Higher cyclic operads

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We introduce a convenient definition for weak cyclic operads, which is based on unrooted trees and Segal conditions. More specifically, we introduce a category Ξ of trees, which carries a tight relationship to the Moerdijk–Weiss category of rooted trees Ω. We prove a nerve theorem exhibiting colored cyclic operads as presheaves on Ξ which satisfy a Segal condition. Finally, we produce a Quillen model category whose fibrant objects satisfy a weak Segal condition, and we consider these objects as an up-to-homotopy generalization of the concept of cyclic operad.

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For certain operads, such as the moduli space of Riemann spheres with labeled punctures or the endomorphism operad of a vector space V equipped with a nondegenerate bilinear form, there is not really a qualitative difference between the notion of input and output. Indeed, in the former case, the "output" of a given element arises solely from our choice of labels and not the underlying geometry, while in the latter case we have natural isomorphisms

$$\text{End}_V(n) = \hom(V^{\otimes n}, V) = \hom(V^{\otimes n}, V^*) = \hom(V^{\otimes n+1}, k).$$
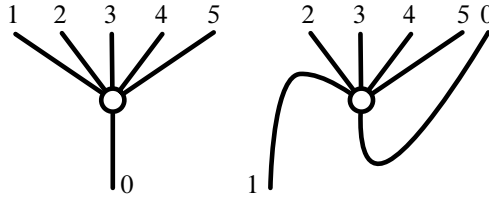
Figure 1: f and $f\tau$

Given a cyclic operad, one can begin talking about graph homology (see Kontsevich [29] as well as the generalizations of Conant and Vogtmann [17]), whereas algebras over cyclic operads admit a cyclic homology theory; see Getzler and Kapranov [22].

This consideration leads directly to the notion of *cyclic operad*, introduced by Getzler and Kapranov [22] (although we add the axiom due to van der Laan; see [30, Section 11] and Markl, Shnider and Stasheff [35, Section II.5.1]). A cyclic operad is an operad O with extra structure, namely an action of the cyclic group $C_{n+1} = \langle \tau \rangle$ on the space $O(n)$. Applied to an element $f \in O(n)$, we should regard $f\tau \in O(n)$ as f with the first input changed to the output and the output changed to the last input, as in Figure 1. To get a feel for how this cyclic operator should act on compositions, one should look at trees with several vertices like the one in Figure 2.

Further examples of cyclic operads include the associative, Lie and commutative operads, (certain models for) the framed little n–disks operad (see Budney [11] and Kimura, Stasheff and Voronov [27]), the A_∞ operad (see Getzler and Kapranov [22]) and also any monoid with involution (see Lindahl and Maserick [31] and Sz-Nagy [43]) — that is, the involution $x \mapsto x^\dagger$ satisfies $x^\dagger y^\dagger = (yx)^\dagger$ — regarded as an operad concentrated in degree 1. The last of these is useful for giving small examples (see Example 8.9 and Proposition 8.10), but also gives a connection with another interesting class of mathematical objects.

A dagger category (see Selinger [41, Definition 2.2]) is a category $\mathcal{C}$ together with an involutive functor $\dagger\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ which is the identity on objects. In other words, a dagger
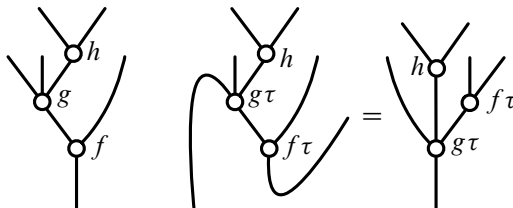
Figure 2: A composition of f, g and h and the action of τ on this composition

structure on $\mathcal{C}$ is an assignment $(f: X \to Y) \mapsto (f^\dagger: Y \to X)$ satisfying $f^{\dagger\dagger} = f$ and $f^\dagger g^\dagger = (gf)^\dagger$; this is the many-objects version of a monoid with involution. As Baez argued eloquently in [2], to understand the similarities between general relativity and quantum theory, one should begin by considering the natural dagger structure on the category of n–cobordisms and on the category of Hilbert spaces, respectively. Other important examples of dagger categories include any groupoid, categories of relations and categories of correspondences.

Colored cyclic operads are a simultaneous generalization of cyclic operads (which we might term "monochrome cyclic operads") and of dagger categories. There are additional examples in the literature (eg Cohen and Voronov [16, Section 5.4] and Tsygan [44, Section 3.11.1]), and the concept provides a bridge to approaches to higher operads based on colored operads. If O is a $\mathcal{C}$–colored operad and $n \geq 0$, then the object $O_n = \coprod_{c,c_1,\ldots,c_n \in \mathcal{C}} O(c_1,\ldots,c_n;c)$ admits a right action by $\Sigma_n = \mathrm{Aut}\{1,\ldots,n\}$ compatible with operadic composition. Write $\Sigma_n^+ = \mathrm{Aut}\{0,1,\ldots,n\}$ and identify Σ_n as the isotropy group of 0.

Definition 0.1 A *cyclic structure* on a $\mathcal{C}$–colored operad O is a collection of maps $- \cdot \sigma \colon O(c_1,\ldots,c_n;c_0) \to O(c_{\sigma(1)},\ldots,c_{\sigma(n)};c_{\sigma(0)})$ for $\sigma \in \Sigma_n^+$ and $c_i \in \mathcal{C}$. These should satisfy two conditions. First, they assemble into a right Σ_n^+ action on O_n, which agrees with the existing Σ_n action. For the second condition, let $\tau_{n+1} \in \Sigma_n^+$ be the element with $\tau_{n+1}(n) = 0$ and $\tau_{n+1}(i) = i + 1$ for $0 \leq i < n$. We insist that if $g \in O(c_1,\ldots,c_k;c_0)$ and $f \in O(d_1,\ldots,d_\ell;c_i)$ are composable at position i, then[1]

$$(1) \qquad (g \circ_i f) \cdot \tau_{k+\ell} = \begin{cases} (g \cdot \tau_{k+1}) \circ_{i-1} f & \text{if } 2 \leq i \leq k, \\ (f \cdot \tau_{\ell+1}) \circ_\ell (g \cdot \tau_{k+1}) & \text{if } i = 1 \text{ and } \ell \neq 0. \end{cases}$$

Informally, when O is equipped with a cyclic structure, we will say that O is a $\mathcal{C}$–*colored cyclic operad*.[2]

The up-to-homotopy cyclic operads that we develop in this paper are a variation on "dendroidal models" for ∞–operads (see Bergner and Hackney [9], Cisinski and

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[1] We exclude the case $i = 1$, $\ell = 0$ from (1), as the formula $(g \circ_1 f) \cdot \tau = (g \cdot \tau^2) \circ_k f$ follows from the first case. Indeed,

$$[(g \cdot \tau_{k+1}^2) \circ_k f] \cdot \tau_k^{k-1} = (g \cdot \tau_{k+1}^{2+k-1}) \circ_{k-(k-1)} f = (g \cdot \tau_{k+1}^{k+1}) \circ_1 f = g \circ_1 f.$$

[2] We should note that our definition is not a symmetric version of the "cyclic multicategories" of Cheng, Gurski and Riehl [12]. A nonsymmetric colored cyclic operad is a nonsymmetric colored operad O together with an action of the subgroup $\langle \tau_{n+1} \rangle \leq \Sigma_n^+$ on O_n such that (1) holds. These form a reflective subcategory of the category of cyclic multicategories.

Moerdijk [13; 14; 15] and Moerdijk and Weiss [37]). The dendroidal category Ω is a category of rooted trees (see Moerdijk and Weiss [36]); each such rooted tree T (with edge set $\mathrm{Ed}(T)$) can be regarded as a free object in the category of $\mathrm{Ed}(T)$–colored operads. The dendroidal category is then defined to be the full subcategory of the category of all colored operads whose objects are the rooted trees. Not only is Ω defined as a subcategory of colored operads, but it turns out that there is a model structure on the category of presheaves of Ω (see [13, Theorem 2.4]) that is Quillen equivalent to a model structure on the category of simplicially enriched colored operads (see [15]). This is an extension of the equivalence between the Joyal model structure on simplicial sets and the Bergner model structure on simplicially enriched categories (see Bergner [8] for references).

It would be a very ambitious project to attempt to do all of the above for colored cyclic operads, and we are skeptical that the full Cisinski–Moerdijk program can be carried out in the cyclic case. A key difficulty is that the adjunction between categories and dagger categories is badly behaved, in particular with respect to equivalences. Thus, in the present paper we limit what is said about colored cyclic operads. It is true that every unrooted tree S freely generates an $\mathrm{Ed}(S)$–colored cyclic operad $C(S)$ (see Section 5), but we do not ever consider the full subcategory of colored cyclic operads spanned by the unrooted trees. The cyclic operad $C(S)$ is nearly always infinite, even when S is a linear tree, and arbitrary maps $C(S) \to C(R)$ do not admit decompositions into cofaces and codegeneracies, as they do in the dendroidal setting. Instead, we directly construct a category Ξ of unrooted trees that is reminiscent of Ω. The assignment $S \mapsto C(S)$ gives a faithful, nonfull functor from Ξ to **Cyc** (Theorem 5.6, Example 5.7). We use this to prove a nerve theorem for colored cyclic operads (Theorem 6.7).

Our main goal is to propose a model for weak monochrome cyclic operads. These are called *Segal cyclic operads* in Section 8, and they are certain reduced presheaves satisfying a Segal condition. The Segal cyclic operads are patterned after the Segal operads appearing in the work of Bergner and the first author [9], which have become important in current work of Boavida, Horel and the second author on profinite completions of the framed little disks operad.

The profinite completion of a product of spaces is weakly equivalent — but in general not isomorphic — to the product of the profinite completions. For this reason, the profinite completion of an operad does not yield an operad on the nose, but rather an ∞–operad. This fact has played a crucial role in work of Horel [25] when he generalized work of Fresse [20] and computed a profinite version of the Grothendieck–Teichmüller

group $\widehat{\mathrm{GT}} \cong \pi_0 \operatorname{End}^h(\hat{D}_2)$, where D_2 is the little 2–disks operad. In work by Boavida, Horel and the second author, they show that considering the framed little 2–disks as an operad, they recover exactly the same result, ie $\widehat{\mathrm{GT}} \cong \pi_0 \operatorname{End}^h(\hat{D}_2) \cong \pi_0 \operatorname{End}^h(\widehat{fD_2})$. Considering fD_2 as a cyclic operad would necessarily result in a smaller set of endomorphisms and conjecturally would provide refinement on these computations; of course one would expect the profinite completion of a cyclic operad to be some type of infinity cyclic operad. Providing a good foundation for this project is one of the major motivations for the present paper.

Overview

We give a brief outline of the paper. Each section begins with a more substantial summary of its contents.

The first section is dedicated to the construction of the category Ξ of unrooted trees. In the second section, we examine exactly how close Ξ is to the category Ω of rooted trees. The third and fourth sections are devoted to two structures on the category Ξ: a generalized Reedy structure and an active/inert (or generic/free) weak factorization system.

The next two sections deal with the relationship of Ξ to colored cyclic operads. In the fifth section we construct the functor $\Xi \to \mathbf{Cyc}$, and in the sixth we prove a nerve theorem for colored cyclic operads.

The final two sections are devoted to model-categorical matters. The penultimate section is about the model structure on diagrams indexed by a generalized Reedy category, and at the beginning of the section we show that this model structure usually has properties which ensure that Bousfield localizations exist. We then restrict to the case when the base category is the category of simplicial sets. In Section 7.1 we discuss certain cases when categories of *reduced* presheaves of simplicial sets admit model structures. In Section 7.2 we show that these model categories are in fact simplicial model categories.

In the last section we prove the existence of a model structure on reduced Ξ–presheaves in simplicial sets whose fibrant objects, the Segal cyclic operads, satisfy a Segal condition. We show that there is a Quillen adjunction (which is not a Quillen equivalence) between this model structure and the model structure for Segal operads from [9].

Finally, in an appendix, we discuss certain additional (co)tensorings by Σ_2–simplicial sets, which exist for Ξ–presheaves which vanish on nonlinear trees.

Notational conventions

If $\mathcal{C}$ is a category, we will write $\mathcal{C}(x, y)$ or $\hom(x, y)$ for the set of morphisms from x to y, depending on if the name of our category is short (eg $\mathcal{C} = \Xi$) or long (eg $\mathcal{C} = \mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$). We will write $\mathrm{Iso}_{\mathcal{C}}(x, y)$ for the isomorphisms from x to y, $\mathrm{Aut}_{\mathcal{C}}(x) := \mathrm{Iso}_{\mathcal{C}}(x, x)$ for the invertible self-maps of x, and $\mathrm{Iso}(\mathcal{C})$ for the wide subcategory of $\mathcal{C}$ consisting of all of the isomorphisms. In all adjunctions $\mathcal{C} \rightleftarrows \mathcal{D}$, the top arrow denotes the left adjoint.

Throughout this paper we use freely the language of Quillen model categories and take the book of Hirschhorn [24] as our standard reference.

Acknowledgements

1 The unrooted tree category Ξ

The main goal of this section is to define a category of unrooted trees Ξ. We will begin with a formalism for general graphs, before defining the objects of Ξ in Definition 1.3. We give two distinct descriptions of the morphisms of Ξ in Definitions 1.12 and 1.13. Each has its own advantage: morphisms in the former sense (here called *complete*) immediately form a category, while morphisms in the latter sense are specified by a smaller set of data and are easier to work with in most situations. We then embark on a sustained study of the nature of these morphisms; key tools are the notions of distance and a (minimal) path in a tree. Along the way, we recover the Moerdijk–Weiss dendroidal category Ω. Finally, in Proposition 1.32, we show that the two definitions of morphisms coincide.

At the heart of this work is the notion of "graph with legs". One can choose several formalisms; for concreteness, let us say that an *undirected graph with legs* consists of two finite sets E and V and a function $\mathrm{Nbhd}\colon V \to \mathcal{P}(E)$ (the set of subsets of E). This data should satisfy one axiom, namely that, for each $e \in E$,

$$\big|\{v \in V \mid e \in \mathrm{Nbhd}(v)\}\big| \leq 2.$$

We will package the triple (E, V, Nbhd) into a single symbol G, and write $\mathrm{Ed}(G) = E$ and $\mathrm{Vt}(G) = V$. Edges actually come in two types, namely interior edges

$$\mathrm{Int}(G) = \{e \in E \mid e \in \mathrm{Nbhd}(v) \cap \mathrm{Nbhd}(w) \text{ for some } v \neq w\}$$

and the set of legs

$$\mathrm{Legs}(G) = \mathrm{Ed}(G) \setminus \mathrm{Int}(G),$$

which are edges incident to at most one vertex.[3] If v is a vertex of G, we also write $|v|$ for the valence of v, or the cardinality of the set $\mathrm{Nbhd}(v)$.

Every graph has an underlying topological space, which can be described as follows. See Figure 3 (left) for an example.

Definition 1.1 (space associated to a graph) Fix an ϵ with $0 < \epsilon < 1$, which we can use to scale the closed unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. Define

$$|\stackrel{\star}{}_0| = \epsilon\mathbb{D} = \{re^{i\theta} \mid 0 \leq r \leq \epsilon, \, 0 \leq \theta \leq 2\pi\} \subsetneq \mathbb{D} \subsetneq \mathbb{C}$$

and, for $n > 0$,

$$|\stackrel{\star}{}_n| = \epsilon\mathbb{D} \cup \bigcup_{k=0}^{n-1}\{re^{(k/n)2\pi i} \mid 0 \leq r \leq 1\},$$

considered as a subspace in the closed unit disc of the complex plane. If G is an undirected graph with legs, fix bijections

$$\kappa_v\colon \mathrm{Nbhd}(v) \stackrel{\cong}{\longrightarrow} \{e^{(k/|v|)2\pi i}\} = S^1 \cap |\stackrel{\star}{}_{|v|}|$$

and define

$$|G| = \left(\frac{\bigsqcup_{v \in \mathrm{Vt}(G)} |\stackrel{\star}{}_{|v|}|}{\kappa_v(e) \sim \kappa_w(e)}\right) \amalg Q \times [0, 1],$$

where $e \in \mathrm{Nbhd}(v) \cap \mathrm{Nbhd}(w)$ and $Q = \mathrm{Ed}(G) \setminus \bigcup_{v \in \mathrm{Vt}(G)} \mathrm{Nbhd}(v)$.

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[3] If $e \in \mathrm{Ed}(G)$ is not incident to any vertex, then one should really think that e appears twice in $\mathrm{Legs}(G)$. Since we are only concerned with connected graphs for the bulk of this paper, only one graph (see Example 1.4) has an edge with this property, so we will just systematically single out that special case.
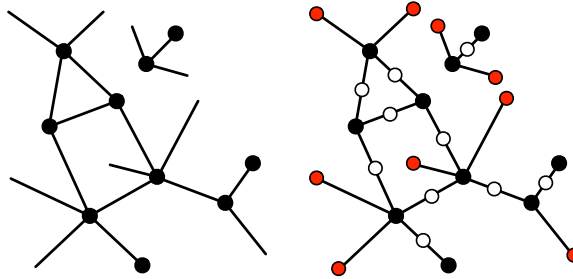
Figure 3: A graph with legs and its corresponding rwb graph

Notice that the homeomorphism type of $|G|$ determines the isomorphism type of G. This would not be the case if we did not add some thickness at the centers of $|\star_n|$ by using the $\epsilon\mathbb{D}$. Indeed, a variation of realization with $\epsilon = 0$ produces the closed unit interval $[0, 1]$ on both the graph G_1 with one edge and no vertices and on the graph G_2 with one vertex v, one edge e and $\mathrm{Nbhd}(v) = \{e\}$.

The following is an alternative, equivalent formalism for graph with legs.

Definition 1.2 (red–white–black formalism)   An *rwb graph* is an ordinary undirected graph (see for example [18, Section 1.1]) where each vertex is colored either red, white or black and such that

- red vertices are univalent,
- white vertices are bivalent and are only adjacent to black vertices, and
- a black vertex is not adjacent to any other black vertex.

From a graph with legs, we can form an rwb graph by coloring all vertices black, adding a white vertex on each interior edge, and adding a red vertex to the loose end of each leg. Each rwb graph determines a graph with legs by deleting the white vertices and joining the edges on either side and deleting all of the red vertices. See Figure 3 for an illustration of this correspondence.

1.1   Trees

The category Ξ governing cyclic dendroidal sets has "unrooted" or "cyclic" trees as objects.

Definition 1.3   An *unpinned tree S* is an undirected graph with legs which is contractible, has at least one leg and is equipped with bijections

$$\mathrm{ord}^v\colon \{0, 1, \ldots, n_v\} \xrightarrow{\;\cong\;} \mathrm{Nbhd}(v),$$
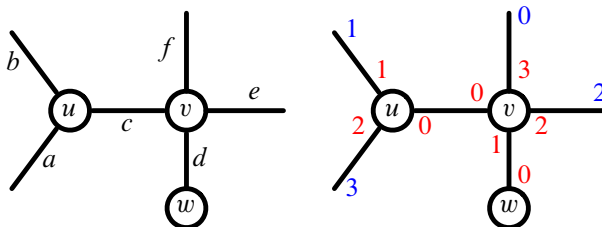
Figure 4: Underlying graph (left) and unpinned and pinned tree data (right)

where $\mathrm{Nbhd}(v) \subseteq \mathrm{Ed}(S)$ is the set of vertices adjacent to v. A *pinned tree*, or just *tree*, has, in addition, a map

$$\mathrm{ord} \colon \{0, 1, \ldots, n\} \to \mathrm{Legs}(S),$$

where $\mathrm{Legs}(S) \subseteq \mathrm{Ed}(S)$ is the set of legs of S, which is a bijection if S contains a vertex and is otherwise the unique map from $\{0, 1\}$ to the single edge.

A typical example of such a graph is found in Figure 4. In pictures of graphs, we will always draw the ordered set of legs $\mathrm{Nbhd}(v)$ in a counterclockwise fashion. Using this convention, to specify the unpinned structure we only need to mark the edges $\{\mathrm{ord}^v(0)\}_v$ in the figures.

Example 1.4 Let us fix several foundational examples of trees (Figure 5):

- The graph with one edge and no vertices, which we write as η.

- For each $n > 0$, the graph $\star_n$. This graph has a single vertex v and n edges $\{0, 1, \ldots, n-1\}$ (and take $\mathrm{ord}^v = \mathrm{ord} = \mathrm{id}$).[4]

- For $n \geq 0$, the linear graph L_n with n distinct vertices $\{v_1, \ldots, v_n\}$; $n+1$ distinct edges $\{e_0, \ldots, e_n\}$; $\mathrm{Nbhd}(v_i) = \{e_{i-1}, e_i\}$, so that $\mathrm{ord}^{v_i}(t) = e_{i-1+t}$; $\mathrm{ord}(0) = e_0$; and $\mathrm{ord}(1) = e_n$. Note that $L_0 \cong \eta$.

- We will call any tree with all vertices bivalent a linear graph.

Remark 1.5 A directed tree is a tree S where each edge has an orientation (see Figure 7); another way to say this when $S \neq \eta$ is to say that there are partitions[5]

$$\mathrm{Nbhd}(v) = \mathrm{out}'(v) \sqcup \mathrm{in}(v), \quad \mathrm{Legs}(S) = \mathrm{out}'(S) \sqcup \mathrm{in}(S),$$

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[4]There is a shift in index compared with [35, page 250]: they use the notation $*_n$ for what we call $\star_{n+1}$.

[5]Outside of this remark, we will only consider directed trees which are rooted, hence we will consider $\mathrm{out}(S)$ and $\mathrm{out}(v)$ as single edges, rather than the one-element sets which contain them; see Definition 1.20.
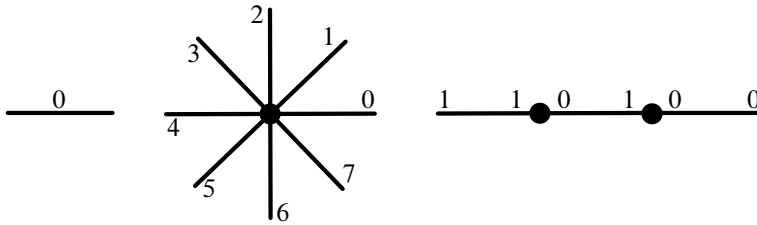
Figure 5: The trees $\eta = L_0$, $\bigstar_8$ and L_2

such that $\operatorname{out}'(v) \cap \operatorname{out}'(w) = \varnothing = \operatorname{in}(v) \cap \operatorname{in}(w)$ for $v \neq w$ and $\operatorname{out}'(v) \cap \operatorname{in}(S) = \varnothing = \operatorname{in}(v) \cap \operatorname{out}'(S)$. This description actually has a little bit more information floating around than we would like, namely a $(|\operatorname{in}(S)|, |\operatorname{out}'(S)|)$–shuffle and, for each $v \in \operatorname{Vt}(S)$, a $(|\operatorname{in}(v)|, |\operatorname{out}'(v)|)$–shuffle. This is simply because for a directed graph we only need separate orderings on the inputs and outputs, not orderings on the entire neighborhoods.

Making a choice for the (p, q)–shuffles above, every directed tree determines a tree. We will use the convention that the total order on $\operatorname{Nbhd}(v)$ is determined by that on $\operatorname{out}'(v)$ and $\operatorname{in}(v)$ by insisting, for $e \in \operatorname{out}'(v)$ and $e' \in \operatorname{in}(v)$, that $e < e'$. Similarly, we get an order on $\operatorname{Legs}(S)$ by saying $\operatorname{out}'(S) < \operatorname{in}(S)$ (unless S consists of a single edge). This convention makes it so that for a rooted tree, the downward edge is always labeled by "0". This gives the map $\operatorname{Ob}(\Omega) \to \operatorname{Ob}(\Xi)$; we will actually define a variant of Ω in Definition 1.20.

1.2 Morphisms of Ξ

When discussing subgraphs of trees, we will always assume that they are nonempty and connected, and contain all edges incident to any of their vertices.

Definition 1.6 A *subgraph of a tree* S consists of a pair of subsets

$$V \subseteq \operatorname{Vt}(S), \quad E \subseteq \operatorname{Ed}(S)$$

such that

- if $v \in V$, then $\operatorname{Nbhd}(v) \subseteq E$ (which means that $R = (V, E, \operatorname{Nbhd})$ constitutes the structure of an undirected graph without orderings),

- the underlying space of the graph $R = (V, E, \operatorname{Nbhd})$ is contractible.

Write $\operatorname{Sbgph}(S)$ for the set of subgraphs of S.

Remark 1.7 Subgraphs of S are naturally *unpinned* trees. The orderings ord^v at each vertex v are inherited from those in S.

Example 1.8 Each edge $e \in S$ constitutes a subgraph with $E = \{e\}$ and $V = \varnothing$. We will write this subgraph as $|_e$.

Example 1.9 For each $v \in \text{Vt}(S)$, there is a subgraph $\star_v$ with $V = \{v\}$ and $E = \text{Nbhd}(v)$. Thus, we have an inclusion $\star \colon \text{Vt}(S) \hookrightarrow \text{Sbgph}(S)$. Notice that $\star_v$ has a preferred ordering with

$$\text{ord}^v_{\star_v} = \text{ord}_{\star_v} = \text{ord}^v_S \colon \{0, 1, \ldots, n\} \xrightarrow{\cong} \text{Nbhd}(v) = \text{Legs}(\star_v).$$

Proposition 1.10 *If R and R' are subgraphs of S and $R \cap R' \neq \varnothing$, then $R \cup R'$ is also a subgraph of S.*

Proof Write $R = (V, E)$ and $R' = (V', E')$. The first condition we need to check for $R \cup R' = (V \cup V', E \cup E')$ is immediate, and does not require the hypothesis. The hypothesis $R \cap R' \neq \varnothing$ means $(V \cap V') \cup (E \cap E') \neq \varnothing$, which implies that the underlying space of $R \cup R'$ is connected (since it is the union of the underlying spaces of R and R'). Thus, it is a connected subspace of a contractible graph, hence is contractible as well. $\qquad\square$

Definition 1.11 (boundary of a subgraph) Suppose that S is a tree.

- If X is a set, let $\mathbf{M}X = \coprod_{n \geq 0} X^{\times n} / \Sigma_n$ be the free commutative unital monoid on X (that is, the set of unordered lists of elements of X).
- There is a function $\eth \colon \text{Sbgph}(S) \to \mathbf{M}(\text{Ed}(S))$ with

$$\eth(R) = \begin{cases} e^2 & \text{if } R = |_e, \\ \prod_{e \in \text{Legs}(R)} e & \text{otherwise.} \end{cases}$$

We say that $\eth(R)$ is the *boundary* of the subgraph R.

If $R, T \in \text{Sbgph}(S)$, then the graph $R \cap T$ is either empty or it is also in $\text{Sbgph}(S)$. Further, $R \cup T \in \text{Sbgph}(S)$ if and only if $R \cap T$ is nonempty. We will say that R and T *overlap* if $R \cap T$ is nonempty (equivalently, if $R \cup T$ is connected).

Definition 1.12 Suppose that S and R are two trees. A *complete morphism* $R \to S$ consists of two functions

- $\alpha_0 \colon \text{Ed}(R) \to \text{Ed}(S)$,
- $\alpha_1 \colon \text{Sbgph}(R) \to \text{Sbgph}(S)$,

that satisfy the following conditions:

(1) The equation $\eth \circ \alpha_1 = (\mathbf{M}\alpha_0) \circ \eth$ holds.

(2) If $T, T' \in \mathrm{Sbgph}(R)$ overlap, then so do $\alpha_1(T)$ and $\alpha_1(T')$. Furthermore,

  (a) $\alpha_1(T \cap T') = \alpha_1(T) \cap \alpha_1(T')$, and

  (b) $\alpha_1(T \cup T') = \alpha_1(T) \cup \alpha_1(T')$.

The set $\mathrm{Sbgph}(R)$ is actually a partial lattice and the second condition just states that α_1 is a map of partial lattices. As the properties above are closed under function composition, there is a category of trees whose morphisms are complete morphisms.

Notice in particular that the existence of the function $\star\colon \mathrm{Vt}(R) \hookrightarrow \mathrm{Sbgph}(R)$ means that every complete morphism has an associated function $\mathrm{Vt}(R) \to \mathrm{Sbgph}(S)$.

Definition 1.13 Suppose that R and S are trees.

- A *morphism* $\phi\colon R \to S$ is defined to be a pair of maps

$$\phi_0\colon \mathrm{Ed}(R) \to \mathrm{Ed}(S), \quad \phi_1\colon \mathrm{Vt}(R) \to \mathrm{Sbgph}(S)$$

  satisfying the following:

  (1) If v is not bivalent (that is, $|\mathrm{Nbhd}(v)| \neq 2$), then $\phi_0|_{\mathrm{Nbhd}(v)}$ is injective.

  (2) For each vertex v, $\phi_0(\mathrm{Nbhd}(v)) = \mathrm{Legs}(\phi_1(v))$ (as unordered sets).

  (3) $\mathrm{Vt}(\phi_1(v)) \cap \mathrm{Vt}(\phi_1(w)) = \varnothing$ for $v \neq w$.

- The *identity map* $\mathrm{id}_R\colon R \to R$ is given by letting $(\mathrm{id}_R)_0 = \mathrm{id}_{\mathrm{Ed}(R)}$ and letting $(\mathrm{id}_R)_1$ be the inclusion $\star\colon \mathrm{Vt}(R) \to \mathrm{Sbgph}(R)$.

- More generally, a morphism $\phi\colon R \to S$ is an *isomorphism* if ϕ_0 is a bijection and if ϕ_1 factors through $\mathrm{Vt}(S)$ as $\phi_1 = \star\widetilde{\phi}_1$ with $\widetilde{\phi}_1$ a bijection:

$$
\begin{array}{ccc}
\mathrm{Vt}(R) & \xrightarrow{\ \phi_1\ } & \mathrm{Sbgph}(S) \\
& \underset{\widetilde{\phi}_1}{\searrow}{\scriptstyle\cong} & \uparrow{\scriptstyle\star} \\
& & \mathrm{Vt}(S)
\end{array}
$$

In Proposition 1.32, we show that precomposition with $\star$ constitutes a bijection between complete morphisms and morphisms. We will also transfer the composition of complete morphisms back to morphisms in Definition 1.33, after which the reader may wish to verify that this definition of isomorphism is correct from a categorical standpoint.

Example 1.14 If $R = R'$ except for orderings, then there is a unique isomorphism $\phi: R \to R'$ with $\phi_0 = \mathrm{id}$. We first note that since Definition 1.13 does not mention orderings, the pair $(\mathrm{id}_{\mathrm{Ed}(R)}, \star)$ constitutes a morphism $R \to R'$ of Ξ.

We now show that the only automorphism ϕ of R which fixes the edges is id_R. For this, we induct on the number of vertices n of R; the case $n = 0$ is clear since there is only one map $\eta \to \eta$. Suppose that uniqueness has been established for all $m < n$; pick any $e_0 \in \mathrm{Legs}(R)$. There is a unique $v_0 \in \mathrm{Vt}(R)$ with $e_0 \in \mathrm{Nbhd}(v_0)$. By assumption, $\phi_0(e_0) = e_0$, hence $\phi_1(v_0) = \star_{v_0}$. For each $e \in \mathrm{Nbhd}(v_0) \setminus e_0$, there is a subgraph R_e of R consisting of all vertices and edges on all paths not containing v but beginning at e. By the induction hypothesis, $\phi|_{R_e} = \mathrm{id}_{R_e}$ is uniquely determined by the fact that ϕ_0 fixes the edges. It follows that $\phi_1(v) = \star_v$ for all $v \in \mathrm{Vt}(R)$, so $\phi = \mathrm{id}_R$.

Remark 1.15 The argument for uniqueness in the previous example fails if we allow graphs without legs. Indeed, the graph $\bullet\!\!-\!\!\bullet$ admits two distinct automorphisms ϕ with $\phi_0 = \mathrm{id}$.

Definition 1.16 (cofaces and codegeneracies) We describe three basic types of morphisms of Ξ. Throughout, S and R will denote objects of Ξ, and $d(R)$ will refer to the number of vertices of R.

- Suppose that S is a subgraph of R and $d(S) = d(R) - 1$. Then we say the inclusion $S \to R$ is an *outer coface*. Up to orderings, the tree S is obtained from R by selecting a pair (v, e) with $v \in \mathrm{Vt}(R)$ and $e \in \mathrm{Nbhd}(v)$ such that $\mathrm{Nbhd}(v) \setminus \{e\} \subseteq \mathrm{Legs}(R)$, and then deleting v and all of the legs in $\mathrm{Nbhd}(v) \setminus \{e\}$. We will write $\delta^v: S \to R$ for such a coface map if $d(R) > 1$, or $i: \eta \to R$ for the map that hits $\mathrm{ord}(i)$ when $d(R) = 1$.

- A map $\phi: S \to R$ is an *inner coface* if $d(S) = d(R) - 1$ and there is a vertex v_0 such that $\phi_1(v_0)$ has exactly two vertices and $\phi_1(v)$ has exactly one vertex for $v \in \mathrm{Vt}(S) \setminus \{v_0\}$. The subgraph $\phi_1(v_0)$ has exactly one inner edge e, and we will often write $\delta^e: S \to R$ for such an inner coface map. The tree S is obtained from R by contracting an inner edge.

- A map $\phi: S \to R$ is a *codegeneracy* if $d(S) = d(R) + 1$ and there is a vertex v_0 (necessarily with $|v_0| = 2$) such that $\phi_1(v_0)$ is an edge and $\phi_1(v)$ has exactly one vertex for $v \in \mathrm{Vt}(S) \setminus \{v_0\}$.

A *coface* is a map which is either an inner coface or an outer coface.

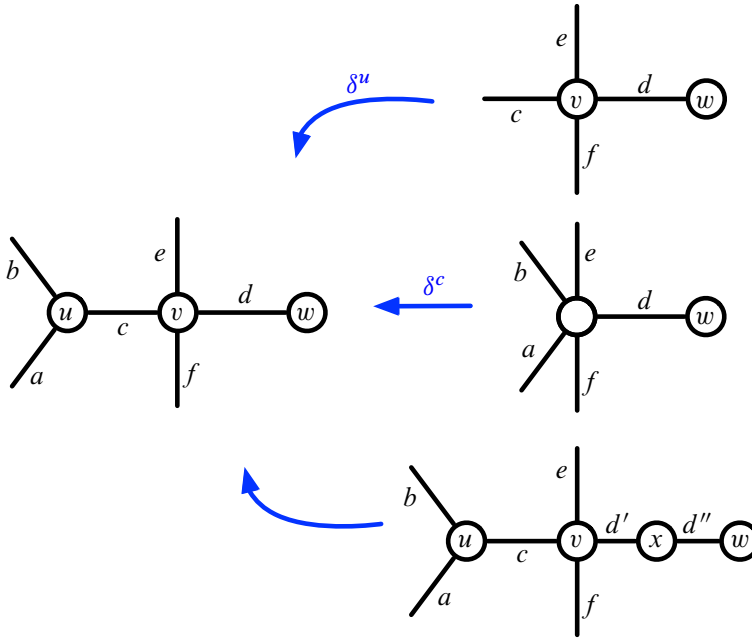An example of each type of map is given in Figure 6.

Figure 6: An outer coface, an inner coface and a codegeneracy

A *path* in a graph G is an alternating word in the alphabet $\mathrm{Ed}(G) \sqcup \mathrm{Vt}(G)$ which can only contain a subword ve or ev if $e \in \mathrm{Nbhd}(v)$. A path from a vertex v to a vertex w is a path of the form

$$P = v e_1 v_1 e_2 \ldots e_{n-1} v_{n-1} e_n w$$

while a path from an edge e to an edge e' is a path of the form

$$P = e v_1 e_1 v_2 \ldots v_{n-1} e_{n-1} v_n e'.$$

The length of a path, denoted by $|P|$, is the length of the word.

We can concatenate paths P and P' if the last letter in P is the first letter in P' or if the last letter in P is adjacent to the first letter in P'. In the former case, we will remove the duplicate letter.

Definition 1.17   Let G be a graph and $v, w \in \mathrm{Vt}(G)$. Define the *distance* from v to w by $d_G(v, w) = d(v, w) = \min_P \frac{1}{2}(|P| - 1)$, where P ranges over all paths in G from v to w. Any path realizing the distance is said to be a *minimal path* (and such exists as long as $d(v, w)$ is defined). Similarly, if $e, f \in \mathrm{Ed}(G)$, one defines $d_G(e, f)$ and minimal paths between edges in G.

If we consider the rwb graph G_{rwb} associated to a graph G, then $\mathrm{Vt}(G)$ is the same as the set of black vertices of G_{rwb} and $\mathrm{Ed}(G)$ is the union of the sets of red and white vertices of G_{rwb}; these distances are then half of the usual graph distance between vertices (see eg [18, Section 1.3]) in G_{rwb}.

Proposition 1.18 *Let S be a tree. If $v, w \in \mathrm{Vt}(S)$, then a minimal path from v to w exists and is unique. A similar statement applies to minimal paths between edges. Minimal paths are characterized as those containing no repeated entries.*

Proof Every vertex and edge of S is a vertex in the corresponding rwb graph S_{rwb}. This statement is then [18, Theorem 1.5.1] applied to the tree S_{rwb}. □

Existence and uniqueness of minimal paths leads to the following result:

Corollary 1.19 *If R is a subgraph of S, then $d_R = d_S|_R$, ie the distance in R is the restriction of the distance in S, for all edges and vertices in R.* □

We now have the necessary tools to define the objects and morphisms of the dendroidal category Ω. Though we do not make substantial use of Ω until Section 2, we include this definition here, rather than after Definition 1.33, to indicate the usefulness of the notion of distance.

Definition 1.20 (dendroidal category) We now define (a variant of) Ω as a subcategory of Ξ.

- A *rooted tree* is a tree R satisfying the following condition: Suppose $r_0 = \mathrm{ord}(0) \in \mathrm{Legs}(R)$. If $v \in \mathrm{Vt}(R)$ and $k > 0$, then

$$d(\mathrm{ord}^v(0), r_0) < d(\mathrm{ord}^v(k), r_0).$$

  If $R \neq \eta$, we set

$$\mathrm{in}(v) = \mathrm{Nbhd}(v) \setminus \mathrm{ord}^v(0), \quad \mathrm{out}(v) = \mathrm{ord}^v(0),$$
$$\mathrm{in}(R) = \mathrm{Legs}(R) \setminus \mathrm{ord}(0), \quad \mathrm{out}(R) = \mathrm{ord}(0) = r_0,$$

  while if $R = \eta$, we set $\mathrm{in}(R) = \{r_0\}$ and $\mathrm{out}(R) = r_0$.

- If R and S are rooted trees and $\phi \colon R \to S$ is a map in Ξ, we say that ϕ is *oriented* if for each $v \in \mathrm{Vt}(R)$ and each $k > 0$,

$$d(\phi_0(\mathrm{ord}^v(0)), s_0) \leq d(\phi_0(\mathrm{ord}^v(k)), s_0).$$
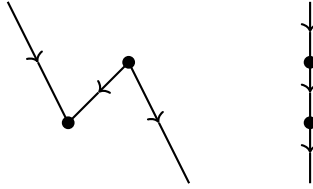
Figure 7: Two different directed structures on the same undirected tree

- The category Ω is the subcategory of Ξ whose objects are rooted trees and whose morphisms are the oriented maps between rooted trees.

- We write $\iota\colon \Omega \to \Xi$ for the subcategory inclusion.

Notice that if ϕ is an oriented map, then $\phi_1(v)$ is a rooted tree (without ordering of the leaves) with root $\phi_0(\mathrm{ord}^v(0)) = \phi_0(\mathrm{out}(v))$.

Remark 1.21 This definition of Ω is analogous to the equivalent category Ω' from [5, Example 2.8]. A rooted tree in our sense is equivalent to a rooted tree together with a planar structure and an ordering of the input edges, and morphisms do not need to preserve the planar structure.

Remark 1.22 In the above definition we were able to *recognize* rooted trees among all trees; we cannot do something similar for general directed trees (and hence for the category Θ from [23, Remark 6.55]). Indeed, graphs which are linear as undirected graphs generally possess many directed structures, even controlling for the number of inputs and outputs. See Figure 7. In short, there is a functor from (a legged variant of) Θ to Ξ, but it is not injective on objects.

Lemma 1.23 *Let $\phi\colon R \to S$ be a morphism of Ξ. Then $\mathrm{Int}(\phi_1(v)) \cap \mathrm{Ed}(\phi_1(w)) = \varnothing$ if $v \neq w$.*

Proof Suppose that $\phi_1(w)$ contains a vertex and $e \in \mathrm{Int}(\phi_1(v)) \cap \mathrm{Ed}(\phi_1(w))$. Since $\phi_1(w) \neq |_e$, there is a vertex $w' \in \mathrm{Vt}(\phi_1(w))$ with $e \in \mathrm{Nbhd}(w')$. Since $e \in \mathrm{Int}(\phi_1(v))$, we know that $w' \in \mathrm{Vt}(\phi_1(v))$, which implies $v = w$ by Definition 1.13(3).

In the general case, we induct on $d(v, w)$. Suppose we have vertices v and w with $d(v, w) = n > 0$ and let $v e_1 v_1 e_2 v_2 \ldots e_{n-1} v_{n-1} e_n w$ be the shortest path from v to w. If $\phi_1(w)$ contains a vertex then we know $\mathrm{Int}(\phi_1(v)) \cap \mathrm{Ed}(\phi_1(w)) = \varnothing$ by the first paragraph. Suppose that $\phi_1(w) = |_e$ is a single edge. If $n = 1$, then $e = \phi_0(e_1) \in \mathrm{Legs}(\phi_1(v))$, which implies $e \notin \mathrm{Int}(\phi_1(v))$ by Definition 1.13(2). Assume

the statement of the lemma is true for vertices of distance equal to $n - 1$. Then $e = \phi_0(e_n) \in \mathrm{Legs}(\phi_1(v_{n-1}))$, which implies that $e \notin \mathrm{Int}(\phi_1(v))$ by the induction hypothesis. □

Lemma 1.24 *Let* $\phi \colon R \to S$ *be a morphism of* Ξ. *For each vertex* $v \in \mathrm{Vt}(R)$, $\mathrm{Int}(\phi_1(v)) \subseteq \mathrm{Ed}(S) \setminus \mathrm{Im}(\phi_0)$.

Proof Suppose that $\phi_0(e) \in \mathrm{Int}(\phi_1(v))$. Since the graph has a vertex v and is connected, every edge is adjacent to at least one vertex. If e is adjacent to v, then $\phi_0(e) \in \phi_0(\mathrm{Nbhd}(v)) = \mathrm{Legs}(\phi_1(v)) \subseteq \mathrm{Ed}(S) \setminus \mathrm{Int}(\phi_1(v))$, so we conclude that e is not adjacent to v. Thus, there exists a $w \neq v$ with $e \in \mathrm{Nbhd}(w)$. But now

$$\phi_0(e) \in \mathrm{Int}(\phi_1(v)) \cap \mathrm{Legs}(\phi_1(w)) \subseteq \mathrm{Int}(\phi_1(v)) \cap \mathrm{Ed}(\phi_1(w)),$$

which is empty by Lemma 1.23. □

Lemma 1.25 *Let* $\phi \colon R \to S$ *be a morphism of* Ξ. *If*

$$|\mathrm{Legs}(\phi_1(v)) \cap \mathrm{Legs}(\phi_1(w))| > 1,$$

then $v = w$.

Proof Suppose $v \neq w$. Let $e, e' \in \mathrm{Legs}(\phi_1(v)) \cap \mathrm{Legs}(\phi_1(w))$. Let P be the shortest path in $\phi_1(v)$ from e to e' and let P' be the shortest path in $\phi_1(w)$ from e to e'. Since P and P' are also distance-minimizing paths in S, uniqueness implies that $P = P'$. If $e \neq e'$, this path contains a vertex, hence $\varnothing \neq \mathrm{Vt}(\phi_1(v)) \cap \mathrm{Vt}(\phi_1(w))$ and we see that $v = w$ by Definition 1.13(3). □

Lemma 1.26 *Suppose that* $\phi \colon R \to S$ *is a morphism of* Ξ. *If* $\phi_0(e) = \phi_0(e')$, *then* e *and* e' *lie on a common linear subgraph* (which may just mean $e = e'$), *all of whose edges map to a common value.*

Proof Induct on $d(e, e')$. If $d(e, e') = 0$ then $e = e'$ and the result follows. If $d(e, e') = 1$ and $\phi_0(e) = \phi_0(e')$, then the vertex adjacent to both e and e' must be bivalent by Definition 1.13(1). Assume the result is known for $d(e, e') < n$. Suppose $\phi_0(e) = \phi_0(e') = s \in \mathrm{Ed}(S)$ with $d(e, e') = n > 1$. Let $e_0 v_1 e_1 v_2 \ldots v_{n-1} e_{n-1} v_n e_n$ be the distance-minimizing path in R from $e = e_0$ to $e' = e_n$.

For each i, let P_i be the shortest path in $\phi_1(v_i)$ from $\phi_0(e_{i-1})$ to $\phi_0(e_i)$. The path $P_1 \ldots P_{n-1}$ contains no repeated entries by Lemma 1.23 and Definition 1.13(3), hence is the unique length-minimizing path (Proposition 1.18) from s to $\phi_0(e_{n-1})$. Both of

these edges are in $\phi_1(v_n)$, hence $P_1 \ldots P_{n-1}$ is a path in $\phi_1(v_n)$. If $s \neq \phi_0(e_{n-1})$, then $P_1 \ldots P_{n-1}$ contains a vertex, violating Definition 1.13(3). Thus, $\phi_0(e_n) = s = \phi_0(e_{n-1})$, so v_n is bivalent. The result now follows from the induction hypothesis since $d(e, e_{n-1}) < d(e, e')$. □

In a moment, we will (in Definition 1.28) define the *image* of a map, which is essentially the union of all of the subgraphs $\phi_1(v)$. We first check that this union actually is a subgraph.

Proposition 1.27 *Suppose that $\phi \colon R \to S$ is a morphism in Ξ and $R \neq \eta$. Then*

$$\bigcup_{v \in \mathrm{Vt}(R)} \phi_1(v)$$

is a subgraph of S.

Proof Suppose that R contains a vertex. Let $P = v_1 e_1 \ldots v_{n-1} e_{n-1} v_n$ be a path in R containing all vertices at least once. Then we have

$$\mathrm{Im}(\phi) = \phi_1(v_1) \cup \phi_1(v_2) \cup \cdots \cup \phi_1(v_n).$$

Use induction. By Proposition 1.10 we know that

$$(\phi_1(v_1) \cup \cdots \cup \phi_1(v_k)) \cup \phi_1(v_{k+1})$$

is a subgraph since $\phi_1(v_1) \cup \cdots \cup \phi_1(v_k)$ and $\phi_1(v_{k+1})$ are (induction hypothesis) and

$$\phi_0(e_k) \in \phi_1(v_k) \cap \phi_1(v_{k+1}). \qquad \square$$

Definition 1.28 Let $\phi \colon R \to S$ be a map in Ξ. Define the *image* of ϕ, denoted by $\mathrm{Im}(\phi)$, to be the subgraph

$$\mathrm{Im}(\phi) = \begin{cases} |_{\phi_0(e)} & \text{if } R = \eta, \\ \bigcup_{v \in \mathrm{Vt}(R)} \phi_1(v) & \text{if } \mathrm{Vt}(R) \neq \varnothing, \end{cases}$$

of S.

Proposition 1.29 *Suppose that $\phi \colon R \to S$ is in Ξ. Then $\phi_0(\mathrm{Legs}(R)) = \mathrm{Legs}(\mathrm{Im}(\phi))$.*

Proof The desired identity is clear when $R = \eta$ is an edge. We show that the desired equality holds when R contains a vertex. If $s \in \mathrm{Ed}(\mathrm{Im}(\phi))$ and $\phi_0^{-1}(s) = \varnothing$, then $s \in \mathrm{Int}(\phi_1(v)) \subseteq \mathrm{Int}(\mathrm{Im}(\phi))$ for some v, and also $s \notin \phi_0(\mathrm{Legs}(R))$.

Hence, for the remainder of the proof, we will only consider edges $s \in \mathrm{Ed}(\mathrm{Im}(\phi))$ such that $\phi_0^{-1}(s)$ is nonempty. We will write L_s for the linear subgraph of R, guaranteed

by Lemma 1.26, with $\mathrm{Ed}(L_s) = \phi_0^{-1}(s)$. Note that if $\mathrm{Nbhd}(v) \cap \mathrm{Ed}(L_s)$ is not empty, then $v \notin \mathrm{Vt}(L_s)$ if and only if $\phi_1(v)$ is not an edge.

To show that $\mathrm{Legs}(\mathrm{Im}(\phi)) \subseteq \phi_0(\mathrm{Legs}(R))$, we prove the equivalent statement: if $s \in \mathrm{Ed}(\mathrm{Im}(\phi))$ and $\phi_0^{-1}(s) \subseteq \mathrm{Int}(R)$, then $s \in \mathrm{Int}(\mathrm{Im}(\phi))$. We have already established this when $\phi_0^{-1}(s) = \varnothing$. If $\varnothing \neq \phi_0^{-1}(s) \subseteq \mathrm{Int}(R)$, then there exist distinct vertices $v_1, v_2 \in \mathrm{Vt}(R)$ with $v_i \notin \mathrm{Vt}(L_s)$ and $\mathrm{Nbhd}(v_i) \cap \mathrm{Ed}(L_s) \neq \varnothing$. As $\phi_1(v_i)$ is not equal to $|_s$, it contains a vertex w_i adjacent to $s \in \mathrm{Legs}(\phi_1(v_i))$. Since $\mathrm{Vt}(\phi_1(v_1)) \cap \mathrm{Vt}(\phi_1(v_2)) = \varnothing$, we know that $w_1 \neq w_2$, hence $s \in \mathrm{Int}(\mathrm{Im}(\phi))$.

Let us turn to the reverse inclusion $\phi_0(\mathrm{Legs}(R)) \subseteq \mathrm{Legs}(\mathrm{Im}(\phi))$. Suppose $r \in \mathrm{Ed}(R)$ and suppose that $s = \phi_0(r) \in \mathrm{Int}(\mathrm{Im}(\phi))$. We must show $r \in \mathrm{Int}(R)$. By assumption, there exist distinct vertices $w_1, w_2 \in \mathrm{Vt}(\mathrm{Im}(\phi))$ with $s \in \mathrm{Nbhd}(w_i)$. There exist unique $v_1, v_2 \in \mathrm{Vt}(R)$ with $w_i \in \mathrm{Vt}(\phi_1(v_i))$. If v_1 were equal to v_2, then we would have $s \in \mathrm{Int}(\phi_1(v_1))$. This is impossible by Lemma 1.24, hence $v_1 \neq v_2$. Now $\mathrm{Nbhd}(v_i) \cap \mathrm{Ed}(L_s) \neq \varnothing$ and $v_i \notin \mathrm{Vt}(L_s)$, hence we have $\mathrm{Ed}(L_s) \subseteq \mathrm{Int}(R)$. Since $r \in \mathrm{Ed}(L_s) \subseteq \mathrm{Int}(R)$, we have completed our proof that $\phi_0^{-1}(\mathrm{Int}(\mathrm{Im}(\phi)) \subseteq \mathrm{Int}(R)$. $\square$

Notice that if $T \subseteq T'$ are subgraphs of R, then $\mathrm{Im}(\phi|_T) \subseteq \mathrm{Im}(\phi|_{T'})$. Also, note that if T and T' are two subgraphs of R with $\mathrm{Vt}(T) = \mathrm{Vt}(T') \neq \varnothing$, then $T = T'$.

Lemma 1.30 *Let* $\phi \colon R \to S$ *be a morphism. Suppose that* $T, T' \in \mathrm{Sbgph}(R)$ *are two subgraphs which overlap. Then*

$$(2) \qquad \mathrm{Im}(\phi|_{T \cap T'}) = \mathrm{Im}(\phi|_T) \cap \mathrm{Im}(\phi|_{T'}),$$

$$(3) \qquad \mathrm{Im}(\phi|_{T \cup T'}) = \mathrm{Im}(\phi|_T) \cup \mathrm{Im}(\phi|_{T'}).$$

Proof We start with the case when T (or T') is a single edge $|_e$. Since T and T' are assumed to overlap, $T \subseteq T'$. Then $T \cap T' = |_e$ and $T \cup T' = T'$. Both sides of (2) are $|_{\phi_0(e)}$, while both sides of (3) are $\mathrm{Im}(\phi|_{T'})$.

It remains to prove the result when T and T' each contain at least one vertex. For (3), we have

$$\mathrm{Im}(\phi|_T) \cup \mathrm{Im}(\phi|_{T'}) = \left(\bigcup_{v \in \mathrm{Vt}(T)} \phi_1(v) \right) \cup \left(\bigcup_{w \in \mathrm{Vt}(T')} \phi_1(w) \right)$$

$$= \bigcup_{v \in \mathrm{Vt}(T) \cup \mathrm{Vt}(T')} \phi_1(v) = \mathrm{Im}(\phi|_{T \cup T'}).$$

Thus, (3) holds.

Since $T \cap T'$ is contained in T (and in T'), it is automatic that $\subseteq$ of (2) holds. We have

$$\mathrm{Vt}(\mathrm{Im}(\phi|_T)) \cap \mathrm{Vt}(\mathrm{Im}(\phi|_{T'})) = \left(\bigcup_{v \in \mathrm{Vt}(T)} \mathrm{Vt}(\phi_1(v)) \right) \cap \left(\bigcup_{w \in \mathrm{Vt}(T')} \mathrm{Vt}(\phi_1(w)) \right)$$

$$= \bigcup_{v \in \mathrm{Vt}(T) \cap \mathrm{Vt}(T')} \mathrm{Vt}(\phi_1(v))$$

by Definition 1.13(3). This last set is of course just $\mathrm{Vt}(\mathrm{Im}(\phi|_{T \cap T'}))$. We have two cases to consider:

- If $\mathrm{Vt}(\mathrm{Im}(\phi|_{T \cap T'})) \neq \varnothing$, then (2) holds since both sides are subgraphs that have the same nonempty set of vertices.

- If $\mathrm{Vt}(\mathrm{Im}(\phi|_{T \cap T'})) = \varnothing$, then both sides of (2) are a single edge. But we already saw that $\subseteq$ of (2) holds, so (2) holds. □

Lemma 1.31 *If* (α_0, α_1) *is a complete morphism* (Definition 1.12) *from* R *to* S, *then* $(\alpha_0, \alpha_1 \circ \star)$ *is a morphism* (Definition 1.13) *from* R *to* S.

Proof For concision, we write $\check{\alpha}_1 := \alpha_1 \circ \star$ in this proof and the next. Conditions (1) and (2) of Definition 1.13 follow immediately from $\eth \circ \alpha_1 = (\mathbf{M}\alpha_0) \circ \eth$. For (3), suppose that $v \neq w$ and induct on $d(v, w)$. If $d(v, w) = 1$, then $\star_v \cap \star_w$ is an edge, hence $\alpha_1(\star_v \cap \star_w) = \alpha_1(\star_v) \cap \alpha_1(\star_w) = \check{\alpha}_1(v) \cap \check{\alpha}_1(w)$ is an edge, thus (3) holds. Assume the result is known for distances less than n, and suppose $ve_1 v_1 e_2 \ldots e_{n-1} v_{n-1} e_n w$ is a minimal path from v to w. Let $T_v = \star_v \cup \bigcup_{i=1}^{n-1} \star_{v_i}$ and $T_w = \star_w \cup \bigcup_{i=1}^{n-1} \star_{v_i}$; by the induction hypothesis, $\mathrm{Vt}(\check{\alpha}_1(w))$ and $\mathrm{Vt}(\check{\alpha}_1(v))$ are both disjoint from $\bigcup_{i=1}^{n-1} \mathrm{Vt}(\check{\alpha}_1(v_i))$. On the other hand,

$$\mathrm{Vt}(\check{\alpha}_1(v)) \cap \mathrm{Vt}(\check{\alpha}_1(w)) \subseteq \mathrm{Vt}(\alpha_1(T_v)) \cap \mathrm{Vt}(\alpha_1(T_w))$$

$$= \mathrm{Vt}(\alpha_1(T_v \cap T_w)) = \bigcup_{i=1}^{n-1} \mathrm{Vt}(\check{\alpha}_1(v_i)),$$

proving the result. □

Proposition 1.32 *Morphisms and complete morphisms from* R *to* S *are in bijective correspondence. Precisely, the assignment that sends* (α_0, α_1) *to* $(\alpha_0, \alpha_1 \circ \star)$ *constitutes a bijective function from the complete morphisms to the morphisms.*

Proof We saw in the previous lemma that this function is well defined. We now construct an inverse. Suppose that $\phi = (\phi_0, \phi_1) \in \Xi(R, S)$. If $T \in \mathrm{Sbgph}(R)$, define $\widehat{\phi}_1(T) = \mathrm{Im}(\phi|_T)$. Let us verify that $(\phi_0, \widehat{\phi}_1)$ is a complete morphism.

If $T = |_e$, then $\eth(T) = e^2$, so

$$\eth\widehat{\phi}_1(|_e) = \eth(|_{\phi_0(e)}) = (\phi_0(e))^2 = (\mathbf{M}\phi_0)e^2 = (\mathbf{M}\phi_0)\eth(|_e).$$

If T contains a vertex, then $\eth\widehat{\phi}_1(T) = (\mathbf{M}\phi_0)\eth(T)$ by Proposition 1.29 applied to $\phi|_T$. Thus, the first condition of Definition 1.12 holds for the pair $(\phi_0, \widehat{\phi}_1)$. The second condition of this definition is guaranteed by Lemma 1.30.

We only need to check that $\phi \mapsto (\phi_0, \widehat{\phi}_1)$ is inverse to $(\alpha_0, \alpha_1) \mapsto (\alpha_0, \check{\alpha}_1) = (\alpha_0, \alpha_1 \circ \star)$. But

$$\check{\widehat{\phi}}_1(v) = \widehat{\phi}_1(\star_v) = \mathrm{Im}(\phi|_{\star_v}) = \phi_1(v),$$

so this function is a right inverse. It only remains to check that this function is a left inverse as well. If $T = |_e$ is an edge, we have

$$\widehat{\check{\alpha}}_1(T) = \mathrm{Im}((\check{\alpha})|_T) = \begin{cases} |_{\alpha_0(e)} & \text{if } T = \eta, \\ \bigcup_{v \in \mathrm{Vt}(T)} \check{\alpha}_1(v) & \text{if } \mathrm{Vt}(T) \neq \varnothing, \end{cases}$$

so if T is an edge, we are done because $\alpha_1(|_e) = |_{\alpha_0(e)}$. If T contains a vertex, then

$$\widehat{\check{\alpha}}_1(T) = \bigcup_{v \in \mathrm{Vt}(T)} \alpha_1(\star_v) = \alpha_1\left(\bigcup_{v \in \mathrm{Vt}(T)} \star_v\right) = \alpha_1(T)$$

since α_1 preserves unions. Thus, $\widehat{\check{\alpha}}_1 = \alpha_1$. □

As the collection of trees together with complete morphisms obviously forms a category, there is then an induced operation $\circ$ such that graphs and morphisms form a category. If one traces through the construction of the inverse in Proposition 1.32, we see that this operation takes the following form:

Definition 1.33 (composition in Ξ) Let $\phi \colon R \to S$ and $\psi \colon S \to T$ be morphisms of Ξ. Define two functions

$$(\psi \circ \phi)_0 \colon \mathrm{Ed}(R) \to \mathrm{Ed}(T), \quad (\psi \circ \phi)_1 \colon \mathrm{Vt}(R) \to \mathrm{Sbgph}(T)$$

by $(\psi \circ \phi)_0 = \psi_0 \circ \phi_0$ and $(\psi \circ \phi)_1(v) = \mathrm{Im}(\psi|_{\phi_1(v)})$.

1.3 Cyclic dendroidal sets

Define the category of *cyclic dendroidal sets* to be the presheaf category $\mathbf{Set}^{\Xi^{\mathrm{op}}}$. Precomposition with the functor $\iota\colon \Omega \to \Xi$ induces a functor

$$\iota^*\colon \mathbf{Set}^{\Xi^{\mathrm{op}}} \to \mathbf{Set}^{\Omega^{\mathrm{op}}}$$

from cyclic dendroidal sets to dendroidal sets. This functor has both adjoints, though we will only need the left adjoint $\iota_!\colon \mathbf{Set}^{\Omega^{\mathrm{op}}} \to \mathbf{Set}^{\Xi^{\mathrm{op}}}$ in this paper.

Given a tree S, we will write $\Xi[S] = \hom_{\Xi}(-, S)$ for the object represented by S.

Definition 1.34 Let $S \in \Xi$ be a tree.

- Suppose that δ is a coface map (Definition 1.16) with codomain S. Then the δ–horn of S is the subobject

$$\Lambda^{\delta}\, \Xi[S] = \bigcup_{\substack{d\,\colon R \to S \\ \delta \not\cong d}} d_* \Xi[R] \subseteq \Xi[S],$$

  where the union is over all coface maps which are not isomorphic (over S) to δ. This is an *inner horn* if δ is an inner coface; otherwise it is an *outer horn*.

- Suppose $S \neq \eta$. The *Segal core* of $\Xi[S]$, denoted by $\mathrm{Sc}[S]$, is defined to be the union

$$\bigcup_{v \in \mathrm{Vt}(S)} \Xi[\maltese_v] \subseteq \Xi[S],$$

  where $\maltese_v$ is regarded as a subobject of S. If $S = \eta$, it is convenient to also define $\mathrm{Sc}[\eta] = \Xi[\eta]$.

Likewise, if $T \in \Omega$ is a rooted tree, we have horns $\Lambda^{\delta} \Omega[T] \subseteq \Omega[T]$ and Segal cores $\mathrm{Sc}[T] = \bigcup \Omega[\maltese_v] \subseteq \Omega[T]$.

2 Using rooting to orient maps in Ξ

In this section we give a careful comparison of the morphism sets of Ξ and Ω. Morally, the category Ξ is built up from the dendroidal category Ω by adding isomorphisms which rotate trees. Thus, every morphism of Ξ should decompose into an oriented map (in Ω) along with some rotation data.

In the present section we make this precise. To each tree S and a choice of root $s_0 \in \mathrm{Legs}(S)$, there is a rooted tree $\mathscr{T}(S, s_0) \in \Omega$ (see Definition 2.1 and Figure 8). Further, given a morphism $\phi\colon R \to S$ we can transform ϕ into an oriented map (Lemma 2.9)

$$\mathscr{L}(\phi)\colon \mathscr{T}(R, r_\phi) \to \mathscr{T}(S, s_0)$$

for some particular choice of root $r_\phi \in \mathrm{Legs}(R)$ (see Definition 2.4). We show that $\mathscr{L}$ respects composition in a certain sense (Proposition 2.11, Remark 2.12). Finally, in Theorem 2.13 (see also Corollaries 2.16 and 2.17) we realize our goal and make clear the idea that (nonconstant) maps $R \to S$ are just certain maps in Ω along with rooting data for S.

**Definition 2.1** (rooting of trees) Suppose we are given a pair (S, s_0) with $S \in \mathrm{Ob}(\Xi)$ and $s_0 \in \mathrm{Legs}(S)$.

- Assume that $S \neq \eta$. We now define a rooted tree $T \in \mathrm{Ob}(\Omega)$ with $\mathrm{Ed}(T) = \mathrm{Ed}(S)$ and $\mathrm{Vt}(T) = \mathrm{Vt}(S)$. For each $v \in \mathrm{Vt}(S)$, let $\mathrm{ord}^v(k_v) = \mathrm{out}(v) \in \mathrm{Nbhd}(v)$ be the element which minimizes the function $d(-, s_0)|_{\mathrm{Nbhd}(v)}$. Setting $\mathrm{in}(v) = \mathrm{Nbhd}(v) \setminus \mathrm{out}(v)$, we have an induced ordering on $\mathrm{in}(v)$ via

$$
\begin{array}{ccc}
\{0, 1, \dots, n_v\} & \xrightarrow[\cong]{\mathrm{ord}^v} & \mathrm{Nbhd}(v) \\
{\scriptstyle +k_v \bmod (n_v+1)}\Big\uparrow & & \Big\| \\
\{0, 1, \dots, n_v\} & & \{\mathrm{out}(v)\} \sqcup \mathrm{in}(v) \\
\Big\uparrow & & \Big\uparrow \\
\{1, \dots, n_v\} & \dashrightarrow[\cong] & \mathrm{in}(v)
\end{array}
$$

Similarly, we have an induced ordering on $\mathrm{in}(T) = \mathrm{Legs}(S) \setminus \{s_0\}$ via

$$
\begin{array}{ccc}
\{0, 1, \dots, n\} & \xrightarrow[\cong]{\mathrm{ord}} & \mathrm{Legs}(S) \\
{\scriptstyle +k \bmod (n+1)}\Big\uparrow & & \Big\| \\
\{0, 1, \dots, n\} & & \{s_0\} \sqcup \mathrm{in}(T) \\
\Big\uparrow & & \Big\uparrow \\
\{1, \dots, n\} & \dashrightarrow[\cong] & \mathrm{in}(T)
\end{array}
$$

where $k = \mathrm{ord}^{-1}(s_0)$. Although ιT may be different from S, since they have different total orderings (though the same *cyclic* orderings [38]) ord and ord^v, there is a *unique* isomorphism $f\colon \iota T \to S$ of Ξ with $f_0 = \mathrm{id}$ by Example 1.14.
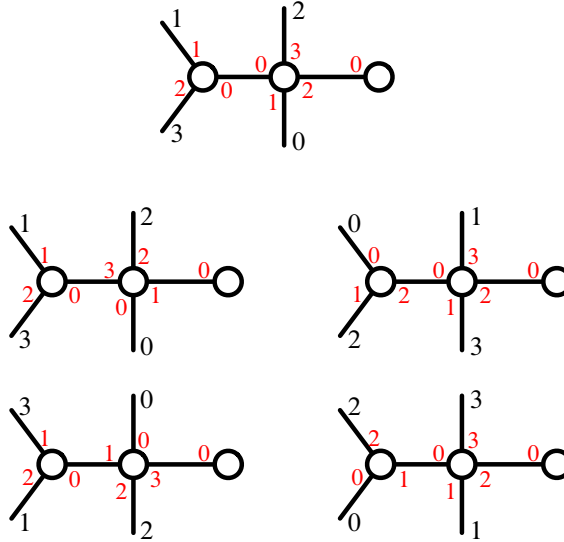
Figure 8: A tree S together with its four associated rooted trees

- We write

$$\mathcal{T}(S, s_0) = (T, f\colon \iota T \xRightarrow{\cong} S)$$

for this construction. The second component is redundant (since we insist that $f_0 = \mathrm{id}$), so we will usually abuse notation and just write $\mathcal{T}(S, s_0) = T$. Since the trivial tree η is already rooted, we also set $\mathcal{T}(\eta, e) = \eta$.

- If $r_0 \in \mathrm{Legs}(R)$ and $s_0 \in \mathrm{Legs}(S)$, define $\mathscr{A}_{r_0, s_0}$ to be the composite

$$\Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \xhookrightarrow{\iota} \Xi(\iota\mathcal{T}(R, r_0), \iota\mathcal{T}(S, s_0)) \xrightarrow[\Xi(f_R^{-1}, f_S)]{\cong} \Xi(R, S).$$

An example is given in Figure 8.

Proposition 2.2 Let $R, U, S \in \mathrm{Ob}(\Xi)$, $r_0 \in \mathrm{Legs}(R)$, $u_0 \in \mathrm{Legs}(U)$ and $s_0 \in \mathrm{Legs}(S)$. Then the diagram

$$
\begin{array}{ccc}
\Omega(\mathcal{T}(U, u_0), \mathcal{T}(S, s_0)) \times \Omega(\mathcal{T}(R, r_0), \mathcal{T}(U, u_0)) & \xrightarrow{\circ} & \Omega(\mathcal{T}(R, r_0), \mathcal{T}(S, s_0)) \\
\downarrow{\scriptstyle \mathscr{A}_{u_0, s_0} \times \mathscr{A}_{r_0, u_0}} & & \downarrow{\scriptstyle \mathscr{A}_{r_0, s_0}} \\
\Xi(U, S) \times \Xi(R, U) & \xrightarrow{\circ} & \Xi(R, S)
\end{array}
$$

commutes.

Proof Let $g \in \Omega(\mathcal{T}(U, u_0), \mathcal{T}(S, s_0))$ and $h \in \Omega(\mathcal{T}(R, r_0), \mathcal{T}(U, u_0))$. The diagram

$$
\begin{array}{ccccc}
& & \iota(g \circ h) & & \\
& \nearrow & & \searrow & \\
\iota(\mathcal{T}(R, r_0)) & \xrightarrow{\iota(h)} & \iota(\mathcal{T}(U, u_0)) & \xrightarrow{\iota(g)} & \iota(\mathcal{T}(S, s_0)) \\
f_R \downarrow \cong & & f_U \downarrow \cong & & f_S \downarrow \cong \\
R & \xrightarrow[\mathscr{A}_{r_0, u_0}(h)]{} & U & \xrightarrow[\mathscr{A}_{u_0, s_0}(g)]{} & S
\end{array}
$$

commutes, hence

$$
\mathscr{A}_{u_0, s_0}(g) \circ \mathscr{A}_{r_0, u_0}(h) = f_S \circ \iota(g \circ h) \circ f_R^{-1} = \mathscr{A}_{r_0, s_0}(g \circ h). \qquad \square
$$

Lemma 2.3 *Suppose that R is a subgraph of S, $s_0 \in \mathrm{Legs}(S)$, and $r_0 \in \mathrm{Ed}(R)$ minimizes $d(-, s_0)|_{\mathrm{Ed}(R)}$. Then $r_0 \in \mathrm{Legs}(R)$ and*

$$
d(r, s_0) = d(r, r_0) + d(r_0, s_0)
$$

for any $r \in \mathrm{Ed}(R)$. In particular, the minimizing element r_0 is unique.

Proof Let P be the minimal path from r_0 to s_0. Then P contains no edges of R other than r_0 by assumption. Let P' be the minimal path from r to r_0. Since P' is actually a path in R, we have that P and P' only have the single edge r_0 in common. Thus, $P'P$ has no repeated edges, and thus is a minimal path from r to s_0 by the characterization in Proposition 1.18. Thus, $|P'P| - 1 = (|P'| - 1) + (|P| - 1)$, which is the claimed equality. $\qquad \square$

Given $R, S \in \mathrm{Ob}(\Xi)$, we write $\Xi^0(R, S) \subseteq \Xi(R, S)$ for the set of maps which factor through the vertex-free graph η. If R is linear then $\Xi^0(R, S) \cong \mathrm{Ed}(S)$, otherwise $\Xi^0(R, S) = \varnothing$. A particular case of Lemma 1.26 says that if $\phi \colon R \to S$ is a map in Ξ with $\phi_0|_{\mathrm{Legs}(R)}$ not injective, then $\phi \in \Xi^0(R, S)$. Combining this fact with $\phi_0(\mathrm{Legs}(R)) = \mathrm{Legs}(\mathrm{Im}(\phi))$ (Proposition 1.29) and Lemma 2.3 applied to $\mathrm{Im}(\phi) \hookrightarrow S$ gives that the following function is well defined:

Definition 2.4 ("find root" function) Let $s_0 \in \mathrm{Legs}(S)$. Define a function $\odot_{s_0}$

$$
\Xi(R, S) \setminus \Xi^0(R, S) \xrightarrow{\odot_{s_0}} \mathrm{Legs}(R), \quad \phi \mapsto r_\phi,
$$

where r_ϕ minimizes the function $d(\phi_0(-), s_0)|_{\mathrm{Legs}(R)}$.

Lemma 2.3 also guarantees that $\odot_{s_0}(\phi)$ minimizes the function $d(\phi_0(-), s_0)$, though of course there may be internal edges which also minimize this function since ϕ_0 need not be injective.

Remark 2.5 If $\phi \in \Xi^0(R, S)$, then R must be a linear tree. The function $d(\phi_0(-), s_0)$ is *constant*, hence is minimized by each of the two extremal edges when $R \cong L_n$ for $n > 0$.

We now turn to several observations which give us effective tools for computing the function $\odot_{s_0}$, in particular with respect to subgraph inclusions and certain compositions.

Remark 2.6 Let $\phi: R \to S$ be a map in Ξ which does not factor through η. If $s_0 \in \mathrm{Legs}(S)$, then $\phi_0(\odot_{s_0}(\phi)) = \odot_{s_0}(\mathrm{Im}(\phi) \hookrightarrow S)$.

In a similar vein, we have the following lemma:

Lemma 2.7 Let $\phi: R \to S \neq \eta$ be a morphism of Ξ with $\mathrm{Im}(\phi) = S$. Suppose $s_0 \in \mathrm{Legs}(S)$ and write $r_0 = \odot_{s_0}(\phi)$ (ie $\phi_0^{-1}(s_0) \cap \mathrm{Legs}(R) = \{r_0\}$). If $v \in \mathrm{Vt}(R)$ is such that $\phi_1(v)$ is not an edge, then

$$\phi_0(\odot_{s_0}(\star_v \hookrightarrow R \xrightarrow{\phi} S)) = \phi_0(\odot_{r_0}(\star_v \hookrightarrow R)).$$

In particular,

$$\odot_{s_0}(\star_v \hookrightarrow R \xrightarrow{\phi} S) = \odot_{r_0}(\star_v \hookrightarrow R).$$

Proof The second statement follows from the first since $\phi_0|_{\mathrm{Nbhd}(v)}$ is injective whenever $\phi_1(v)$ is not an edge.

Suppose that the elements $e_0 = \odot_{s_0}(\star_v \to R \to S)$ and $e_1 = \odot_{r_0}(\star_v \to R)$ are distinct. Let $e_1 v_2 e_2 \ldots e_{n-1} v_n e_n$ be the minimal path from e_1 to $e_n = r_0 = \odot_{s_0}(\phi)$. Since $d(e_1, r_0) < d(e_0, r_0)$, the path $e_0 v_1 e_1 v_2 \ldots v_n e_n$, where $v_1 = v$, is a minimal path from e_0 to e_n. For $i = 1, \ldots, n$, let P_i be the minimal path from $\phi_0(e_{i-1})$ to $\phi_0(e_i)$; the path P_i is a path in $\phi_1(v_i)$. Since the v_i are distinct, $P_1 P_2 \ldots P_n$ contains no repeated entries, hence is the minimal path from $\phi_0(e_0)$ to $\phi_0(r_0) = s_0$. Likewise, $P_2 \ldots P_n$ is the minimal path from $\phi_0(e_1)$ to s_0. Since $d(\phi_0(e_0), s_0) \leq d(\phi_0(e_1), s_0)$, the paths $P_1 P_2 \ldots P_n$ and $P_2 \ldots P_n$ are equal. This implies that the path P_1 from $\phi_0(e_0)$ to $\phi_0(e_1)$ does not contain a vertex, hence $\phi_0(e_0) = \phi_0(e_1)$.   □

Lemma 2.8 *Suppose that $A \neq \eta$ and that*

$$A \hookrightarrow B \hookrightarrow C$$

is a pair of subgraph inclusions. If $c_0 \in \mathrm{Legs}(C)$ and $b_0 = \odot_{c_0}(B \hookrightarrow C) \in \mathrm{Legs}(B)$, then $\odot_{b_0}(A \hookrightarrow B) = \odot_{c_0}(A \hookrightarrow C)$.

Proof   Write $a_0 = \odot_{b_0}(A \hookrightarrow B)$ and apply Lemma 2.3 twice to get

$$(4) \qquad\qquad d(a, b_0) = d(a, a_0) + d(a_0, b_0),$$

$$(5) \qquad\qquad d(a, c_0) = d(a, b_0) + d(b_0, c_0),$$

for any $a \in \mathrm{Ed}(A) \subseteq \mathrm{Ed}(B)$. For the particular case when $a = a_0$, (5) becomes $d(a_0, c_0) = d(a_0, b_0) + d(b_0, c_0)$. Combining with (4) we have

$$d(a, b_0) = d(a, a_0) + d(a_0, c_0) - d(b_0, c_0),$$

hence

$$d(a, c_0) = d(a, b_0) + d(b_0, c_0) = d(a, a_0) + d(a_0, c_0).$$

Then a_0 is the element of $\mathrm{Ed}(A)$ which minimizes this function, hence

$$a_0 = \odot_{c_0}(A \hookrightarrow C). \qquad\qquad \square$$

2.1   Orientation of maps

Lemma 2.9   *Let $r_0 \in \mathrm{Legs}(R)$ and $s_0 \in \mathrm{Legs}(S)$. There is a (unique) function $\mathscr{L} = \mathscr{L}_{r_0, s_0}$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
\odot_{s_0}^{-1}(r_0) & \xrightarrow{\ \ \mathscr{L}\ \ } & \Omega(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0)) \\
\downarrow & & \ \downarrow{\scriptstyle \mathscr{A}_{r_0, s_0}} \\
\Xi(R, S) \setminus \Xi^0(R, S) & \xrightarrow{\hspace{2cm}} & \Xi(R, S)
\end{array}
$$

If $\phi \in \Xi(R, S) \setminus \Xi^0(R, S)$, we will use $\mathscr{L}_{s_0}(\phi)$ as shorthand for $\mathscr{L}_{\odot_{s_0}(\phi), s_0}(\phi)$.

Proof   Let $\phi \colon R \to S$ be in $\odot_{s_0}^{-1}(r_0)$. To see that the functions

$$\phi_0 \colon \mathrm{Ed}(R) \to \mathrm{Ed}(S), \quad \phi_1 \colon \mathrm{Vt}(R) \to \mathrm{Sbgph}(S)$$

determine a map $\mathscr{T}(R, r_0) \to \mathscr{T}(S, s_0)$ in Ω, we just need to establish that, for each $v \in \mathrm{Vt}(R)$, we have $\phi_0(\mathrm{out}(v)) = \mathrm{out}(\phi_1(v))$. There is nothing to prove in the case $\phi_1(v)$ is a single edge. For concision, write $\bar{r}_0$ for the element (see Remark 2.6)

$$\bar{r}_0 := \phi_0(r_0) = \phi_0(\odot_{s_0}(\phi)) = \odot_{s_0}(\mathrm{Im}(\phi) \hookrightarrow S) \in \mathrm{Legs}(\mathrm{Im}(\phi)) = \phi_0(\mathrm{Legs}(R)).$$

In the case when $\phi_1(v)$ is not an edge, we have

$$
\begin{aligned}
\mathrm{out}(\phi_1(v)) &= \odot_{s_0}(\phi_1(v) \hookrightarrow S) \\
&= \odot_{\bar{r}_0}(\phi_1(v) \hookrightarrow \mathrm{Im}(\phi)) && \text{(Lemma 2.8)} \\
&= \phi_0(\odot_{\bar{r}_0}(\star_v \hookrightarrow R) \to \mathrm{Im}(\phi)) && \text{(Remark 2.6)} \\
&= \phi_0(\odot_{r_0}(\star_v \hookrightarrow R)) && \text{(Lemma 2.7)} \\
&= \phi_0(\mathrm{out}(v)) && \text{(Definition 2.1).} \qquad \square
\end{aligned}
$$

Lemma 2.10 *If $r_0 \neq r_1 \in \mathrm{Legs}(R)$, then*

$$\mathscr{A}_{r_0,s_0}\big(\Omega(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0))\big) \cap \mathscr{A}_{r_1,s_0}\big(\Omega(\mathscr{T}(R, r_1), \mathscr{T}(S, s_0))\big) \subseteq \Xi^0(R, S).$$

Proof If $\phi \in \mathscr{A}_{r_i,s_0}\big(\Omega(\mathscr{T}(R, r_i), \mathscr{T}(S, s_0))\big) \setminus \Xi^0(R, S)$, then $\odot_{s_0}(\phi) = r_i$. $\qquad \square$

The following proposition describes precisely how the functions $\mathscr{L}_{r_0,s_0}$ behave with respect to composition in Ξ. Special cases of the first part have already appeared in Lemmas 2.7 and 2.8.

Proposition 2.11 *Consider a composite*

$$R \xrightarrow{\ \psi\ } U \xrightarrow{\ \phi\ } S$$

in Ξ, and let $s_0 \in \mathrm{Legs}(S)$. Suppose that $\phi \circ \psi \notin \Xi^0(R, S)$. If $u_0 = \odot_{s_0}(\phi)$, then

$$\odot_{u_0}(\psi) = \odot_{s_0}(\phi \circ \psi)$$

and

$$\mathscr{L}_{s_0}(\phi) \circ \mathscr{L}_{u_0}(\psi) = \mathscr{L}_{s_0}(\phi \circ \psi).$$

Proof Set $u_0 = \odot_{s_0}(\phi)$ and $r_0 = \odot_{u_0}(\psi)$. Omitting the subscripts, we know that the diagram

$$\odot_{s_0}^{-1}(u_0) \times \odot_{u_0}^{-1}(r_0)$$

$$\downarrow{\scriptstyle \mathscr{L}\times\mathscr{L}}$$

$$\Omega(\mathscr{T}(U, u_0), \mathscr{T}(S, s_0)) \times \Omega(\mathscr{T}(R, r_0), \mathscr{T}(U, u_0)) \xrightarrow{\mathscr{A}\times\mathscr{A}} \Xi(U, S) \times \Xi(R, U)$$

$$\downarrow{\scriptstyle \circ} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \circ}$$

$$\Omega(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0)) \xrightarrow{\qquad\quad \mathscr{A} \qquad\quad} \Xi(R, S)$$

commutes by Proposition 2.2 and Lemma 2.9. Further, (ϕ, ψ) is an element in the apex. Thus,

$$\phi \circ \psi \in \mathscr{A}_{r_0, s_0}\big(\Omega(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0))\big),$$

so $\odot_{s_0}(\phi \circ \psi) = r_0$. Thus, we have established the first statement.

The second statement is now immediate since the underlying maps of $\mathscr{L}_{s_0}(\phi)$, $\mathscr{L}_{u_0}(\psi)$ and $\mathscr{L}_{s_0}(\phi \circ \psi)$ are just ϕ_i, ψ_i and $(\phi \circ \psi)_i$ by the proof of Lemma 2.9. $\qquad\square$

Remark 2.12 Let $S \in \Xi$, and let $\mathcal{C} \subseteq \Xi \downarrow S$ be the full subcategory with object set

$$\coprod_{R \in \Xi} \Xi(R, S) \setminus \Xi^0(R, S).$$

A restatement of the last part of the proof of Proposition 2.11 is that for each $s_0 \in S$, there is a functor $\mathcal{C} \to \Omega \downarrow \mathscr{T}(S, s_0)$ which on objects sends ϕ to $\mathscr{L}_{s_0}(\phi)$. It is possible to extend this functor to the larger full subcategory $\mathcal{C} \subseteq \mathcal{C}' \subseteq \Xi \downarrow S$ which includes the objects $\Xi(\eta, S)$, o that $\mathcal{C}' \to \Omega \downarrow \mathscr{T}(S, s_0) \xrightarrow{\iota} \Xi \downarrow S$ is isomorphic to the inclusion. It is not generally possible to extend the functor to all of $\Xi \downarrow S$.

If T and T' are rooted trees, let $\Omega^0(T, T') \subseteq \Omega(T, T')$ denote the subset of oriented maps which factor through η. Notice that every morphism of Ξ from T to T' that factors through η is automatically oriented, so $\Omega^0(T, T') = \Xi^0(T, T')$; we make the distinction in notation only for emphasis.

For the remainder of the section, if $s_0 \in \mathrm{Legs}(S)$, we will write

$$\mathscr{A}_{s_0} \colon \coprod_{r \in \mathrm{Legs}(R)} \Omega(\mathscr{T}(R, r), \mathscr{T}(S, s_0)) \to \Xi(R, S)$$

for the coproduct of the $\mathscr{A}_{r, s_0}$.

Theorem 2.13 *Suppose that $s_0 \in \mathrm{Legs}(S)$. The function $\mathscr{A}_{s_0}$ restricts to a bijection*

$$\mathscr{A}_{s_0}^{\mathrm{nc}}\colon \coprod_{\mathrm{Legs}(R)} \left(\Omega(\mathscr{T}(R,r), \mathscr{T}(S,s_0)) \setminus \Omega^0(\mathscr{T}(R,r), \mathscr{T}(S,s_0)) \right)$$

$$\to \Xi(R,S) \setminus \Xi^0(R,S) = \coprod_{\mathrm{Legs}(R)} \odot_{s_0}^{-1}(r).$$

Proof There is a diagram

$$
\begin{array}{ccc}
\coprod_{\mathrm{Legs}(R)} \left(\Omega(\mathscr{T}(R,r), \mathscr{T}(S,s_0)) \setminus \Omega^0(\mathscr{T}(R,r), \mathscr{T}(S,s_0)) \right) & \xrightarrow{\;\mathscr{A}_{s_0}^{\mathrm{nc}}\;} & \Xi(R,S) \setminus \Xi^0(R,S) \\
\downarrow & & \downarrow \\
\coprod_{\mathrm{Legs}(R)} \Omega(\mathscr{T}(R,r), \mathscr{T}(S,s_0)) & \xrightarrow[\;\;\;\;\mathscr{A}_{s_0}\;\;\;\;]{} & \Xi(R,S)
\end{array}
$$

with $\mathscr{L}_{s_0}$ a diagonal map.

The bottom triangle commutes by Lemma 2.9. Given $\psi\colon R \to S$,

$$\mathscr{L}_{s_0}(\psi)\colon \mathscr{T}(R,r_0) \to \mathscr{T}(S,s_0)$$

is the unique map of Ω such that $\mathscr{A}_{s_0}(\mathscr{L}_{s_0}(\psi)) = \psi$. If $\psi = \mathscr{A}_{s_0}(\phi\colon \mathscr{T}(R,r_0) \to \mathscr{T}(S,s_0))$, then certainly ϕ satisfies the condition to be $\mathscr{L}_{s_0}(\mathscr{A}_{s_0}(\phi))$. Thus, the top triangle commutes.

Since the left vertical map is injective, so is $\mathscr{A}_{s_0}^{\mathrm{nc}}$. Further, if $\phi\colon R \to S$ is not constant, then

$$\mathscr{L}_{s_0}(\phi) \in \Omega(\mathscr{T}(R, \odot_{s_0}(\phi)), \mathscr{T}(S,s_0))$$

is also not constant, so $\mathscr{A}_{s_0}^{\mathrm{nc}}$ is surjective. $\qquad\square$

Corollary 2.14 *We have, for each $s_0 \in \mathrm{Legs}(S)$,*

$$\mathrm{Iso}_{\Xi}(R,S) \cong \coprod_{\mathrm{Legs}(R)} \mathrm{Iso}_{\Omega}(\mathscr{T}(R,r), \mathscr{T}(S,s_0))$$

via $\mathscr{A}_{s_0}$. Specializing to the case $R = S$, we have

$$\mathrm{Aut}_{\Xi}(S) \cong \coprod_{s \in \mathrm{Legs}(S)} \mathrm{Iso}_{\Omega}(\mathscr{T}(S,s), \mathscr{T}(S,s_0)).$$

Proof The statement is trivial if S (and hence R) does not have a vertex. Otherwise, isomorphisms are not constant, so this follows from Theorem 2.13 by taking subsets. $\quad\square$

Example 2.15 Let S be the tree with $\mathrm{Ed}(S) = \mathrm{Legs}(S) = \{e_1, e_2, e_3\}$ and $\mathrm{Vt}(S) = \{v\}$. We can apply Corollary 2.14 to reveal some of the structure of $\mathrm{Aut}_{\Xi}(S) = \Sigma_3$, but

the $\mathscr{A}_{e_i}$ as i varies do not behave well together. For example, under the composite

$$\mathrm{Iso}_\Omega(\mathscr{T}(S, e_1), \mathscr{T}(S, e_2)) =\!=\!=\!= \{\phi, \psi\}$$

$$\coprod_{i=1}^{3} \mathrm{Iso}_\Omega(\mathscr{T}(S, e_i), \mathscr{T}(S, e_2)) \xrightarrow[\cong]{\mathscr{A}_{e_2}} \mathrm{Aut}_\Xi(S) \xrightarrow[\cong]{\mathscr{A}_{e_1}^{-1}} \coprod_{j=1}^{3} \mathrm{Iso}_\Omega(\mathscr{T}(S, e_j), \mathscr{T}(S, e_1))$$

(where, say, $\phi_0(e_1) = e_2$, $\phi_0(e_2) = e_1$, $\phi_0(e_3) = e_3$, $\psi_0(e_1) = e_2$, $\psi_0(e_2) = e_3$ and $\psi_0(e_3) = e_1$), ϕ and ψ map to different coproduct summands. The morphism ϕ lands in the $j = 2$ component and ψ lands in the $j = 3$ component.

Let us rephrase this. The bottom line of this diagram may be identified with

$$\coprod_{[\pi] \in (13)\backslash \Sigma_3} [\pi] \xrightarrow{\cong} \Sigma_3 \xrightarrow{\cong} \coprod_{[\pi'] \in (23)\backslash \Sigma_3} [\pi'],$$

where the coproducts are indexed by right cosets. So the preceding paragraph reflects that $\mathscr{A}_{e_2}$ and $\mathscr{A}_{e_1}$ perform right coset decompositions for different stabilizer subgroups.

Corollary 2.16 *If R is nonlinear, then*

$$\mathscr{A}_{s_0}: \coprod_{\mathrm{Legs}(R)} \Omega(\mathscr{T}(R, r), \mathscr{T}(S, s_0)) \to \Xi(R, S)$$

is an isomorphism, with inverse given by $\mathscr{L}_{s_0}$.

Proof If R is nonlinear, then $\Xi^0(R, S) = \varnothing$. $\qquad\square$

Corollary 2.17 *If $R \neq \eta$ is a linear graph with at least one vertex, then $\mathrm{Legs}(R) = \{r_0, r_1\}$ and*

$$\mathscr{A}_{s_0}: \Omega(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0)) \amalg \Omega(\mathscr{T}(R, r_1), \mathscr{T}(S, s_0)) \to \Xi(R, S)$$

satisfies

$$|\mathscr{A}_{s_0}^{-1}(\phi)| = \begin{cases} 2 & \text{if } \phi \in \Xi^0(R, S), \\ 1 & \text{if } \phi \notin \Xi^0(R, S). \end{cases}$$

If $R = \eta = L_0$, then $\mathscr{A}_{s_0}$ is a bijection.

Proof In general, $\mathscr{A}_{s_0}$ splits as a coproduct of $\mathscr{A}_{s_0}^{\mathrm{nc}}$ with

$$\mathscr{A}_{s_0}^{\mathrm{c}}: \coprod_{\mathrm{Legs}(R)} \Omega^0(\mathscr{T}(R, r), \mathscr{T}(S, s_0)) \to \Xi^0(R, S).$$

If R is linear and has at least one vertex, then

$$\Omega^0(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0)) \amalg \Omega^0(\mathscr{T}(R, r_1), \mathscr{T}(S, s_0)) \xrightarrow{\mathscr{A}_{s_0}^{\mathrm{c}}} \Xi^0(R, S)$$

$$\downarrow \cong \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \cong$$

$$\mathrm{Ed}(S) \amalg \mathrm{Ed}(S) \xrightarrow{\mathrm{id} \amalg \mathrm{id}} \mathrm{Ed}(S)$$

is two-to-one and $\mathscr{A}_{s_0}^{\mathrm{nc}}$ is injective by Theorem 2.13. Hence, the first statement is proved.

For the second statement, if $R = \eta$ then all maps $R \to S$ are constant, $\mathrm{Legs}(R) = \{r\}$ has one element, and

$$\mathscr{A}_{s_0} = \mathscr{A}_{s_0}^{\mathrm{c}} \colon \Omega^0(\mathscr{T}(R, r), \mathscr{T}(S, s_0)) \to \Xi^0(R, S)$$

is isomorphic to the identity map on $\mathrm{Ed}(S)$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 A generalized Reedy structure on Ξ

The category Δ of nonempty finite ordered sets is the prototypical example of a *Reedy category*. The surjective (resp. injective) maps form a wide subcategory (ie a subcategory which contains all of the objects of the ambient category) Δ^- (resp. Δ^+) of morphisms which lower (resp. raise) degrees such that any map has a unique factorization $f = f^+ f^-$. Numerous inductive techniques used in the theory of (co)simplicial objects actually work in diagrams indexed by arbitrary Reedy categories.

Generalized Reedy categories were introduced in [5] and capture the dendroidal category Ω as an example, highlighting its similarities to Δ. We will return to the theory of model structures on diagram categories index by a generalized Reedy category $\mathbb{R}$ in Section 7, but for now we show that Ξ admits such a structure.

Definition 3.1 [5, Definition 1.1] A *generalized Reedy structure* on a small category $\mathbb{R}$ consists of

- wide subcategories $\mathbb{R}^+$ and $\mathbb{R}^-$, and
- a degree function $d \colon \mathrm{Ob}(\mathbb{R}) \to \mathbb{N}$

satisfying the following four axioms:

(i) Noninvertible morphisms in $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) raise (resp. lower) the degree. Isomorphisms in $\mathbb{R}$ preserve the degree.

(ii) $\mathbb{R}^+ \cap \mathbb{R}^- = \mathrm{Iso}(\mathbb{R})$.

(iii) Every morphism f of $\mathbb{R}$ factors as $f = gh$ with $g \in \mathbb{R}^+$ and $h \in \mathbb{R}^-$. This factorization is unique up to isomorphism in the sense that if $g'h'$ is another such factorization, then there is an isomorphism θ such that $\theta h = h'$ and $g = g'\theta$.

(iv) If $\theta f = f$ for $\theta \in \mathrm{Iso}(\mathbb{R})$ and $f \in \mathbb{R}^-$, then θ is an identity.

If, moreover, the condition

(iv′) if $f\theta = f$ for $\theta \in \mathrm{Iso}(\mathbb{R})$ and $f \in \mathbb{R}^+$, then θ is an identity

holds, then we call this a generalized *dualizable* Reedy structure.

An ordinary Reedy category is a generalized Reedy category where there are no isomorphisms other than the identity maps.

Definition 3.2 Consider the following structures on Ξ:

- The degree function $d\colon \mathrm{Ob}(\Xi) \to \mathbb{N}$ with $d(S) = |\mathrm{Vt}(S)|$.

- The wide subcategory Ξ^+ consisting of all maps $\phi\colon R \to S$ such that the map $\phi_0\colon \mathrm{Ed}(R) \to \mathrm{Ed}(S)$ is injective.

- The wide subcategory Ξ^- consisting of all maps $\phi\colon R \to S$ such that ϕ_0 is surjective and, for each vertex $v \in \mathrm{Vt}(S)$, there exists a vertex $w \in \mathrm{Vt}(R)$ with $v \in \mathrm{Vt}(\phi_1(w))$.

This definition is chosen to be compatible with the known generalized Reedy structure[6] on the dendroidal category Ω from Definition 1.20, in the sense that the equalities $\Omega^+ = \Omega \cap \Xi^+$ and $\Omega^- = \Omega \cap \Xi^-$ hold, and the degree functions agree. Both inner and outer cofaces from Definition 1.16 are in Ξ^+, while codegeneracies are in Ξ^-. In fact, one can show that Ξ^+ is generated by the cofaces and Ξ^- is generated by the codegeneracies, though we do not need this here. Notice that

(6) $$\Xi^0(R, S) \cap \Xi^+(R, S) \cong \begin{cases} \mathrm{Ed}(S) & \text{if } R = \eta, \\ \varnothing & \text{otherwise,} \end{cases}$$

is nonempty if and only if $R = \eta$. The following lemma is immediate:

---

[6]See [5, Example 2.8] and the minor correction in [10, page 216].

Lemma 3.3 *Given $s_0 \in \mathrm{Ed}(S)$, the map*

$$\mathscr{L}\colon \Xi(R, S) \setminus \Xi^0(R, S) \to \coprod_{r \in \mathrm{Legs}(R)} \Omega(\mathscr{T}(R, r), \mathscr{T}(S, s_0))$$

restricts to maps

$$\Xi^+(R, S) \setminus \Xi^0(R, S) \to \coprod_{r \in \mathrm{Legs}(R)} \Omega^+(\mathscr{T}(R, r), \mathscr{T}(S, s_0)),$$

$$\Xi^-(R, S) \setminus \Xi^0(R, S) \to \coprod_{r \in \mathrm{Legs}(R)} \Omega^-(\mathscr{T}(R, r), \mathscr{T}(S, s_0)).$$

Proposition 3.4 *With the structure from Definition 3.2, Ξ is a dualizable generalized Reedy category.*

Proof For (i), note that isomorphisms preserve degree. If $\phi\colon R \to S \in \Xi^+$, then $\{\mathrm{Vt}(\phi_1(v))\}_{\mathrm{Vt}(R)}$ is a collection of pairwise disjoint, nonempty subsets of $\mathrm{Vt}(S)$. Thus,

$$d(R) = |\mathrm{Vt}(R)| = \sum_{\mathrm{Vt}(R)} 1 \leq \sum_{v \in \mathrm{Vt}(R)} |\mathrm{Vt}(\phi_1(v))| \leq |\mathrm{Vt}(S)| = d(S).$$

If $\phi\colon R \to S \in \Xi^-$, then surjectivity of ϕ_0 implies (by Lemma 1.23) that $|\mathrm{Vt}(\phi_1(v))| \leq 1$ for all $v \in \mathrm{Vt}(R)$. Let $\mathrm{Vt}(S) \to \mathrm{Vt}(R)$ be the map which sends $v \in \mathrm{Vt}(S)$ to the (unique, by Definition 1.13(3)) vertex w with $v \in \mathrm{Vt}(\phi_1(w))$. Since there is at most one v in a given $\phi_1(w)$, the map $\mathrm{Vt}(S) \to \mathrm{Vt}(R)$ is injective, hence $d(S) \leq d(R)$.

For (ii), it is clear that $\mathrm{Iso}(\Xi)$ is contained in $\Xi^+ \cap \Xi^-$. For the reverse inclusion, suppose that $\phi\colon R \to S$ is in both Ξ^+ and Ξ^-. If ϕ is constant, then by (6) we must have $R = \eta$; since ϕ is also in Ξ^-, S is also an edge, and ϕ is an isomorphism. If ϕ is not constant, choose a root s_0 for S. Then, by Lemma 3.3, we know that $\mathscr{L}(\phi)$ is in

$$\Omega^+(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0)) \cap \Omega^-(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0)) = \mathrm{Iso}_\Omega(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0)),$$

where $r_0 = \odot_{s_0}(\phi)$. By Corollary 2.14, we thus have that ϕ is an isomorphism in Ξ. Since ϕ was arbitrary, we have $\Xi^+ \cap \Xi^- \subseteq \mathrm{Iso}(\Xi)$ as well.

For (iv), note that if θ is an isomorphism, $\phi\colon R \to S \in \Xi^-$ and $\theta\phi = \phi$, then $\theta_0\phi_0 = \phi_0$. Since θ_0 is a bijection of sets and ϕ_0 is a surjection of sets, it follows that θ_0 is an identity. There is only one isomorphism $S \to S$ in Ξ which is the identity on edges (Example 1.14), hence $\theta = \mathrm{id}_S$. The proof that Ξ satisfies (iv′) follows similarly.

We finally turn to (iii). We first *construct* a factorization of a given morphism of Ξ. We may assume that $\phi \in \Xi(R, S) \setminus \Xi^0(R, S)$. Pick a root s_0 for S, and consider

$$\mathscr{L}(\phi) \in \Omega(\mathscr{T}(R, r_0), \mathscr{T}(S, s_0))$$

where $r_0 = \odot_{s_0}(\phi)$. There is a decomposition $\mathscr{L}(\phi) = g \circ h$ with $g \in \Omega^+(T, \mathscr{T}(S, s_0))$ and $h \in \Omega^-(\mathscr{T}(R, r_0), T)$. We have

$$\mathscr{T}(R, r_0) \xrightarrow{\ h\ } T \xrightarrow{\ g\ } \mathscr{T}(S, s_0);$$

apply the functor ι to this diagram to get

$$
\begin{array}{ccc}
\iota(\mathscr{T}(R, r_0)) \xrightarrow{\ \iota h\ } \iota T \xrightarrow{\ \iota g\ } \iota(\mathscr{T}(S, s_0)) \\
\downarrow{\cong} \qquad\qquad\qquad\qquad \downarrow{\cong} \\
R \xrightarrow{\ \ \mathscr{A}(\mathscr{L}(\phi)) = \phi\ \ } S
\end{array}
$$

Since $\iota h \in \Xi^-$, $\iota g \in \Xi^+$, and isomorphisms are in $\Xi^+ \cap \Xi^-$, we have provided the desired decomposition of ϕ.

Suppose that $\phi^1 \circ \psi^1 = \phi^2 \circ \psi^2$ with $\phi^i \in \Xi^+(U_i, S)$ and $\psi^i \in \Xi^-(R, U_i)$. Let $u_i = \odot_{s_0}(\psi^i)$ and $r_0 = \odot_{u_1}(\phi^1) = \odot_{u_2}(\phi^2)$ by Proposition 2.11. We have, by Proposition 2.11,

$$\mathscr{L}_{u_1}(\phi^1) \circ \mathscr{L}_{s_0}(\psi^1) = \mathscr{L}_{u_2}(\phi^2) \circ \mathscr{L}_{s_0}(\psi^2).$$

Now $\mathscr{L}_{u_i}(\phi^i) \in \Omega^+$ and $\mathscr{L}_{s_0}(\psi^i) \in \Omega^-$, so there exists an isomorphism a making the diagram

$$
\begin{array}{ccc}
\mathscr{T}(R, r_0) & \xrightarrow{\mathscr{L}_{u_1}(\psi^1)} & \mathscr{T}(U_1, u_1) & \xrightarrow{\mathscr{L}_{s_0}(\phi^1)} & \mathscr{T}(S, S_0) \\
\Big\| {\scriptstyle \mathrm{id}} & & \cong \Big\downarrow {\scriptstyle a} & & \Big\| {\scriptstyle \mathrm{id}} \\
\mathscr{T}(R, r_0) & \xrightarrow{\mathscr{L}_{u_2}(\psi^2)} & \mathscr{T}(U_2, u_2) & \xrightarrow{\mathscr{L}_{s_0}(\phi^2)} & \mathscr{T}(S, S_0)
\end{array}
$$

commute. Applying ι gives the back square of the diagram

and there exists a dashed map making the diagram commute. This map is necessarily an isomorphism. Thus, the front face establishes uniqueness of decompositions in Ξ. $\square$

A stronger statement is true, namely that Ξ is an EZ–category in the sense of [5, Definition 6.7]. We delay a proof of this fact (Theorem 4.9) until the end of the next section, as we should first learn a bit more about maps in Ξ^-.

4   The active/inert weak factorization system on Ξ

In this section we exhibit a weak factorization system on the category Ξ. Given a class I of morphisms in a category $\mathcal{C}$, write $I^{\boxtimes}$ for the maps which have the right lifting property with respect to every element of I. In other words, $f\colon X \to Y$ is in $I^{\boxtimes}$ if, and only if, every commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
{\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
B & \longrightarrow & Y
\end{array}
$$

with $i \in I$ admits a lift $B \to X$. Similarly, $^{\boxtimes}I$ is the class of maps having the left lifting property with respect to every element of I.

A weak factorization system (see [40, Definition 11.2.1]) consists of two classes of maps $\mathcal{L}$ and $\mathcal{R}$ such that every morphism factors into a map in $\mathcal{L}$ followed by one in $\mathcal{R}$, and such that $^{\boxtimes}\mathcal{R} = \mathcal{L}$ and $\mathcal{L}^{\boxtimes} = \mathcal{R}$.

Remark 4.1   We have actually already encountered one weak factorization system in this paper, namely the one whose left class is Ξ^- and whose right class is Ξ^+. In fact, the following is true: if $\mathbb{R}$ is a generalized Reedy category, then $(\mathbb{R}^-, \mathbb{R}^+)$ is an *orthogonal factorization system* (see eg [21, 2.2]), that is, a weak factorization system in which all of the liftings are unique. This is mentioned in [42, Remark 8.28]; its

proof is an exercise using only axioms (iii)–(iv) and closure of $\mathbb{R}^+$ and $\mathbb{R}^-$ under composition.

Any weak factorization system whose right class $\mathcal{R}$ is contained in the monomorphisms will be orthogonal. This will be true, in particular, of the weak factorization system in Proposition 4.7 (using Theorem 4.9 and Remark 4.3).

Definition 4.2 A morphism $\phi\colon R \to S$ in Ξ is called *active* if $\mathrm{Im}(\phi) = S$. It is called *inert* if $\phi_0\colon \mathrm{Ed}(R) \to \mathrm{Ed}(S)$ is injective and if $w \in \mathrm{Vt}(\phi_1(v))$ then $\mathrm{Nbhd}(w) \subseteq \mathrm{Im}(\phi_0)$.

Outer cofaces (see Definition 1.16) are inert maps, while codegeneracies and inner cofaces are active maps. Notice that every map in Ξ^- (see Definition 3.2) is an active map. Further, every inert map is contained in Ξ^+ by the following remark.

Remark 4.3 Suppose that ϕ is inert. Then, since ϕ_0 is injective, $0 < |\mathrm{Vt}(\phi_1(v))|$ for any $v \in \mathrm{Vt}(R)$. Further, if $\mathrm{Vt}(\phi_1(v))$ contains two (adjacent) vertices w_1 and w_2, connected by an edge e, then $e \in \mathrm{Int}(\phi_1(v)) \cap \mathrm{Nbhd}(w_1) \subseteq \mathrm{Int}(\phi_1(v)) \cap \mathrm{Im}(\phi_0) = \varnothing$ by Lemma 1.24. Thus, $|\mathrm{Vt}(\phi_1(v))| \leq 1$. In other words, ϕ is (isomorphic to) a subgraph inclusion.

Let us spell out an alternative characterization of the active maps. The following lemma will be useful in the proof of Proposition 4.5:

Lemma 4.4 *Suppose that $R \in \mathrm{Sbgph}(S)$ is not an edge. If $\mathrm{Legs}(R) = \mathrm{Legs}(S)$, then $R = S$.*

Proof Suppose that $v \in \mathrm{Vt}(S)$ is connected to a different vertex $w \in \mathrm{Vt}(R)$ by an edge e. Then $e \notin \mathrm{Legs}(S) = \mathrm{Legs}(R)$, so $v \in \mathrm{Vt}(R)$. Since S is connected and $\mathrm{Vt}(R) \neq \varnothing$, we have $\mathrm{Vt}(R) = \mathrm{Vt}(S)$. The definition of subgraph implies $\mathrm{Ed}(R) = \mathrm{Ed}(S)$ as well. $\square$

Proposition 4.5 *Suppose $\phi\colon R \to S$ is in Ξ.*

(1) *Suppose $\phi \notin \Xi^0(R, S)$. Then ϕ is active if and only if $\phi_0(\mathrm{Legs}(R)) = \mathrm{Legs}(S)$.*

(2) *Suppose $\phi \in \Xi^0(R, S)$. Then ϕ is active if and only if $S = \eta$.*

Proof The forward direction of (1) follows immediately from Proposition 1.29. For the reverse direction, by hypothesis and Proposition 1.29 we have $\mathrm{Legs}(S) = \mathrm{Legs}(\mathrm{Im}(\phi))$, so by Lemma 4.4 we have $\mathrm{Im}(\phi) = S$. For (2), simply note that if $\phi \in \Xi^0(R, S)$, then $\mathrm{Im}(\phi)$ is an edge. $\square$

Lemma 4.6 *The set of active morphisms has the left lifting property with respect to the set of inert morphisms.*

Proof Suppose we are given a commutative diagram in Ξ

(7)
$$
\begin{array}{ccc}
R & \xrightarrow{\;\alpha\;} & P \\
\phi\downarrow & \gamma\nearrow & \downarrow\psi \\
S & \xrightarrow[\;\beta\;]{} & Q
\end{array}
$$

with ϕ active and ψ inert (so, in particular, P is a subgraph of Q); we wish to show that a lift $\gamma\colon S \to P$ exists. Temporarily write $\alpha_0^L\colon \mathrm{Legs}(R) \to \mathrm{Ed}(P)$ and $\phi_0^L\colon \mathrm{Legs}(R) \to \mathrm{Legs}(S)$ for the restrictions of α_0 and ϕ_0 to legs.

If $S = \eta$ consists of the single edge e, then R is linear. We have a diagram

$$
\begin{array}{ccc}
\mathrm{Legs}(R) & \xrightarrow{\;\alpha_0^L\;} & \mathrm{Ed}(P) \\
\phi_0\downarrow & & \downarrow\psi_0 \\
\{e\} & \xrightarrow[\;\beta_0\;]{} & \mathrm{Ed}(Q)
\end{array}
$$

with ψ_0 an injection. Thus, $\mathrm{Im}(\alpha_0^L) = |_p$ is a single edge, and we define $\gamma\colon S \to P$ by $\gamma_0(e) = p \in \mathrm{Ed}(P)$.

If S contains a vertex, then $\phi_0^L\colon \mathrm{Legs}(R) \to \mathrm{Legs}(S)$ is a bijection. Define $\gamma_0^L = \alpha_0^L \circ (\phi_0^L)^{-1}\colon \mathrm{Legs}(S) \to \mathrm{Ed}(P)$. We wish to extend this to $\mathrm{Int}(S)$, which we will do in a moment. Since ϕ is active, every $w \in \mathrm{Vt}(S)$ is in $\mathrm{Vt}(\phi_1(v))$ for some $v \in \mathrm{Vt}(R)$. Notice that

$$
\bigcup_{w\in\phi_1(v)} \beta_1(w) = (\beta \circ \phi)_1(v) = (\psi \circ \alpha)_1(v) = \bigcup_{t\in\alpha_1(v)} \psi_1(t) \in \mathrm{Sbgph}(\mathrm{Im}(\psi))
$$

and, since $P \to \mathrm{Im}(\psi)$ is an isomorphism, $\mathrm{Sbgph}(\mathrm{Im}(\psi)) \cong \mathrm{Sbgph}(P)$. Thus, for $w \in \mathrm{Vt}(\phi_1(v))$, there is a unique subgraph $G_w \in \mathrm{Sbgph}(P)$ which maps to $\beta_1(w) \in \mathrm{Sbgph}(\mathrm{Im}(\psi)) \subseteq \mathrm{Sbgph}(Q)$ under ψ.

Suppose that $s \in \mathrm{Int}(S)$ is an interior edge. The edge s is adjacent to two distinct vertices w and w'. We have $\beta_1(w) \cap \beta_1(w') = \{\beta_0(s)\}$, hence $G_w \cap G_{w'} = \{p\}$ for some edge p. Set $\gamma_0(s) = p$; by definition we have $\phi_0\gamma_0(s) = \beta_0(s)$. Further, since ϕ is active, it sends legs to legs, so if $s = \phi_0(r)$, then r is an internal edge between distinct vertices v and v', with $w \in \mathrm{Vt}(\phi_1(v))$ and $w' \in \mathrm{Vt}(\phi_1(v'))$. Since $G_w \subseteq \alpha_1(v)$

and $G_{w'} \subseteq \alpha_1(v')$, we have $\{\gamma_0(s)\} = G_w \cap G_{w'} \subseteq \alpha_1(v) \cap \alpha_1(v') = \{\alpha_0(r)\}$. Thus, $\gamma_0\phi_0(r) = \gamma_0(s) = \alpha_0(r)$. In conclusion, $\alpha_0 = \gamma_0\phi_0$ and $\psi_0\gamma_0 = \beta_0$.

Next, define $\gamma_1(w) = G_w \in \mathrm{Sbgph}(\alpha_1(v))$. The pair $\gamma = (\gamma_0, \gamma_1)$ is a morphism of Ξ, which quickly follows from inertness of ψ and the fact that β is a morphism of Ξ. By definition, we have $(\psi \circ \gamma)_1(w) = \bigcup_{v \in \gamma_1(w)} \psi_1(v) = \beta_1(w)$, hence $\psi \circ \gamma = \beta$. Then $\psi \circ \gamma \circ \phi = \beta \circ \phi = \psi \circ \alpha$, hence the injective map $\psi \colon \mathrm{Sbgph}(P) \hookrightarrow \mathrm{Sbgph}(Q)$ takes $(\gamma \circ \phi)_1(v)$ and $\alpha_1(v)$ to the same element. It follows that $(\gamma \circ \phi)_1(v) = \alpha_1(v)$, so $\gamma \circ \phi = \alpha$. Thus, we have shown that (7) always has a lift. $\square$

Proposition 4.7 *The active and inert morphisms form a weak factorization system.*

Proof Given a map $\phi \colon R \to S$, we can factor ϕ as $R \to \mathrm{Im}(\phi) \to S$. The map $R \to \mathrm{Im}(\phi)$ is active since $\mathrm{Legs}(\mathrm{Im}(\phi)) = \phi_0(\mathrm{Legs}(R))$, while $\mathrm{Im}(\phi) \to S$ is a subgraph inclusion, hence inert.

For the remainder of the proof, write $\mathcal{L}$ for the set of active morphisms and $\mathcal{R}$ for the set of inert morphisms. In Lemma 4.6 we showed that $\mathcal{R} \subseteq \mathcal{L}^{\boxempty}$ (or, equivalently, $\mathcal{L} \subseteq {}^{\boxempty}\mathcal{R}$).

For the reverse inclusions, suppose that $\phi \colon R \to S$ is in $\mathcal{L}^{\boxempty}$. Consider the decomposition

$$R \xrightarrow{\phi^-} T \xrightarrow{\phi^+} S$$

coming from the generalized Reedy structure on Ξ. We know that ϕ^- is an active morphism, hence we can lift in the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\mathrm{id}} & R \\
{\scriptstyle\phi^-}\downarrow & {\scriptstyle\psi}\nearrow & \downarrow{\scriptstyle\phi} \\
T & \xrightarrow[\phi^+]{} & S
\end{array}
$$

which implies $\psi\phi^- = \mathrm{id}$, hence ϕ^- is an isomorphism. Thus, ϕ_0 is injective. Further, the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\mathrm{id}} & R \\
{\scriptstyle\bar\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\
\mathrm{Im}(\phi) & \xrightarrow{i} & S
\end{array}
$$

admits a lift; if w is a vertex in $\phi_1(v) \subseteq \mathrm{Im}(\phi)$, then

$$\mathrm{Nbhd}(w) = i_0(\mathrm{Nbhd}(w)) = \phi_0\psi_0(\mathrm{Nbhd}(w)) \subseteq \mathrm{Im}(\phi_0).$$

Hence, ϕ is inert, and we see that $\mathcal{L}^{\boxempty} \subseteq \mathcal{R}$.

Now suppose that $\phi\colon R \to S$ is in $^{\boxtimes}\mathcal{R}$. Then

$$\begin{array}{ccc} R & \xrightarrow{\bar{\phi}} & \mathrm{Im}(\phi) \\ \phi \downarrow & \quad\psi\quad\nearrow & \downarrow i \\ S & \xrightarrow{\mathrm{id}} & S \end{array}$$

admits a lift $\psi\colon S \to \mathrm{Im}(\phi)$. Factor $\psi = \psi^{+}\psi^{-}$ with $\psi^{+} \in \Xi^{+}$, $\psi^{-} \in \Xi^{-}$. Since $\mathrm{id}_S = (i\psi^{+})\psi^{-}$ with $i\psi^{+} \in \Xi^{+}$, we have $i\psi^{+}, \psi^{-} \in \mathrm{Iso}(\Xi)$ by Definition 3.1(iii). It follows that $i \in \Xi^{+}$ preserves degree, hence is an isomorphism. Thus, ϕ is active, and we find that $^{\boxtimes}\mathcal{R} \subseteq \mathcal{L}$. □

The active and inert maps which are oriented form a weak factorization system on the category Ω. This was proved in [28, Proposition 1.3.13], where inert maps are called "free" and active maps are called "boundary-preserving".

Proposition 4.8 *The inclusion $\iota\colon \Omega \to \Xi$ respects the weak factorization structures.*
□

As promised at the conclusion of the previous section, we have the following:

Theorem 4.9 *The category Ξ, together with the degree function $d(S) = |\mathrm{Vt}(S)|$ from Definition 3.2, is an EZ–category in the sense of [5, Definition 6.7].*

Proof As we have already established that Ξ is a dualizable generalized Reedy category (Proposition 3.4), it is enough to show that

(a) Ξ^{+} is the subcategory of monomorphisms,

(b) Ξ^{-} is the subcategory of split epimorphisms, and

(c) any pair of split epimorphisms with common domain has an absolute pushout [39] (that is, can be extended to a commutative square which becomes a pushout square after applying any functor).

For efficiency, we rely on the fact that Ω is an EZ–category [5, Examples 6.8]. Every map $\phi\colon R \to S$ in Ξ is isomorphic (in the arrow category $\Xi^{[1]}$) to at least one map $\iota(\widetilde{\phi}\colon \widetilde{R} \to \widetilde{S})$ by Corollaries 2.16 and 2.17, and the fact that $\mathrm{Legs}(S) \neq \varnothing$. Suppose that ϕ is a map and we have chosen an isomorphism $\gamma\colon \iota\widetilde{\phi} \cong \phi$ in $\Xi^{[1]}$. The following hold, which then imply that (a) and (b) hold by the corresponding properties of Ω:

(i) ϕ is a monomorphism (resp. split epimorphism) if and only if $\widetilde{\phi}$ is a monomorphism (resp. split epimorphism).

(ii) ϕ is in Ξ^+ (resp. in Ξ^-) if and only if $\widetilde{\phi}$ is in Ω^+ (resp. in Ω^-).

The only point that is perhaps not immediate is that if ϕ is a split epimorphism, so is $\widetilde{\phi}$. Suppose α is a section of ϕ. If ϕ is constant then $S = \widetilde{S} = \eta$ and the composite $\eta \xrightarrow{\alpha} R \xrightarrow{\gamma^{-1}} \iota\widetilde{R}$ is automatically oriented and is a section for $\widetilde{\phi}$. If ϕ is not constant, then $\widetilde{\phi}$ is isomorphic in the arrow category $\Omega^{[1]}$ to $\mathscr{L}_{s_0,r_0}(\phi)\colon \mathscr{T}(R,r_0) \to \mathscr{T}(S,s_0)$, where s_0 and r_0 are the images under γ of the roots of $\widetilde{S}$ and $\widetilde{R}$, respectively. We then have

$$\mathrm{id}_{\mathscr{T}(S,s_0)} = \mathscr{L}_{s_0}(\mathrm{id}_S) = \mathscr{L}_{s_0}(\phi \circ \alpha) = \mathscr{L}_{s_0}(\phi) \circ \mathscr{L}_{r_0}(\alpha)$$

by Proposition 2.11. Since $\mathscr{L}_{s_0}(\phi)$ admits a section, so does $\widetilde{\phi}$.

Let us turn to point (c). Suppose that $\phi\colon R \to S$ and $\psi\colon R \to U$ are two maps in Ξ^- (ie two split epimorphisms). Pick $r_0 \in \mathrm{Legs}(R)$, and let $s_0 = \phi_0(r_0) \in \mathrm{Legs}(S)$ and $u_0 = \psi_0(r_0) \in \mathrm{Legs}(U)$ (using that ϕ and ψ are active). There are maps $\widetilde{\phi}\colon \mathscr{T}(R,r_0) \to \mathscr{T}(S,s_0)$ and $\widetilde{\psi}\colon \mathscr{T}(R,r_0) \to \mathscr{T}(U,u_0)$ and an isomorphism of diagrams of shape $\bullet \leftarrow \bullet \to \bullet$,

$$
\begin{array}{ccccc}
\iota\mathscr{T}(U,u_0) & \xleftarrow{\;\iota\widetilde{\psi}\;} & \iota\mathscr{T}(R,r_0) & \xrightarrow{\;\iota\widetilde{\phi}\;} & \iota\mathscr{T}(S,s_0) \\
\cong \downarrow{\scriptstyle\gamma_U} & & \cong \downarrow{\scriptstyle\gamma_R} & & \cong \downarrow{\scriptstyle\gamma_S} \\
U & \xleftarrow{\quad\psi\quad} & R & \xrightarrow{\quad\phi\quad} & S
\end{array}
$$

in Ξ. Let

(8)
$$
\begin{array}{ccc}
\mathscr{T}(R,r_0) & \xrightarrow{\;\widetilde{\phi}\;} & \mathscr{T}(S,s_0) \\
\widetilde{\psi}\downarrow & & \downarrow h \\
\mathscr{T}(U,u_0) & \xrightarrow{\quad g\quad} & T
\end{array}
$$

be an absolute pushout of $\widetilde{\phi}, \widetilde{\psi} \in \Omega^-$. The diagram (8) remains an absolute pushout after applying ι, and the resulting square is isomorphic to the square

(9)
$$
\begin{array}{ccc}
R & \xrightarrow{\;\phi\;} & S \\
\psi\downarrow & & \downarrow{(\iota h)\gamma_S^{-1}} \\
U & \xrightarrow{\;(\iota g)\gamma_U^{-1}\;} & \iota T
\end{array}
$$

Thus, (9) is an absolute pushout as well, and (c) is established. $\qquad\square$

5 A functor from Ξ to the category of cyclic operads

In this section, we will show that given a tree R, there is a cyclic (colored) operad $C(R) \in \mathbf{Cyc}$ with $\mathrm{Col}(C(R)) = \mathrm{Ed}(R)$. Further, the assignment $R \mapsto C(R)$ is the object part of a functor $C \colon \Xi \to \mathbf{Cyc}$. This functor is faithful but not full.

5.1 A monadic description

Fix a color set $\mathfrak{C}$. A $\mathfrak{C}$–colored tree is a (pinned) tree S together with a function $\xi \colon \mathrm{Ed}(S) \to \mathfrak{C}$. If $\underline{c} = c_0, c_1, \ldots, c_n$ is a profile in $\mathfrak{C}$, we will write $\mathtt{tree}(\underline{c}) = \mathtt{tree}^{\mathfrak{C}}(\underline{c})$ for the groupoid of all $\mathfrak{C}$–colored trees S such that

$$\{0, 1, \ldots, n\} \xrightarrow{\mathrm{ord}} \mathrm{Legs}(S) \hookrightarrow \mathrm{Ed}(S) \xrightarrow{\xi} \mathfrak{C}$$

takes i to c_i for $0 \le i \le n$. The isomorphisms $(S, \xi) \to (S', \xi')$ in $\mathtt{tree}(\underline{c})$ are those isomorphisms $\phi \colon S \to S'$ in Ξ such that $\xi = \xi' \circ \phi_0$ and $\phi_0 \circ \mathrm{ord}_S = \mathrm{ord}_{S'}$.

Consider the groupoid $\Sigma_{\mathfrak{C}}^+$ whose objects are finite, nonempty, ordered lists $\underline{c} = c_0, c_1, \ldots, c_n$ ($n \ge 0$) of elements of $\mathfrak{C}$, and morphisms are $\underline{c}\sigma \xrightarrow{\sigma} \underline{c}$, where $\sigma \in \Sigma_m^+ = \mathrm{Aut}\{0, 1, \ldots, m\}$ and $\underline{c}\sigma = (c_{\sigma(0)}, c_{\sigma(1)}, \ldots, c_{\sigma(m)})$. Such a morphism of $\Sigma_{\mathfrak{C}}^+$ determines a morphism $\mathtt{tree}(\underline{c}) \to \mathtt{tree}(\underline{c}\sigma)$ sending S to $S\sigma$, which has all of the same structure as S except that the leg ordering $\mathrm{ord}_{S\sigma}$ is the composite

$$\{0, 1, \ldots, m\} \xrightarrow{\sigma} \{0, 1, \ldots, m\} \xrightarrow{\mathrm{ord}_S} \mathrm{Legs}(S).$$

This is, in fact, a contravariant functor $\mathtt{tree}(-) = \mathtt{tree}^{\mathfrak{C}}(-) \colon \Sigma_{\mathfrak{C}}^+ \to \mathbf{Gpd}$.

Every object $X \in \mathbf{Set}^{\Sigma_{\mathfrak{C}}^+}$ determines a functor $\mathtt{tree}^{\mathfrak{C}}(\underline{c}) \to \mathbf{Set}$, given on an object $S \in \mathtt{tree}^{\mathfrak{C}}(\underline{c})$ by

$$X[S] = \prod_{v \in \mathrm{Vt}(S)} X(\xi(\mathrm{Nbhd}(v))).$$

Remark 5.1 There is a monad $T^+ \colon \mathbf{Set}^{\Sigma_{\mathfrak{C}}^+} \to \mathbf{Set}^{\Sigma_{\mathfrak{C}}^+}$ given on objects by

$$T^+(X)(\underline{c}) = \operatorname*{colim}_{S \in \mathtt{tree}(\underline{c})} X[S].$$

The category of algebras for T^+ is $\mathbf{Cyc}_{\mathfrak{C}}$, the subcategory of $\mathbf{Cyc}$ consisting of those cyclic operads with color set $\mathfrak{C}$ and morphisms which are the identity on colors. An analogous statement appears in [35, Section 5.1] in the monochrome case (see also [26] for the colored case).

Notice that if $f\colon \mathfrak{C} \to \mathfrak{D}$ is a map of sets, then restriction along $\Sigma_{\mathfrak{C}}^+ \to \Sigma_{\mathfrak{D}}^+$ induces a functor $f^*\colon \mathbf{Cyc}_{\mathfrak{D}} \to \mathbf{Cyc}_{\mathfrak{C}}$. A map $\alpha\colon P \to Q$ in $\mathbf{Cyc}$ is the same thing as a pair (α_0, α_1), where $\alpha_0\colon \mathrm{Col}(P) \to \mathrm{Col}(Q)$ is a map of sets and $\alpha_1\colon P \to \alpha_0^* Q$ is a map in $\mathbf{Cyc}_{\mathfrak{C}}$.

5.2 The functor $C\colon \Xi \to \mathbf{Cyc}$

Let $\mathrm{Prof}(\mathfrak{C})$ be the set of nonempty ordered lists of elements in $\mathfrak{C}$; it is the set of objects of the groupoid $\Sigma_{\mathfrak{C}}^+$. There is a forgetful functor $\mathbf{Cyc}_{\mathfrak{C}} \to \mathbf{Set}^{\Sigma_{\mathfrak{C}}^+} \to \mathbf{Set}^{\mathrm{Prof}(\mathfrak{C})}$. Write

$$F_{\mathfrak{C}}\colon \mathbf{Set}^{\mathrm{Prof}(\mathfrak{C})} \rightleftarrows \mathbf{Cyc}_{\mathfrak{C}} :U_{\mathfrak{C}}$$

for the corresponding adjunction.

Definition 5.2 Let S be a tree and $\mathfrak{C} = \mathrm{Ed}(S)$. Then S determines an object $Z = Z^S$ of $\mathbf{Set}^{\mathrm{Prof}(\mathfrak{C})}$ with

(10)
$$Z_{\underline{c}} = \begin{cases} \{v\} & \text{if } \mathrm{Nbhd}(v) = \underline{c} \text{ as ordered lists,} \\ \varnothing & \text{otherwise.} \end{cases}$$

The cyclic operad $C(S)$ is defined as $F_{\mathfrak{C}}(Z)$.

Applying the left adjoint of $\mathbf{Set}^{\Sigma_{\mathfrak{C}}^+} \to \mathbf{Set}^{\mathrm{Prof}(\mathfrak{C})}$ (for $\mathfrak{C} = \mathrm{Ed}(S)$) to the object $Z = Z^S$ gives a new object Z'. This object has the property that $|Z'_{\underline{c}}| = 1$ if and only if $\underline{c}$ contains no repetitions and $\underline{c} = \mathrm{Nbhd}(v)$ as *unordered* lists for some v. Otherwise, $|Z'_{\underline{c}}| = 0$.

Notice that if $T \in \Omega$ is a rooted tree and $F\colon \mathbf{Op} \to \mathbf{Cyc}$ is left adjoint to the forgetful functor, then

$$C(\iota(T)) = F(\Omega(T)).$$

Remark 5.3 If $R \in \mathrm{Sbgph}(S)$ and R is a pinned graph, with

$$\{0, 1, \ldots, n\} \xrightarrow{\mathrm{ord}_R} \mathrm{Legs}(R) \hookrightarrow \mathrm{Ed}(S)$$

sending i to c_i, then R determines an element of $C(S)(c_1, \ldots, c_n; c_0)$. There may, of course, be many elements of this set that do not come from subgraphs of S. Varying the vertex orders ord^v on R does not change the element produced in this way. The cyclic operad $C(S)$ is nearly always infinite (see Example 5.7 below). Specifically, $C(S)$ is infinite whenever $S \notin \{\eta, \star_1\}$, as then S has a vertex v with $|v| > 1$.

Example 5.4 Let S be the graph from Figure 4. Then $C(S)$ is the cyclic operad O generated by three operations $u \in O(b, a; c)$, $v \in O(d, e, f; c)$ and $w \in O(; d)$. We see there is an element $v \circ_1 w \in O(e, f; c)$. Notice that there are no elements of the form $q \circ_i r$, where $r \in \{u, v\}$ and $q \in \{u, v, w\}$, as c is not an input for such a q. But we can first apply the rotation to get, say, $v \cdot \tau \in O(e, f, c; d)$, and then compose to get $(v \cdot \tau) \circ_3 u \in O(e, f, b, a; d)$. Finally, there is an element $((v \cdot \tau^2) \circ_2 u) \circ_4 w \in O(f, b, a; e)$; these are all of the elements given by subgraphs of S. Notice there are many other elements, for example $u \circ_1 (u \cdot \tau) \in O(a, c, a; c)$.

We wish to extend $S \mapsto C(S)$ to a functor $\Xi \to \mathbf{Cyc}$. Defining maps out of $C(S)$ is easy, as this object is free in $\mathbf{Cyc}_{\mathrm{Ed}(S)}$. We utilize Remark 5.3 in order to regard subgraphs of S as elements in $C(S)$.

Definition 5.5 (C as a functor) Suppose $\phi \in \Xi(R, S)$ and write $\mathfrak{C} = \mathrm{Col}(C(R)) = \mathrm{Ed}(R)$. Set $C(\phi)_0 = \phi_0 \colon \mathfrak{C} = \mathrm{Ed}(R) \to \mathrm{Ed}(S)$. Since $C(R)$ is free in $\mathbf{Cyc}_{\mathfrak{C}}$, it is enough to define $C(\phi)_1$ on generators. Let $\phi_1^{\natural} \colon Z^R \to U_{\mathfrak{C}} \phi_0^* C(S)$ in $\mathbf{Set}^{\mathrm{Prof}(\mathfrak{C})}$ by $v \mapsto \phi_1(v) \in \mathrm{Sbgph}(S)$, endowed with the ordering

$$\{0, 1, \dots, n_v\} \xrightarrow{\mathrm{ord}^v} \mathrm{Nbhd}(v) \xrightarrow{\phi_0} \mathrm{Legs}(\phi_1(v)).$$

Define $C(\phi)_1 \colon C(R) \to \phi_0^* C(S)$ to be the adjoint of $\phi_1^{\natural}$.

Theorem 5.6 *The functor $C \colon \Xi \to \mathbf{Cyc}$ is faithful.*

Proof Let $\phi, \psi \in \Xi(R, S)$, and suppose that $C(\phi) = C(\psi)$. Write $f = \phi_0 = \psi_0$ for the common function on color sets. By assumption, for each $v \in \mathrm{Vt}(R)$, we have that $C(\phi)_1(v)$ is equal to $C(\psi)_1(v)$. But $C(\phi)_1(v)$ comes from $\phi_1(v)$ (with appropriate choice of pinned structure) as in Remark 5.3, and likewise for $C(\psi)_1(v)$. It follows then that the subgraphs $\phi_1(v)$ and $\psi_1(v)$ are the same, hence $\phi = \psi$.   □

Example 5.7 ($C \colon \Xi \to \mathbf{Cyc}$ is not full) Consider the graph $L_1 = \text{—•—}$ with edges 0 and 1 and vertex v. There are exactly four elements in $\Xi(L_1, L_1)$, corresponding to the four maps of edge sets $\{0, 1\} \to \{0, 1\}$. But there are infinitely many maps in $\mathrm{hom}_{\mathbf{Cyc}}(C(L_1), C(L_1))$. One example which is not in $C(\Xi(L_1, L_1))$ is the map $f \colon C(L_1) \to C(L_1)$ which on color sets is $f(i) = 0$ and on morphisms is specified by $f(v) = v \circ_1 v$:

$$\underline{\overset{0}{\bullet}\,\overset{1}{\bullet}\,\overset{0}{}} \; .$$

In Section 1.3, we saw several examples of cyclic dendroidal sets. Another class of examples are the nerves of cyclic operads.

Definition 5.8 (nerve) There is a functor $N_c\colon \mathbf{Cyc} \to \mathbf{Set}^{\Xi^{\mathrm{op}}}$ defined, on an object $O \in \mathbf{Cyc}$, by

$$N_c(O) = \mathbf{Cyc}(C(-), O) \in \mathbf{Set}^{\Xi^{\mathrm{op}}}.$$

We will refer to N_c as the *cyclic dendroidal nerve*.

Recall there is an analogous *dendroidal* nerve $N_d\colon \mathbf{Op} \to \mathbf{Set}^{\Omega^{\mathrm{op}}}$ defined by $N_d(O)_T = \mathbf{Op}(\Omega(T), O)$, where $\Omega(T)$ is the $\mathrm{Ed}(T)$–colored operad freely generated by T. This functor is a fully faithful embedding, and the essential image may be characterized using *inner horns* and *Segal cores* from Definition 1.34.

Theorem 5.9 (Moerdijk–Weiss; Cisinski–Moerdijk) *Let $X \in \mathbf{Set}^{\Omega^{\mathrm{op}}}$ be a dendroidal set. The following are equivalent:*

(i′) $X \cong N_d(O)$ *for some operad O.*

(ii′) $X_T = \hom(\Omega[T], X) \to \hom(\Lambda^\delta \Omega[T], X)$ *is a bijection for every inner coface map δ.*

(iii′) $X_T = \hom(\Omega[T], X) \to \hom(\mathrm{Sc}[T], X)$ *is a bijection for every rooted tree T.*

Proof The equivalence of the first two was established in [37, Proposition 5.3 and Theorem 6.1], while the third was shown to be equivalent to the first two in [14, Corollary 2.6]. □

The next section is dedicated to establishing an analogous theorem for the cyclic dendroidal nerve.

Remark 5.10 Example 5.7 shows, in particular, that the composite

$$\Xi \to \mathbf{Cyc} \to \mathbf{Set}^{\Xi^{\mathrm{op}}}, \quad S \to N_c C(S),$$

is not the Yoneda embedding $S \mapsto \Xi[S]$, since the Yoneda embedding is fully faithful.

6 Cyclic operads and the nerve theorem

Our goal in this section is to prove Theorem 6.7. The method of proof is to reduce to the dendroidal case and apply Theorem 5.9. To do this, we first need to understand the relationship between dendroidal Segal cores (resp. dendroidal inner horns) and their cyclic dendroidal analogues from Definition 1.34. The other main ingredient is to show that if X is a cyclic dendroidal set such that $\iota^* X$ is the nerve of an operad, then X was already the nerve of a cyclic operad. This is Theorem 6.3.

Lemma 6.1 *If $T \in \Omega$ is a rooted tree, then*

$$\iota_!(\mathrm{Sc}[T] \to \Omega[T]) \cong (\mathrm{Sc}[\iota T] \to \Xi[\iota T]).$$

Proof Given $S \in \Xi$, define a category $\mathcal{C}^S$ with $\mathrm{Ob}(\mathcal{C}^S) = \mathrm{Ed}(S) \sqcup \mathrm{Vt}(S)$, and nonidentity maps $\{e \to v \mid e \in \mathrm{Nbhd}(v)\}$. There is a functor $F \colon \mathcal{C}^S \to \mathbf{Set}^{\Xi^{\mathrm{op}}}$ which on objects is given by $e \mapsto \Xi[\eta]$ and $v \mapsto \Xi[\maltese_{|v|}]$. Define $F(\mathrm{ord}^v(k) \to v)$ to be the inclusion $k_* \colon \Xi[\eta] \to \Xi[\maltese_{|v|}]$ which hits $k \in \Xi[\maltese_{|v|}]_\eta = \{0, 1, \dots, |v| - 1\}$. Then $\mathrm{colim}_{\mathcal{C}^S} F \cong \mathrm{Sc}[S]$. A similar consideration applies for $\mathrm{Sc}[T] \subseteq \Omega[T]$; namely if T is a rooted tree in Ω, then there is a functor (using the same domain category from above) $F' \colon \mathcal{C}^T \to \mathbf{Set}^{\Omega^{\mathrm{op}}}$ with $\mathrm{colim}_{\mathcal{C}^T} F' \cong \mathrm{Sc}[T] \subseteq \Omega[T]$.

Notice that $\iota_! \Omega[T] \cong \Xi[\iota T]$, since, for any $X \in \mathbf{sSet}^{\Xi^{\mathrm{op}}}$, they are the same on $\mathrm{hom}(-, X)$:

$$\mathrm{hom}(\iota_! \Omega[T], X) = \mathrm{hom}(\Omega[T], \iota^* X) = (\iota^* X)_T = X_{\iota T} = \mathrm{hom}(\Xi[T], X).$$

In particular, $\iota_!(\Omega[\maltese_v]) = \Xi[\maltese_v]$ and $\iota_!(\Omega[\eta]) = \Xi[\eta]$. Since $\iota_!$ is a left adjoint, it commutes with colimits, and we have

$$\iota_!(\mathrm{Sc}[T] \to \Omega[T]) \cong (\mathrm{Sc}[\iota T] \to \Xi[\iota T]). \qquad \square$$

Lemma 6.2 *Let $\delta \colon R \to T$ be any coface map in Ω. Then*

$$\iota_!(\Lambda^\delta \Omega[T] \hookrightarrow \Omega[T]) \cong (\Lambda^\delta \Xi[T] \hookrightarrow \Xi[T]).$$

Proof Our strategy is the same as in the previous lemma: we write both sides as a colimit of representables and then use that $\iota_!$ commutes with colimits and takes representables to representables.

Suppose that $S \in \Xi$. Let V^S be a skeleton of the comma category $\Xi^+ \downarrow S$; there is an evident functor $A^S \colon V^S \to \mathbf{Set}^{\Xi^{\mathrm{op}}}$ which sends $\phi \colon R \to S$ to the object $\Xi[R]$ and

$$R' \xrightarrow{\quad \alpha \quad} R$$
$$\phi' \searrow \quad \swarrow \phi$$
$$S$$

to $\alpha_* \colon \Xi[R'] \to \Xi[R]$. Since V^S has a terminal object (the one isomorphic to id_S), $\mathrm{colim}_{V^S} A^S = \Xi[S]$. If $\delta \colon R \to S$ is a coface map, write V_δ^S for the full subcategory

of V^S that excludes the two objects isomorphic to id_S and δ. Then there is an isomorphism

$$\mathrm{colim}_{V_\delta^S} A^S|_{V_\delta^S} \xrightarrow{\ \cong\ } \Lambda^\delta \Xi[S]$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\mathrm{colim}_{V^S} A^S \xrightarrow{\ \cong\ } \Xi[S]$$

This isomorphism arises from the fact that $\Xi[R] \to \Xi[S]$ is a monomorphism whenever $R \to S$ is in Ξ^+ (Theorem 4.9) and the fact that all elements of $\Xi[S]$ (in particular, all elements of $\Lambda^\delta \Xi[S]$) factor through a minimal face by Definition 3.1(iii).

Suppose that T is a rooted tree (with root t_0), and let $\phi\colon R \to T$ be any map in Ξ^+. If $R = \eta$, then ϕ is automatically oriented. Assume that R has at least one vertex. Then $\mathscr{L}_{t_0}(\phi)\colon \mathscr{T}(R, \odot_{t_0}(\phi)) \to \mathscr{T}(T, t_0) = T$ is isomorphic to ϕ over T and is also in Ξ^+. Thus, we may as well have assumed that ϕ was oriented in the first place. In particular, we may assume that all objects in V^T are oriented maps. With this assumption, all morphisms in V^T are also oriented maps, for if $\phi\alpha = \phi'$ in Ξ^+ with ϕ and ϕ' oriented maps, then α is also oriented by Proposition 2.11. Thus, we have a functor $B^T\colon V^T \to \mathbf{Set}^{\Omega^{\mathrm{op}}}$ sending $R \to T$ to $\Omega[R]$, and with $\iota_! \circ B^T \cong A^T$. Further, as above, $\mathrm{colim}_{V_\delta^T} B^T|_{V_\delta^T}$ is isomorphic to $\Lambda^\delta \Omega[T]$ over $\Omega[T]$. Concluding the proof, we have

$$\iota_! \Lambda^\delta \Omega[T] \cong \iota_! \Big(\mathrm{colim}_{V_\delta^T} B^T|_{V_\delta^T}\Big) \cong \mathrm{colim}_{V_\delta^T}(\iota_! \circ B^T|_{V_\delta^T}) \cong \mathrm{colim}_{V_\delta^T} A^T|_{V_\delta^T} \cong \Lambda^\delta \Xi[T]. \quad \square$$

Theorem 6.3 *Let $X \in \mathbf{Set}^{\Xi^{\mathrm{op}}}$ be a cyclic dendroidal set and O be a colored operad. If $\alpha\colon \iota^* X \xrightarrow{\cong} N_d(O)$ is an isomorphism of dendroidal sets, then there is a unique cyclic structure on O such that α lifts to an isomorphism between X and the cyclic dendroidal nerve of O. In other words, there is a unique object $\tilde{O} \in \mathbf{Cyc}$ and an isomorphism $\tilde{\alpha}\colon X \to N_c(\tilde{O})$ such that $U(\tilde{O}) = O$ and $\tilde{\alpha}_{\iota T} = \alpha_T\colon X_{\iota T} = (\iota_* X)_T \to N_d(O)_T = N_d(U\tilde{O})_T = N_c(\tilde{O})_{\iota T}$ for every rooted tree $T \in \Omega$.*

Further, if $\beta\colon \iota^* Y \to N_d(P)$ is another such isomorphism and $f\colon X \to Y$ is a morphism of $\mathbf{Set}^{\Xi^{\mathrm{op}}}$, then there is a (unique) morphism $f'\colon \tilde{O} \to \tilde{P}$ in $\mathbf{Cyc}$ making the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \tilde{\alpha}\ } & N_c(\tilde{O}) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle N_c(f')} \\
Y & \xrightarrow{\ \tilde{\beta}\ } & N_c(\tilde{P})
\end{array}
$$

commute.

In order to prove this theorem, we need to delve a bit into how the operad structure of O is manifested in the dendroidal set $N_d(O)$.

Note that $\star_{n+1}$ from Example 1.4 is a rooted tree (which in the dendroidal setting is usually called $C_n \in \Omega$). Let $i\colon \eta \to \star_{n+1}$ be the (oriented) map with image the single edge i; if $X \in \mathbf{Set}^{\Xi^{\mathrm{op}}}$, write $\mathscr{P} = \prod_{i=0}^{n} i^*\colon X_{\star_{n+1}} \to \prod_{i=0}^{n} X_\eta$. If O is a colored operad and $W = N_d(O)$, then $W_\eta = \mathrm{Col}(O)$ and $W_{\star_{n+1}} = \coprod_{c,c_1,\ldots,c_n} O(c_1, \ldots, c_n; c)$ with $\mathscr{P}$ retrieving the profile $(c, c_1, \ldots, c_n)$. Now that we have the elements of O, let us examine the composition. For that, we will utilize the following trees:

Definition 6.4 Suppose that $m \geq 1$, $n \geq 0$ and $1 \leq i \leq m$. Define a (rooted) tree $\mathbf{Z}_{m,n}^i$ with two vertices a and b, $\mathrm{Legs}(\mathbf{Z}_{m,n}^i) = 0, 1, \ldots, m+n-1$ (in that order) and one internal edge e. We further declare that the vertex neighborhoods are the ordered sets

$$\mathrm{Nbhd}(a) = 0, 1, \ldots, i-1, e, n+i, \ldots, m+n-1,$$
$$\mathrm{Nbhd}(b) = e, i, i+1, \ldots, n+i-1.$$

See Figure 9 for examples. This graph admits one inner coface map $\delta^e\colon \star_{n+m} \to \mathbf{Z}_{m,n}^i$ with $\delta^e(k) = k$ for all edges k, and two outer cofaces

$$\delta^a\colon \star_{n+1} \to \mathbf{Z}_{m,n}^i, \qquad \delta^a(k) = \begin{cases} e & \text{if } k = 0, \\ i+k-1 & \text{otherwise,} \end{cases} \qquad \delta^a(v) = \star_b,$$

$$\delta^b\colon \star_{m+1} \to \mathbf{Z}_{m,n}^i, \qquad \delta^b(k) = \begin{cases} k & \text{if } 0 \leq k < i, \\ e & \text{if } k = i, \\ n-1+k & \text{if } i < k \leq m, \end{cases} \qquad \delta^b(v) = \star_a.$$

All three of these cofaces are oriented.

Remark 6.5 Let O be a $\mathfrak{C}$–colored operad and $W = N_d(O)$ its dendroidal nerve. Then the operadic multiplication $\circ_i$ is recovered from

$$
\begin{array}{ccc}
O(c_1, \ldots, c_m; c_0) \times O(d_1, \ldots, d_n; c_i) & \longrightarrow & W_{\star_{m+1}} \times_{W_\eta} W_{\star_{n+1}} \\
\Big\downarrow {\scriptstyle \circ_i} & & {\scriptstyle (d_b \times d_a)^{-1}} \Big\downarrow {\scriptstyle \cong} \\
& & W_{\mathbf{Z}_{m,n}^i} \\
& & \Big\downarrow {\scriptstyle d_e} \\
O(c_1, \ldots, c_{i-1}, d_1, \ldots, d_n, c_{i+1}, \ldots, c_m; c_0) & \longrightarrow & W_{\star_{m+n}}
\end{array}
$$

where the pullback in the top right is obtained from the maps $\star_{m+1} \xleftarrow{\ i\ } \eta \xrightarrow{\ 0\ } \star_{n+1}$.
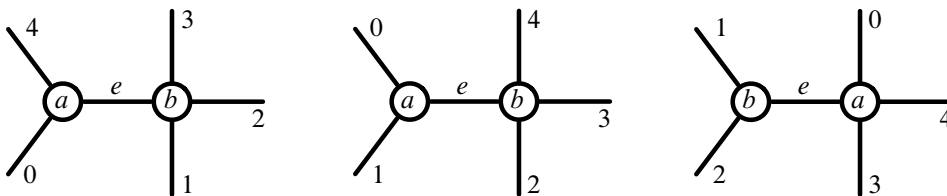
Figure 9: The rooted trees $\mathbf{Z}_{2,3}^1$, $\mathbf{Z}_{2,3}^2$ and $\mathbf{Z}_{3,2}^1$

Given $\sigma \in \Sigma_n^+ = \mathrm{Aut}\{0, 1, \ldots, n\}$, there is a morphism $\psi_\sigma \colon \star_{n+1} \to \star_{n+1}$ determined by $(\psi_\sigma)_0 = \sigma$ on edges. The diagram

$$
\begin{array}{ccc}
X_{\star_{n+1}} & \xrightarrow{\ \psi_\sigma^*\ } & X_{\star_{n+1}} \\
\downarrow{\scriptstyle \mathscr{P}} & & \downarrow{\scriptstyle \mathscr{P}} \\
\prod_{i=0}^n X_\eta & \xrightarrow{\ (-)\cdot\sigma\ } & \prod_{i=0}^n X_\eta
\end{array}
$$

commutes. In particular, ψ_σ^* restricts to a function

$$\mathscr{P}^{-1}(x_0, \ldots, x_n) \to \mathscr{P}^{-1}(x_{\sigma(0)}, \ldots, x_{\sigma(n)}).$$

For the purposes of the next lemma, let us fix some notation. For $q \geq 1$, let $\tau_q \in \mathrm{Aut}\{0, 1, \ldots, q-1\}$ be given by $\tau_q(k) = k + 1 \bmod q$. Let $\psi^q = \psi_{\tau_q} \colon \star_q \to \star_q$ be the associated map. If $1 \leq i \leq m - 1$, define a map $\phi^{m,n} \colon \mathbf{Z}_{m,n}^i \to \mathbf{Z}_{m,n}^{i+1}$ by

$$e \mapsto e, \quad a \mapsto \star_a, \quad k \mapsto \tau_{m+n}(k), \quad b \mapsto \star_b.$$

Likewise, if $n \geq 1$, define a map $\phi^{m,n} \colon \mathbf{Z}_{m,n}^m \to \mathbf{Z}_{n,m}^1$ by

$$e \mapsto e, \quad a \mapsto \star_b, \quad k \mapsto \tau_{m+n}(k), \quad b \mapsto \star_a.$$

The following lemma is a diagrammatic repackaging of Definition 0.1(1).

Lemma 6.6 *If $1 \leq i \leq m - 1$, then the diagram*

commutes. If $i = m$ and $n \geq 1$, then the diagram

commutes.

Proof One merely needs to check that the composite functions are identical on edge sets, which is straightforward. □

Proof of Theorem 6.3 We begin by defining the cyclic structure on O. Let c_0, c_1, ..., $c_n \in \mathrm{Col}(O)$, and write $x_i := \alpha^{-1}(c_i) \in X_\eta$. Then α determines the isomorphism

on the left. We define, for $\sigma \in \Sigma_n^+$,

$$(-) \cdot \sigma \colon O(c_1, \ldots, c_n; c_0) \to O(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c_{\sigma(0)})$$

as the restriction of the endomorphism $\alpha^{-1} \circ \psi_\sigma^* \circ \alpha$ of $N_d(O)_{\star_{n+1}}$. When σ fixes 0, this is precisely the usual symmetric group action.

Now suppose that we are working with τ_{n+1} and consider

$$(-) \cdot \tau_{n+1} \colon O(c_1, \ldots, c_n; c_0) \to O(c_2, \ldots, c_n, c_0; c_1),$$

which is the restriction of $\alpha^{-1} \circ \psi_{n+1} \circ \alpha$. By Theorem 5.9, we simply need to show that $\{(-) \cdot \tau_{n+1}\}$ are compatible with operadic multiplication; using the notation from Remark 6.5, we have that $\circ_i = d_e \circ (d_b \times d_a)^{-1}$ on $W = N_d(O)$.

In the diagrams in Figures 10 and 11, we have $W_\eta = \mathrm{Col}(O)$ and

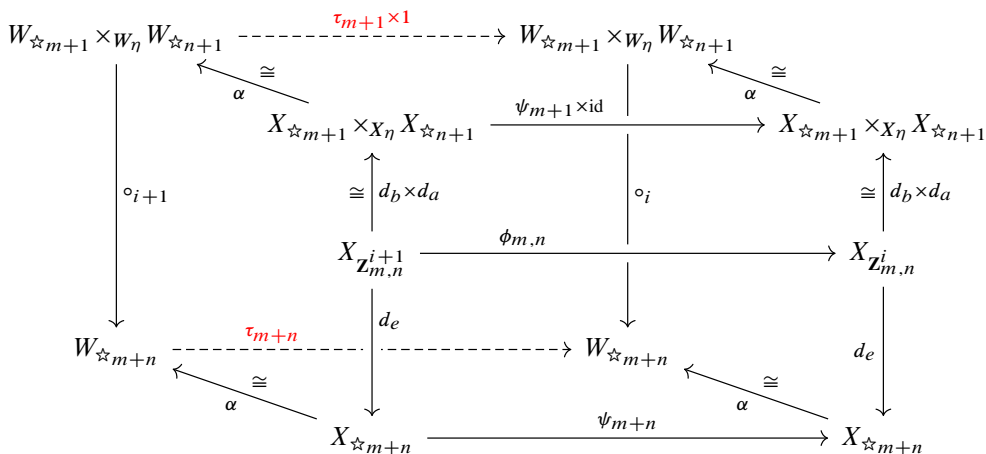$$W_{\star_{m+1}} = O_m = \coprod_{W_\eta^{\times m+1}} O(c_1, \ldots, c_m; c_0)$$

Figure 10: Compatibility with operadic composition

since $W = N_d(O)$. By Lemma 6.6, the diagram in Figure 10 commutes for $1 \le i \le m-1$ and the diagram in Figure 11 commutes for $i = m$ (where σ interchanges the two factors). This shows that the proposed Σ_n^+ actions really turn O into a cyclic operad. Uniqueness of the structure is clear from the requirement that α lift to an isomorphism of cyclic dendroidal sets. Finally, the existence of a unique f' in the second part is just a characterization of cyclic operad maps as operad maps which respect the extra symmetry, the fact that N_d is fully faithful, and that the f' guaranteed by fullness of N_d preserves the extra symmetry by inspection. $\qquad\square$
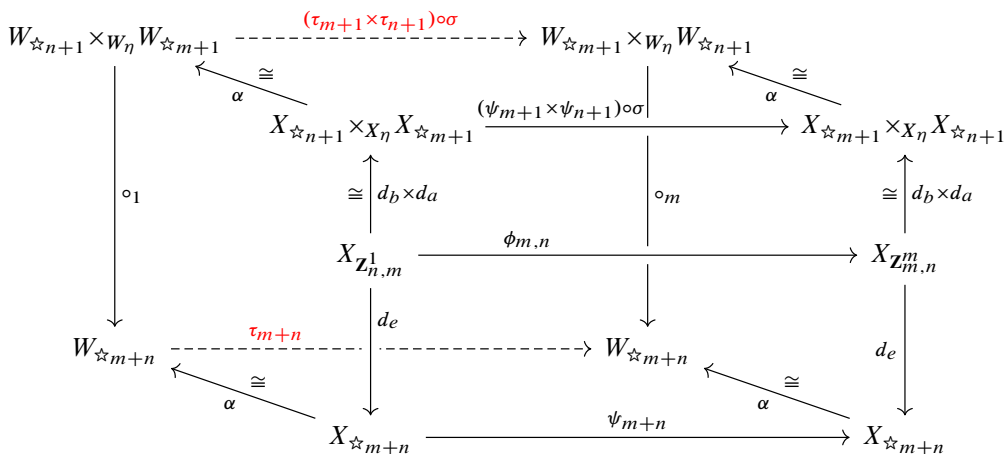

Figure 11: Compatibility with operadic composition

We are now ready to prove the main theorem about the cyclic dendroidal nerve. Note that our nerve theorem does not fit into the general monadic framework for nerve theorems from [45; 3], as, for the reasons outlined in the introduction, we have chosen a nonfull subcategory Ξ of **Cyc** as our indexing category (contrast with the paragraph after [3, Definition 2.3], where the indexing category Θ_T is always a full subcategory).

Theorem 6.7 *The functor* $N_c \colon \mathbf{Cyc} \to \mathbf{Set}^{\Xi^{\mathrm{op}}}$ *is fully faithful. Further, if* $X \in \mathbf{Set}^{\Xi^{\mathrm{op}}}$ *is a cyclic dendroidal set, then the following are equivalent:*

(i) $X \cong N_c(O)$ *for some cyclic operad* O.

(ii) $X_S = \hom(\Xi[S], X) \to \hom(\Lambda^\delta \Xi[S], X)$ *is a bijection for every inner coface map* δ.

(iii) $X_S = \hom(\Xi[S], X) \to \hom(\mathrm{Sc}[S], X)$ *is a bijection for every rooted tree* S.

Proof   That N_c is fully faithful follows from the existence and uniqueness of f' in the second part of Theorem 6.3.

The remainder of the proof relies on Theorem 5.9, and goes by comparing properties of X to properties of $\iota^* X$. Notice, for instance, that Theorem 6.3 implies that $\iota^* X$ satisfies (i$'$) if and only if X satisfies (i).

Since every tree $S \in \Xi$ is isomorphic to a rooted tree $T \in \Omega$, it suffices to examine the remaining condition only for rooted trees T. Likewise, in (ii), we may assume that the inner coface map $\delta \colon R \to T$ is an oriented map. So, to see that (ii) holds for X if and only if (ii$'$) holds for $\iota^* X$, we merely need to note that the commutative diagram

$$
\begin{array}{ccc}
\hom(\Omega[T], \iota^* X) & \longrightarrow & \hom(\Lambda^\delta \Omega[T], \iota^* X) \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
\hom(\iota_! \Omega[T], X) & \longrightarrow & \hom(\iota_! \Lambda^\delta \Omega[T], X) \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
\hom(\Xi[T], X) & \longrightarrow & \hom(\Lambda^\delta \Xi[T], X)
\end{array}
$$

has the indicated isomorphisms by Lemma 6.2. A similar argument shows that (iii) holds for X if and only if (iii$'$) holds for $\iota^* X$, this time applying Lemma 6.1. $\square$

7 Berger–Moerdijk Reedy model structure

In this section, we first recall the Berger–Moerdijk Reedy model structure on categories of diagrams indexed by a generalized Reedy category $\mathbb{R}$ (Definition 3.1) and investigate properties of this model structure. The basic reference is [5], while many of these results in the case of classical Reedy categories may be found in [24, Chapter 15]. We then turn to the special case of simplicial $\mathbb{R}$–presheaves, and consider the full subcategory $\mathbf{sSet}_*^{\mathbb{R}^{op}} \subseteq \mathbf{sSet}^{\mathbb{R}^{op}}$ of diagrams which are a point in degree zero. For certain $\mathbb{R}$, this category admits a model structure (Theorem 7.16) which is simplicial (Theorem 7.24), left proper, and cellular (Proposition 7.18). The results of this section are likely well known among experts, at least in certain special cases.

A model category $\mathcal{M}$ is called $\mathbb{R}$–*projective* if, for every $r \in \mathbb{R}$, the category of (right) $\mathrm{Aut}(r)$–equivariant maps $\mathcal{M}^{\mathrm{Aut}(r)}$ admits the model structure where weak equivalences and fibrations are detected in $\mathcal{M}$. This occurs, for instance, if $\mathcal{M}$ is cofibrantly generated (see [24, 11.6.1]) or if $\mathrm{Aut}(r) = \{e\}$ for all $r \in \mathbb{R}$.

If $r \in \mathbb{R}$, let $\mathbb{R}^+(r)$ be the category whose objects are *noninvertible* maps in $\mathbb{R}^+$ with codomain r, and whose morphisms $\alpha \to \beta$ are commutative triangles

$$
\begin{array}{ccc}
s & \xrightarrow{\ \sigma\ } & s' \\
& \alpha \searrow \quad \swarrow \beta & \\
& r &
\end{array}
$$

in $\mathbb{R}^+$. This is a full subcategory of $\mathbb{R}^+ \downarrow r$. Similarly, for each $r \in \mathbb{R}$, there is a full subcategory $\mathbb{R}^-(r) \subseteq r \downarrow \mathbb{R}^-$ whose objects are noninvertible morphisms with domain r.

If $Z \in \mathcal{M}^{\mathbb{R}}$, define, for each $r \in \mathbb{R}$, *latching* and *matching* objects

$$
L_r Z = \operatornamewithlimits{colim}_{\substack{\mathbb{R}^+(r) \\ \alpha:\, s \to r}} Z_s, \quad M_r Z = \operatornamewithlimits{lim}_{\substack{\mathbb{R}^-(r) \\ \alpha:\, r \to s}} Z_s,
$$

which come equipped with maps

$$
L_r Z \to Z_r \to M_r Z
$$

in $\mathcal{M}^{\mathrm{Aut}(r)}$.

Theorem 7.1 [5, Theorem 1.6] *If $\mathbb{R}$ is a generalized Reedy category and $\mathcal{M}$ is an $\mathbb{R}$–projective model category, then the diagram category $\mathcal{M}^{\mathbb{R}}$ admits a model structure, where $f: X \to Y$ is a*

- *weak equivalence if, for each $r \in \mathbb{R}$, $f_r: X_r \to Y_r$ is a weak equivalence in $\mathcal{M}$;*

- *cofibration if, for each $r \in \mathbb{R}$,*

$$X_r \amalg_{L_r X} L_r Y \to Y_r$$

  is a cofibration in $\mathcal{M}^{\mathrm{Aut}(r)}$; and

- *fibration if, for each $r \in \mathbb{R}$,*

$$X_r \to M_r X \times_{M_r Y} Y_r$$

  is a fibration in $\mathcal{M}$.

We will call this model structure the *Berger–Moerdijk Reedy* model structure on $\mathcal{M}^{\mathbb{R}}$. It would be convenient to know when this model structure is left proper and cellular, so that we can guarantee the existence of left Bousfield localizations [24, 4.1.1].

Theorem 7.2 *If $\mathbb{R}$ is a generalized Reedy category and $\mathcal{M}$ is left proper and $\mathbb{R}$–projective, then the Berger–Moerdijk Reedy model structure on $\mathcal{M}^{\mathbb{R}}$ is left proper.*

A key component of the proof is that Reedy cofibrations are levelwise cofibrations (Lemma 7.4); in establishing this lemma, we will utilize the fact that $\mathcal{M}^G \to \mathcal{M}$ preserves cofibrations when G is a finite group.

Remark 7.3 Suppose that G is any discrete group and $\mathcal{M}$ is a category such that products indexed by G exist. The forgetful functor $f \colon \mathcal{M}^G \to \mathcal{M}$ has a right adjoint r such that the composite $\mathcal{M} \xrightarrow{r} \mathcal{M}^G \xrightarrow{f} \mathcal{M}$ takes an object X to $\prod_{g \in G} X$, and similarly for morphisms. If $\mathcal{M}$ is a model category, then the composite fr takes acyclic fibrations to acyclic fibrations. Suppose further that $\mathcal{M}^G$ admits the projective model structure, that is, the model structure such that f creates fibrations and weak equivalences. In this case, r preserves acyclic fibrations, hence the left adjoint $f \colon \mathcal{M}^G \to \mathcal{M}$ preserves cofibrations.

Lemma 7.4 *If $\mathbb{R}$ is a generalized Reedy category, $\mathcal{M}$ is $\mathbb{R}$–projective and $f \colon A \to B$ is a Reedy cofibration, then for each $r \in \mathbb{R}$ the map $f_r \colon A_r \to B_r$ is a cofibration in $\mathcal{M}$.*

Proof A variation on [5, Lemma 5.3] shows that, for each $r \in \mathbb{R}$, the map $L_r A \to L_r B$ is a cofibration in $\mathcal{M}$. As it is a bit simpler than that lemma, let us give a sketch of the argument. The category $\mathbb{S} = \mathbb{R}^+(r)$ is again generalized Reedy, with $\mathbb{S} = \mathbb{S}^+$, and the domain functor $\varphi \colon \mathbb{S} \to \mathbb{R}$ is a morphism of generalized Reedy categories. For any

$s \to r$ in $\mathbb{S}$, the diagram

$$
\begin{array}{ccc}
L_{s \to r}(\varphi^* A) & \xrightarrow{\ \cong\ } & L_s(A) \\
\downarrow & & \downarrow \\
(\varphi^* A)_{s \to r} & \xrightarrow{\ =\ } & A_s
\end{array}
$$

commutes, where the top isomorphism is a consequence of [5, Lemma 4.4(i)]. Thus, $\varphi^*(f)$ is a cofibration in $\mathcal{M}^{\mathbb{S}}$. Further, since $\mathrm{colim}_{\mathbb{S}}(\varphi^* A) = L_r(A)$, we have

$$
(L_r A \xrightarrow{L_r f} L_r B) = \mathrm{colim}_{\mathbb{S}}(\varphi^* A \xrightarrow{\varphi^*(f)} \varphi^* B),
$$

which is a cofibration in $\mathcal{M}$ by [5, Corollary 1.7].

The map $A_r \to A_r \amalg_{L_r A} L_r B$ is a pushout of a cofibration

$$
\begin{array}{ccc}
L_r A & \longrightarrow & A_r \\
\downarrow & & \downarrow \\
L_r B & \longrightarrow & A_r \amalg_{L_r A} L_r B
\end{array}
$$

hence is a cofibration [24, Proposition 7.2.12(a)]. Since f is a Reedy cofibration,

$$
A_r \amalg_{L_r A} L_r B \to B_r
$$

is a cofibration in $\mathcal{M}^{\mathrm{Aut}(r)}$. By Remark 7.3, it is also a cofibration in $\mathcal{M}$. Now $f_r \colon A_r \to B_r$ is the composite of two cofibrations in $\mathcal{M}$,

$$
A_r \to A_r \amalg_{L_r A} L_r B \to B_r,
$$

hence is also a cofibration in $\mathcal{M}$. $\qquad\square$

Proof of Theorem 7.2 Suppose we have a pushout diagram

(11)
$$
\begin{array}{ccc}
A & \rightarrowtail & B \\
\downarrow{\scriptstyle \simeq} & & \downarrow \\
X & \longrightarrow & X \amalg_A B
\end{array}
$$

in $\mathcal{M}^{\mathbb{R}}$; we wish to show that $B \to X \amalg_A B$ is a weak equivalence. Evaluating (11) at $r \in \mathbb{R}$ gives the pushout square

$$
\begin{array}{ccc}
A_r & \rightarrowtail & B_r \\
\downarrow{\scriptstyle \simeq} & & \downarrow \\
X_r & \longrightarrow & X_r \amalg_{A_r} B_r
\end{array}
$$

By Lemma 7.4, $A_r \to B_r$ is a cofibration in $\mathcal{M}$; since $\mathcal{M}$ is left proper, this implies that $B_r \to X_r \amalg_{A_r} B_r$ is a weak equivalence. Since r was arbitrary, $B \to X \amalg_A B$ is a weak equivalence. $\qquad\square$

Proposition 7.5 *If $\mathcal{M}$ is cellular, then so is $\mathcal{M}^{\mathbb{R}}$.*

Proof The proof given in [24, 15.7] goes through, with the caveat that in the proof of [24, 15.7.1], one should use Lemma 7.4 in place of [24, 15.3.11]. $\qquad\square$

Remark 7.6 Implicit in the explanation for Proposition 7.5 is the fact that the generating (acyclic) cofibrations of $\mathcal{M}^{\mathbb{R}}$ are described as in [24, Definition 15.6.23]. Indeed, the proof of cofibrant generation in the first paragraph of [5, Theorem 7.6] only relies on $\mathbb{R}$ being a dualizable generalized Reedy category and $\mathcal{M}$ being cofibrantly generated. We will need this explicit description of the generating (acyclic) cofibrations for the proof of Theorem 7.16 (in the special case when $\mathcal{M} = \mathbf{sSet}$).

7.1 Reduced presheaves in simplicial sets

Let $\mathbb{R}$ be a generalized dualizable Reedy category. We assume that $\mathbb{R}$ has a unique object η of degree zero and, for any $r \in \mathbb{R}$, the set $\mathbb{R}(r, \eta)$ is either empty or has exactly one element. Further, we assume that if $\mathbb{R}(r, \eta) = \{f\}$, then $\mathbb{R}(\eta, r) \neq \varnothing$ as well; any inhabitant of $\mathbb{R}(\eta, r)$ is automatically a section of f. In symbols, this says

$$(12) \qquad\qquad |\mathbb{R}(r, \eta)| \leq \min(1, |\mathbb{R}(\eta, r)|) \quad \text{for all } r \in \mathbb{R},$$

which we take as a standing assumption for all that follows. Examples of such categories include Δ, Ω and Ξ.

Let $\mathbf{sSet}_*^{\mathbb{R}^{op}}$ be the full subcategory of $\mathbf{sSet}^{\mathbb{R}^{op}}$ consisting of those X such that $X_\eta = \Delta[0]$. As this category has been thoroughly analyzed in [7] in the case of $\mathbb{R} = \Delta$ and [9] in the case of $\mathbb{R} = \Omega$, many of the arguments and constructions in the remainder of this section should look familiar. Write

$$\mathscr{I} \colon \mathbf{sSet}_*^{\mathbb{R}^{op}} \hookrightarrow \mathbf{sSet}^{\mathbb{R}^{op}}$$

for the inclusion. This functor admits a left adjoint

$$\mathscr{R} \colon \mathbf{sSet}^{\mathbb{R}^{op}} \to \mathbf{sSet}_*^{\mathbb{R}^{op}},$$

which we now describe explicitly.

Definition 7.7 We define the *reduction* of an object $X \in \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$.

- A map $\mathbb{R}[\eta] \times X_\eta \to X$ is given in degree r by either $X_\eta \xrightarrow{f^*} X_r$ if $\mathbb{R}(r, \eta) = \{f\}$ or else the unique map $\varnothing \to X_r$ if $\mathbb{R}(r, \eta) = \varnothing$. Since such an f admits a section, this map is a monomorphism, and we write $X^{(0)} \subseteq X$ for its image. Notice that there is a unique map $X^{(0)} \to \mathbb{R}[\eta]$ and that

$$X_r^{(0)} \cong \begin{cases} X_\eta & \text{if } \mathbb{R}(r, \eta) = *, \\ \varnothing & \text{if } \mathbb{R}(r, \eta) = \varnothing. \end{cases}$$

- Define $\mathscr{R}(X)$ as the pushout

$$
\begin{array}{ccc}
X^{(0)} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathbb{R}[\eta] & \longrightarrow & \mathscr{R}(X)
\end{array}
$$

Suppose $X \in \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$. If there is not a map $r \to \eta$, then $\mathscr{R}(X)_r = X_r$. Suppose $Z \in \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$. If there is a map from r to η then it is unique: $\mathbb{R}(r, \eta) = \{f\}$; this implies that Z_r has a natural basepoint given by $f^*(t)$, where $t \in Z_\eta$ is the unique element.

Remark 7.8 Many results of this section hold with a weaker assumption than (12), namely one could just assume $|\mathbb{R}(r, \eta)| \leq 1$ for all $r \in \mathbb{R}$. In this case, $\mathbb{R}[\eta] \times X_\eta \to X$ may not be a monomorphism, and we should define $X^{(0)}$ to be $\mathbb{R}[\eta] \times X_\eta$ instead of its image in X. We refrain from giving further details, as our main application is to the categories Ξ and Ω (which do satisfy (12)).

The category $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ is bicomplete: Limits and directed colimits are computed in the larger category $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$, while finite coproducts are given by the formula

(13) $$(X \amalg Y)_r = \begin{cases} X_r \amalg Y_r & \text{if } \mathbb{R}(r, \eta) = \varnothing, \\ X_r \vee Y_r & \text{if } \mathbb{R}(r, \eta) = *. \end{cases}$$

Equivalently, $X \amalg Y = \mathscr{R}(\mathscr{I}(X) \amalg \mathscr{I}(Y))$. Though $\mathscr{I}$ does not preserve coproducts, it does preserve pushouts.

Lemma 7.9 *The inclusion functor* $\mathscr{I} \colon \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}} \to \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ *preserves connected colimits.*

Proof Suppose $F\colon \mathcal{C} \to \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ is a functor from a connected category and $A = \operatorname{colim}_{\mathcal{C}} \mathscr{I} F$. Then

$$A_\eta = (\operatorname*{colim}_{c\in\mathcal{C}} \mathscr{I} F(c))_\eta = \operatorname*{colim}_{c\in\mathcal{C}} (\mathscr{I} F(c)_\eta) = \operatorname*{colim}_{c\in\mathcal{C}} \Delta[0] = \Delta[0].$$

Since A is already in $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ and $\mathscr{I}$ is fully faithful, A is also the colimit of F in $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$. $\qquad\square$

Given an object $r \in \mathbb{R}$, recall that the *boundary of r* is the subobject $\partial\mathbb{R}[r] \subseteq \mathbb{R}[r] = \mathbb{R}(-, r)$ such that $\partial\mathbb{R}[r]_s \subseteq \mathbb{R}(s, r)$ consists of those maps $s \to r$ which factor through an object of degree less than $d(r)$. In particular, if $d(s) < d(r)$, then $\partial\mathbb{R}[r]_s = \mathbb{R}[r]_s$.

Definition 7.10 We define two sets of maps in $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$. The set I consists of all inclusions

$$(\partial\mathbb{R}[r] \times \Delta[n]) \cup_{\partial\mathbb{R}[r]\times\partial\Delta[n]} (\mathbb{R}[r] \times \partial\Delta[n]) \to \mathbb{R}[r] \times \Delta[n]$$

for $n \geq 0$ and $r \in \mathbb{R}$. The set J consists of all inclusions

$$(\partial\mathbb{R}[r] \times \Delta[n]) \cup_{\partial\mathbb{R}[r]\times\Lambda^k[n]} (\mathbb{R}[r] \times \Lambda^k[n]) \to \mathbb{R}[r] \times \Delta[n].$$

The set I (resp. J) is a set of generating (acyclic) cofibrations for the Berger–Moerdijk Reedy model structure on $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ (see Remark 7.6).

Lemma 7.11 *All objects of $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ and $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ are small* [24, Definition 10.4.1].

Proof In any locally presentable category, every object is small in this sense. Any diagram category $\mathbf{Set}^{\mathcal{C}}$ (with $\mathcal{C}$ small) is locally presentable [1, Example 1.12], hence $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}} \cong \mathbf{Set}^{\Delta^{\mathrm{op}}\times\mathbb{R}^{\mathrm{op}}}$ is. Finally, $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ is a full, reflective subcategory of $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ which is further closed under λ–directed colimits (λ a regular cardinal) by Lemma 7.9, hence is locally presentable by [1, Representation theorem 1.46]. $\qquad\square$

Lemma 7.12 *Let A be a domain or codomain of an element of $I \cup J$, with $A \subseteq \mathbb{R}[r]\times\Delta[n]$. If $d(r) > 0$, then $A^{(0)} = \mathbb{R}[r]^{(0)} \times \Delta[n] \cong \coprod_{\mathbb{R}(\eta, r)} \mathbb{R}[\eta] \times \Delta[n]$. If $r = \eta$, then $A^{(0)} = A$, hence $\mathscr{R}(A) = \mathbb{R}[\eta]$.*

Proof The important point is that

$$(\partial\mathbb{R}[r])^{(0)} = \begin{cases} \mathbb{R}[r]^{(0)} & \text{if } d(r) > 0, \\ \varnothing & \text{if } d(r) = 0. \end{cases}$$

If $d(r) > 0$, then

$$\mathbb{R}[r]^{(0)} \times \Delta[n] = (\partial \mathbb{R}[r] \times \Delta[n])^{(0)} \subseteq A^{(0)} \subseteq (\mathbb{R}[r] \times \Delta[n])^{(0)} = \mathbb{R}[r]^{(0)} \times \Delta[n].$$

If $r = \eta$, then A is one of $\mathbb{R}[\eta] \times \partial \Delta[n]$, $\mathbb{R}[\eta] \times \Lambda^k[n]$ or $\mathbb{R}[\eta] \times \Delta[n]$, hence $A^{(0)} = A$. $\qquad \square$

Proposition 7.13 *Suppose that* $j \colon A \to \mathbb{R}[r] \times \Delta[n]$ *is in* I *or* J *and* $d(r) > 0$. *If* $j \in I$, *then* $\mathscr{R}(j)$ *is a cofibration in* $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$, *while if* $j \in J$, *then* $\mathscr{R}(j)$ *is an acyclic cofibration in* $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$.

Proof The cube with front and back squares pushouts

reduces to a rectangle

where the big rectangle and the left square are pushouts, hence so is the right square.

The result now follows since the class of (acyclic) cofibrations is closed under pushouts. $\qquad \square$

Proposition 7.14 *Suppose that* $j \colon A \to \mathbb{R}[\eta] \times \Delta[n]$ *is in* I *or* J. *Then* $\mathscr{R}(j)$ *is the identity map on* $\mathbb{R}[\eta]$.

Proof This is a direct consequence of the second part of Lemma 7.12 and the fact that $\hom(\mathbb{R}[\eta], \mathbb{R}[\eta]) = \mathbb{R}(\eta, \eta) = *$. $\qquad \square$

Lemma 7.15 *If* $j \in J$, *then* $\mathscr{R}(j)_s$ *is an acyclic cofibration in* $\mathbf{sSet}$ *for every* $s \in \mathbb{R}$. *If* $j \in I$, *then* $\mathscr{R}(j)_s$ *is a cofibration in* $\mathbf{sSet}$ *for every* $s \in \mathbb{R}$.

Proof By the previous two propositions, we know that $\mathscr{R}(j)$ is an (acyclic) cofibration in the Berger–Moerdijk Reedy model structure on $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$. The result then follows from Lemma 7.4. □

Theorem 7.16 *Suppose that $\mathbb{R}$ is a generalized Reedy category with unique element η in degree zero and $|\mathbb{R}(r, \eta)| \leq \min(1, |\mathbb{R}(\eta, r)|)$ for all $r \in \mathbb{R}$. Then the Berger–Moerdijk Reedy model structure lifts along the adjunction*

$$\mathscr{R} \colon \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}} \rightleftarrows \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}} \colon \mathscr{I}.$$

In other words, $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ admits a cofibrantly generated model structure with weak equivalences (resp. fibrations) those maps which are weak equivalences (resp. fibrations) in the larger category $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$.

Proof We apply Kan's lifting theorem [24, 11.3.2]. We know that $\mathscr{R}I$ and $\mathscr{R}J$ satisfy the small object argument by Lemma 7.11, hence condition (1) is satisfied. For condition (2) we know that $\mathscr{R}(j)_s$ is an acyclic cofibration in $\mathbf{sSet}$ for any $s \in \mathbb{R}$ by Lemma 7.15. Then, given any relative $\mathscr{R}J$–cell complex $X \to Y$ in $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$, we also have that $X_s \to Y_s$ is an acyclic cofibration for each $s \in \mathbb{R}$. Thus, $\mathscr{I}X \to \mathscr{I}Y$ is a weak equivalence, and condition (2) is established. □

Notice that the inclusion functor does not admit a right adjoint (as it does not preserve finite coproducts), hence cannot be a left Quillen functor. Nevertheless, we have the following, which hinges on the fact that $\mathscr{I}$ preserves pushouts and transfinite composition (Lemma 7.9).

Proposition 7.17 *The inclusion functor $\mathscr{I} \colon \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}} \to \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ preserves (acyclic) cofibrations.*

Proof The inclusion functor preserves all weak equivalences, so it is enough to show that it preserves cofibrations. By Proposition 7.13 and 7.14, we know that $\mathscr{I}$ takes generating cofibrations in $\mathscr{R}I$ to cofibrations in $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$. The inclusion functor preserves pushouts (Lemma 7.9) and transfinite composition, so takes relative $\mathscr{R}I$–cell complexes to cofibrations. Since every cofibration in $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ is a retract of a relative $\mathscr{R}I$–cell complex, every cofibration in $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ is again a cofibration in $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$. □

The following has precursors elsewhere in special cases, for instance in the proof of [9, 4.3].

Proposition 7.18 *With the hypotheses of Theorem 7.16, the model structure on* $\mathbf{sSet}_*^{\mathbb{R}^{op}}$ *is left proper and cellular.*

Proof For left properness, it is enough to show that the pushout of a weak equivalence along a *generating* cofibration $\mathscr{R}(i) \in \mathscr{R}I$ is again a weak equivalence; this follows from Lemma 7.15, left properness of $\mathbf{sSet}$ and the fact that weak equivalences are levelwise weak equivalences.

We now turn to cellularity, and aim to verify the three conditions from [24, 12.1.1]. We know that (2) holds by Lemma 7.11. Recall from Proposition 7.5 that $\mathbf{sSet}^{\mathbb{R}^{op}}$ is a cellular model category. Cofibrations in $\mathbf{sSet}_*^{\mathbb{R}^{op}}$ are also cofibrations (Proposition 7.17) in the cellular model category $\mathbf{sSet}^{\mathbb{R}^{op}}$, hence are effective monomorphisms. Thus, (3) holds.

It remains to show (1). All elements of the form $\mathbb{R}[r] \times \Delta[n]$ are compact relative to I since they are codomains of elements of I and $\mathbf{sSet}^{\mathbb{R}^{op}}$ is cellular. If A is a domain or a codomain of an element of I, then $A^{(0)}$ is compact by Lemma 7.12. It follows that $\mathscr{R}(A)$ is compact (relative to I) by [24, 10.8.8], since A, $A^{(0)}$ and $\mathbb{R}[\eta] = \mathbb{R}[\eta] \times \Delta[0]$ are compact. This shows that the set $\mathscr{R}(I)$ is a set of cofibrations of $\mathbf{sSet}^{\mathbb{R}^{op}}$ with compact domains. By [24, 11.4.9], if $W \in \mathbf{sSet}^{\mathbb{R}^{op}}$ is compact relative to I, then W is compact relative to $\mathscr{R}(I)$. In particular, $\mathscr{R}(A)$ is compact relative to $\mathscr{R}(I)$. Presented relative $\mathscr{R}(I)$–cell complexes in $\mathbf{sSet}^{\mathbb{R}^{op}}$ with $X_0 \in \mathbf{sSet}_*^{\mathbb{R}^{op}}$ are the same thing as presented relative $\mathscr{R}(I)$–cell complexes in $\mathbf{sSet}_*^{\mathbb{R}^{op}}$, hence all domains and codomains of elements in $\mathscr{R}(I) \cup \mathscr{R}(J)$ are compact in $\mathbf{sSet}_*^{\mathbb{R}^{op}}$. Thus, [24, 12.1.1(1)] holds, and we conclude that $\mathbf{sSet}_*^{\mathbb{R}^{op}}$ is a cellular model category. □

7.2 Simplicial model structures

As in the case of an ordinary Reedy category, the Berger–Moerdijk Reedy model structure on $\mathbf{sSet}^{\mathbb{R}^{op}}$ is a *simplicial model structure* (see [24, Chapter 9]). The structure is given as follows:

Definition 7.19 Suppose that $X, Y \in \mathbf{sSet}^{\mathbb{R}^{op}}$ and $K \in \mathbf{sSet}$.

- The object $X \otimes K$ is defined on objects by

$$(X \otimes K)_r = X_r \times K \in \mathbf{sSet}.$$

- The object Y^K is defined on objects by

$$(Y^K)_r = \mathrm{map}_{\mathbf{sSet}}(K, Y_r) \in \mathbf{sSet},$$

  where $\mathrm{map}_{\mathbf{sSet}}$ is the simplicial mapping space, that is, $\mathrm{map}_{\mathbf{sSet}}(A, B)_n = \mathrm{hom}_{\mathbf{sSet}}(A \times \Delta[n], B)$.

- The (simplicial) mapping spaces $\mathrm{map}(X, Y)$ are defined in degree n by

$$\mathrm{map}(X, Y)_n = \mathrm{hom}(X \otimes \Delta[n], Y).$$

The simplicially enriched category $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ satisfies the two conditions to be a simplicial model category [24, 9.1.6], namely

(M6) For every two objects X and Y and every $K \in \mathbf{sSet}$,

$$\mathrm{map}(X \otimes K, Y) \cong \mathrm{map}_{\mathbf{sSet}}(K, \mathrm{map}(X, Y)) \cong \mathrm{map}(X, Y^K)$$

  (with isomorphisms natural in the three variables).

(M7) If $A \to B$ is a cofibration and $X \to Y$ is a fibration, then

(14) $$\mathrm{map}(B, X) \to \mathrm{map}(A, X) \times_{\mathrm{map}(A,Y)} \mathrm{map}(B, Y)$$

  is a fibration of simplicial sets. If either map is a weak equivalence, then (14) is also a weak equivalence.

Assuming (M6), condition (M7) is equivalent (see eg [33, Remark A.3.1.6] or [19, Section 3]) to the following statement:

(M7′) If $X \to Y$ is a fibration in our model category and $K \to L$ is a cofibration of simplicial sets, then

(15) $$X^L \to X^K \times_{Y^K} Y^L$$

  is a fibration. If either map is a weak equivalence, then so is (15).

Any full subcategory of a simplicially enriched category is simplicially enriched; we will now work towards Theorem 7.24, where we show that $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ is a simplicial model category.

Notation 7.20 For the remainder of this section, we will write

$$\mathcal{M} = \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}, \quad \mathcal{N} = \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$$

for these two model categories. For $X, Y \in \mathcal{M}$ and $K \in \mathbf{sSet}$, we will write $X \otimes_{\mathcal{M}} K := X \otimes K$ and $\mathrm{map}_{\mathcal{M}}(X, Y) := \mathrm{map}(X, Y)$.

Recall that we have a Quillen adjunction

$$\mathscr{R} \colon \mathcal{M} \rightleftarrows \mathcal{N} \colon \mathscr{I}.$$

Definition 7.21 Suppose that $X, Y \in \mathcal{N} = \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ and $K \in \mathbf{sSet}$.

- The object $X \otimes_{\mathcal{N}} K$ is defined to be $X \otimes_{\mathcal{N}} K = \mathscr{R}(\mathscr{I}(X) \otimes_{\mathcal{M}} K)$.
- The object Y^K is defined as $Y^K = \mathscr{R}[(\mathscr{I}Y)^K]$.
- We define $\mathrm{map}_{\mathcal{N}}(X, Y) = \mathrm{map}_{\mathcal{M}}(\mathscr{I}X, \mathscr{I}Y)$.

Remark 7.22 The object $Z = (\mathscr{I}Y)^K$ already has $Z_\eta = \Delta[0]$, which explains why we've elected not to distinguish between the exponential in the two categories. Indeed,

$$Z_\eta = ((\mathscr{I}Y)^K)_\eta = \mathrm{map}_{\mathbf{sSet}}(K, Y_\eta) = \mathrm{map}_{\mathbf{sSet}}(K, \Delta[0]) = \Delta[0].$$

Lemma 7.23 *There is an isomorphism*

$$\mathrm{map}_{\mathcal{N}}(\mathscr{R}(-), -) \cong \mathrm{map}_{\mathcal{M}}(-, \mathscr{I}(-))$$

of bifunctors $\mathcal{N} \times \mathcal{M} \to \mathbf{sSet}$.

Proof Let $Z \in \mathcal{N}$, $Y \in \mathcal{M}$ and $n \geq 0$. We have

$$
\begin{aligned}
\mathrm{map}_{\mathcal{N}}(\mathscr{R}(Z), Y)_n &= \mathrm{map}_{\mathcal{M}}(\mathscr{I}\mathscr{R}(Z), \mathscr{I}Y)_n \\
&= \hom_{\mathcal{M}}(\mathscr{I}\mathscr{R}(Z) \otimes_{\mathcal{M}} \Delta[n], \mathscr{I}Y) \\
&= \hom_{\mathcal{M}}(\mathscr{I}\mathscr{R}(Z), (\mathscr{I}Y)^{\Delta[n]}) \\
&= \hom_{\mathcal{M}}(\mathscr{I}\mathscr{R}(Z), \mathscr{I}\mathscr{R}(\mathscr{I}Y)^{\Delta[n]}) \quad (\text{Remark 7.22}) \\
&= \hom_{\mathcal{N}}(\mathscr{R}(Z), \mathscr{R}(\mathscr{I}Y)^{\Delta[n]}) \\
&= \hom_{\mathcal{M}}(Z, \mathscr{I}\mathscr{R}(\mathscr{I}Y)^{\Delta[n]}) \\
&= \hom_{\mathcal{M}}(Z, (\mathscr{I}Y)^{\Delta[n]}) = \mathrm{map}_{\mathcal{M}}(Z, \mathscr{I}Y)_n,
\end{aligned}
$$

with all isomorphisms natural in Z, Y and n. $\qquad\square$

Theorem 7.24 *With the structure from Definition 7.21,* $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ *is a simplicial model category.*

Proof By Remark 7.22, the fact that (M7$'$) holds for $\mathcal{M} = \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$, and the fact that $\mathscr{I}$ creates (acyclic) fibrations, (M7$'$) holds for $\mathcal{N} = \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$. Thus, it is enough to check (M6).

Let $X, Y \in \mathcal{N}$ and $K \in \mathbf{sSet}$. First,

$$\mathrm{map}_{\mathcal{N}}(X \otimes_{\mathcal{N}} K, Y) = \mathrm{map}_{\mathcal{N}}(\mathcal{R}(\mathcal{I}(X) \otimes_{\mathcal{M}} K), Y) = \mathrm{map}_{\mathcal{M}}(\mathcal{I}(X) \otimes_{\mathcal{M}} K, \mathcal{I}Y)$$

by Lemma 7.23. Thus, using (M6) for $\mathcal{M}$, the simplicial set $\mathrm{map}_{\mathcal{N}}(X \otimes_{\mathcal{N}} K, Y)$ is isomorphic to, on the one hand,

$$\mathrm{map}_{\mathcal{M}}(\mathcal{I}(X), (\mathcal{I}Y)^K) = \mathrm{map}_{\mathcal{M}}(\mathcal{I}X, \mathcal{I}\mathcal{R}(\mathcal{I}Y)^K) = \mathrm{map}_{\mathcal{N}}(X, Y^K),$$

and on the other to

$$\mathrm{map}_{\mathbf{sSet}}(K, \mathrm{map}_{\mathcal{M}}(\mathcal{I}X, \mathcal{I}Y)) = \mathrm{map}_{\mathbf{sSet}}(K, \mathrm{map}_{\mathcal{N}}(X, Y)). \qquad \square$$

Remark 7.25 We are grateful to an anonymous referee for observing that the above results reflect a general pattern. The adjunction $\mathcal{I} \dashv \mathcal{R}$ is monadic since $\mathcal{R}$ is conservative and preserves connected colimits; thus, $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ is equivalent to the category of $\mathcal{I}\mathcal{R}$–algebras. Remark 7.22 should come as no surprise, as the simplicial cotensorings are an enriched limit, hence should be computed in the ground category $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$. The rest of the structure (simplicial tensorings and simplicial homs) are then forced by the two-variable adjunctions of (M6). Provided these exist, the simplicial structure is guaranteed since (M7) is equivalent to (M7′).

8 Segal cyclic operads

In this section we define Segal cyclic operads as certain fibrant objects in $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$ which satisfy a Segal condition (Definition 8.8). The Segal cyclic operads may be identified as the fibrant objects after we have localized the model structure on $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$ with respect to Segal core inclusions.

We begin this section by specializing the work of Section 7 to the cases $\mathbb{R} = \Omega$ or Ξ. We show that the Quillen adjunctions from Theorem 7.16 fit into a diagram (18). Afterward, we discuss the left Bousfield localizations, and show that after localization we still have a diagram of Quillen adjunctions. Finally, we check in Proposition 8.10 that the homotopy theory for Segal cyclic operads is distinct from that for Segal operads, and speculate on the possibility of rigidification theorems.

Proposition 8.1 *Using the Berger–Moerdijk Reedy model structures, the adjunction*

$$\iota_! : \mathbf{sSet}^{\Omega^{\mathrm{op}}} \rightleftarrows \mathbf{sSet}^{\Xi^{\mathrm{op}}} : \iota^*$$

is a Quillen adjunction.

Proof The map ι^* preserves weak equivalences since those are defined levelwise, hence it is enough to show that ι^* preserves fibrations. If T is a rooted tree, we will show that

(16)
$$
\begin{array}{ccc}
(\Omega^{\mathrm{op}})^-(T) & \longrightarrow & (\Xi^{\mathrm{op}})^-(\iota T) \\
\| & & \| \\
(\Omega^+(T))^{\mathrm{op}} & & (\Xi^+(\iota T))^{\mathrm{op}}
\end{array}
$$

is an initial functor (see [34, Section IX.3]). This implies that the natural map

$$
M_{\iota T} X = \lim_{(\Xi^{\mathrm{op}})^-(\iota T)} X_S \to \lim_{(\Omega^{\mathrm{op}})^-(T)} (\iota^* X)_{T'} = M_T(\iota^* X)
$$

is an isomorphism. Hence, if $X \to Y$ is a map in $\mathbf{sSet}^{\Xi^{\mathrm{op}}}$ and $T \in \Omega$, the map on the right of the commutative diagram

$$
\begin{array}{ccc}
(\iota^* X)_T & \longrightarrow & M_T(\iota^* X) \times_{M_T(\iota^* Y)} (\iota^* Y)_T \\
\| & & \uparrow \\
X_{\iota T} & \longrightarrow & M_{\iota T}(X) \times_{M_{\iota T}(Y)} Y_{\iota T}
\end{array}
$$

is an isomorphism. In particular, if $X \to Y$ is a fibration, then so is $\iota^*(X \to Y)$.

As promised, we will now show that (16) is an initial functor; this is equivalent to the induced functor

(17)
$$
\iota_T \colon \Omega^+(T) \to \Xi^+(\iota T)
$$

being final. Suppose that $S \xrightarrow{\phi} \iota T$ is an object of $\Xi^+(\iota T)$, that is, ϕ is an element of $\Xi^+(S, \iota T) \setminus \mathrm{Iso}(\Xi)$. Our goal is to show that $\phi \downarrow \iota_T$ is nonempty and connected. We first explain the case when ϕ is not constant (that is, when $S \neq \eta$), and indicate later the changes for the simpler case when $S = \eta$. Write $s_0 = \odot_{t_0}(\phi)$, where t_0 is the root of T; we have a morphism

$$
\mathscr{L}_{t_0}(\phi) \colon \mathscr{T}(S, s_0) \to \mathscr{T}(\iota T, t_0) = T.
$$

Using the structure map $f \colon \iota \mathscr{T}(S, s_0) \xrightarrow{\cong} S$ from Definition 2.1, the commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\quad f^{-1} \quad} & \iota \mathscr{T}(S, s_0) \\
& {\scriptstyle \phi} \searrow \quad \swarrow {\scriptstyle \iota \mathscr{L}_{t_0}(\phi)} & \\
& \iota T &
\end{array}
$$

in Ξ constitutes an object

$$\phi \xrightarrow{f^{-1}} \iota_T(\mathscr{L}_{t_0}(\phi))$$

in $\phi \downarrow \iota_T$. To show that this category is connected, suppose that we have an arbitrary object

$$\phi \xrightarrow{\gamma} \iota_T(R \xrightarrow{\alpha} T)$$

of $\phi \downarrow \iota_T$, where $\alpha \in \Omega^+(T)$. Such an object corresponds to a commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\quad\gamma\quad} & \iota R \\
 & \phi \searrow \quad \swarrow \iota\alpha & \\
 & \iota T &
\end{array}
$$

with $\gamma \in \Xi^+$. Lifting all maps to Ω, we have the diagram

$$
\begin{array}{ccccc}
\mathscr{T}(S, s_0) & \xrightarrow{\quad\mathscr{L}_{r_0}(\gamma)\quad} & \mathscr{T}(\iota R, r_0) & =\!=\!= & R \\
 & \mathscr{L}_{t_0}(\phi) \searrow \quad \swarrow \mathscr{L}_{t_0}(\iota\alpha) & & \searrow \alpha & \\
 & \mathscr{T}(\iota T, t_0) & =\!=\!=\!=\!= & T &
\end{array}
$$

in Ω, which commutes by Proposition 2.11. Commutativity of the diagram

$$
\begin{array}{ccccc}
 & & \gamma & & \\
S & \xrightarrow{f^{-1}} & \iota\mathscr{T}(S, s_0) & \xrightarrow{\iota(\mathscr{L}_{r_0}(\gamma))} & \iota R \\
 & \phi \searrow & \Big\downarrow \iota(\mathscr{L}_{t_0}(\phi)) & \swarrow \iota(\alpha) & \\
 & & \iota T & &
\end{array}
$$

in Ξ shows that $\iota(\mathscr{L}_{r_0}(\gamma))$ constitutes a morphism $f^{-1} \to \gamma$ in $\phi \downarrow \iota_T$; thus this category is connected.

A similar proof holds when ϕ is constant, that is, when $S = \eta$. In this case, $\mathscr{T}(S, s_0)$ should be replaced by η, f should be taken to be the identity map, and $\mathscr{L}_{t_0}(\phi)$ (resp. $\mathscr{L}_{r_0}(\gamma)$) should be replaced by the unique lift $\eta \to T$ of ϕ (resp. the unique lift $\eta \to R$ of γ). Since $\phi \downarrow \iota_T$ is connected for every object $\phi \in \Xi^+(\iota T)$, (17) is a final functor. $\square$

Corollary 8.2 *The adjunction*

$$\iota_! : \mathbf{sSet}_*^{\Omega^{\mathrm{op}}} \rightleftarrows \mathbf{sSet}_*^{\Xi^{\mathrm{op}}} : \iota^*$$

is a Quillen adjunction.

Proof We have a commutative diagram

$$
\begin{array}{ccc}
\mathbf{sSet}^{\Omega^{\mathrm{op}}} & \xleftarrow{\quad \iota^* \quad} & \mathbf{sSet}^{\Xi^{\mathrm{op}}} \\
\Big\uparrow{\scriptstyle \mathscr{I}} & & \Big\uparrow{\scriptstyle \mathscr{I}} \\
\mathbf{sSet}_*^{\Omega^{\mathrm{op}}} & \xleftarrow{\quad \iota^* \quad} & \mathbf{sSet}_*^{\Xi^{\mathrm{op}}}
\end{array}
$$

of right adjoints where all but the bottom map ι^* are known to be right Quillen functors. Suppose that $X \to Y$ is an (acyclic) fibration in $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$, which implies $\mathscr{I}\iota^*(X \to Y) = \iota^*\mathscr{I}(X \to Y)$ is an (acyclic) fibration in $\mathbf{sSet}^{\Omega^{\mathrm{op}}}$. Since $\mathscr{I}\colon \mathbf{sSet}_*^{\Omega^{\mathrm{op}}} \to \mathbf{sSet}^{\Omega^{\mathrm{op}}}$ detects fibrations and weak equivalences, we know that $\iota^*(X \to Y)$ is an (acyclic) fibration, implying $\iota^*\colon \mathbf{sSet}_*^{\Xi^{\mathrm{op}}} \to \mathbf{sSet}_*^{\Omega^{\mathrm{op}}}$ is a right Quillen functor. $\qquad\square$

We now have a diagram of Quillen adjunctions

$$
(18) \qquad
\begin{array}{ccc}
\mathbf{sSet}^{\Omega^{\mathrm{op}}} & \overset{\iota_!}{\underset{\iota^*}{\rightleftarrows}} & \mathbf{sSet}^{\Xi^{\mathrm{op}}} \\
{\scriptstyle \mathscr{R}}\Big\updownarrow{\scriptstyle \mathscr{I}} & & {\scriptstyle \mathscr{R}}\Big\updownarrow{\scriptstyle \mathscr{I}} \\
\mathbf{sSet}_*^{\Omega^{\mathrm{op}}} & \overset{\iota_!}{\underset{\iota^*}{\rightleftarrows}} & \mathbf{sSet}_*^{\Xi^{\mathrm{op}}}
\end{array}
$$

and we wish to localize each of these model structures.

8.1 Localizations

Roughly speaking, a left localization of a model category $\mathcal{M}$ at a set of maps C is a left Quillen functor from $\mathcal{M}$ which is initial among all left Quillen functors which take elements of C to weak equivalences. Recall from [24, 3.1.4] that an object W of $\mathcal{M}$ is called C–local if W is fibrant and, for each $f\colon A \to B$ which is an element of C, the map $\mathrm{map}^h(B, W) \to \mathrm{map}^h(A, W)$ is a weak equivalence of simplicial sets. A map $g\colon X \to Y$ is called a C–local equivalence if $\mathrm{map}^h(Y, W) \to \mathrm{map}^h(X, W)$ is a weak equivalence for every C–local W. The left Bousfield localization of $\mathcal{M}$ with respect to C [24, 3.3.1], denoted by $\mathcal{L}_C \mathcal{M}$, is then a model structure (which may or may not exist) on $\mathcal{M}$ with weak equivalences the C–local equivalences and cofibrations the ordinary cofibrations in $\mathcal{M}$. The fibrant objects in this model structure (if it exists) are precisely the C–local objects, and the identity functor $\mathcal{M} \to \mathcal{L}_C \mathcal{M}$ is a left localization of $\mathcal{M}$.

In order to show that the diagram (18) remains a diagram of Quillen adjunctions after localization, we will apply the following lemma several times:

Lemma 8.3 *Let* $L\colon \mathcal{M} \rightleftarrows \mathcal{N} \colon R$ *be a Quillen adjunction. Suppose that* $C \subseteq \mathcal{M}$ *and* $D \subseteq \mathcal{N}$ *are sets of maps with the domain and codomain of each element of* C *cofibrant. Suppose further that the left Bousfield localizations* $\mathcal{L}_C \mathcal{M}$ *and* $\mathcal{L}_D \mathcal{N}$ *exist. If, for each* $c \in C$*, the map* $L(c) \in \mathcal{N}$ *is isomorphic to some* $d \in D$*, then*

$$L\colon \mathcal{L}_C \mathcal{M} \rightleftarrows \mathcal{L}_D \mathcal{N} \colon R$$

is a Quillen adjunction.

Proof   There is a left Quillen functor

$$F\colon \mathcal{M} \to \mathcal{N} \to \mathcal{L}_D \mathcal{N}.$$

If $c \in C$, then c is a cofibrant approximation to itself. Further, $F(c) \cong d \in D$ is a weak equivalence, hence F takes any cofibrant approximation of c to a weak equivalence by [24, 8.1.24(1)]. By [24, 3.3.18(1)], the functor F is then a left Quillen functor when regarded as a functor $\mathcal{L}_C \mathcal{M} \to \mathcal{L}_D \mathcal{N}$. $\square$

Let $\mathbb{R}$ be either Ω or Ξ. Define $\mathscr{S}_{\mathbb{R}}$ to be the set of Segal core inclusions

$$\mathscr{S}_{\mathbb{R}} = \{\mathrm{Sc}[r] \to \mathbb{R}[r]\}_{r \neq \eta}.$$

Remark 8.4   According to Section 7.2, $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ and $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ are *simplicial model categories*. Thus, if A is cofibrant and X is fibrant, we may use the simplicial mapping space $\mathrm{map}(A, X)$ as a model for the homotopy mapping space $\mathrm{map}^h(A, X)$ (by, for example, [24, Example 17.2.4]). Since $\mathrm{Sc}[r]$, $\mathbb{R}[r]$, $\mathscr{R}(\mathrm{Sc}[r])$ and $\mathscr{R}(\mathbb{R}[r])$ are all cofibrant,[7] it suffices to work with map rather than map^h when discussing $\mathscr{S}_{\mathbb{R}}$ or $\mathscr{R}(\mathscr{S}_{\mathbb{R}})$ locality.

Since $\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ is left proper and cellular by Theorem 7.2 and Proposition 7.5, the left Bousfield localization

$$\mathcal{L}_{\mathscr{S}_{\mathbb{R}}} \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$$

exists by [24, 4.1.1]. Since $\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}$ is left proper and cellular by Proposition 7.18, we can take the left Bousfield localization with respect to the set of maps $\mathscr{R}(\mathscr{S}_{\mathbb{R}})$. We call the resulting model structure

$$\mathcal{L}_{\mathscr{R}(\mathscr{S}_{\mathbb{R}})} \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}}.$$

---

[7]To establish cofibrancy of $\Xi[S]$, simply pick a rooted tree with $\iota T = S$. Then, since $\mathrm{Sc}[T]$ is cofibrant [13, Corollary 1.7], so is $\iota_!(\mathrm{Sc}[T]) \cong \mathrm{Sc}[\iota T] \cong \mathrm{Sc}[S]$ by Lemma 6.1 and Proposition 8.1.

For $\mathbb{R} = \Omega$, these two model structures appear in [14, Definition 5.4; 9, Proposition 4.3], respectively.

Proposition 8.5 *Let $\mathbb{R}$ be either Ω or Ξ. Then the Quillen adjunction*

$$\mathscr{R}\colon \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}} \rightleftarrows \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}} \colon \mathscr{I}$$

induces a Quillen adjunction

$$\mathscr{R}\colon \mathcal{L}_{\mathscr{S}_\mathbb{R}}\mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}} \rightleftarrows \mathcal{L}_{\mathscr{R}(\mathscr{S}_\mathbb{R})}\mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}} \colon \mathscr{I}$$

after taking left Bousfield localization.

Proof If $r \neq \eta$ is an object of $\mathbb{R}$, then both $\mathrm{Sc}[r]$ and $\mathbb{R}[r]$ are cofibrant. Thus, Lemma 8.3 applies. □

The following is a variation on Lemma 6.1:

Proposition 8.6 *If $T \in \Omega$ is a rooted tree, then*

$$\iota_!(\mathscr{R}(\mathrm{Sc}[T]) \to \mathscr{R}(\Omega[T])) \cong (\mathscr{R}(\mathrm{Sc}[\iota T]) \to \mathscr{R}(\Xi[\iota T])).$$

Proof We have

$$\iota_! \mathscr{R}(\mathrm{Sc}[T] \to \Omega[T]) = \mathscr{R}\iota_!(\mathrm{Sc}[T] \to \Omega[T]) \quad \text{(by (18))}$$
$$\cong \mathscr{R}(\mathrm{Sc}[\iota T] \to \Xi[\iota T]) \quad \text{(Lemma 6.1).} \qquad \square$$

Proposition 8.7 *The diagram (18) gives a diagram*

$$
\begin{array}{ccc}
\mathcal{L}_{\mathscr{S}_\Omega}\mathbf{sSet}^{\Omega^{\mathrm{op}}} & \underset{\iota^*}{\overset{\iota_!}{\rightleftarrows}} & \mathcal{L}_{\mathscr{S}_\Xi}\mathbf{sSet}^{\Xi^{\mathrm{op}}} \\
\mathscr{R}\Big\updownarrow\mathscr{I} & & \mathscr{R}\Big\updownarrow\mathscr{I} \\
\mathcal{L}_{\mathscr{R}(\mathscr{S}_\Omega)}\mathbf{sSet}_*^{\Omega^{\mathrm{op}}} & \underset{\iota^*}{\overset{\iota_!}{\rightleftarrows}} & \mathcal{L}_{\mathscr{R}(\mathscr{S}_\Xi)}\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}
\end{array}
$$

of Quillen adjunctions after localizing.

Proof In light of Proposition 8.5, we only need to show that the horizontal adjunctions are Quillen adjunctions. The objects $\mathrm{Sc}[T]$ and $\Omega[T]$ are cofibrant in $\mathbf{sSet}^{\Omega^{\mathrm{op}}}$, so the top adjunction is a Quillen adjunction by Lemmas 8.3 and 6.1. Since $\mathscr{R}$ is a left Quillen functor, $\mathscr{R}(\mathrm{Sc}[T])$ and $\mathscr{R}(\Omega[T])$ are cofibrant in $\mathbf{sSet}_*^{\Omega^{\mathrm{op}}}$. Thus, the bottom adjunction is a Quillen adjunction by Lemma 8.3 and Proposition 8.6. □

Definition 8.8 A *Segal cyclic operad* is a fibrant object in the model category

$$\mathcal{L}_{\mathcal{R}(\mathcal{S}_{\Xi})}\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}.$$

Notice that every Segal cyclic operad has an underlying Segal operad via the functor ι^*.

The following example gives two cyclic operad structures on the same underlying operad. In fact, this hints at a whole class of examples: if A is an abelian group, then cyclic structures on the operad given by A are in bijection with the order 1 and 2 elements of $\mathrm{Aut}(A)$.

Example 8.9 Let M be the group $\mathbb{Z}/2 \times \mathbb{Z}/2 = \{0, 1\} \times \{0, 1\}$, whose elements are written as $00, 01, 10, 11$. Then M determines a (monochrome) operad O with

$$O(n) = \begin{cases} \{00, 01, 10, 11\} & \text{if } n = 1, \\ \varnothing & \text{if } n \neq 1, \end{cases}$$

with the operadic multiplication given by the group operation.

The operad O admits distinct cyclic structures C and C'. For the first, the action of $\Sigma_1^+ = \Sigma_2$ interchanges 01 and 10 and fixes 00 and 11, while in the second, the action fixes every element. These two structures are not isomorphic as cyclic operads because the Σ_2–sets $C(1) = \Sigma_2 \amalg * \amalg *$ and $C'(1) = * \amalg * \amalg * \amalg *$ are not isomorphic.

Proposition 8.10 *The Quillen adjunction*

$$\iota_!\colon \mathcal{L}_{\mathcal{R}(\mathcal{S}_{\Omega})}\mathbf{sSet}_*^{\Omega^{\mathrm{op}}} \rightleftarrows \mathcal{L}_{\mathcal{R}(\mathcal{S}_{\Xi})}\mathbf{sSet}_*^{\Xi^{\mathrm{op}}} \colon \iota^*$$

does not induce an equivalence of homotopy categories. In particular, this adjunction is not a Quillen equivalence.

Proof Consider the two cyclic operads C and C' in **Set** from Example 8.9; recall from that example that $UC = UC'$. Let $X = N_c(C)$ and $X' = N_c(C')$, and note that $\iota^* X = \iota^* X'$. Additionally, let A be the operad with

$$A(n) = \begin{cases} \mathbb{Z}/2 = \{e, x \mid x^2 = e\} & \text{if } n = 1, \\ \varnothing & \text{if } n \neq 1. \end{cases}$$

The operad A admits a unique cyclic operad structure where the Σ_2 action on $A(1)$ fixes x. There are exactly two maps of cyclic operads $A \to C$, while there are four maps $A \to C'$; we will show that this remains true once we pass to the homotopy category of $\mathcal{L}_{\mathcal{R}(\mathcal{S}_{\Xi})}\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$. Write $W = N_c(A)$, and note that $W_S = \varnothing$ if S is nonlinear, while

W_{L_m} is the set of words of length m in the alphabet e, x. The objects W, X and X' are fibrant in $\mathcal{L}_{\mathcal{R}(\mathscr{S}_\Xi)}\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$ by Theorems 7.16 and 6.7(iii), and the fact that all maps between discrete simplicial sets are Kan fibrations.

Let $E\Sigma_2 \to \Delta[0]$ be a cofibrant resolution of the terminal object of $\mathbf{sSet}^{\Sigma_2}$; there is a cofibrant resolution of W in the unlocalized model structure $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$ that is isomorphic levelwise to the tensor product $W \otimes E\Sigma_2$ from Definition 7.21. The object $W \otimes E\Sigma_2$ does not take into account the Σ_2–structure on $E\Sigma_2$, so we must modify the presheaf structure a bit. As this seems interesting in its own right, we discuss for $K \in \mathbf{sSet}^{\Sigma_2}$ tensoring $(- \circledast K)$ and cotensoring $(K \pitchfork -)$ in detail in the appendix. We now show that a cofibrant resolution of W is given by $\widetilde{W} = W \circledast E\Sigma_2 \to W$. We have $(W \circledast E\Sigma_2)_S = (W \otimes E\Sigma_2)_S$ for every tree S. To see that $\widetilde{W} \simeq W$, notice that at L_m we have

$$\widetilde{W}_{L_m} = ((W_{L_m} \setminus \{e^{\times m}\}) \times E\Sigma_2)_+ \simeq ((W_{L_m} \setminus \{e^{\times m}\}))_+ = W_{L_m}$$

and $\widetilde{W}_S = \varnothing$ if S is nonlinear.

To see that $\widetilde{W}$ is cofibrant, notice that W and $\widetilde{W}$ admit a filtration with $W^{(k)}$ consisting of those words which have x appearing at most k times and $\widetilde{W}^{(k)} = W^{(k)} \circledast E\Sigma_2$. Then $\widetilde{W}^{(k)}$ is the pushout in $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$,

$$
\begin{array}{ccc}
\mathscr{R}(\partial\Xi[L_k]) \circledast E\Sigma_2 & \longrightarrow & \widetilde{W}^{(k-1)} \\
\downarrow & & \downarrow \\
\mathscr{R}(\Xi[L_k]) \circledast E\Sigma_2 & \longrightarrow & \widetilde{W}^{(k)}
\end{array}
$$

By Propositions A.1 and A.2, $\widetilde{W}^{(k-1)} \to \widetilde{W}^{(k)}$ is a cofibration in $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$. Since $\widetilde{W}^{(0)} = W^{(0)} = \Xi[\eta]$ is the initial object of this category, it follows that $\widetilde{W} = \operatorname{colim} \widetilde{W}^{(k)}$ is cofibrant.

We now have

$$\hom(\widetilde{W}, X) = \hom(N_c A \circledast E\Sigma_2, N_c C) = \hom(N_c A, E\Sigma_2 \pitchfork N_c C);$$

but $N_c C$ is levelwise discrete, so $E\Sigma_2 \pitchfork N_c C \cong N_c C$. Using Theorem 6.7, we see

$$\hom(\widetilde{W}, X) = \hom(N_c A, N_c C) = \hom(A, C)$$

and similarly $\hom(\widetilde{W}, X') = \hom(A, C')$. But these sets are especially easy to understand. Maps of cyclic operads from A to any other cyclic operad are determined by

where we send x, and we have

$$\mathrm{hom}(\widetilde{W}, X) = \mathrm{hom}(A, C) = \{f_{00}, f_{11}\},$$
$$\mathrm{hom}(\widetilde{W}, X') = \mathrm{hom}(A, C') = \{f_{00}, f_{01}, f_{10}, f_{11}\},$$

where $f_a(x) = a$.

Since $\widetilde{W}$ is cofibrant in the unlocalized model structure and the objects X and X' are fibrant in the localized model structure, we can compute the homotopy classes of maps from $\widetilde{W}$ to X (or X') in either the unlocalized or localized model structure and will get the same answer by [24, 3.5.2]. We now work in the unlocalized model structure $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$, which is a simplicial model category. Since our objects are levelwise discrete, $\widetilde{W}$ is cofibrant, and X is fibrant, we have

$$\mathrm{hom}(\widetilde{W}, X) = \pi_0 \, \mathrm{hom}(\widetilde{W}, X) = \pi_0 \, \mathrm{map}(\widetilde{W}, X) = \pi_0 \, \mathrm{map}^h(\widetilde{W}, X) = \pi(\widetilde{W}, X)$$

by Remark 8.4 and Propositions 9.5.3 and 9.5.24 of [24] (see [24, Section 7.5] for π–notation). Similarly, $\mathrm{hom}(\widetilde{W}, X') = \pi(\widetilde{W}, X')$. We thus have $|\pi(\widetilde{W}, X')| = 4 > 2 = |\pi(\widetilde{W}, X)|$, which shows that X and X' are not isomorphic in the homotopy category of $\mathcal{L}_{\mathscr{R}(\mathscr{S}_\Xi)}\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$. □

There is a model structure on the category of (monochrome) simplicial cyclic operads where the weak equivalences and fibrations are defined as those maps which are level-wise weak equivalences. This follows by considering either the multisorted algebraic theory of cyclic operads or the colored operad controlling cyclic operads (for the latter, see [32, Section 1.6.4]), and then applying [6, Theorem 4.7] or [4, Theorem 2.1].

Conjecture 8.11 The model structure for simplicial cyclic operads is Quillen equivalent to $\mathcal{L}_{\mathscr{R}(\mathscr{S}_\Xi)}\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$.

Analogous results for simplicial monoids and for simplicial operads appear in [7] and [9], respectively.

Appendix   Tensoring and cotensoring with Σ_2 simplicial sets

As inspiration for this section, recall the cartesian closed structure on the category $\mathbf{Set}^{\Sigma_2}$ of involutive sets. The cartesian product $X \times Y$ has the diagonal Σ_2 action, and the internal hom is just the set of ordinary functions, together with the conjugation Σ_2 action $(\sigma \cdot f)(x) = \sigma \cdot f(\sigma \cdot x)$. The fixed points of the action on the internal hom are precisely the Σ_2–equivariant functions $X \to Y$.

Let ∇ be the full subcategory of Ξ spanned by the objects L_n (Example 1.4), so that ∇ is equivalent to $\Xi \downarrow \eta$. If $n > 0$, write $\vartheta \colon L_n \to L_n$ for the unique nonoriented isomorphism, and if $n = 0$, write $\vartheta = \mathrm{id}_{L_0}$. Each morphism in ∇ is of the form

$$(19) \qquad\qquad L_n \xrightarrow{\vartheta^i} L_n \xrightarrow{\gamma} L_m,$$

where γ is oriented and $i \in \{0, 1\}$. Moreover, the data (i, γ) for the decomposition (19) of a map is unique if and only if that map is not constant (otherwise there are two such decompositions; see Corollary 2.17). Left Kan extension along $\nabla \hookrightarrow \Xi$ gives a functor $\mathbf{sSet}_*^{\nabla^{\mathrm{op}}} \to \mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$ which allows us to identify the former category as being equivalent to the full subcategory consisting of those $X \in \mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$ such that $X_R = \varnothing$ whenever R is nonlinear. Notice that if $X \in \mathbf{sSet}_*^{\nabla^{\mathrm{op}}}$, then X_{L_n} may be considered as an object of $\mathbf{sSet}^{\Sigma_2}$ using the action of ϑ^*.

The category $\mathbf{sSet}^{\nabla^{\mathrm{op}}}$ is tensored and cotensored over $\mathbf{sSet}^{\Sigma_2}$. For our purposes we are interested only in the reduced case, so we will reserve notation for that. We have, for $X \in \mathbf{sSet}_*^{\nabla^{\mathrm{op}}}$ and $K \in \mathbf{sSet}^{\Sigma_2}$, that $X \circledast K$ is given on objects as the pushout

$$
\begin{array}{ccc}
K = X_{L_0} \times K & \longrightarrow & X_{L_n} \times K \\
\downarrow & & \downarrow \\
\Delta[0] & \longrightarrow & (X \circledast K)_{L_n}
\end{array}
$$

in $\mathbf{sSet}^{\Sigma_2}$, where the top map is induced from the unique morphism $f_n \colon L_n \to L_0$. The presheaf structure is given for a morphism (19) by the induced maps on pushouts coming from the commutative diagram

$$
\begin{array}{ccccc}
\Delta[0] & \longleftarrow & X_{L_0} \times K & \xrightarrow{f_m^* \times \mathrm{id}} & X_{L_m} \times K \\
\downarrow = & & \downarrow \mathrm{id} \times \sigma^i & & \downarrow (\gamma \vartheta^i)^* \times \sigma^i \\
\Delta[0] & \longleftarrow & X_{L_0} \times K & \xrightarrow{f_n^* \times \mathrm{id}} & X_{L_n} \times K
\end{array}
$$

Note that this is well defined on constant maps, though the decomposition (19) need not be unique in this case.

We write $K \pitchfork -$ for the right adjoint to $- \circledast K$. If $Y \in \mathbf{sSet}_*^{\nabla^{\mathrm{op}}}$ then

$$(K \pitchfork Y)_{L_n} = \mathrm{map}_{\mathbf{sSet}}(K, Y_{L_n});$$

in particular, if $n = 0$, this is just $\Delta[0]$. Given a morphism (19), the map $(K \pitchfork Y)_{L_m} \to (K \pitchfork Y)_{L_n}$ is given by

$$\mathrm{map}(\sigma^i, (\gamma \vartheta^i)^*) \colon \mathrm{map}(K, Y_{L_m}) \to \mathrm{map}(K, Y_{L_n}).$$

As simplicial sets, we have (natural in X, Y)

$$(X \circledast K)_{L_n} = (X \otimes K)_{L_n}, \quad (K \pitchfork Y)_{L_n} = (Y^K)_{L_n},$$

where the objects on the right are those from Definition 7.21 after forgetting the Σ_2 action on K. Note that if $\gamma \colon L_n \to L_m$ is an *oriented* map, then the above definitions also give equalities of the maps γ^*,

$$(X \circledast K)_\gamma = (X \otimes K)_\gamma, \quad (K \pitchfork Y)_\gamma = (Y^K)_\gamma.$$

Where these functors differ is exactly on the ϑ^*.

Proposition A.1 *If $K \in \mathbf{sSet}^{\Sigma_2}$, then the functor $- \circledast K \colon \mathbf{sSet}_*^{\nabla^{\mathrm{op}}} \to \mathbf{sSet}_*^{\nabla^{\mathrm{op}}}$ preserves Reedy cofibrations.*

Proof To prove this statement, it is of course equivalent to prove that the right adjoint $K \pitchfork -$ preserves acyclic fibrations. Notice by (M7′) and the fact that every simplicial set is cofibrant, we have that $(-)^K$ preserves (acyclic) fibrations. Since $(-)^K$ preserves acyclic fibrations, $((-)^K)_{L_m} \cong (K \pitchfork -)_{L_m}$ for all $m \geq 0$, and Reedy weak equivalences are levelwise, we see that $K \pitchfork -$ sends acyclic fibrations to weak equivalences.

We will conclude by showing that $K \pitchfork -$ preserves Reedy fibrations. Recall the definition of fibration from Theorem 7.1. The key point to check is that we have an isomorphism of matching objects

$$M_{L_m}(K \pitchfork Y) \cong M_{L_m}(Y^K)$$

for every $m \geq 0$. For an arbitrary presheaf Z, we have

$$M_{L_m} Z = \lim_{\substack{(\nabla^{\mathrm{op}})^-(L_m) \\ \alpha \colon L_m \to L_n}} Z_{L_n} = \lim_{\substack{\nabla^+(L_m) \\ \beta \colon L_n \to L_m}} Z_{L_n}.$$

Let $\Delta^+(m) \subseteq \nabla^+(L_m)$ be the full subcategory whose objects are the *oriented* maps $L_n \to L_m$ in $\nabla^+ \subseteq \Xi^+$. Note that if we have a morphism

$$
\begin{array}{ccc}
L_n & \xrightarrow{\ \phi\ } & L_p \\
 & \gamma \searrow \ \swarrow \gamma' & \\
 & L_m &
\end{array}
$$

with $\gamma, \gamma' \in \Delta^+(m)$, then ϕ is an oriented map. Thus, every morphism in $\Delta^+(m)$ is an oriented map as well, which explains the choice of notation. (This argument is much like one appearing in the proof of Lemma 6.2.) Further, we have that $\Delta^+(m) \subseteq \nabla^+(L_m)$

is essentially surjective: a nonoriented map $\gamma\vartheta$ in $\nabla^+(L_m)$ is isomorphic to γ. Thus, the inclusion map is an equivalence of categories.

It follows that

$$M_{L_m} Z \cong \lim_{\substack{\Delta^+(m) \\ \beta \colon L_n \to L_m}} Z_{L_n}.$$

Since $K \pitchfork Y$ and Y^K are equal when applied to oriented maps, it follows that we have, for each K, an isomorphism $M_{L_m}(K \pitchfork Y) \cong M_{L_m}(Y^K)$, natural in Y. In fact, we have

$$((K \pitchfork Y)_{L_m} \to M_{L_m}(K \pitchfork Y)) \cong ((Y^K)_{L_m} \to M_{L_m}(Y^K)),$$

so since $(-)^K$ preserves fibrations, $K \pitchfork -$ also preserves fibrations. $\qquad\square$

Proposition A.2 *Suppose that* $\mathbf{sSet}_*^{\nabla^{\mathrm{op}}}$ *and* $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$ *are endowed with the model structures from Theorem 7.16. Consider the adjunction*

$$\mathbf{sSet}_*^{\nabla^{\mathrm{op}}} \rightleftarrows \mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$$

induced from the full-subcategory inclusion $\nabla \to \Xi$. *Then the left adjoint preserves and detects cofibrations.*

Proof We identify $\mathbf{sSet}_*^{\nabla^{\mathrm{op}}}$ as the full subcategory whose objects A satisfy $A_R = \varnothing$ if R is nonlinear. By Theorem 7.16, Proposition 7.17 and $\mathscr{R}\mathscr{I} \cong \mathrm{id}$, the functor $\mathscr{I} \colon \mathbf{sSet}_*^{\mathbb{R}^{\mathrm{op}}} \to \mathbf{sSet}^{\mathbb{R}^{\mathrm{op}}}$ creates cofibrations (where $\mathbb{R} = \nabla, \Xi$). Thus, it is enough to work with Reedy cofibrations in the unreduced categories.

Consider the indexing category for the latching object of A. If R is any object of Ξ, then $(\Xi^{\mathrm{op}})^+(R) = \Xi^-(R)$ has objects of the form $R \to S$ in $\Xi^- \setminus \mathrm{Iso}(\Xi)$. Given a map $R \to S$, if S is linear, so is R. It follows that

$$L_R^\Xi(A) = \operatorname*{colim}_{\substack{\Xi^-(R) \\ R \to S}} A_S = \varnothing$$

when R is a nonlinear tree. If $R = L_m$, then

$$L_{L_m}^\Xi(A) = \operatorname*{colim}_{\substack{\Xi^-(L_m) \\ L_m \to S}} A_S = \operatorname*{colim}_{\substack{\nabla^-(L_m) \\ L_m \to L_n}} A_{L_n} = L_{L_m}^\nabla(A)$$

since $A_S = \varnothing$ when S is not linear. Thus, a map $A \to B$ in $\mathbf{sSet}_*^{\nabla^{\mathrm{op}}}$ is a cofibration in $\mathbf{sSet}_*^{\nabla^{\mathrm{op}}}$ if and only if it is a cofibration in $\mathbf{sSet}_*^{\Xi^{\mathrm{op}}}$. $\qquad\square$

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