\mathbf{Z}\times_{\mu_r} H$, hence $\frac{1}{r}$ acts as $\varphi^{-1}(\zeta)$ on A. Identifying $\frac{1}{r}\mathbf{Z}$ with $\mathbf{Z}$ via $x \mapsto rx$ for $x \in \frac{1}{r}\mathbf{Z}$, we obtain the assertion of the proposition. $\square$

Now let us take $m = 11$, then $A = \mathbf{Z}/11\mathbf{Z}$. We take $B = (\mathbf{Z}/11\mathbf{Z})^{*2}$, the group of nonzero quadratic residues modulo 11. The group B is a cyclic group of order 5, namely, $B = \{\bar{1},\bar{4},\bar{9},\bar{5},\bar{3}\}$. Then $H = A \rtimes B$ is a finite nonabelian group of order 55. Let $n \geq 5$, then there exists an embedding $\alpha\colon H \hookrightarrow \mathrm{SL}(n,\mathbf{C})$. For $b \in B$, $b \neq \bar{1}$, let φ_b denote the embedding $B \hookrightarrow \mathbf{C}^*$ taking the generator b of B to ζ, then $\varphi_b^{-1}(\zeta) = b$. We write $X_{\alpha,b}$ for X_{α,φ_b}. Let σ be any field automorphism of $\mathbf{C}$ taking ζ to ζ^2. Consider the conjugate variety $\sigma X_{\alpha,b}$.

Theorem 4.2. *For $A = \mathbf{Z}/11\mathbf{Z}$, $B = (\mathbf{Z}/11\mathbf{Z})^{*2}$, $\sigma \in \mathrm{Aut}(\mathbf{C})$ taking ζ to ζ^2, the groups $\pi_1(X_{\alpha,4})$ and $\pi_1(\sigma X_{\alpha,4})$ are not isomorphic.*

Proof. We have $\sigma(\zeta) = \zeta^2$. The homomorphism $\sigma \circ \varphi\colon B \to \mathbf{C}^*$ takes $\bar{4}$ to $\sigma(\zeta) = \zeta^2$, hence it takes $\bar{4}^3 = \bar{9}$ to $(\zeta^2)^3 = \zeta$. Thus $\sigma \circ \varphi_4 = \varphi_9$.

For our group G defined over $\mathbf{Q}$ and for $X = G/\Gamma$, we have $\sigma X = G/\sigma(\Gamma)$, where σ acts on $\mathrm{SL}(n,\mathbf{C})$ and on $\mathbf{C}^*$ via the action on $\mathbf{C}$. For an embedding $\varphi\colon B \hookrightarrow \mathbf{C}^*$ we have

$$\sigma X_{\alpha,\varphi} = G/(\sigma \circ \psi_{\alpha,\varphi})(H) = G/\psi_{\sigma\circ\alpha,\sigma\circ\varphi}(H) = X_{\sigma\circ\alpha,\sigma\circ\varphi}\,.$$

We obtain that

$$\sigma X_{\alpha,4} = \sigma X_{\alpha,\varphi_4} = X_{\sigma\circ\alpha,\sigma\circ\varphi_4} = X_{\sigma\circ\alpha,\varphi_9} = X_{\sigma\circ\alpha,9}.$$

By Proposition 4.1 we have

$$\pi_1(X_{\alpha,b}) \simeq (\mathbf{Z}/11\mathbf{Z}) \rtimes_b \mathbf{Z},$$

hence

$$\pi_1(X_{\alpha,4}) \simeq (\mathbf{Z}/11\mathbf{Z}) \rtimes_4 \mathbf{Z} \quad \text{and} \quad \pi_1(\sigma X_{\alpha,4}) = \pi_1(X_{\sigma\circ\alpha,9}) \simeq (\mathbf{Z}/11\mathbf{Z}) \rtimes_9 \mathbf{Z}.$$

Now the theorem follows from the next Lemma 4.3. $\square$

Lemma 4.3. $(\mathbf{Z}/11\mathbf{Z}) \rtimes_4 \mathbf{Z} \not\simeq (\mathbf{Z}/11\mathbf{Z}) \rtimes_9 \mathbf{Z}$.

We first need the following group-theoretic fact.

Lemma 4.4. *Let A be any group without nonzero homomorphisms into $\mathbf{Z}$. When κ is an automorphism of A, we write $A \rtimes_\kappa \mathbf{Z}$ for the semidirect product of A and $\mathbf{Z}$, where the generator $t = 1$ of $\mathbf{Z}$ acts on A by κ. Fix two automorphisms $\kappa_1, \kappa_2 \in \mathrm{Aut}(A)$, and denote by $\overline{\kappa}_i$ the image of κ_i in the group of outer automorphisms $\mathrm{Out}(A)$. If $\overline{\kappa}_1$ is* *conjugate to neither $\overline{\kappa}_2$ nor $\overline{\kappa}_2^{-1}$ in* $\mathrm{Out}(A)$*, then the semidirect products $G_1 = A \rtimes_{\kappa_1} \mathbf{Z}$ and $G_2 = A \rtimes_{\kappa_2} \mathbf{Z}$ are not isomorphic.*

Proof. By contraposition, let $\lambda\colon G_1 \xrightarrow{\sim} G_2$ be an isomorphism. Since for each of $i = 1,2$, the subgroup A is equal to the kernel of some/any nonzero homomorphism $G_i \to \mathbf{Z}$, we have $\lambda(A) = A$. Let $\kappa \in \mathrm{Aut}(A)$ denote the restriction of λ to A. For the generator $t \in \mathbf{Z} \subset G_1$, write $\lambda(t)$ as $at^e \in G_2$ with $a \in A$ and $e \in \mathbf{Z}$. Since t generates G_1/A, we see that $\lambda(t)$ generates G_2/A and hence $e = \pm 1$. Then for all $a' \in A$, writing $\gamma_a(a') = aa'a^{-1}$ we have

$$\kappa(\kappa_1(a')) = \lambda(ta't^{-1}) = at^e\kappa(a')t^{-e}a^{-1} = \gamma_a(\kappa_2^e(\kappa(a'))),$$

whence $\kappa_1 = \kappa^{-1}\gamma_a\kappa_2^e\kappa$. Hence $\overline{\kappa}_1 = \overline{\kappa}^{-1}\ \overline{\kappa}_2^e\ \overline{\kappa}$. Thus $\overline{\kappa}_1$ and $\overline{\kappa}_2^e$ are conjugate in $\mathrm{Out}(A)$. ☐

Proof of Lemma 4.3. We use Lemma 4.4 when $A = \mathbf{Z}/11\mathbf{Z}$, in which case $\mathrm{Aut}(A) = \mathrm{Out}(A)$, is abelian and can be identified with $(\mathbf{Z}/11\mathbf{Z})^*$. Hence the assumption of Lemma 4.4 in this case is just that κ_1 and $\kappa_2^{\pm 1}$ are distinct as elements of $(\mathbf{Z}/11\mathbf{Z})^*$. Here $\kappa_1 = \overline{4}$ and $\kappa_2 = \overline{9}$. Since modulo 11 we have $\overline{9} \neq \overline{4}$ and $\overline{9}^{-1} = \overline{5} \neq \overline{4}$, Lemma 4.4 applies and we see that $(\mathbf{Z}/11\mathbf{Z}) \rtimes_4 \mathbf{Z} \not\simeq (\mathbf{Z}/11\mathbf{Z}) \rtimes_9 \mathbf{Z}$. This completes the proofs of Lemma 4.3 and Theorem 4.2. ☐

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