

SEMI-INFINITE PLÜCKER RELATIONS AND WEYL MODULES

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ABSTRACT. The goal of this paper is twofold. First, we write down the semi-infinite Plücker relations, describing the Drinfeld-Plücker embedding of the (formal version of) semi-infinite flag varieties in type A. Second, we study the homogeneous coordinate ring, i.e. the quotient by the ideal generated by the semi-infinite Plücker relations. We establish the isomorphism with the algebra of dual global Weyl modules and derive a new character formula.

INTRODUCTION

The goal of this paper is to study the homogeneous coordinate ring of the formal version of the semi-infinite flag varieties of type A (see [FiMi]). More precisely, we consider the Drinfeld–Plücker embedding of the semi-infinite flag variety, compute explicitly the reduced scheme structure and study the quotient algebra with respect to the ideal of defining relations. Before giving more detailed summary of our results, we describe the finite-dimensional analogue of the story (see e.g. [F]).

Let $G = SL_n = SL_n(\mathbb{K})$ (\mathbb{K} is an algebraically closed field of characteristic 0) and let $B \subset G$ be a Borel subgroup. The quotient G/B is known to be isomorphic to the variety of complete flags in an n -dimensional vector space. The Plücker embedding realizes the flag variety inside the product of projective spaces of all fundamental representations of SL_n , which are isomorphic to the wedge powers of the vector representation in type A . In particular, the coordinates X_I on the k -th fundamental representation are labeled by the cardinality k subsets of the set $\{1, \dots, n\}$. The (quadratic) Plücker relations describe the image of this embedding, i.e. they generate the ideal J_n of all multi-homogeneous polynomials vanishing on the image of the Plücker embedding. By definition, J_n is the ideal of relations satisfied by the minors of the matrices from SL_n . A very important property of J_n is that the quotient of the polynomial ring R_n in variables X_I modulo the ideal J_n is isomorphic to the direct sum of (dual) irreducible finite-dimensional representations of SL_n . Finally, let us mention that there exists a remarkable basis of R_n consisting of monomials in X_I . The monomials in this basis are parametrized by the semi-standard tableaux.

Our goal is to generalize the finite-dimensional picture to the semi-infinite settings [FiMi, BF1, BF2, Kat]. This means that the group G is replaced with the group $G[[z]]$ and all the representations V are replaced with the

infinite-dimensional spaces $V[[z]] = V \otimes \mathbb{K}[[z]]$. The main geometric object \mathbf{Q} we are interested in is defined as follows. Let $V(\omega_k)$, $k = 1, \dots, n-1$ be the fundamental irreducible modules of SL_n . Let $\mathring{\mathbf{Q}} \subset \prod_{k=1}^{n-1} \mathbb{P}(V(\omega_k)[[z]])$ be the $G[[z]]$ -orbit through the product of highest weight lines (with the reduced scheme structure). Then the scheme \mathbf{Q} is defined as the Zariski closure of $\mathring{\mathbf{Q}}$ inside the product of projective spaces. We note that the (formal) Drinfeld-Plücker data (see [FiMi]) defines the (non-reduced) scheme with the same support (a.k.a. the projectivization of the arc scheme of the closure of the basic affine scheme). Our main results are as follows:

- we determine the reduced scheme structure of \mathbf{Q} , i.e. we write down the generators of the ideal of relations of $\mathring{\mathbf{Q}}$ (Proposition 2.9);
- we derive a new explicit formula for the characters of the Weyl modules (Theorem 3.21);
- we show that the homogeneous coordinate ring of \mathbf{Q} is isomorphic to the direct sum of dual global Weyl modules (Corollary 3.22);
- we give a monomial basis of the homogeneous coordinate ring; the labeling set for the monomials extends the semi-standard tableaux (Corollary 2.12).

Our main players are two algebras. The first algebra \mathcal{M} is generated by the coefficients of minors of the elements of the group $G[[z]]$ (to be precise, we consider the minors supported on the first k rows for some k ; each such a minor is a Taylor series in z and \mathcal{M} is generated by all the coefficients of these series). The second algebra \mathbb{W} is defined as the direct sum of dual global Weyl modules for all dominant integral weights (see [Kat]). Our strategy is as follows: we first write down an explicit set of semi-infinite Plücker relations for the algebra \mathcal{M} , then show that \mathcal{M} surjects onto \mathbb{W} and finally prove that the character of \mathbb{W} coincides with the character of the quotient by the ideal generated by the semi-infinite Plücker relations.

Let us add several comments on our results. First, we recall that the classical Plücker relations are linear combinations of the quadratic monomials of the form $X_{I_1} X_{I_2}$. We note that the coordinates on the semi-infinite space $V(\omega_k)[[z]]$ are of the form $X_I^{(l)}$, where I is a cardinality k subset of $\{1, \dots, n\}$ and l is a non-negative integer. To a variable X_I we attach the generating function $X_I(s) = \sum_{l \geq 0} X_I^{(l)} s^l$. Then a part of semi-infinite Plücker relations can be obtained in the following way: one takes a classical Plücker relation, replaces each variable X_I with the generating function $X_I(s)$ and collect all coefficients of the resulting formal series in the variable s (this is exactly what one is doing in order to pass to the arc scheme, see e.g. [Mu]). Then these coefficients belong to the ideal vanishing on $\mathring{\mathbf{Q}}$. However, we show that these relations do not generate the whole vanishing ideal. The complete list of generators is given in Theorem 2.3.

Second, recall that in the classical theory an important role is played by the semi-standard Young tableaux. In short, a tableau is semi-standard if its columns increase from left to right (the longer columns are on the left).

To compare two columns one compares the entries belonging to one row: an entry of the smaller column can not exceed the corresponding entry of the larger one (for each row). In particular, this order is partial and there are many uncomparable pairs of columns. We introduce a measure $k(\sigma, \tau)$ of how much uncomparable the columns σ and τ are. Roughly speaking, the $k(\sigma, \tau)$ is equal to the number of times the signs " $<$ " and " $>$ " got changed when one compares the entries of σ and τ row by row moving from bottom to top. The complete definition is given in Definition 2.6. The quantity $k(\sigma, \tau)$ is used in our description of the semi-infinite Plücker relations as well as in the character formula for the Weyl modules.

Third, let us comment on the character formula for the global Weyl modules. The characters of global (infinite-dimensional) and local (finite-dimensional) Weyl modules differ by a simple factor ([CFK, N]). We derive a character formula for the local Weyl modules for the Lie algebra \mathfrak{sl}_n by embedding it into the Weyl module for \mathfrak{sl}_{2n} . We give a construction of a basis of this embedded module and compute its character. The character of the local Weyl module $W(\mu)$ coincides with the specialization of nonsymmetric Macdonald polynomial $E_{w_0(\mu)}(X, q, 0)$ (see [M, Ch, N, S, I, CI]). Thus we obtain a combinatorial formula for such specializations for arbitrary dominant weight μ . Note that in [FM] we specialized the combinatorial formula of Haglund, Haiman and Loehr (see [HHL]) and gave a recurrent formula for the nonsymmetric Macdonald polynomials at $t = 0$. However, we were able to obtain an explicit character formula only for the Weyl modules $W(\mu)$, where μ is a linear combination of ω_1 and ω_{n-1} .

Finally let us add a remark due to Michael Finkelberg. In the works [BF1, BF2, BF3], the authors study the rings of functions on zastava schemes (for the curve \mathbb{A}^1) and the spaces of sections of line bundles over quasimaps' schemes (for the curve \mathbb{P}^1); in particular, the characters of these spaces are computed. The results of this paper imply that these zastava and quasimaps as schemes representing the corresponding moduli problems, are *non reduced* (for $G = SL(n)$, $n \geq 5$). The zastava and quasimaps' spaces studied in [BF1, BF2, BF3] are defined as the corresponding *reduced* schemes, i.e. *varieties*. In particular, the character formulas of [BF1, BF2, BF3] hold for the spaces of sections over the *reduced* zastava and quasimaps' varieties.

The paper is organized as follows. In Section 1 we collect the notation and give generalities on the representation theoretic, combinatorial and geometric structures we use in the paper. In Section 2 we write down the generators of the ideal of relations for the semi-infinite flag varieties; as a consequence we derive an estimate for the character of the global Weyl modules. In Section 3 we prove that formula from Section 2 for the characters of the Weyl modules holds and thus finalize the proof. In Appendix A we describe the whole story for the simplest symplectic algebra \mathfrak{sp}_4 .

1. GENERALITIES

1.1. Finite-dimensional picture. Let \mathfrak{g} be a simple finite dimensional Lie algebra over an algebraically closed field \mathbb{K} of characteristic 0. Let Δ be the root system of \mathfrak{g} , $\Delta = \Delta_+ \sqcup \Delta_-$ the union of positive and negative roots. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the Cartan decomposition of \mathfrak{g} . For a positive root α let $e_\alpha \in \mathfrak{n}_+$ and $f_{-\alpha} \in \mathfrak{n}_-$ be the Chevalley generators. The weight lattice X contains the positive part X_+ , containing all fundamental weights ω_k , $k = 1, \dots, \text{rk}(\mathfrak{g})$. For $\lambda \in X_+$ we denote by $V(\lambda)$ the irreducible highest weight \mathfrak{g} -module with highest weight λ .

For any two weights $\lambda, \mu \in X_+$ there is an embedding

$$V(\lambda + \mu) \hookrightarrow V(\lambda) \otimes V(\mu).$$

Dualizing this injection we obtain the surjection:

$$V(\lambda)^* \otimes V(\mu)^* \twoheadrightarrow V(\lambda + \mu)^*.$$

These surjections define a multiplication on the space $\mathbb{V} = \bigoplus_{\lambda \in X_+} V(\lambda)^*$ and make \mathbb{V} into an associative and commutative algebra.

Take the corresponding algebraic groups $G \supset B \supset U$, where B is a Borel subgroup and N is a unipotent subgroup. Then we have:

$$G/U \simeq \text{Spec} \mathbb{V}.$$

The algebra \mathbb{V} has the natural X_+ -grading. The projective spectrum of \mathbb{V} with respect to this grading is isomorphic to the flag variety G/B (see e.g. [Kum]).

Let $G = SL_n$, $\mathfrak{g} = sl_n$. We take n^2 variables z_{ij} , $1 \leq i, j \leq n$ and consider the subalgebra of $\mathbb{K}[z_{i,j}]_{i,j=1}^n$ generated by determinants

$$\left| \begin{pmatrix} z_{1i_1} & z_{1i_2} & \cdots & z_{1i_k} \\ z_{2i_1} & z_{2i_2} & \cdots & z_{2i_k} \\ \vdots & \vdots & \ddots & \vdots \\ z_{ki_1} & z_{ki_2} & \cdots & z_{ki_k} \end{pmatrix} \right|$$

for all $k \geq 1$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Then this algebra is isomorphic to \mathbb{V} . The algebra \mathbb{V} is generated by dual fundamental modules $V(\omega_k)^*$ which satisfy (quadratic) Plücker relations. There is a remarkable basis of a module $V(\lambda)^*$ given by semistandard tableaux of shape λ (see [F]).

In Section 2 we describe the semi-infinite analogue of these constructions.

1.2. Weyl modules. In this paper we are mainly interested in representations of the current algebra $\mathfrak{g} \otimes \mathbb{K}[t]$ which is a maximal parabolic subalgebra of the affine Kac-Moody Lie algebra attached to \mathfrak{g} . For $x \in \mathfrak{g}$ we sometimes denote the element $x \otimes 1$ simply by x .

Definition 1.1. [CP] *Let $\lambda \in X_-$. Then the global Weyl module $\mathbb{W}(\lambda)$ is the cyclic $\mathfrak{g} \otimes \mathbb{K}[t]$ module with a generator v_λ and the following defining*

relations:

$$(1.1) \quad (e_\alpha \otimes t^k)v_\lambda = 0, \quad \alpha \in \Delta_+, \quad k \geq 0;$$

$$(1.2) \quad (f_{-\alpha} \otimes 1)^{(\alpha^\vee, \lambda)+1}v_\lambda = 0, \quad \alpha \in \Delta_+.$$

Local Weyl modules $W(\lambda)$ are defined by previous conditions and one additional condition:

$$(1.3) \quad h \otimes t^k v_\lambda = 0 \text{ for all } h \in \mathfrak{h}, k > 0.$$

Weyl modules are the natural analogues of finite-dimensional simple \mathfrak{g} -modules $V(\lambda)$. They are graded by the degree of t :

$$\mathbb{W}(\mu) = \bigoplus_{k=0}^{\infty} \mathbb{W}(\mu)^{(k)}$$

with finite-dimensional homogeneous components. Therefore we can define the restricted dual module:

$$\mathbb{W}(\mu)^* = \bigoplus_{k=0}^{\infty} (\mathbb{W}(\mu)^{(k)})^*.$$

Global Weyl modules have the following properties.

Lemma 1.2. $\mathbb{W}(\mu)^*$ is cocyclic, i. e. there exists an element (cogenerator) $v^* \in \mathbb{W}(\mu)^*$ such that for any element $u \in \mathbb{W}(\mu)^*$ there exists an element $f \in U(\mathfrak{g} \otimes \mathbb{K}[t])$ such that $fu = v^*$. The set of cogenerators coincides with $(\mathbb{W}(\mu)^{(0)})^* \simeq V(\mu)^*$.

Proof. This is a direct consequence of the fact that Weyl module is cyclic. \square

For a dominant weight $\lambda = \sum_{k=1}^{\text{rk}(\mathfrak{g})} m_k \omega_k$ we define

$$(q)_\lambda = \prod_{k=1}^{\text{rk}(\mathfrak{g})} \prod_{i=1}^{m_k} (1 - q^i).$$

Each Weyl module is graded by the \mathfrak{h} -weights and by t -degree. For any $\mathfrak{g} \otimes \mathbb{K}[t]$ -module U with such a grading let $U(\nu, m)$, $\nu \in \mathfrak{h}^*$, $m \in \mathbb{Z}$ be the weight space of the corresponding weight.

Definition 1.3.

$$\text{ch}U = \sum_{\nu, m} \dim U(\nu, m) x^\nu q^m.$$

Proposition 1.4. [CFK, N]

$$\text{ch}\mathbb{W}(\mu) = \frac{\text{ch}W(\mu)}{(q)_\mu}.$$

Lemma 1.5. [Kat] The $\mathfrak{g} \otimes \mathbb{K}[t]$ -submodule of $\mathbb{W}(\lambda) \otimes \mathbb{W}(\mu)$ generated by $v_\lambda \otimes v_\mu$ is isomorphic to $\mathbb{W}(\lambda + \mu)$.

Corollary 1.6. *There exists a surjection of dual Weyl modules:*

$$(1.4) \quad \mathbb{W}^*(\lambda) \otimes \mathbb{W}^*(\mu) \rightarrow \mathbb{W}^*(\lambda + \mu),$$

inducing the structure of associative and commutative algebra on the space $\bigoplus_{\lambda \in \Delta_+} \mathbb{W}^*(\lambda)$. We denote this algebra by $\mathbb{W} = \mathbb{W}(\mathfrak{g})$.

Remark 1.7. *The algebra \mathbb{W} is an analogue of the algebra $\mathbb{V} = \bigoplus_{\lambda \in \Delta_-} V(\lambda)^*$. The algebra \mathbb{V} is generated by the space $\bigoplus_{k=1}^{\text{rk}(\mathfrak{g})} V(\omega_k)^*$ and there are only quadratic relations (see [F, Kum]).*

The following proposition is a direct consequence of Corollary 1.6.

Proposition 1.8. *\mathbb{W} is generated by the space $\bigoplus_{k=1}^{\text{rk}(\mathfrak{g})} \mathbb{W}(\omega_k)^*$.*

Let us describe the structure of fundamental Weyl modules $\mathbb{W}(\omega_k)$.

Lemma 1.9. *Assume that $W(\omega_k)|_{\mathfrak{g} \otimes 1} \simeq V(\omega_k)$. Then for the global Weyl module we have:*

$$\mathbb{W}(\omega_k) \simeq V(\omega_k) \otimes \mathbb{K}[t]$$

with the action of the current algebra given by the following rule:

$$x \otimes t^l . u \otimes t^k = x . u \otimes t^{l+k}.$$

Proof. We note that all defining relations for $\mathbb{W}(\omega_k)$ hold on the constructed module. Therefore we have a surjection $\mathbb{W}(\omega_k) \rightarrow V(\omega_k) \otimes \mathbb{K}[t]$. Now it suffices to note that

$$\text{ch}(V(\omega_k) \otimes \mathbb{K}[t]) = \frac{\text{ch}W(\omega_k)}{(q)_{\omega_k}} = \frac{\text{ch}W(\omega_k)}{1 - q}.$$

□

Lemma 1.10. *Under assumptions of Lemma 1.9 we have:*

$$\mathbb{W}(\omega_k)^* \simeq V(\omega_k)^* \otimes \mathbb{K}[t]$$

with the action of the current algebra given by the following rule:

$$(1.5) \quad x \otimes t^l . u \otimes t^k = \begin{cases} x . u \otimes t^{k-l}, & \text{if } k \geq l; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1.11. *Note that the conditions of Lemma 1.9 hold for all fundamental weights in types A and C, vector and spinor representations in types B and D, two 27-dimensional representations in type E_6 , 56-dimensional representation in type E_7 , 26-dimensional representation in type F_4 and 7-dimensional representation in type G_2 .*

From now on we assume that $W(\omega_k)|_{\mathfrak{g} \otimes 1} \simeq V(\omega_k)$ for all fundamental weights. Let $\{X_{i,1}, \dots, X_{k,l_k}\}$ be a basis of $V(\omega_k)^*$. Then $\{X_{kj}^{(l)} = X_{kj} \otimes t^l\}$ is a basis of $\mathbb{W}(\omega_k)^*$. We consider the formal series $X_{kj}(s) = \sum_{l=0}^{\infty} X_{kj}^{(l)} s^l$.

Proposition 1.12. *Assume that a relation*

$$r := \sum_{\substack{j_1=1,\dots,l_{k_1} \\ j_2=1,\dots,l_{k_1}}} c_{j_1,j_2} X_{k_1,j_1} X_{k_2,j_2} = 0$$

holds in \mathbb{V} for some constants $c_{j_1,j_2} \in \mathbb{K}$. Then

$$r(s) := \sum_{\substack{j_1=1,\dots,l_{k_1} \\ j_2=1,\dots,l_{k_1}}} c_{j_1,j_2} X_{k_1,j_1}(s) X_{k_2,j_2}(s) = 0$$

in the algebra $\mathbb{W}[[s]]$, i. e. all the coefficients of this series form relations in \mathbb{W} .

Proof. We know that for any relation r and any element $e \in \mathfrak{g}$ the element $e.r = r_1$ is a relation. We have $e \otimes 1.r(s) = r_1(s)$. Let $r(s) = \sum_{l=0}^{\infty} r^{(l)} s^l$. If $h_\alpha.r = ar$, $a \in \mathbb{K}$, then $h_\alpha \otimes t^l.r^{(k)} = ar^{(k-l)}$. Thus all coefficients of series of the form $r(s)$ form a submodule in $\mathbb{W}(\omega_{k_1} + \omega_{k_2})^*$. Note that this submodule has zero intersection with the zero level submodule $V(\omega_{k_1} + \omega_{k_2})^*$. However $\mathbb{W}^*(\omega_{k_1} + \omega_{k_2})$ is cocyclic. Therefore this submodule is zero. \square

However in general these are not only relations in the algebra \mathbb{W} .

1.3. Semi-infinite flag varieties. Let $G[[t]]$ be the group of the Lie algebra $\mathfrak{g} \otimes \mathbb{K}[[t]]$, $B[[t]] \supset U[[t]]$ be the groups of $\mathfrak{b} \otimes \mathbb{K}[[t]] \supset \mathfrak{n} \otimes \mathbb{K}[[t]]$ respectively. In this paper we deal with $\mathfrak{g} = \mathfrak{sl}_n$. In this case $G[[t]] = SL(\mathbb{K}[[t]])$, $B[[t]]$ and $U[[t]]$ are upper triangular and unitriangular matrices over this ring. We consider the varieties $G[[t]]/U[[t]]$ and $G[[t]]/H \cdot U[[t]]$. They have the following realization due to Drinfeld [FiMi, BG].

For any $\lambda \in X_+$ consider the space $V(\lambda)[[t]] = V(\lambda) \otimes \mathbb{K}[[t]]$. Then we have the family of embeddings $m_{\lambda,\mu} : V(\lambda + \mu)[[t]] \hookrightarrow V(\lambda)[[t]] \otimes V(\mu)[[t]]$. A Drinfeld-Plücker data is a set of lines $l_\lambda \in V(\lambda)[[t]]$ such that $m_{\lambda,\mu} l_{\lambda+\mu} = l_\lambda \otimes l_\mu$. Such a collection of lines is fixed by the lines $l_{\omega_k} \in V(\omega_k)[[t]]$ for all fundamental weights ω_k . Take a basis $\{v_{kj}\}$ of $V(\omega_k)$. Put $l_{\omega_k} = \mathbb{K} \sum_j a_{kj} v_{kj}$, where $a_{kj} \in \mathbb{K}[[t]]$. Then the family of lines $\{l_{\omega_k}\}$ gives a Drinfeld-Plücker data iff the series (a_{kj}) satisfy the Plücker relations (thus this realization of $\overline{G[[t]]/U[[t]]}$ is the projectivized arc scheme of $\overline{G/U}$, see e.g. [Mu]). The set of points satisfying these relations contains $G[[t]]/H \cdot U[[t]]$ as an open dense subset. Indeed, take a product of the highest weight lines $\mathbb{K}v_{\omega_1} \otimes \dots \otimes \mathbb{K}v_{\omega_{n-1}}$ in $V(\omega_1)[[t]] \otimes \dots \otimes V(\omega_{n-1})[[t]]$. Then the set of Drinfeld-Plücker data is the closure of the $G[[t]]$ -orbit of this element and its stabilizer is $H \cdot U[[t]]$.

In type A this construction can be written down in a very explicit way. We denote the coordinates in $V(\omega_k)[[t]]$ by $X_I^{(l)}$, where $I = (1 \leq i_1 < \dots < i_k < n)$ and $l \in \mathbb{Z}_{\geq 0}$ (recall that $V(\omega_k)$ is the k -th wedge power of the vector representation; the coordinate $X_I^{(l)}$ is dual to $v_{i_1} \wedge \dots \wedge v_{i_k} \otimes t^l$). An element $g \in SL_n[[t]]$ is given by $n \times n$ matrix whose entries are Taylor series

in t . Therefore a coordinate $X_I^{(l)}$ of a point $g \cdot \mathbb{K}v_{\omega_1} \otimes \cdots \otimes \mathbb{K}v_{\omega_{n-1}}$ is equal to the coefficient in front of t^l of the minor of the matrix g located on the intersection of the first k rows and i_1, \dots, i_k columns.

Finally, let us note that the Plücker relations $m_{\lambda, \mu} l_{\lambda + \mu} = l_\lambda \otimes l_\mu$ give the non-reduced scheme structure on the $G[[t]]$ -orbit closure for $n \geq 5$. Corollary 4.27 from [KNS] says that $\overline{G[[t]]/U[[t]]} \simeq \text{Spec } \mathbb{W}$ and $\overline{G[[t]]/H \cdot U[[t]]}$ is the spectrum of multi-homogenous ideals of \mathbb{W} . In Section 2 we describe explicitly the difference between the reduced scheme structure provided by \mathbb{W} and the non-reduced scheme structure defined by $\mathbb{K}[t]$ Plücker relations. We close this section with an example.

Example 1.13. *Let $G = SL_5$. Then one can check that the coordinates of a point in the open $G[[t]]$ -orbit satisfy the relation*

$$(1.6) \quad X_{12}^{(1)} X_{345}^{(0)} - X_{13}^{(1)} X_{245}^{(0)} + X_{14}^{(1)} X_{235}^{(0)} - X_{15}^{(1)} X_{234}^{(0)} + X_{23}^{(1)} X_{145}^{(0)} \\ - X_{24}^{(1)} X_{135}^{(0)} + X_{25}^{(1)} X_{134}^{(0)} + X_{34}^{(1)} X_{125}^{(0)} - X_{35}^{(1)} X_{124}^{(0)} + X_{45}^{(1)} X_{123}^{(0)}.$$

We note that (1.6) does not belong to the ideal generated by the coefficients of $\mathbb{K}[t]$ Plücker relations, because (in particular) in each Plücker relation the term $X_{12}^{(1)} X_{345}^{(0)}$ shows up together with its companion $X_{12}^{(0)} X_{345}^{(1)}$. For example, the sum of (1.6) with

$$X_{12}^{(0)} X_{345}^{(1)} - X_{13}^{(0)} X_{245}^{(1)} + X_{14}^{(0)} X_{235}^{(1)} - X_{15}^{(0)} X_{234}^{(1)} + X_{23}^{(0)} X_{145}^{(1)} \\ - X_{24}^{(0)} X_{135}^{(1)} + X_{25}^{(0)} X_{134}^{(1)} + X_{34}^{(0)} X_{125}^{(1)} - X_{35}^{(0)} X_{124}^{(1)} + X_{45}^{(0)} X_{123}^{(1)}$$

does belong to the ideal of Plücker relations.

2. SEMI-INFINITE PLÜCKER RELATIONS

2.1. Algebra of minors. Consider the set of variables $z_{ij}^{(l)}$, $1 \leq i, j \leq n$, $l \geq 0$. Let e_{pq} be the (p, q) -th matrix unit, $h_{pq} = e_{pp} - e_{qq}$. We define a derivation action of $\mathfrak{sl}_n \otimes \mathbb{K}[t]$ on the polynomials in these variables by the following action of generators:

$$e_{qp} \otimes t^l z_{ij}^{(k)} = \begin{cases} -z_{ip}^{(k-l)}, & \text{if } q = j, k \geq l; \\ 0, & \text{otherwise.} \end{cases}, \quad p \neq q$$

$$h_{qp} \otimes t^l z_{ij}^{(k)} = \begin{cases} -z_{ij}^{(k-l)}, & \text{if } p = j, k \geq l; \\ z_{ij}^{(k-l)}, & \text{if } q = j, k \geq l; \\ 0, & \text{otherwise.} \end{cases}$$

We define the formal series $z_{ij}(s) = \sum_{k=0}^{\infty} z_{ij}^{(k)} s^k$. Let

$$(2.1) \quad m_{i_1, \dots, i_k}(s) = \det \begin{pmatrix} z_{1, i_1}(s) & z_{1, i_2}(s) & \cdots & z_{1, i_k}(s) \\ z_{2, i_1}(s) & z_{2, i_2}(s) & \cdots & z_{2, i_k}(s) \\ \vdots & \vdots & \ddots & \vdots \\ z_{k, i_1}(s) & z_{k, i_2}(s) & \cdots & z_{k, i_k}(s) \end{pmatrix}$$

Note that $m_{i_1, \dots, i_k}(s)$ is alternating with respect to permutation of indexes.

Let us write the decomposition $m_{i_1, \dots, i_k}(s) = \sum_{l=0}^{\infty} m_{i_1, \dots, i_k}^{(l)} s^l$. We consider the subalgebra $\mathcal{M} = \mathbb{K}[m_{i_1, \dots, i_k}^{(l)}] \subset \mathbb{K}[z_{ij}^{(l)}]$ and define $\deg m_{i_1, \dots, i_k}^{(l)} = \omega_k$. Then \mathcal{M} is graded by the semigroup of dominant weights:

$$\mathcal{M} = \bigoplus_{\lambda \in X_+} \mathcal{M}(\lambda).$$

Each $\mathcal{M}(\lambda)$ is $\mathfrak{sl}_n \otimes \mathbb{K}[t]$ -submodule by the degree of t ,

$$\mathcal{M}(\lambda) = \bigoplus_{l=0}^{\infty} \mathcal{M}(\lambda)^{(l)}.$$

Proposition 2.1. *The set of elements*

$$\{m_{i_1, \dots, i_k}^{(l)}, i_1 < i_2 < \dots < i_k, l = 0, 1, \dots\}$$

form a basis of $\mathfrak{sl}_n \otimes \mathbb{K}[t]$ -module $\mathcal{M}(\omega_k) \simeq \mathbb{W}(\omega_k)^*$.

Proof. We have the standard basis of $V(\omega_k)^* \simeq V(\omega_{n-k})$ consisting of elements $\{X_{i_1, \dots, i_k}\}$ defined by:

$$X_{i_1, \dots, i_k} = v_{i_1}^* \wedge \dots \wedge v_{i_k}^*, i_1 < \dots < i_k.$$

Therefore Lemma 1.10 tells us that $\{X_{i_1, \dots, i_k} \otimes t^l\}, l = 0, 1, \dots$ is a basis of $\mathbb{W}(\omega_k)^*$. It is easy to see that the map $\{X_{i_1, \dots, i_k} \otimes t^l\} \mapsto \{m_{i_1, \dots, i_k}^{(l)}\}$ defines the surjection of modules. Moreover the set $\{m_{i_1, \dots, i_k}^{(0)}\}$ forms a basis of $V(\omega_k)^*$ and the dual Weyl module is cocyclic. Thus all the elements $m_{i_1, \dots, i_k}^{(l)}$ are linearly independent. \square

Let $X_{i_1, \dots, i_p, i_{p+1}, \dots, i_k} = -X_{i_1, \dots, i_{p+1}, i_p, \dots, i_k}$, $X_{i_1, \dots, i_p, i_p, \dots, i_k} = 0$. Using this rule we define X_I for arbitrary k -tuple I .

Proposition 2.2. *There exists the surjection of algebras $\mathcal{M} \twoheadrightarrow \mathbb{W}$ sending $m_{i_1, \dots, i_k}^{(l)}$ to $X_{i_1, \dots, i_k}^{(l)}$ and $\mathcal{M}(\lambda)$ to $\mathbb{W}(\lambda)^*$.*

Proof. Degree zero elements $m_{i_1, \dots, i_k}^{(0)}$ satisfy usual Plücker relations (see, for example [MS]). Therefore degree zero submodule $\mathcal{M}(\lambda)^{(0)}$ is isomorphic to $V(\lambda)^*$. Consider a polynomial algebra generated by auxiliary variables $\chi_{i_1, \dots, i_k}^{(l)}$. We define the action of $\mathfrak{sl}_n \otimes \mathbb{K}[t]$ on $\langle \chi_{i_1, \dots, i_k}^{(l)} \rangle$ in such a way that the map $\chi_{i_1, \dots, i_k}^{(l)} \mapsto m_{i_1, \dots, i_k}^{(l)}$ is an isomorphism of $\langle \chi_{i_1, \dots, i_k}^{(l)} \rangle$ and $\bigoplus_{k=1}^{\text{rk}(\mathfrak{g})} \mathcal{M}(\omega_k)$. We attach the homogeneous degree ω_k and t -degree l to any variable of the form $\chi_{i_1, \dots, i_k}^{(l)}$. Then we have the following surjections of algebras with derivations:

$$\phi : \mathbb{K}[\chi_{i_1, \dots, i_k}^{(l)}] \twoheadrightarrow \mathcal{M};$$

$$\psi : \mathbb{K}[\chi_{i_1, \dots, i_k}^{(l)}] \twoheadrightarrow \mathbb{W}.$$

Let F_λ be the submodule of $\mathbb{K}[\chi_{i_1, \dots, i_k}^{(l)}]$ of degree λ . Take a homogeneous polynomial $P \in \mathbb{K}[\chi_{i_1, \dots, i_k}^{(l)}]$ of degree λ . Assume that $\phi(P) = 0$. Then $U(\mathfrak{sl}_n \otimes \mathbb{K}[t]).P \cap F_\lambda^0$ does not contain an element of weight λ . Therefore $\psi(P) = 0$ because $\mathbb{W}(\lambda)^*$ is cocyclic with cogenerator of weight λ in t-degree 0. \square

Now we construct a family of (quadratic) relations for the elements $m_{i_1, \dots, i_k}^{(l)}$. We take two sets of numbers $\sigma = \{i_1, \dots, i_q\}$ and $\tau = \{j_1, \dots, j_p\}$, $i_1 < i_2 < \dots < i_q$, $j_1 < j_2 < \dots < j_p$, $q \leq p$. For $k \in \mathbb{N}$ we consider $p + k = a + b$ pairwise distinct numbers $i_{f_1}, \dots, i_{f_a}, j_{g_1}, \dots, j_{g_b}$. Let

$$P = \{i_{f_1}, \dots, i_{f_a}\} \sqcup \{j_{g_1}, \dots, j_{g_b}\}.$$

Denote $\sigma \setminus P = \{i_{f'_1}, \dots, i_{f'_{q-a}}\}$, $\tau \setminus P = \{j_{g'_1}, \dots, j_{g'_{p-b}}\}$. For any a cardinality a subset $A \subset P$,

$$A = \{y_1, \dots, y_a\}, \quad P \setminus A = \{y_{a+1}, \dots, y_{a+b}\}$$

with $y_1 < \dots < y_a$ and $y_{a+1} < \dots < y_{a+b}$ we consider

$$\eta_1(\sigma, \tau, P, A) = (i_{f'_1}, \dots, i_{f'_{q-a}}, y_1, \dots, y_a),$$

$$\eta_2(\sigma, \tau, P, A) = (j_{g'_1}, \dots, j_{g'_{p-b}}, y_{a+1}, \dots, y_{a+b}).$$

Denote by $\text{sign}(\sigma, \tau, P, A)$ the sign of the permutation shuffling the tuple $(y_1, \dots, y_a, y_{a+1}, \dots, y_{a+b})$ into $\{i_{f_1}, \dots, i_{f_a}, j_{g_1}, \dots, j_{g_b}\}$.

Proposition 2.3. *For any $0 \leq k' \leq k - 1$ we have the following equality in $\mathbb{W}[[s]]$:*

$$(2.2) \quad \sum_{A \subset P, |A|=a} (-1)^{\text{sign}(\sigma, \tau, P, A)} \frac{\partial^{k'} m_{\eta_1(\sigma, \tau, P, A)}(s)}{\partial s^{k'}} m_{\eta_2(\sigma, \tau, P, A)}(s) = 0.$$

Proof. For any minor $m_{i_1, \dots, i_q}(s)$ we have the Leibniz rule:

$$(2.3) \quad \frac{\partial^{k'} m_{i_1, \dots, i_q}(s)}{\partial s^{k'}} = \sum_{k_1 + \dots + k_q = k'} \det \begin{pmatrix} \frac{\partial^{k_1} z_{1, i_1}(s)}{\partial s^{k_1}} & \frac{\partial^{k_1} z_{1, i_2}(s)}{\partial s^{k_1}} & \dots & \frac{\partial^{k_1} z_{1, i_q}(s)}{\partial s^{k_1}} \\ \frac{\partial^{k_2} z_{2, i_1}(s)}{\partial s^{k_2}} & \frac{\partial^{k_2} z_{2, i_2}(s)}{\partial s^{k_2}} & \dots & \frac{\partial^{k_2} z_{2, i_q}(s)}{\partial s^{k_2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^{k_q} z_{q, i_1}(s)}{\partial s^{k_q}} & \frac{\partial^{k_q} z_{q, i_2}(s)}{\partial s^{k_q}} & \dots & \frac{\partial^{k_q} z_{q, i_q}(s)}{\partial s^{k_q}} \end{pmatrix}.$$

Hence equation (2.2) can be rewritten in the following way:

$$(2.4) \quad \sum_{k_1 + \dots + k_q = k'} \sum_{A \subset P, |A|=a} (-1)^{\text{sign}(\sigma, \tau, P, A)} \det \begin{pmatrix} \frac{\partial^{k_1} z_{1, i_{f'_1}}(s)}{\partial s^{k_1}} & \dots & \frac{\partial^{k_1} z_{1, y_a}(s)}{\partial s^{k_1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^{k_q} z_{q, i_{f'_1}}(s)}{\partial s^{k_q}} & \dots & \frac{\partial^{k_q} z_{q, y_a}(s)}{\partial s^{k_q}} \end{pmatrix} m_{\eta_2(\sigma, \tau, P, A)}(s) = 0.$$

We consider a summand of the left hand side of equation (2.4):

$$(2.5) \quad S(k_1, \dots, k_q) = \sum_{A \subset P, |A|=a} (-1)^{\text{sign}(\sigma, \tau, P, A)} \det \begin{pmatrix} \frac{\partial^{k_1} z_{1, i_{f'_1}}(s)}{\partial s^{k_1}} & \dots & \frac{\partial^{k_1} z_{1, y_a}(s)}{\partial s^{k_1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^{k_q} z_{q, i_{f'_1}}(s)}{\partial s^{k_q}} & \dots & \frac{\partial^{k_q} z_{q, y_a}(s)}{\partial s^{k_q}} \end{pmatrix} m_{\eta_2(\sigma, \tau, P, A)}(s).$$

We note that there is less than or equal to $k - 1$ numbers j such that $k_j > 0$. For any $1 \leq i \leq n$ define the column

$$\left(z_{1, i}(s), \dots, z_{p, i}(s), \frac{\partial^{k_1} z_{1, i}(s)}{\partial s^{k_1}}, \dots, \frac{\partial^{k_q} z_{q, i}(s)}{\partial s^{k_q}} \right)^t,$$

where the element $\frac{\partial^{k_j} z_{j, i}(s)}{\partial s^{k_j}}$ is skipped if $k_j = 0$. All these columns lie in free $\mathbb{K}[z_{ij}^{(l)}][[s]]$ -module of rank less than or equal to $p + k - 1$. It is easy to see that $S(k_1, \dots, k_q)$ is multilinear function in such columns and alternating with respect to transposition of $p + k$ columns numbered by $y_i \in P$. Therefore $S(k_1, \dots, k_q) = 0$. \square

Corollary 2.4. *The following relation holds in \mathbb{W} :*

$$(2.6) \quad \sum_{A \subset P, |A|=a} (-1)^{\text{sign}(\sigma, \tau, P, A)} \frac{\partial^{k'} X_{\eta_1(\sigma, \tau, P, A)}(s)}{\partial s^{k'}} X_{\eta_2(\sigma, \tau, P, A)}(s) = 0.$$

Now we have the set of relations on generators of \mathbb{W} . We want to write down a linearly independent set of these relations. We prepare the following definitions.

Definition 2.5. *For two strictly increasing columns of numbers*

$$\sigma = (\sigma_1, \dots, \sigma_{l_\sigma})^t, \quad \tau = (\tau_1, \dots, \tau_{l_\tau})^t$$

we write $\sigma < \tau$ if $l_\sigma > l_\tau$ or $l_\sigma = l_\tau$ and for some j for any $j' > j$ $\sigma_{j'} = \tau_{j'}$ and $\sigma_j < \tau_j$.

Definition 2.6. Assume that for some $\sigma < \tau$ we have the following set of inequalities:

$$\begin{aligned} \sigma_{l_\tau} &\leq \tau_{l_\tau}, \dots, \sigma_{j_1+1} \leq \tau_{j_1+1}, \\ \sigma_{j_1} &> \tau_{j_1}, \sigma_{j_1-1} \geq \tau_{j_1-1}, \dots, \sigma_{j_2+1} \geq \tau_{j_2+1}, \\ \sigma_{j_2} &< \tau_{j_2}, \sigma_{j_2-1} \leq \tau_{j_2-1}, \dots \end{aligned}$$

We define strictly decreasing sequence of elements

$$P(\sigma, \tau) = (\sigma_{l_\sigma}, \sigma_{l_\sigma-1}, \dots, \sigma_{l_\tau+1}, \dots, \sigma_{j_1+1}, \sigma_{j_1}, \tau_{j_1}, \tau_{j_1-1}, \dots, \tau_{j_2}, \sigma_{j_2}, \dots).$$

We set

$$k(\sigma, \tau) = |P(\sigma, \tau)| - l_\sigma.$$

Example 2.7. In the following examples the numbers $k(\sigma, \tau)$ are equal to 2, 1 and 0 respectively. The elements of the sets $P(\sigma, \tau)$ are highlighted by the boldface font and are given by $(7, 6, 5, 4, 3, 1)$, $(7, 5, 4, 3, 1)$, $(7, 5, 3, 2)$ respectively.

$$\begin{array}{ccc} \mathbf{1} & \leq & \mathbf{2} & & \mathbf{2} & \geq & \mathbf{1} & & \mathbf{2} & \leq & \mathbf{2} \\ \mathbf{3} & < & \mathbf{4} & & \mathbf{4} & > & \mathbf{3} & & \mathbf{3} & \leq & \mathbf{4} \\ \mathbf{6} & > & \mathbf{5} & , & \mathbf{5} & \leq & \mathbf{6} & , & \mathbf{5} & \leq & \mathbf{5} \\ \mathbf{7} & < & \mathbf{8} & & \mathbf{7} & & & & \mathbf{7} & \leq & \end{array}$$

The algebra \mathbb{W} for \mathfrak{sl}_n is generated by the elements $X_\tau^{(l)}$, where $\tau = (\tau_1, \dots, \tau_{l_\tau})$, $1 \leq \tau_1 < \dots < \tau_{l_\tau} \leq n$. We define the following partial order on monomials in $X_\tau^{(l)}$. Let us consider two monomials

$$u_1 = X_{\tau_1}^{(a_1)} \dots X_{\tau_o}^{(a_o)}, \quad u_2 = X_{\mu^1}^{(b_1)} \dots X_{\mu^p}^{(b_p)},$$

where $l_{\tau_1} \geq \dots \geq l_{\tau_o}$, $l_{\mu^1} \geq \dots \geq l_{\mu^p}$. If $p > o$ then $u_2 \succ u_1$. Assume that $p = o$. If $(l_{\tau_1}, \dots, l_{\tau_p}) > (l_{\mu^1}, \dots, l_{\mu^p})$ in lexicographic order then $u_2 \succ u_1$. Assume that $l_{\tau_j} = l_{\mu^j}$, $1 \leq i, j \leq n$. Let $L = l_{\tau_1}$ and write $\tau_j^k = 0$, if $j > l_{\tau^k}$. We consider two sequences:

$$\begin{aligned} s(u_1) &= \left(\sum_{i=1}^p \tau_L^i, \sum_{i=1}^p \tau_{L-1}^i, \dots, \sum_{i=1}^p \tau_1^i \right); \\ s(u_2) &= \left(\sum_{i=1}^p \mu_L^i, \sum_{i=1}^p \mu_{L-1}^i, \dots, \sum_{i=1}^p \mu_1^i \right). \end{aligned}$$

Then if $s(u_2) > s(u_1)$ in usual lexicographic order, then $u_2 \succ u_1$. Assume that $s(u_2) = s(u_1)$. We define the following sequence:

$$(2.7) \quad sd(u_1) = \left(\sum_{i=1}^p (\tau_L^i - \tau_{L-1}^i)^2, \sum_{i=1}^p (\tau_{L-1}^i - \tau_{L-2}^i)^2, \sum_{i=1}^p (\tau_L^i - \tau_{L-2}^i)^2, \right. \\ \left. \sum_{i=1}^p (\tau_{L-2}^i - \tau_{L-3}^i)^2, \sum_{i=1}^p (\tau_{L-1}^i - \tau_{L-3}^i)^2, \dots \right)$$

and analogously define $sd(u_2)$. Then if $sd(u_2) > sd(u_1)$ in usual lexicographic order then $u_2 \succ u_1$.

Remark 2.8. *The order \succ is not sensitive to the upper indices of the variables $X_\tau^{(l)}$. Forgetting these upper indices, the order \succ is the total term order.*

Let $\sigma \leq \tau$ be two strictly increasing columns of numbers from 1 to n .

Proposition 2.9. *a). For any $k' \leq k(\sigma, \tau) - 1$ we have the following equality in $\mathbb{W}[[s]]$:*

$$(2.8) \quad \sum_{A \subset P(\sigma, \tau), |A|=|\sigma \cap P|} (-1)^{\text{sign}(\sigma, \tau, P(\sigma, \tau), A)} \frac{\partial^{k'} X_{\eta_1(\sigma, \tau, P(\sigma, \tau), A)}(s)}{\partial s^{k'}} X_{\eta_2(\sigma, \tau, P(\sigma, \tau), A)}(s) = 0.$$

b). We have $X_\sigma^{(a)} X_\tau^{(b)} \succeq X_{\eta_1(\sigma, \tau, P(\sigma, \tau), A)}^{(c)} X_{\eta_2(\sigma, \tau, P(\sigma, \tau), A)}^{(d)}$ for any a, b, c, d and the strict inequality holds for all monomials except for the monomials of the form $X_\sigma^{(c)} X_\tau^{(d)}$.

Proof. To prove the first claim we note that equality (2.8) is a particular case of equality (2.4).

Now let us prove part b). Assume that $\eta_1(\sigma, \tau, P(\sigma, \tau), A) \neq \sigma$. Since $P(\sigma, \tau)$ is decreasing, we have

$$s(X_\sigma^{(a)} X_\tau^{(b)}) \geq s(X_{\eta_1(\sigma, \tau, P(\sigma, \tau), A)}^{(c)} X_{\eta_2(\sigma, \tau, P(\sigma, \tau), A)}^{(d)})$$

and equality holds if $\eta_1(\sigma, \tau, P(\sigma, \tau), A)_j \in \{\sigma_j, \tau_j\}$, $\eta_2(\sigma, \tau, P(\sigma, \tau), A)_j \in \{\sigma_j, \tau_j\}$. Assume that for some j $\eta_2(\sigma, \tau, P(\sigma, \tau), A)_{j'} = \sigma_{j'}$ for all $j' > j$ and $\eta_2(\sigma, \tau, P(\sigma, \tau), A)_j = \tau_j$. Then $\sigma_j, \tau_j \in P(\sigma, \tau)$. Thus for some j_1 either

$$\sigma_j > \tau_j, \sigma_{j+1} = \tau_{j+1}, \dots, \sigma_{j_1-1} = \tau_{j_1-1}, \sigma_{j_1} < \tau_{j_1}$$

or

$$\sigma_j < \tau_j, \sigma_{j+1} = \tau_{j+1}, \dots, \sigma_{j_1-1} = \tau_{j_1-1}, \sigma_{j_1} > \tau_{j_1}.$$

This implies that $sd(X_\sigma^{(a)} X_\tau^{(b)}) > sd(X_{\eta_1(\sigma, \tau, P(\sigma, \tau), A)}^{(c)} X_{\eta_2(\sigma, \tau, P(\sigma, \tau), A)}^{(d)})$. \square

Note that all relations (2.8) are homogenous with respect to the Cartan weight, with respect to the t -degree (counting the sum of the upper indices of the variables) and with respect to the homogenous degree. Consider degenerations of these relations with respect to the partial order " \succ ":

$$(2.9) \quad \frac{\partial^{k'} X_\tau(s)}{\partial s^{k'}} X_\sigma(s) = 0.$$

for any $k' \leq k(\sigma, \tau) - 1$. We denote this degenerate algebra by $\widetilde{\mathbb{W}}$.

Proposition 2.10. $\text{ch} \widetilde{\mathbb{W}} \geq \text{ch} \mathbb{W}$.

Proof. We consider the auxiliary algebra $\widetilde{\mathbb{W}}$ defined as the quotient of the polynomial algebra in variables $X_\tau^{(l)}$ by the ideal generated by relations (2.8). We know that $\widetilde{\mathbb{W}}$ surjects onto \mathbb{W} and hence $\text{ch}\widetilde{\mathbb{W}} \geq \text{ch}\mathbb{W}$.

The term order \succ defines the filtration on the algebra $\widetilde{\mathbb{W}}$. The associated graded algebra is the quotient of the polynomial algebra in variables $X_\tau^{(l)}$ by the ideal, which contains all the relations (2.9). We conclude that $\text{ch}\widetilde{\mathbb{W}} \geq \text{ch}\mathbb{W}$. \square

2.2. The character formula. We compute the character of the degenerated algebra $\widetilde{\mathbb{W}}$. Recall the notation $(q)_r = \prod_{i=1}^r (1 - q^i)$. For a non-empty subset $\sigma = (\sigma_1, \dots, \sigma_k) \subset \{1, \dots, n\}$ we introduce a variable r_σ .

Proposition 2.11.

$$(2.10) \quad \text{ch}\widetilde{\mathbb{W}}(m_1\omega_1 + \dots + m_{n-1}\omega_{n-1}) = \sum_{\sum_{|\sigma|=k} r_\sigma = m_k} \frac{q^{\sum_{\sigma < \tau} k(\sigma, \tau) r_\sigma r_\tau} \prod_{i=1}^n x_i^{\sum_{\sigma \ni i} r_\sigma}}{\prod_{\sigma} (q)_{r_\sigma}}.$$

Proof. We first note that the space $\widetilde{\mathbb{W}}(\lambda)$ has several gradings. First of all, attaching degree l to a variable $X_\sigma^{(l)}$ one gets the standard degree grading. As usual, the power of the variable q takes care of this grading in the character formula. Now, $\widetilde{\mathbb{W}}(\lambda)$ is additionally graded by the group $\mathbb{Z}_{\geq 0}^{2^n - 2}$ with the coordinates labeled by the proper non-empty subsets $\sigma \subsetneq \{1, \dots, n\}$ (since the relations (2.9) are homogeneous). More concretely, the homogeneous part of degree $\mathbf{r} = (r_\sigma)_\sigma$ is spanned by the monomials in $X_\tau^{(l)}$ such that the number of factors of the form $X_\sigma^{(l)}$ is exactly r_σ . So it suffices to find the q -character of homogeneous part $\widetilde{\mathbb{W}}(\mathbf{r})$ of $\widetilde{\mathbb{W}}$ of degree \mathbf{r} . We note that this part sits in $\widetilde{\mathbb{W}}(\lambda)$ if and only if $\sum_{|\sigma|=k} r_\sigma = m_k$ for all $k = 1, \dots, n-1$.

We consider a functional realization of the dual space of $\widetilde{\mathbb{W}}(\mathbf{r})$. Namely, given a linear function ξ on the space $\widetilde{\mathbb{W}}(\mathbf{r})$ we attach to it the polynomial f_ξ in variables $Y_{\sigma, j}$, $\sigma \subset \{1, \dots, n\}$, $1 \leq j \leq r_\sigma$ defined as follows. Recall $X_\sigma(s) = \sum_{l \geq 0} X_\sigma^{(l)} s^l$. Then

$$(2.11) \quad f_\xi = \xi\left(\prod_{\sigma} X_\sigma(Y_{\sigma, 1}) \dots X_\sigma(Y_{\sigma, r_\sigma})\right).$$

We claim that formula (2.11) defines an isomorphism between the space of functionals on $\widetilde{\mathbb{W}}(\mathbf{r})$ and the space $\text{Pol}(\mathbf{r})$ of polynomials f in variables $Y_{\sigma, j}$ subject to the following conditions:

- f is symmetric in variables $Y_{\sigma, j}$ for each σ ,
- f is divisible by $(Y_{\sigma, j_1} - Y_{\tau, j_2})^{k(\sigma, \tau)}$ for all σ, τ, j_1, j_2 .

The first condition is obvious and the second one comes from the relations (2.9). We note that the q -degree on $\widetilde{\mathbb{W}}(\mathbf{r})$ attaching degree l to a variable

$X_\sigma^{(l)}$ is now translated into the counting of the total degree in all variables $Y_{\sigma,j}$. Now the q -character of the space $\text{Pol}(\mathbf{r})$ is given by

$$(2.12) \quad \frac{q^{\sum_{\sigma < \tau} k(\sigma, \tau) r_\sigma r_\tau} \prod_{i=1}^n x_i^{\sum_{\sigma \ni i} r_\sigma}}{\prod_{\sigma} (q)_{r_\sigma}}.$$

Indeed, $(q)_r^{-1}$ is the character of the space of symmetric polynomials in r variables and the factor $\prod_{\sigma < \tau} (Y_{\sigma, j_1} - Y_{\tau, j_2})^{k(\sigma, \tau)}$ produces the numerator of the above formula. \square

Corollary 2.12. *The homogeneous component $\widetilde{\mathbb{W}}(\mathbf{r}) \subset \widetilde{\mathbb{W}}$ has the basis consisting of monomials of the form*

$$(2.13) \quad \prod_{\sigma \in \{1, \dots, 2n\}} X_\sigma^{(l_{1, \sigma})} \dots X_\sigma^{(l_{r_\sigma, \sigma})}, \quad 0 \leq l_{1, \sigma} \leq \dots \leq l_{r_\sigma, \sigma}$$

such that for any pair $\sigma < \tau$ one has $l_{1, \sigma} \geq k(\sigma, \tau) r_\tau$.

Proof. We note that the character of the set of monomials (2.13) is equal to (2.12). Hence it suffices to show that the elements (2.13) span the space $\widetilde{\mathbb{W}}(\mathbf{r})$.

Assume that there exists an element $\xi \in (\widetilde{\mathbb{W}}(\mathbf{r}))^*$ vanishing on all the monomials (2.13). We want to show that in this case ξ is zero; equivalently, we need to prove that $f_\xi = 0$. A non-zero polynomial divisible by $\prod_{\sigma < \tau} (Y_{\sigma, j_1} - Y_{\tau, j_2})^{k(\sigma, \tau)}$ contains a monomial

$$(2.14) \quad \prod_{\sigma \in \{1, \dots, 2n\}} Y_\sigma^{l_{1, \sigma}} \dots Y_\sigma^{l_{r_\sigma, \sigma}}, \quad 0 \leq l_{1, \sigma} \leq \dots \leq l_{r_\sigma, \sigma}$$

such that $l_{1, \sigma} \geq k(\sigma, \tau) r_\tau$ for any pair $\sigma < \tau$ (coming from the choice of the term $Y_{\sigma, j_1}^{k(\sigma, \tau)}$ in each bracket $(Y_{\sigma, j_1} - Y_{\tau, j_2})^{k(\sigma, \tau)}$). However, the coefficient in front of monomial (2.14) is equal to $\xi(X_\sigma^{(l_{1, \sigma})} \dots X_\sigma^{(l_{r_\sigma, \sigma})})$. Therefore, if ξ vanishes on all the elements (2.13), then f_ξ is zero. \square

Corollary 2.13. *In Section 3 we show that the characters of the algebras \mathbb{W} and its degenerate version $\widetilde{\mathbb{W}}$ coincide. This implies that the union of monomials (2.13) over all \mathbf{r} such that $\sum_{|\sigma|=k} r_\sigma = m_k$ for all $k = 1, \dots, n-1$ is a basis of $\mathbb{W}(\lambda)^*$.*

Remark 2.14. *The t -degree zero part of the basis (2.13) is given by the semi-standard tableaux. In fact, one needs the monomials (2.13) with zero $l_{a, \sigma}$ for all a and σ . This is possible if and only if for any two distinct σ, τ such that $r_\sigma > 0$ and $r_\tau > 0$ one has $k(\sigma, \tau) = 0$. In other words, the indices σ of a degree zero basis vector can be packed into the semi-standard tableau.*

Remark 2.15. *In contrast to the degenerate algebra $\widetilde{\mathbb{W}}$, the algebra \mathbb{W} does not have that many gradings. Instead of the group $\mathbb{Z}_{\geq 0}^{2^n - 2}$ the nondegenerate algebra has only Cartan $\mathbb{Z}_{\geq 0}^{n-1}$ grading.*

Let us introduce the notation for the q -multinomial coefficient

$$\binom{m_k}{[\mathbf{r}]_k}_q = \frac{(q)_{m_k}}{\prod_{|\sigma|=k} (q)_{r_\sigma}}.$$

Recall $(q)_\lambda = \prod_{k=1}^{n-1} (q)_{m_k}$.

Corollary 2.16.

(2.15)

$$(q)_\lambda^{-1} \sum_{\sum_{|\sigma|=k} r_\sigma = m_k} q^{\sum_{\sigma < \tau} k(\sigma, \tau) r_\sigma r_\tau} \prod_{i=1}^n x_i^{\sum_{\sigma \ni i} r_\sigma} \prod_{k=1}^{n-1} \binom{m_k}{[\mathbf{r}]_k}_q \geq \text{ch} \mathbb{W}(\lambda).$$

Proof. This is a reformulation of inequality (2.10) using the definition of q -multinomial coefficient. \square

Recall the relation between the characters of the local and global Weyl modules:

$$(2.16) \quad \text{ch} \mathbb{W}(\lambda) = (q)_\lambda^{-1} \text{ch} W(\lambda).$$

Let $W(\lambda) = \bigoplus_{l \geq 0} W(\lambda)^{(l)}$ be the q -degree decomposition of the local Weyl module. In particular, the character of each $W(\lambda)^{(l)}$ is a polynomial in x_1, \dots, x_n . Let $\sum_{l \geq 0} q^l C_l(x_1, \dots, x_n)$ be the q -expansion of the left hand side of (2.15) multiplied by $(q)_\lambda$. In particular, each C_l is a Laurent polynomial in x_i .

Lemma 2.17. *Let j be the smallest number such that $\text{ch} W(\lambda)^{(j)} \neq C_j$. Then $\text{ch} W(\lambda)^{(j)} < C_j$ coefficient-wise.*

Proof. Note that the claim is true for global Weyl modules instead of local. Thus the proposition follows from relation (2.16) between the characters. \square

3. EVALUATION MODULES

In this section we consider representations of the Lie algebras $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g} = \mathfrak{sl}_{2n}$.

3.1. Fusion construction. For any $k = 1, \dots, n-1$ let $v_{\omega_k} \in V(\omega_k)$ be a highest weight vector. Given $\zeta \in \mathbb{K}$ and $k = 1, \dots, n-1$ we consider the evaluation $\mathfrak{g} \otimes \mathbb{K}[t]$ -module $V(\omega_k)_\zeta$, which is the cyclic module with cyclic vector v_{ω_k} and the following action of current algebra:

$$(3.1) \quad x \otimes t^a \cdot v = \zeta^a x v, \quad x \in \mathfrak{g}, v \in V(\omega_k).$$

Let $\lambda = \sum_{k=1}^{n-1} m_k \omega_k$ and let $(\zeta_{k,i})$, $1 \leq k \leq n-1$, $1 \leq i \leq m_k$ be a tuple of pairwise distinct elements. We consider the tensor product:

$$V(\lambda)_{(\zeta_{k,i})} = \bigotimes_{k=1}^{n-1} \bigotimes_{i=1}^{m_k} V(\omega_k)_{\zeta_{k,i}}.$$

Proposition 3.1. [CL, FL1, FL2, N] *The module $V(\lambda)_{(\zeta_{k,i})}$ is cyclic with cyclic vector v of weight λ . The t -degree grading on $U(\mathfrak{g} \otimes \mathbb{K}[t])$ gives a filtration on $V(\lambda)_{(\zeta_{k,i})}$:*

$$(3.2) \quad \{0\} = F_{-1} \subset U(\mathfrak{g} \otimes 1)v = F_0 \subset F_1 \subset F_2 \subset \dots$$

The corresponding graded module $\bigoplus_{i=0}^{\infty} F_i/F_{i-1}$ is isomorphic to $W(\lambda)$.

Assume that we have a basis $\{f_{\gamma_{o,1}} \otimes t^{l_{o,1}} \dots f_{\gamma_{o,q_o}} \otimes t^{l_{o,q_o}} v\}$ of the space of elements of weight μ in $V(\lambda)_{(\zeta_{k,i})}$, $\gamma_{o,j} \in \Delta_-$, o runs from 1 to the dimension of this space. Its character is a polynomial

$$\sum_o q^{\sum_{j=1}^{q_o} l_{o,j}} = \sum_{i \geq 0} a_{\mu,i} q^i.$$

Let $\sum_{i \geq 0} b_{\mu,i} q^i$ be the character of the space of elements of weight μ in $W(\lambda)$.

Lemma 3.2. *Let j be the smallest number such that $a_{\mu,j} \neq b_{\mu,j}$. Then $a_{\mu,j} < b_{\mu,j}$.*

Proof. An element

$$(3.3) \quad f_{\gamma_{o,1}} \otimes t^{l_{o,1}} \dots f_{\gamma_{o,q_o}} \otimes t^{l_{o,q_o}} v$$

belongs to $F_{\sum_{i=1}^{q_o} l_{o,i}}$. Then the t -degree of the image of this element in the adjoint graded space (aka Weyl module) is less than or equal to $\sum_{i=1}^{q_o} l_{o,i}$. If j is the smallest number such that $a_{\mu,j} \neq b_{\mu,j}$, then all the elements (3.3) with $\sum_{i=1}^{q_o} l_{o,i} = j$ do not belong to the filtration space $F_{j'}$ for $j' < j$. In addition, there exists a basis element (3.3) with $\sum_{i=1}^{q_o} l_{o,i} > j$ which belongs to F_j . Hence $b_{\mu,j} > a_{\mu,j}$. \square

Corollary 3.3. *If there exists a basis of the tensor product $V(\lambda)_{(\zeta_{k,i})}$ such that its character is equal to the left hand side of (2.15) multiplied by $(q)_\lambda$, then the inequality (2.15) is in fact an equality.*

Proof. The Corollary is implied by Proposition 1.4, Lemma 2.17 and Lemma 3.2. \square

In the rest of the section we construct a basis with the desired property. The construction uses an embedding of a Weyl module attached to \mathfrak{sl}_n into a Weyl module attached to \mathfrak{sl}_{2n} .

3.2. Embedding of Weyl modules. Let $\mathfrak{g} = \mathfrak{sl}_{2n}$ and let v_1, \dots, v_{2n} be the standard basis of the vector representation of \mathfrak{sl}_{2n} . We denote the weight of v_j by ϵ_j . For a cardinality k set $J = (j_1 < \dots < j_k) \subset \{1, \dots, 2n\}$ we denote by $v_J \in V(\omega_k)$ the wedge product $v_{j_1} \wedge \dots \wedge v_{j_k}$. We also denote by $f_{pq} \in \mathfrak{sl}_{2n}$, $p < q$ the matrix unit sending v_p to v_q .

We consider the inclusion $\mathfrak{sl}_n \subset \mathfrak{sl}_{2n}$ via tautological map $e_{ij} \mapsto e_{ij}$, $1 \leq i, j \leq n$, where the first matrix unit is an element of \mathfrak{sl}_n and the second one is an element of \mathfrak{sl}_{2n} . Let $\lambda = \sum_{k=1}^{n-1} m_k \omega_k$ be a dominant weight; in

what follows we consider λ as both \mathfrak{sl}_n and \mathfrak{sl}_{2n} weight. To distinguish these cases we write $\bar{\lambda}$ in the \mathfrak{sl}_{2n} case.

Let v be a cyclic vector of the $\mathfrak{sl}_{2n} \otimes \mathbb{K}[t]$ Weyl module $W(\bar{\lambda})$. The following proposition can be extracted from the results of [CL], but for the readers convenience we give a short proof below.

Proposition 3.4. *The $\mathfrak{sl}_n \otimes \mathbb{K}[t]$ -module $U(\mathfrak{sl}_n)v \subset W(\bar{\lambda})$ is isomorphic to the Weyl module $W(\lambda)$. It is equal to the set of elements of $W(\bar{\lambda})$ with weights in $\mathbb{Z}\langle \epsilon_1, \dots, \epsilon_n \rangle$.*

Proof. Note that all relations of the Weyl module $W(\lambda)$ hold in $U(\mathfrak{sl}_n)v$, so we only need to compare the dimensions. The Weyl module $W(\bar{\lambda})$ is generated from v with the action of the operators $f_{pq} \otimes t^l$. We note that a vector $\{\prod_{i=1}^d f_{i_j i} \otimes t^{x_i} \cdot v\}$ with some $j_a > n$ never has a weight in $\mathbb{Z}\langle \epsilon_1, \dots, \epsilon_n \rangle$. However the dimension of the space of elements of weight in $\mathbb{Z}\langle \epsilon_1, \dots, \epsilon_n \rangle$ is exactly equal to $\dim W(\lambda)$ because of Proposition 3.1. \square

3.3. Combinatorial construction. For an \mathfrak{sl}_{2n} -weight $\lambda = \sum_{k=1}^{n-1} m_k \omega_k$ we consider a tuple of pairwise distinct elements $(\zeta_{k,i})$, $k = 1, \dots, n-1$, $1 \leq i \leq m_k$ and the tensor product:

$$V(\lambda)_{(\zeta_{k,i})} = \bigotimes_{k=1}^{n-1} \bigotimes_{i=1}^{m_k} V(\omega_k)_{\zeta_{k,i}}.$$

We construct a basis of this module.

Let \mathbb{B} be the set consisting of collections $(\mathcal{B}_{k,i})$ labeled by pairs $k = 1, \dots, n-1$, $1 \leq i \leq m_k$, where each $\mathcal{B}_{k,i}$ is the set consisting of elements f_{pq} , $1 \leq p < q \leq n$ satisfying two following conditions:

F1. If $f_{pq} \in \mathcal{B}_{k,i}$, then $p \leq k < q$.

F2. If $f_{p_1 q_1} \in \mathcal{B}_{k,i}$ and $f_{p_2 q_2} \in \mathcal{B}_{k,i}$ then either $p_1 < p_2$ and $q_2 < q_1$ or $p_1 > p_2$ and $q_2 > q_1$.

Example 3.5. For $n = 5$ the set \mathbb{B} contains the element $B = (\mathcal{B}_{2,1}, \mathcal{B}_{3,1})$ defined by $\mathcal{B}_{2,1} = \{f_{1,9}, f_{2,8}\}$, $\mathcal{B}_{3,1} = \{f_{1,10}, f_{2,7}, f_{3,6}\}$.

Lemma 3.6. *For any pair k, i the sets $\mathcal{B}_{k,i}$ satisfying F1. and F2. are in natural bijection with certain basis of $V(\omega_k)$.*

Proof. The basis of $V(\omega_k)$ we need consists of vectors

$$\prod_{j=1}^l f_{p_j q_j} v = v_{\{1, \dots, k\} \setminus \{p_1, \dots, p_l\} \cup \{q_1, \dots, q_l\}},$$

where $p_j > p_{j'}$, $q_j < q_{j'}$ for $j < j'$ (see e.g. [FFL1]). \square

For a pair (k, i) let $v_{k,i}$ be a highest weight vector of the module $V(\omega_k)_{\zeta_{k,i}}$.

Corollary 3.7. *We have a bijection \mathcal{P} between the set \mathbb{B} and the basis of $V(\lambda)_{(\zeta_{k,i})}$ defined by the formula*

$$\begin{aligned} \mathcal{P}((\mathcal{B}_{k,i})) &= \left(\prod_{f_{p,q} \in \mathcal{B}_{1,1}} f_{pq} \right) v_{1,1} \otimes \left(\prod_{f_{p,q} \in \mathcal{B}_{1,2}} f_{pq} \right) v_{1,2} \otimes \dots \\ &\quad \otimes \left(\prod_{f_{p,q} \in \mathcal{B}_{n-1,m_{n-1}}} f_{pq} \right) v_{n-1,m_{n-1}}. \end{aligned}$$

Note that all $f_{p,q}$ showing up in $\mathcal{B}_{k,i}$ commute because of condition $F1$. Therefore this basis is well defined. To a set $\mathcal{B}_{k,i}$ we attach the set $J_{k,i} \subset \{1, \dots, 2n\}$ defined by

$$(3.4) \quad \prod_{f_{pq} \in \mathcal{B}_{k,i}} f_{pq} v_{\omega_k} = v_{J_{k,i}} = \pm \bigwedge_{j \in J_{k,i}} v_j.$$

Then

$$\mathcal{P}((\mathcal{B}_{k,i})) = \pm \bigotimes_{\substack{1 \leq k \leq n-1 \\ 1 \leq i \leq m_k}} v_{J_{k,i}}.$$

We consider the following order on the elements f_{pq} , $1 \leq p < q \leq 2n$:

$$(3.5) \quad f_{12} < f_{13} < f_{23} < f_{14} < f_{24} < f_{34} < f_{15} < \dots$$

We fix an element $B = (\mathcal{B}_{k,i}) \in \mathbb{B}$. For any element $f_{pq} \in \mathcal{B}_{k,i}$ we attach a degree in the following way.

First, we consider the restriction of B to elements less than f_{pq} , i.e. $B' = (\mathcal{B}'_{k,i})$, where $\mathcal{B}'_{k,i} = \mathcal{B}_{k,i} \cap \{f_{ab}, f_{ab} < f_{pq}\}$.

Second, we consider all pairs (k, i) such that $\mathcal{B}'_{k,i} \cup f_{pq}$ satisfies conditions $F1$ and $F2$. We call these pairs (p, q) -admissible.

Third, we define the following order on (p, q) -admissible pairs. Let (k_1, i_1) and (k_2, i_2) be two (p, q) -admissible pairs. Let

$$\mathcal{B}'_{k_1, i_1} = \{f_{p_1, j, q_1, j}, 1 \leq j \leq l_1\}, \quad \mathcal{B}'_{k_2, i_2} = \{f_{p_2, j, q_2, j}, 1 \leq j \leq l_2\}.$$

If $k_1 - l_1 < k_2 - l_2$, then $(k_1, i_1) < (k_2, i_2)$. Assume that $k_1 - l_1 = k_2 - l_2$. We consider vectors

$$\bar{q}_1 = (q_{1, l_1}, \dots, q_{1, 1}) \text{ and } \bar{q}_2 = (q_{2, l_2}, \dots, q_{2, 1}).$$

If $\bar{q}_1 < \bar{q}_2$ in lexicographic order (comparing from left to right, i.e. we first compare q_{1, l_1} with q_{2, l_2} , then $q_{1, l_1 - 1}$ with $q_{2, l_2 - 1}$, etc.), then $(k_1, i_1) < (k_2, i_2)$. Finally if $k_1 - l_1 = k_2 - l_2$, $\bar{q}_1 = \bar{q}_2$ (therefore $k_1 = k_2$) and $i_1 < i_2$, then $(k_1, i_1) < (k_2, i_2)$.

Fourth, we consider all (p, q) -admissible pairs with the above order. For an element $f_{p,q} \in \mathcal{B}_{k,i}$ we attach t -degree $d(p, q, k, i)$ according to the following definition.

Definition 3.8. *$d(p, q, k, i)$ is equal to the number of all (p, q) -admissible pairs (k', i') such that $(k', i') < (k, i)$ and $f_{p,q} \notin \mathcal{B}_{k', i'}$.*

Finally, we define the following element of the universal enveloping algebra $U(\mathfrak{sl}_{2n} \otimes \mathbb{K}[t])$:

$$(3.6) \quad \Pi(B) = \prod_{f_{pq} \in \mathcal{B}_{k,i}} f_{pq} \otimes t^{d(p,q,k,i)}.$$

The product in the formula above is taken in such a way that smaller f_{pq} are applied first (i.e. the smaller f_{pq} show up on the right).

Definition 3.9. We define $\mathcal{F}(B) \in V(\lambda)_{(\zeta_{k,i})}$ as $\Pi(B)v$.

3.4. Basis. Let us write a weight over \mathfrak{sl}_{2n} as a linear combination of ϵ_i , $i = 1, \dots, 2n$; a root vector f_{pq} has the weight $\epsilon_q - \epsilon_p$ and the highest weight of a fundamental representation $V(\omega_k)$ is equal to $\sum_{i=1}^k \epsilon_i$.

Let \mathbb{B}_μ be the set of elements $B \in \mathbb{B}$ such that the weight of $\mathcal{F}(B)$ is equal to μ (equivalently the weight of $\mathcal{P}(B)$ is equal to μ).

Remark 3.10. In what follows we only consider weights $\mu \in \mathbb{Z}_{\geq 0}\langle \epsilon_{n+1}, \dots, \epsilon_{2n} \rangle$. For such a weight let B be an element in \mathbb{B}_μ . Then conditions *F1* and *F2* imply that for each pair (k, i) the set $\mathcal{B}_{k,i}$ consists of elements $f_{p,q}$ satisfying the following conditions: $1 \leq p < n < q \leq 2n$ and $p + q \leq 2n + 1$.

Definition 3.11. A subset $J \subset \{1, \dots, 2n\}$ is called dense if $J \cap \{1, \dots, n\} = \{1, \dots, a\}$ for some a . An element $B \in \mathbb{B}$ is called dense, if all entries $\mathcal{B}_{k,i}$ correspond to dense sets $J_{k,i}$ in the sense of (3.4). A vector $\mathcal{P}(B)$ is called dense if B is dense.

Remark 3.12. We note that B is not dense if and only if there exist a, b, k, i , $a < k$ such that $f_{ab} \in \mathcal{B}_{k,i}$ and there is no element c such that $f_{a+1,c} \in \mathcal{B}_{k,i}$.

Now let us take any f_{pq} and put $\mathcal{B}'_{k,i} = \mathcal{B}_{k,i} \cap \{f_{ab}, f_{ab} < f_{pq}\}$, $B' = (\mathcal{B}'_{k,i})$. We call B' a strict (p, q) -restriction of B . In the following lemma we fix p, q with $1 \leq p < n < q \leq 2n$.

Lemma 3.13. For a \mathfrak{sl}_{2n} -weight $\mu \in \mathbb{Z}_{\geq 0}\langle \epsilon_{n+1}, \dots, \epsilon_{2n} \rangle$ and any $B \in \mathbb{B}_\mu$ its strict (p, q) -restriction B' is dense.

Proof. Follows from the explicit form of the order (3.5) and property *F2*. \square

Recall that we consider irreducible highest weight representations $V(\lambda)$ of \mathfrak{sl}_{2n} with $\lambda = \sum_{k=1}^{n-1} m_k \omega_k$.

Proposition 3.14. For an \mathfrak{sl}_{2n} weight $\mu \in \mathbb{Z}_{\geq 0}\langle \epsilon_{n+1}, \dots, \epsilon_{2n} \rangle$ the elements $\{\mathcal{F}(B)\}$, $B \in \mathbb{B}_\mu$ constitute a basis of the μ -weight space of

$$V(\lambda)_{(\zeta_{k,i})} = \bigotimes_{k=1}^{n-1} \bigotimes_{i=1}^{m_k} V(\omega_k)_{\zeta_{k,i}}.$$

Proof. We divide the proof into four steps. In *Step 1* we consider the decomposition of an element $\mathcal{F}(B)$ in the basis $\mathcal{P}(\mathbb{B}_\mu)$. In *Step 2* we restrict to the "top" summands from the decomposition of *Step 1*. In *Step 3* we further

restrict to the dense summands of the "top" part from *Step 2*. In *Step 4* we finalize the proof by proving the non-degeneracy of the transition matrix from the set $\mathcal{F}(\mathbb{B}_\mu)$ to the dense terms from *Step 3*.

Step 1. Recall that we have order (3.5) on elements f_{pq} . We consider the lexicographic order on sequences $(f_{p_j q_j}, j = 1, \dots, a)$ with $f_{p_j q_j} \geq f_{p_{j+1} q_{j+1}}$. In other words, we first compare the largest elements of two sequences, then the next to the largest elements, etc. This order gives an order on multisets of elements f_{pq} .

Note that the weight of an element $\mathcal{P}((\mathcal{B}_{k,i}))$ depends only on the multiset $M(B) = \sqcup_{k,i} \mathcal{B}_{k,i}$ (i.e. each appearance of an $f_{p,q}$ in a $\mathcal{B}_{k,i}$ increases the multiplicity of $f_{p,q}$ in $M(B)$). For $B \in \mathbb{B}_\mu$ with $\mu \in \mathbb{Z}_{\geq 0} \langle \epsilon_{n+1}, \dots, \epsilon_{2n} \rangle$, the multiset $M(B)$ consists of elements f_{pq} with the property $p < n < q$ (recall that $\lambda \in \mathbb{Z}_{\geq 0} \langle \epsilon_1, \dots, \epsilon_n \rangle$). We denote the weight of $\mathcal{P}((\mathcal{B}_{k,i}))$ by $wt(M(B))$. Note that if $wt(M(B)) \in \mathbb{Z}_{\geq 0} \langle \epsilon_{n+1}, \dots, \epsilon_{2n} \rangle$ then $|M(B)| = \sum_{k=1}^{n-1} km_k$.

We first claim that $\mathcal{F}(B) \in \bigoplus_{M(\tilde{B}) \leq M(B)} \mathbb{K}\mathcal{P}(\tilde{B})$. Indeed, take any summand of $\mathcal{F}(B)$ in the basis $\mathcal{P}(\mathbb{B})$. It is of the form

$$\bigotimes_{\substack{1 \leq k \leq n-1 \\ 1 \leq i \leq m_k}} \prod_{j=1}^{l_{i,k}} f_{p_{j,i,k} q_{j,i,k}} v_{\omega_k} = \mathcal{P}(\tilde{B})$$

and the multiset $\{f_{p_{j,i,k}}\}$ is equal to $M(B)$. Assume that

$$\prod_{j=1}^{l_{i,k}} f_{p_{j,i,k} q_{j,i,k}} v_{\omega_k} = \pm \prod_{j=1}^{l_{i,k}} f_{p'_{j,i,k} q'_{j,i,k}} v_{\omega_k}$$

and the set $\{f_{p'_{j,i,k} q'_{j,i,k}}, j = 1, \dots, l_{i,k}\}$ satisfies condition *F2*. Then it is easy to see that $\{f_{p_{j,i,k} q_{j,i,k}}, j = 1, \dots, l_{i,k}\} \geq \{f_{p'_{j,i,k} q'_{j,i,k}}, j = 1, \dots, l_{i,k}\}$. Therefore $M(\tilde{B}) < M(B)$.

Step 2. Let us fix a multiset M . We prove that the classes of elements $\mathcal{F}(B)$, $M(B) = M$, form a basis of $\bigoplus_{M(B) \leq M} \mathbb{K}\mathcal{P}(B) / \bigoplus_{M(B) < M} \mathbb{K}\mathcal{P}(B)$.

We fix a pair (p, q) and consider the set $\mathcal{B}'_{k,i} = \mathcal{B}_{k,i} \cap \{f_{ab}, f_{ab} < f_{pq}\}$, $B' = (\mathcal{B}'_{k,i})$. Consider the decomposition of $\mathcal{F}(B')$ in the basis $\mathcal{P}(\mathbb{B})$. Take a non-dense summand of this decomposition of the form $\mathcal{P}(\tilde{B})$, $\tilde{B} = (\tilde{\mathcal{B}}_{k,i})$ (see Definition 3.13 and Remark 3.12). Then we have:

$$(3.7) \quad \left(\prod_{\substack{f_{p'q'} \in \mathcal{B}_{k,i} \\ f_{p'q'} \geq f_{pq}}} f_{p'q'} \otimes t^{d(p,q,k,i)} \right) \mathcal{P}(\tilde{B}) \in \bigoplus_{\substack{M(B) < M \\ B \in \mathbb{B}_\mu}} \mathbb{K}\mathcal{P}(B).$$

Indeed, any nonzero summand of the decomposition of the left hand side of (3.7) in the basis $\mathcal{P}(\mathbb{B})$ has (k, i) -th tensor factor of the form $f_{ab} f_{a+1, b'} \prod_{xy} f_{xy} v_{\omega_k}$, $b' > b$ (since $f_{a+1, b'} > f_{a,b}$). Therefore the set $\{f_{ab} f_{a+1, b'}\} \cup \{f_{xy}\}$ does not satisfy condition *F2*, which implies $M(B) < M$.

Step 3. Recall that we have fixed a multiset M and a pair of indices p, q with $1 \leq p < n < q \leq 2n$. Let $\mathbb{B}_\mu^{<pq}$ be the set of strict (p, q) -restrictions of the elements $B \in \mathbb{B}_\mu$ with $M(B) = M$. We denote the elements of $\mathbb{B}_\mu^{<pq}$ by B'_1, \dots, B'_g , so there exist elements $B_1, \dots, B_g \in \mathbb{B}_\mu$, $M(B_j) = M$ such that the strict (p, q) -restriction of B_j is B'_j . For an element B'_j , $1 \leq j \leq g$ we consider the decomposition $\mathcal{F}(B'_j) = X + Y$ in the basis $\mathcal{P}(\mathbb{B})$, where X is the linear combination of non-dense summands and Y is the linear combination of dense summands. We claim that

$$(3.8) \quad Y = \sum_{l=1}^g a_{jl} \mathcal{P}(B'_l)$$

and the matrix (a_{jl}) is non-degenerate. This claim would imply Proposition 3.14 thanks to Step 2 above. We note that Proposition 3.14 follows from the non-degeneracy of the unrestricted matrix (a_{jl}) . We introduce the pq -restriction in order to be able to use the induction on f_{pq} .

We first explain the existence of the decomposition (3.8). Let $(B_j)_{k,i} = (B'_j)_{k,i} \sqcup (\bar{B}_j)_{k,i}$, so $(\bar{B}_j)_{k,i}$ consists of $f_{a,b}$ such that $f_{a,b} \geq f_{p,q}$. Let $\mathcal{P}(B')$ be a dense summand of Y . It suffices to show that there exists a permutation σ of the set $\{(k, i) : 1 \leq k \leq n-1, 1 \leq i \leq m_k\}$ such that for each pair k, i the set $B'_{k,i} \sqcup (B_j)_{\sigma(k,i)}$ satisfies conditions $F1$ and $F2$. Such a σ is constructed as follows: let $a = 1, \dots, n-1$ be the minimal number such that $f_{a,\bullet} \in B'_{k,i}$. We fix a pair k', i' such that a is the minimal number with $f_{a,\bullet} \in (B_j)_{k',i'}$. Then $\sigma(k, i) = (k', i')$. We note that such a pair (k', i') may not be unique, however, there is one for each (k, i) with a as above.

Step 4. In the rest of the proof we show by induction on (p, q) that the matrix (a_{jl}) is non-degenerate. The base of induction $(p, q) = (1, n+1)$ is trivial. Assume that (a_{jl}) is non-degenerate for the strict (p, q) -restriction.

Several observations are in order. First, for a (dense) summand $\mathcal{P}(B'_l)$, $B'_l = ((\mathcal{B}_l)_{k,i})$ there are exactly

$$L = |\{(k, i), f_{p+1,q'} \in \mathcal{B}_{k,i}, q' < q\}| - |\{(k, i), f_{p,\tilde{q}} \in \mathcal{B}_{k,i}, \tilde{q} < q\}|$$

pairs k, i such that $(\mathcal{B}'_l)_{k,i} \cup \{f_{pq}\}$ is still dense. It is important to note that L does not change when we vary l , i.e. L is completely determined by M .

Second, the same number L is the cardinality of the set S_l of pairs (k, i) such that one can add $f_{p,q}$ to $(\mathcal{B}'_l)_{k,i}$ with (k, i) from S_l in such a way that the result is still a restriction of an element $B \in \mathbb{B}_\mu$ with $M(B) = M$. Moreover, the set S_l coincides with the set of pairs (k, i) from the first observation above.

Third, we note that all pairs $(k, i) \in S_l$ are (p, q) -admissible. We observe that if $(k, i) \in S_l$ and $(k', i') \notin S_l$ are two (p, q) -admissible positions, then $(k, i) < (k', i')$ (since $(k', i') \notin S_l$ means that the difference between k' and the cardinality of $(\mathcal{B}'_l)_{k',i'}$ is greater than p , while the same difference for the pair (k, i) is equal to p).

Fourth, we fix $l = 1, \dots, g$ and consider the tensor product of L copies of evaluation fundamental \mathfrak{sl}_2 modules $\bigotimes_{(k,i) \in S(l)} V(\omega)_{\zeta_{k,i}}$. The role of $f \in \mathfrak{sl}_2$ is played by the element $f_{p,q}$. The set \mathbb{B} in this case has 2^L elements (recall that $L = |S_l|$) and the transition matrix $c_{u,u'}(l)$ from $\mathcal{F}(\mathbb{B})$ to $\mathcal{P}(\mathbb{B})$ is non-degenerate. We conclude that the transition matrix from the \mathcal{F} -basis to the \mathcal{P} -basis after $f_{p,q}$ is added (i.e. for $(p+1, q)$ -restriction if $p \leq n-1$ or for the $(n+1, q+1)$ -restriction, if $p = n-1$) is equal to $a_{j,l} c_{u,u'}(l)$. The determinant of this matrix is equal to $\det(a_{j,l})^L \prod_l \det(c_{u,u'}(l))$. This completes the proof of the Proposition. \square

3.5. Generating function. Let w_0 be a permutation interchanging i and $2n+1-i$ and let $\mathbb{B}_{>n} = \cup_{\mu \in \mathbb{N}(\epsilon_{n+1}, \dots, \epsilon_{2n})} \mathbb{B}_\mu$. We also fix the notation $wt(f_{a,c}) = \epsilon_c - \epsilon_a$ and

$$wt(B) = \sum_{\substack{k=1, \dots, n-1, \\ i=1, \dots, m_k}} \sum_{f_{a,c} \in \mathcal{B}_{k,i}} wt(f_{ac}).$$

In particular, wtB is always non positive (negative if B is nonempty). In the rest of the section we prove the following equality:

$$(3.9) \quad \sum_{B \in \mathbb{B}_{>n}} q^{\sum_{k,i} \sum_{f_{ac} \in \mathcal{B}_{k,i}} d(a,c,k,i)} x^{w_0(\lambda + wt(B))} \\ = \sum_{\sum_{|\sigma|=k} r_\sigma = m_k} q^{\sum_{\sigma < \tau} k(\sigma, \tau) r_\sigma r_\tau} \prod_{i=1}^n x_i^{\sum_{\sigma \ni i} r_\sigma} \prod_{k=1}^{n-1} \binom{m_k}{[\mathbf{r}]_k}_q$$

(to be compared with (2.16)).

Recall that we are working with fundamental representations of the Lie algebra \mathfrak{sl}_{2n} . More precisely, we are only interested in the weight spaces corresponding to the weights being linear combinations of $\epsilon_{n+1}, \dots, \epsilon_{2n}$. In a fundamental module $V(\omega_k)$ such vectors are parametrized by

$$(3.10) \quad f_{1,l_1} \dots f_{k,l_k} v_{\omega_k}, \quad n+1 \leq l_k < \dots < l_1 \leq 2n$$

(an important observation is that the number of factors is exactly k). In order to prove (3.9) let us generalize the formula. Namely, we consider the order on the elements f_{ac} , $1 \leq a < n < c \leq 2n$ induced by (3.5):

$$(3.11) \quad f_{1,n+1}, f_{2,n+1}, \dots, f_{n-1,n+1}, f_{1,n+2}, f_{2,n+2}, \dots, f_{n-1,n+2}, \dots, f_{1,2n}, \dots, f_{n-1,2n}.$$

Remark 3.15. *To be precise, we only need the elements f_{pq} with $p+q \leq 2n+1$ (see Remark 3.10).*

Now for a pair a, c we denote by $\mathbb{B}_{>n}^{\leq ac}$ the set of all (a, c) -restrictions of the elements from $\mathbb{B}_{>n}$. In other words, $\mathbb{B}_{>n}^{\leq ac}$ consists of all collections $(\mathcal{B}'_{k,i})$ obtained from a collection $(\mathcal{B}_{k,i}) \in \mathbb{B}_{>n}$ by forgetting all elements greater than $f_{a,c}$ (in the order (3.11)).

The generalization of the formula (3.9) is as follows.

Let I be a subset of the set $\{1, \dots, 2n\}$. We define $wt(I) = \sum_{l \in I} \epsilon_l$. For a pair (a, c) such that $1 \leq a < n < c \leq 2n$ we say that $I \subset \{1, \dots, 2n\}$ is (a, c) -completable if there exists a product $f_{1, l_1} \dots f_{k, l_k}$, $k = |I|$, $l_1 > \dots > l_k$ such that its (a, c) -restriction (in order (3.11)) being applied to the highest weight vector v_{ω_k} is equal (up to a sign) to v_I .

We note that if I is (a, c) -completable, then I is dense, i.e. $I \cap \{1, \dots, n\} = \{1, \dots, r\}$ for some r . Two examples are in order:

- any $I \subset \{n+1, \dots, 2n\}$ is $(n-1, 2n)$ -completable;
- if $(a, c) = (1, n+1)$, then the completable I 's are as follows: $I = \{1, \dots, |I|\}$ if $|I| > 1$; $I = \{1\}$ or $I = \{n+1\}$, if $|I| = 1$.

Remark 3.16. *A dense $I = (1, \dots, r, i_{r+1}, \dots, i_k)$, $r < n$, $2n \geq i_{r+1} > \dots > i_k > n$ is (a, c) -completable if and only if*

- $i_{r+1} \leq c$,
- if $i_{r+1} = c$, then $r+1 \leq a$.

In order to state the generalization of formula (3.9) we need one more piece of notation. Let

$$\begin{aligned} I &= (1, \dots, r_1, i_{r_1+1}, \dots, i_{l(I)}), \quad r_1 < n < i_{l(I)} < \dots < i_{r_1+1} \leq 2n, \\ J &= (1, \dots, r_2, j_{r_2+1}, \dots, j_{l(J)}), \quad r_2 < n < j_{l(J)} < \dots < j_{r_2+1} \leq 2n \end{aligned}$$

be two (a, c) -completable sets. We consider the following sequence $P(I, J) = (p_1, \dots, p_u)$ (to be compared with $P(\sigma, \tau)$, see Definition 2.6).

First, we replace the elements $1, \dots, r_1$ in I and $1, \dots, r_2$ in J with the number $2n+1$.

Second, if $l(I) > l(J)$, then $p_1 = i_{l(I)}$. If $l(I) < l(J)$, then $p_1 = j_{l(J)}$. If $l(I) = l(J)$, then $p_1 = \max(i_{l(I)}, j_{l(J)})$. We put the set $(I$ or $J)$ containing p_1 to the left and the other set to the right. If $l(I) = l(J)$ and $i_{l(I)} = j_{l(J)}$, then we compare $i_{l(I)-1}$ and $j_{l(J)-1}$, etc. until we find $i_m \neq j_m$. Then we put to the left the set containing $\max(i_m, j_m)$. We consider the sets I and J as columns. We note that the numbers in both columns (non-strictly) decrease from top to bottom.

Now we move upstairs in the left column writing the elements to $P(I, J)$ provided the left element is no smaller than the right one. If at some point the sign got changed, we change the column and write the corresponding element to $P(I, J)$. We then continue moving upstairs until the sign is preserved adding the elements we pass to $P(I, J)$. If at some point the sign got changed, we change the column, etc.

Definition 3.17. *We define $k^{ac}(I, J)$ as the number of times we change the columns in $P(I, J)$.*

Example 3.18. *Let $I = \{1, 2, n+4, n+2, n+1\}$, $J = \{1, n+5, n+4, n+3\}$, n large enough. Assume that both I and J are (a, c) -completable. Then we*

change 1 and 2 to $2n + 1$ and we get the following columns

$$\begin{array}{rcl} \mathbf{2n} + \mathbf{1} & \geq & 2n + 1 \\ \mathbf{2n} + \mathbf{1} & > & \mathbf{n} + \mathbf{5} \\ n + 4 & \leq & \mathbf{n} + \mathbf{4} \\ \mathbf{n} + \mathbf{2} & < & \mathbf{n} + \mathbf{3} \\ \mathbf{n} + \mathbf{1} & & \end{array}$$

The set $P(I, J)$ is equal to $\{n + 1, n + 2, n + 3, n + 4, n + 5, 2n + 1, 2n + 1\}$ and $k^{ac}(I, J) = 2$.

For a collection of numbers $\rho = (\rho_I)_I$ labeled by the cardinality k subsets of $\{1, \dots, 2n\}$ and summing up to m_k we use the notation

$$\binom{m_k}{[\rho]_k}_q = \frac{(q)_{m_k}}{\prod_I (q)_{\rho_I}}.$$

We prove the following combinatorial identities parametrized by pairs (a, c) :

Proposition 3.19.

$$(3.12) \quad \sum_{B \in \mathbb{B}_{>n}^{ac}} q^{\sum_{k,i} \sum_{f_{jl} \in \mathcal{B}_{k,i}} d(j,l,k,i)} x^{\lambda + wt(B)}$$

$$= \sum_{\substack{\rho_I \geq 0 \\ I: (a,c)\text{-completable} \\ \sum_{|I|=k} \rho_I = m_k}} q^{\sum_{\{I,J\}} k^{ac}(I,J) \rho_I \rho_J} \prod_{k=1}^{n-1} \binom{m_k}{[\rho]_k}_q x^{\sum_I \rho_I wt(I)}.$$

Proof. The sum in the quadratic form in the right hand side is taken over all unordered pairs of I, J , i.e. each pair appears only once (we note that $k^{ac}(I, J) = k^{ac}(J, I)$ and $k^{ac}(I, I) = 0$). We prove formula (3.12) by induction on a, c in the order (3.11). We note that the $(n - 1, 2n)$ -formula coincides with the desired identity (3.9).

Let us start with the base of induction $(a, c) = (1, n + 1)$. We note that $\mathbb{B}_{>n}^{1,n+1}$ consists of collections $(\mathcal{B}_{k,i})$ with the following entries: all entries $\mathcal{B}_{k,i}$ with $k > 1$ are empty and the entries of the form $\mathcal{B}_{1,i}$ are either empty or equal to $f_{1,n+1}$. Indeed, a non empty entry $\mathcal{B}_{k,i}$ is equal to $f_{1,n+1}$. If k is greater than one, there is no way to extend it to an element of the form (3.10). Now let $1 \leq i_1 < \dots < i_s \leq m_1$ be the numbers such that $\mathcal{B}_{1,i} = \{f_{1,n+1}\}$. Then by Definition 3.8 for i such that $\mathcal{B}_{1,i} = \{f_{1,n+1}\}$ the quantity $d(1, n + 1, k, i)$ is equal to the number of $i' < i$ such that $\mathcal{B}_{1,i'} = \emptyset$. Therefore,

$$\sum_{B \in \mathbb{B}_{>n}^{1,n+1}} q^{\sum_{k,i} \sum_{f_{jl} \in \mathcal{B}_{k,i}} d(j,l,k,i)} x^{\lambda + wt(B)}$$

$$= \sum_{\rho_1 + \rho_{n+1} = m_1} \binom{m_1}{\rho_1, \rho_{n+1}}_q x^{\lambda + \rho_{n+1} wt(f_{1,n+1})}.$$

Here ρ_{n+1} denotes the number of nonempty $\mathcal{B}_{1,i}$ (i.e. the number of i such that $\mathcal{B}_{1,i} = \{f_{1,n+1}\}$). We note that $f_{1,n+1}$ maps v_1 to v_{n+1} , so ρ_{n+1} is equal to the number of times the basis vector v_{n+1} shows up.

Before going to the general induction step, let us write down explicitly the formula for the case $(a, c) = (n-1, n+1)$. In this case the (a, c) -completable B are as follows: either $\mathcal{B}_{k,i} = \emptyset$ or $\mathcal{B}_{k,i} = \{f_{k,n+1}\}$. In other words, (a, c) -completable sets I are either $\{1, \dots, k\}$ or $\{1, \dots, k-1, n+1\}$. It is easy to see that for all such I, J we have $k^{(n-1, n+1)}(I, J) = 0$. According to Definition 3.8 we obtain the following equality

$$\begin{aligned} & \sum_{B \in \mathbb{B}_{>n}^{n-1, n+1}} q^{\sum_{k,i} \sum_{f_{jl} \in \mathcal{B}_{k,i}} d(j,l,k,i)} x^{\lambda + \text{wt}(B)} \\ &= \sum_{\substack{\rho_1 + \rho_{n+1} = m_1 \\ \rho_{12} + \rho_{1,n+1} = m_2 \\ \dots \\ \rho_{1,\dots,n-1} + \rho_{1,\dots,n-2,n+1} = m_{n-1}}} \binom{m_1}{\rho_1, \rho_{n+1}}_q \cdots \binom{m_{n-1}}{\rho_{1,2,\dots,n-1}, \rho_{1,\dots,n-2,n+1}}_q \\ & \quad \times x^{\lambda + \sum_{k=1}^{n-1} \rho_{1,\dots,k-1,n+1} (\epsilon_{n+1} - \epsilon_k)}. \end{aligned}$$

In fact, according to Definition 3.8 in order to compute the left hand side of the above formula we count the number of pairs i_1, i_2 such that $i_1 < i_2$ and $\mathcal{B}_{k,i_1} = \emptyset$, $\mathcal{B}_{k,i_2} = \{f_{k,n+1}\}$. Clearly, the generating function is equal to the product of q -binomial coefficients given in the right hand side of the formula.

We now proceed by induction. Assume that equality (3.12) holds for a pair (a, c) . There are two separate cases: $a + c < 2n + 1$ and $a + c = 2n + 1$. We work out the first case, the second is very similar.

The element $f_{a,c}$ is followed by the element $f_{a+1,c}$ (in our case $a + 1 + c$ is still no larger than $2n + 1$). We consider all admissible places $\mathcal{B}_{k,i}$. The corresponding vectors v_I satisfy the following properties: $I \cap \{1, \dots, n\} = \{1, \dots, a+1\}$ and $I \cap \{n+1, \dots, 2n\} \subset \{n+1, \dots, c-1\}$. Let us consider the set of (a, c) -completable I and the corresponding variables ρ_I showing up in the formula (3.12). We also consider all sets $B = (\mathcal{B}_{k,i})$ showing up in the left hand side of (3.12) (for the pair (a, c)) with fixed numbers ρ_I (for all (a, c) -completable I). In order to pass to the $(a+1, c)$ case we apply the operator $f_{a+1,c}$ to some of the $\mathcal{B}_{k,i}$. As a result some of the (a, c) -completable and $(a+1, c)$ -admissible I got replaced with $I' = I \setminus \{a+1\} \cup \{c\}$. By induction we know that the sum of the terms

$$q^{\sum_{k,i} \sum_{f_{jl} \in \mathcal{B}_{k,i}} d(j,l,k,i)} x^{\lambda + \text{wt}(B)}$$

for all B with the fixed numbers ρ_I for all (a, c) -completable I satisfying $\sum_{|I|=k} \rho_I = m_k$ is equal to

$$(3.13) \quad q^{\sum_{\{I,J\}} k^{ac}(I,J) \rho_I \rho_J} \prod_{k=1}^{n-1} \binom{m_k}{[\rho]_k}_q x^{\sum_I \rho_I \text{wt}(I)}$$

(the sum in the quadratic form is taken over all unordered pairs of (a, c) -completable I, J , i.e. each pair appears only once). We want to control what is happening with the expression (3.13) after we pass from the (a, c) -case to the $(a + 1, c)$ -case. For each (a, c) -completable and $(a + 1, c)$ -admissible I the ρ_I positions of B with $\mathcal{B}_{k,i}$ producing vector v_I are divided into two parts: the first part consists of the positions where $\mathcal{B}_{k,i}$ does not change (hence, I does not change); the second part consists of positions with $\mathcal{B}'_{k,i} = \mathcal{B}_{k,i} \cup f_{a+1,c}$ or, equivalently, $I' = I \setminus \{a + 1\} \cup \{c\}$. We denote the number of the positions from the second group by $\rho_{I'}$; thus, the number of positions of the first group is $\rho_I - \rho_{I'}$. The change from (a, c) to $(a + 1, c)$ amounts in multiplication by the expression

$$(3.14) \quad \prod_I \binom{\rho_I}{\rho_{I'}}_q q^{\sum_{J < I} \rho_{I'}(\rho_J - \rho_{J'})},$$

where the product is taken over all (a, c) -completable and $(a + 1, c)$ -admissible I and the sum in the exponent of q is taken over all pairs of (a, c) -completable and $(a + 1, c)$ -admissible I, J with the condition $J < I$ coming from the rule formulated above Definition 3.8. In fact, $\rho_J - \rho_{J'}$ is the number of $(a + 1, c)$ -admissible places smaller than I , which are not occupied by $f_{a+1,c}$. Multiplying the q -binomial part of (3.14) with the corresponding term of the (a, c) formula (3.12) we obtain

$$(3.15) \quad \prod_{k=1}^{n-1} \binom{m_k}{[\rho]_k}_q \prod_{\substack{I:(a,c)\text{-completable} \\ I:(a+1,c)\text{-admissible}}} \binom{\rho_I}{\rho_{I'}}_q = \prod_{k=1}^{n-1} \binom{m_k}{[\bar{\rho}]_k}_q,$$

where

- in the product $\prod_{k=1}^{n-1} \binom{m_k}{[\rho]_k}_q$ the component of the vector $[\rho]_k$ are numbers ρ_I parametrized by (a, c) -completable I ;
- in the product $\prod_{k=1}^{n-1} \binom{m_k}{[\bar{\rho}]_k}_q$ the component of the vector $[\bar{\rho}]_k$ are numbers $\bar{\rho}_I$ parametrized by $(a + 1, c)$ -completable I .

We note that

$$(3.16) \quad \bar{\rho}_J = \begin{cases} \rho_J - \rho_{J'}, & J \text{ is } (a + 1, c) \text{ admissible,} \\ \rho_J, & J \text{ is not } (a + 1, c) \text{ - admissible.} \end{cases}$$

Therefore,

$$(3.17) \quad q^{\sum_{J < I} \rho_{I'}(\rho_J - \rho_{J'})} = q^{\sum_{J < I} \bar{\rho}_{I'} \bar{\rho}_J}.$$

Let us now write down the function $k^{a+1,c}$. First, if I and J are not $(a + 1, c)$ -admissible, then $k^{a+1,c}(I, J) = k^{a,c}(I, J)$. Second, if I is $(a + 1, c)$ -admissible, but J is not $(a + 1, c)$ -admissible, then

$$k^{a+1,c}(I, J) = k^{a+1,c}(I', J) = k^{a,c}(I, J).$$

Third, if both I and J are $(a+1, c)$ -admissible, then

$$\begin{aligned} k^{a+1, c}(I, J) &= k^{a, c}(I, J), \\ k^{a+1, c}(I', J') &= k^{a, c}(I, J), \\ k^{a+1, c}(I', J) &= \begin{cases} k^{a, c}(I, J), & I < J \\ k^{a, c}(I, J) + 1, & I > J. \end{cases} \end{aligned}$$

Using formula (3.17), we derive the $(a+1, c)$ -case of formula (3.12). \square

Corollary 3.20. *Equality (3.9) holds true.*

Proof. Formula (3.9) is obtained from formula (3.12) for $(a, c) = (n-1, 2n)$ via substitution $\rho_I = r_{w_0}\sigma$, where w_0 is the longest element in the symmetric group S_{2n} (note that all $(n-1, 2n)$ -completable I satisfy $I \subset \{n+1, \dots, 2n\}$). \square

Theorem 3.21. *We have the following formula for the character of the Weyl module $W(\lambda)$ over \mathfrak{sl}_n :*

$$(3.18) \quad \text{ch}W(\lambda) = \sum_{\sum_{|\sigma|=k} r_\sigma = m_k} q^{\sum_{\sigma < \tau} k(\sigma, \tau) r_\sigma r_\tau} \prod_{i=1}^n x_i^{\sum_{\sigma \ni i} r_\sigma} \prod_{k=1}^{n-1} \binom{m_k}{[\mathbf{r}]_k}_q.$$

Proof. The character of \mathcal{F} -basis is equal to the character of the left hand side of (2.15) multiplied by $(q)_\lambda$. Therefore Lemma 2.17 and Lemma 3.2 imply our Theorem. \square

Corollary 3.22. *$\mathbb{W} \simeq \mathcal{M}$ and this algebra is generated by the dual fundamental Weyl modules with the set of quadratic relations (2.8).*

APPENDIX A. SYMPLECTIC RANK TWO CASE

The methods from the main body of this paper can be applied to Weyl modules over Lie algebras of some other types. To illustrate this let us compute the characters of Weyl modules in type C_2 . Both fundamental modules satisfy conditions of Lemma 1.9. The fundamental module $V(\omega_1)^*$ has a basis $\{X_{\pm\epsilon_1}, X_{\pm\epsilon_2}\}$, the fundamental module $V(\omega_2)^*$ has a basis $\{X_{\pm\epsilon_1 \pm \epsilon_2}, X_0\}$. Hence the Weyl module $W(\omega_1)^*$ has a basis $\{X_{\pm\epsilon_1}^{(l)}, X_{\pm\epsilon_2}^{(l)}\}$, the Weyl module $W(\omega_2)^*$ has a basis $\{X_{\pm\epsilon_1 \pm \epsilon_2}^{(l)}, X_0^{(l)}\}$, $l \in \mathbb{N} \cup \{0\}$. Let

$$X_\mu(s) = \sum_{l \geq 0} X_\mu^{(l)} s^l.$$

The algebra $\mathbb{V} = \bigoplus_{\lambda \in P_+} V(\lambda)^*$ satisfies the following relations:

$$(A.1) \quad X_{\epsilon_1 + \epsilon_2} X_{-\epsilon_1 - \epsilon_2} + X_{\epsilon_1 - \epsilon_2} X_{-\epsilon_1 + \epsilon_2} + X_0^2 = 0;$$

$$(A.2) \quad X_{\epsilon_1} X_{-\epsilon_1 + \epsilon_2} + X_{-\epsilon_1} X_{\epsilon_1 + \epsilon_2} + X_{\epsilon_2} X_0 = 0$$

and three relations obtained from (A.2) by the Weyl group. Therefore we have the following relations in algebra \mathbb{W} :

$$(A.3) \quad X_{\epsilon_1+\epsilon_2}(s)X_{-\epsilon_1-\epsilon_2}(s) + X_{\epsilon_1-\epsilon_2}(s)X_{-\epsilon_1+\epsilon_2}(s) + X_0(s)^2 = 0;$$

$$(A.4) \quad X_{\epsilon_1}(s)X_{-\epsilon_1+\epsilon_2}(s) + X_{-\epsilon_1}(s)X_{\epsilon_1+\epsilon_2}(s) + X_{\epsilon_2}(s)X_0(s) = 0$$

and three relations obtained from (A.4) by the Weyl group. Let \mathcal{R} be the set of nine-tuples of numbers $(r_{\pm\epsilon_1}, r_{\pm\epsilon_2}, r_{\pm\epsilon_1\pm\epsilon_2}, r_0)$ such that $r_{\epsilon_1} + r_{-\epsilon_1} + r_{\epsilon_2} + r_{-\epsilon_2} = m_1$, $\sum r_{\pm\epsilon_1\pm\epsilon_2} + r_0 = m_2$. For $\bar{r} \in \mathcal{R}$ define:

$$b(\bar{r}) = r_{\epsilon_2-\epsilon_1}r_{-\epsilon_2+\epsilon_1} + (r_{\epsilon_2+\epsilon_1} + r_{-\epsilon_2+\epsilon_1})r_{-\epsilon_1} + (r_{\epsilon_2+\epsilon_1} + r_{\epsilon_2-\epsilon_1})r_{-\epsilon_2},$$

$$s_1(\bar{r}) = r_{\epsilon_1} - r_{-\epsilon_1} + r_{\epsilon_1+\epsilon_2} + r_{\epsilon_1-\epsilon_2} - r_{-\epsilon_1+\epsilon_2} - r_{-\epsilon_1-\epsilon_2},$$

$$s_2(\bar{r}) = r_{\epsilon_2} - r_{-\epsilon_2} + r_{\epsilon_1+\epsilon_2} - r_{\epsilon_1-\epsilon_2} + r_{-\epsilon_1+\epsilon_2} - r_{-\epsilon_1-\epsilon_2}$$

Therefore for $\lambda = m_1\omega_1 + m_2\omega_2$ we have the following inequality of characters:

$$(A.5) \quad (q)_\lambda^{-1} \sum_{\bar{r} \in \mathcal{R}} \prod q^{b(\bar{r})} x_1^{s_1(\bar{r})} x_2^{s_2(\bar{r})} \binom{m_1}{r_{\epsilon_1}, r_{-\epsilon_1}, r_{\epsilon_2}, r_{-\epsilon_2}}_q \binom{m_2}{r_{\pm\epsilon_1\pm\epsilon_2}, r_0}_q \geq \text{ch}\mathbb{W}(\lambda).$$

For a tuple of pairwise distinct elements $(\zeta_{k,i})$, $1 \leq i \leq m_k$, $k = 1, 2$ we consider the tensor product:

$$V(\lambda)_{(\zeta_{k,i})} = \bigotimes_{k=1}^2 \bigotimes_{i=1}^{m_k} V(\omega_k)_{\zeta_{k,i}}.$$

Let v be the highest weight vector of this tensor product.

The negative root vectors of \mathfrak{sp}_4 are of the form $f_{-2\epsilon_1}, f_{-\epsilon_2-\epsilon_1}, f_{-2\epsilon_2}, f_{-\epsilon_2+\epsilon_1}$. We consider the set \mathbb{B} of collections $B = (\mathcal{B}_{k,i})$, $k = 1, 2$, $1 \leq i \leq m_k$, where $\mathcal{B}_{1,i}$ is either empty or contains one of the following elements: $f_{-\epsilon_2-\epsilon_1}, f_{-2\epsilon_2}$, $f_{-\epsilon_2+\epsilon_1}$; $\mathcal{B}_{2,i}$ is equal to one of five following sets:

$$\emptyset, \{f_{-2\epsilon_1}\}, \{f_{-\epsilon_2-\epsilon_1}\}, \{f_{-2\epsilon_2}\}, \{f_{-2\epsilon_1}, f_{-2\epsilon_2}\}.$$

Define the following order on elements f_α :

$$f_{-2\epsilon_1} < f_{-\epsilon_2-\epsilon_1} < f_{-2\epsilon_2} < f_{-\epsilon_2+\epsilon_1}.$$

For any $B \in \mathbb{B}$ the strict α -restriction B' of B is defined by $B' = (\mathcal{B}'_{k,i})$, where $\mathcal{B}'_{k,i} = \mathcal{B}_{k,i} \cap \{f_\gamma, f_\gamma < f_\alpha\}$. We define α -admissible pairs in the following way. A pair (k, i) is α -admissible if the α -restriction $\mathcal{B}'_{k,i}$ is empty or $\alpha = -2\epsilon_2$, $k = 2$ and $\mathcal{B}'_{k,i} = \{f_{-2\epsilon_1}\}$. We define an order on α -admissible pairs in the following way: $(k, i) < (k, j)$ if $i < j$ and $(2, i) < (1, j)$. Then for $f_\alpha \in \mathcal{B}_{k,i}$ define $d(\alpha, k, i)$ as the number of α -admissible pairs $(k', i') < (k, i)$ such that $f_\alpha \notin \mathcal{B}_{k',i'}$. Then:

$$\mathcal{F}(B) = \prod_{f_\alpha \in \mathcal{B}_{k,i}} f_\alpha \otimes t^{d(\alpha, k, i)} v.$$

Proposition A.1. *The set $\{\mathcal{F}(B)\}$ is a basis of $V(\lambda)_{(\zeta_{k,i})}$.*

Proof. The proof is analogous to the proof of Proposition 3.14. \square

Moreover by direct computations we obtain that the character of the set $\mathcal{F}(\mathbb{B})$ is equal to the left hand side of (A.5) multiplied by $(q)_\lambda$. Thus we get the character formula for Weyl modules over Lie algebras of type C_2 . In order to generalize the combinatorial construction to the higher rank symplectic algebras one will need the monomial bases constructed in [FFL2].

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